Simple Semi-Analytical Expression of the Lightning Base Current in the Frequency-Domain

D. Assante* and C. Cesarano

Faculty of Engineering, International Telematic University UNINETTUNO
Corso Vittorio Emanuele II n° 39, Rome, Italy,

___

Abstract

A simple procedure to express the lightning base current in the frequency domain is presented. The formula, based on a piecewise approximation of the lightning base current in time domain, allows obtaining an expression in the frequency domain in terms of elementary functions. The presented procedure is fast and general, since it can be used with any current waveshape.

Keywords: Lightning Base Current, Heidler model, piecewise functions

___

1. Introduction

The proper modeling of the lightning current is a key aspect for the analysis of all the effects produced by the lighting phenomenon itself. The computation of the radiated electromagnetic field and of the voltages and currents induced on transmission lines is strongly affected by the lighting channel modeling.

The topic has been widely discussed in literature [1] and nowadays the most accredited model is the so-called *engineering model*, defined by means of the expression:

$$i(z,t) = i(0,t\text{-}z/v)P(z) \ , \tag{1}$$

describing the current at any height z along the lightning channel, where $i(0,t)$ is the channel base current, v represents the return stroke velocity ($\sim 2c/3$), and $P(z)$ is a height-dependent attenuation function [2].

Several models have been adopted in literature to describe the attenuation function $P(z)$, and several studies have been performed to validate the proposed models [3]. More recently, inverse procedures have also been proposed in literature in order to identify the attenuation function starting from the measured electromagnetic field [4-6]. However, the debate is still open in literature.

A more uniform opinion is found in literature about the lightning base current, thanks to the several measurements performed on natural [7] and artificial lightning [8] at the hit points. Nowadays the most accredited and widely used model to describe the lightning base current is the one proposed by Heidler [9] according to the expression

$$i(t) = \frac{I_{01}}{\eta_1}\frac{(t/\tau_{11})^{n_1}}{1+(t/\tau_{11})^{n_1}}e^{-t/\tau_{12}} + \frac{I_{02}}{\eta_2}\frac{(t/\tau_{21})^{n_2}}{1+(t/\tau_{21})^{n_2}}e^{-t/\tau_{22}}, \tag{2}$$

The parameters appearing in (2) depend by the specific

lighting characteristics and their proper choice allows to fit very well any measured lightning current.

However, a problem occurs in frequency domain, since the Fourier transform of the Heidler current can't be expressed in terms of elementary functions [10]. This is a limitation, since in several cases it may be easier to perform calculations in frequency domain [11]. Also, in the frequency domain it is easier to take into account finite conductivity ground [12] or even multilayered structures.

This problem is usually overcome by performing a FFT of (2). However, this solution too has some disadvantages. Since the lightning base current is usually an input quantity used for further computations and the FFT returns a purely numerical result, this limits the possibility to carry out analytical calculations. Then, it has to be considered that the lightning base current typically has an initial rising time of few μs and a descending part of hundreds of μs. So, in order to properly describe the lightning base current, since the FFT requires a linear sampling of the signal, a high number of samples is required.

In order to overcome this problem, other lightning base current expressions have been proposed in literature, that can be analytically expressed in the frequency domain too [13-14]. However, they are not so popular in literature and the Heidler model is currently the most used and accepted one.

So, in order to find a practical way to express the Heidler base current in the frequency domain, an alternative approach is proposed. Recently, a simple procedure has been presented in order to represent the lightning base current in terms of piecewise functions in time-domain [15]. In this paper we show how to extend the procedure to find a simple representation of the lightning base current in the frequency-domain as sum of elementary functions. The method is tested on the Heidler base current, but can be applied to any lightning current waveform.

The paper is divided in two parts: the procedure to represent the lighting base current in time-domain is recalled at first, and the efficiency of this method is evaluated. Then, the expression in the frequency-domain is obtained and

___

2. Piecewise Representation

Let us consider a function *f(t)* continuous in a given domain. The method could be also adopted for discontinuous functions with some small complications; however it is not necessary since the lightning base current can be assumed as a continuous function.

It is possible to divide the entire time domain of interest in *N* non-uniform intervals, over each of which the function *f(t)* is represented by means of a polynomial functions. It is worth to take into consideration the use for fitting linear and quadratic piecewise functions. Higher order piecewise functions could be considered too, but the advantages of using them would be minimal in terms of quality of the fitting or reduction of the fitting intervals. Constant piecewise functions offer a poor fitting.

The quadratic piecewise functions allow to obtain a more accurate fitting. In addition, the use of quadratic piecewise functions allows to have in the first interval a representation whose derivative can be zero for *t* = 0. This is a characteristic of the lightning base current, appearing also in (2), that can't be achieved by means of linear functions, unless a null function is considered in the first interval.

2.1 Polynomial approximation

The function *f(t)* can be approximated as $\widetilde{f}$ by means of quadratic piecewise functions according to the following expression

$$\widetilde{f}(t) = \sum_{n=1}^{N} \left(u_n + v_n t + w_n t^2 \right) \left[U(t - t_n) - U(t - t_{n+1}) \right], \quad (3)$$

where t_n is the starting point of the *n*-th interval and $U(t - t_n)$ is the unit step function shifted by t_n. In each interval these coefficients can be computed as

$$\begin{cases} u_n = f_n - v_n t_n - w_n t_n^2 \,, \\[2mm] v_n = \dfrac{f_{n+1} - f_n}{t_{n+1} - t_n} - w_n \left(t_{n+1} + t_n \right), \\[2mm] w_n = 2\dfrac{f_n + f_{n+1} - 2f_{n+1/2}}{\left(t_{n+1} - t_n \right)^2} \,, \end{cases} \quad (4)$$

where $f_n = f(t_n)$ and $f_{n+1/2} = f((t_n+t_{n+1})/2)$.

It is worth noting to observe that (3) can be rewritten in a more compact form as

$$\widetilde{f}(t) = \sum_{n=1}^{N+1} \left(a_n + b_n t + c_n t^2 \right) U(t - t_n), \quad (5)$$

where

$$\begin{cases} a_1 == u_1 \,, \ a_n = u_n - u_{n+1} \,, \ a_{N+1} = -u_N \,, \\[2mm] b_1 == v_1 \,, \ b_n = v_n - v_{n+1} \,, b_{N+1} = -v_N \,, \\[2mm] c_1 == w_1 \,, \ c_n = w_n - w_{n+1} \,, \ c_{N+1} = -w_N \,. \end{cases} \quad (6)$$

It is also useful to observe that. in order to use linear

piecewise functions to perform the fitting of the function *f(t)*, it is enough to cancel the coefficients w_n in (4) or c_n in (6). However, note that cancelling these coefficients, the values of u_n, v_n, a_n and b_n change.

2.2 Optimal computation of the sampling intervals

If we assume to fit the function *f(t)* with a representation such as (4–6), it is trivial that the accuracy of the fitting is better by increasing the number of intervals. However, the number of coefficients increases too, and so the computational effort to use the representation in further expressions.

In principle, it would be desirable to have a method in order to find the optimal initial points t_n that allows to have minimal set of intervals ensuring a given fitting accuracy. However it is not a simple task since the number and the widths of the intervals have to be optimized at the same time.

In order to have a quick estimation of the fitting error in each interval, it is possible to choose two check points

$$\begin{aligned} t_{n+1/4} &= t_n + 0.25\left(t_{n+1} - t_n \right) \\ t_{n+3/4} &= t_n + 0.75\left(t_{n+1} - t_n \right) \end{aligned}, \quad (7)$$

and define the relative fitting error as

$$err = \max\left\{ \frac{\left| f(t_{n+1/4}) - \widetilde{f}(t_{n+1/4}) \right|}{1 + \left| f(t_{n+1/4}) \right|}, \frac{\left| f(t_{n+3/4}) - \widetilde{f}(t_{n+3/4}) \right|}{1 + \left| f(t_{n+3/4}) \right|} \right\}. \quad (8)$$

This error estimation is suitable if piecewise quadratic or linear functions are used. It is efficient, since approaches to the relative error for big values of the function ($|f(t)| \gg 1$) and to the absolute error for small values of the function ($|f(t)| \ll 1$), this is important in our case.

In order to obtain the fitting with a required error, an efficient procedure (*method (a)*, hereafter) can be to start with a set of intervals, compute the fitting coefficients according to (4), evaluate the fitting error and then split each interval where the fitting error is higher than the whished threshold [16].

Another procedure (*method (b)*, hereafter) consists of adaptively building the intervals consecutively. At first, a trial interval is chosen and then it is reduced or enlarged, until the maximum interval width satisfying the requested error is found. Then, starting from the end of the first interval, the operation is repeated for a second one and so on, until all the interval of interest is covered. This second method should ensure to obtain the same quality of the fitting with a lower number of intervals with respect to the first method. However, its computational effort is affected by the parameters chosen to adaptively enlarge or contract the intervals.

Both these methods will be tested in a practical case and their performances will be discussed.

2.3 Numerical results

In order to prove the efficiency of piecewise approximation, we perform the fitting for a typical lightning first and subsequent return stroke [17]. The adopted parameters are shown in Table 1. The analysis is performed in order to discuss the two methods proposed in Section 2.2.

Table 1. Lightning current parameters for first and subsequent strokes

	I_{01} (kA)	n_1	τ_{11} (µs)	T_{21} (µs)	I_{02} (kA)	n_2	τ_{12} (µs)	τ_{22} (µs)
First	28	2	1.8	95	-	-	-	-
Subseq.	10.7	2	0.25	2.5	6.5	2	2.1	230

We consider the lightning base current computed in interval of 100 µs and 10 ms and we evaluate the required number of piecewise functions and the computational time, for different values of the approximation error. The results are shown in Tables 2 and 3, for a first and a subsequent stroke, respectively.

The comparison is performed considering two different time intervals because, considering the behavior of the lightning base current, an interval of 100 µs is enough in time domain, for instance for computing the electromagnetic field radiated by the lightning, while an interval of 10 ms is required at least if it is necessary to operate in the frequency domain.

Table 2. Fitting results for f**irst stroke**

t_{max}	err	Method (a)		Method(b)	
		N. coeff.	CPU time	N. coeff.	CPU time
0.1 ms	0.1	7	8 ms	4	7 ms
	0.01	13	16 ms	9	9 ms
10 ms	0.1	21	21 ms	19	8 ms
	0.01	31	27 ms	30	19 ms

Table 3. Fitting results for subsequent stroke

t_{max}	err	Method (a)		Method(b)	
		N. coeff.	CPU time	N. coeff.	CPU time
0.1 ms	0.1	9	11 ms	6	10 ms
	0.01	15	18 ms	10	14 ms
10 ms	0.1	23	23 ms	16	18 ms
	0.01	34	30 ms	28	22 ms

The interesting result emerging from simulations is that method (b) requires a smaller number of coefficients to perform the fitting. Although the number of coefficient is not so high for both methods, even a small difference may be useful when the lightning base current is used in further computations.

Also the computational time is shown in Tables 2 and 3, however both methods appear to be fast and so the computational effort is not a crucial aspect for these methods.

The results in Table 2 and 3 are comparable, so the methods are efficient for both kinds of lightning.

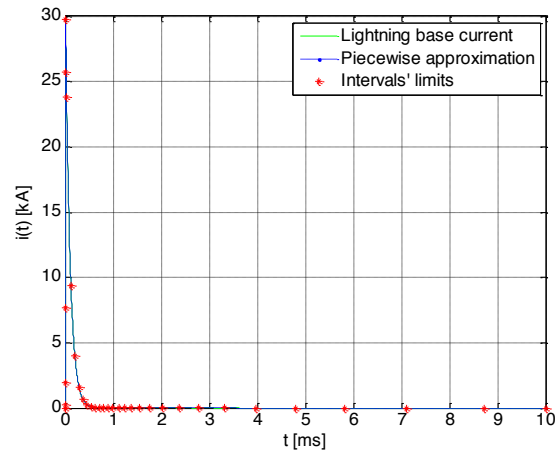Then, adopting method (b), in Fig. 1 and 2 we show the lightning base current and its piecewise representation.

Fig. 1 First stroke lightning base current in time domain and piecewise representation (err = 0.01).
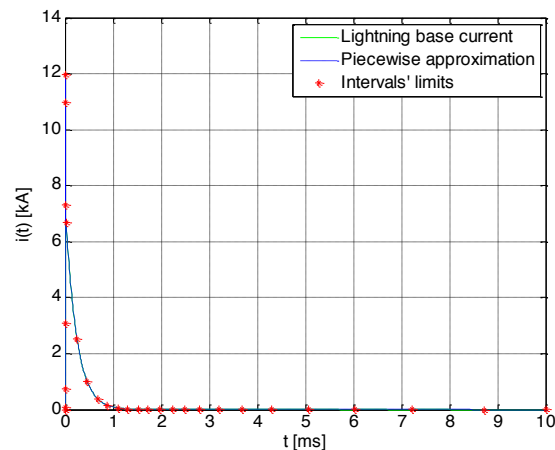

Fig. 2 Subsequent stroke lightning base current in time domain and piecewise representation (err = 0.01).

In Fig. 3 we also show a detail of Fig 2, in order to better appreciate the fitting.
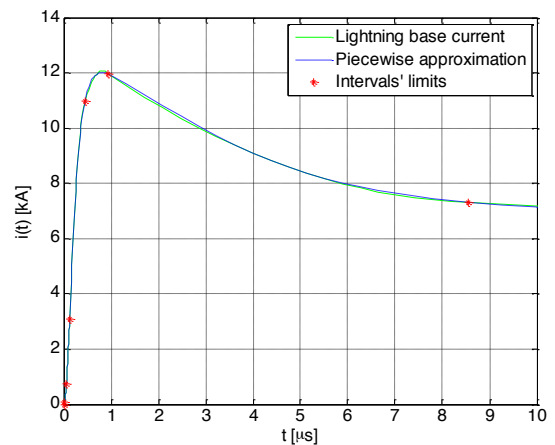

Fig. 3 Subsequent stroke lightning base current in time domain and piecewise representation (err = 0.01) – detail.

3. Lightning base current in the frequency-domain

Once the lightning base current is represented as sum of piecewise polynomial functions, that is to say once the

expansion coefficients are computed in (4–6), its expression in the frequency-domain can be easily found.

If quadratic piecewise functions are adopted, the lightning base current in the frequency domain can be represented as

$$I(\omega) = \sum_{n=1}^{N} \left(u_n \left[\frac{e^{-j\omega t_n} - e^{-j\omega t_{n+1}}}{j\omega} \right] + \right.$$
$$-v_n \left[\frac{(1+j\omega t_n)e^{-j\omega t_n} - (1+j\omega t_{n+1})e^{-j\omega t_{n+1}}}{\omega^2} \right] +$$
$$\left. -w_n \left[\frac{-(2+2j\omega t_n - \omega^2 t_n^2)e^{-j\omega t_n} + (2+2j\omega t_{n+1} - \omega^2 t_{n+1}^2)e^{-j\omega t_{n+1}}}{j\omega^3} \right] \right), \quad (9)$$

or, by means of the (6),

$$I(\omega) = \sum_{n=1}^{N+1} \frac{a_n \omega^2 - b_n \omega (j - \omega t_n) - c_n (2 + 2j\omega t_n - \omega^2 t_n^2)}{j\omega^3} e^{-j\omega t_n}. \quad (10)$$

Again, in case of linear piecewise function representation of the lightning base current, is it enough to cancel the terms multiplied by w_n in (9) and by c_n in (9), remembering that this operation affects the values of the coefficients of u_n, v_n, a_n and b_n.

The previous representation has the relevant advantage to express the lightning base current as sum of elementary functions. So this representation is powerful if the lightning base current has to be used in the frequency-domain for further analytical manipulations.

It is worth noting that, despite the representation (10) is more concise, a pole for $\omega = 0$ is introduced. It is a false pole, since the single terms of the sum diverge but the sum itself has a finite value. This aspect has to be accurately taken into account for the numerical implementation of the formula.

Instead, the equivalent representation (9) is less concise but has the advantage that each term of the sum approaches to finite value for ω going toward zero.

It is interesting and useful to compute the limit of $I(\omega)$ for ω going toward 0, that is

$$I(\omega) = \sum_{n=1}^{N} u_n (t_{n+1} - t_n) + \frac{v_n}{2}(t_{n+1}^2 - t_n^2) + \frac{w_n}{2}(t_{n+1}^3 - t_n^3) =$$
$$= \sum_{n=1}^{N} a_n (t_{N+1} - t_n) + \frac{v_n}{2}(t_{N+1}^2 - t_n^2) + \frac{w_n}{2}(t_{N+1}^3 - t_n^3). \quad (11)$$

This limit proves that the lightning base current representation assumes a finite value in the frequency domain, for $\omega = 0$. Also, physically (11) represents the integral of $i(0,t)$, so the total charge flowing at the base during the lightning discharge.

Finally, in Fig. 4 we show the lightning base current in the frequency domain obtained implementing the (9), computed with the first stroke parameters adopted in Table 1, and we compare the result with the frequency behavior obtained with the FFT. A very good agreement is found.
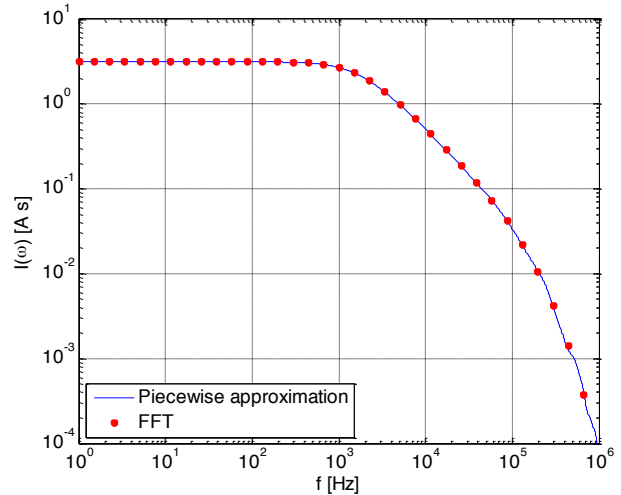

Fig. 4 First stroke lightning base current in frequency domain: comparison between piecewise representation and FFT.

The plot is obtained by representing the lightning base current in time domain with quadratic piecewise functions, using the fitting method (b) in an interval of 10 ms and requiring a fitting error of 0.01. Using these parameters, the result is acceptable in a frequency until 1 MHz, which anyway is the most useful interval by the practical point of view.

In order to better appreciate the effectiveness of the method, in Fig. 5 we also show the percentage error between the FFT and the piecewise representation.
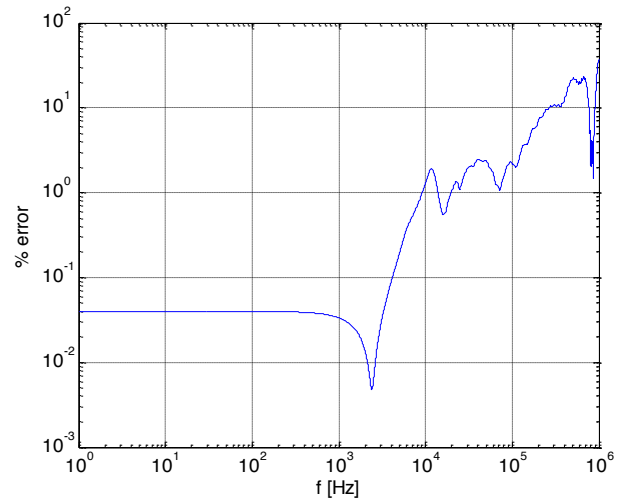

Fig. 5 First stroke lightning base current in frequency domain: percentage error

The same simulations are performed for a subsequent stroke lightning base current in Figs. 6 and 7, with similar results.
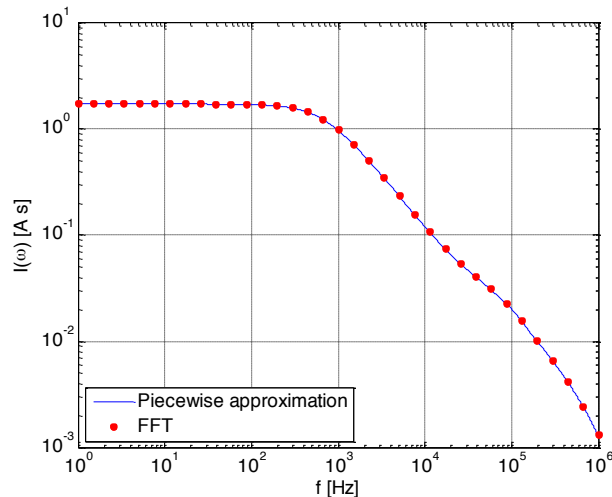
Fig. 6 Subsequent stroke lightning base current in frequency domain: comparison between piecewise representation and FFT.
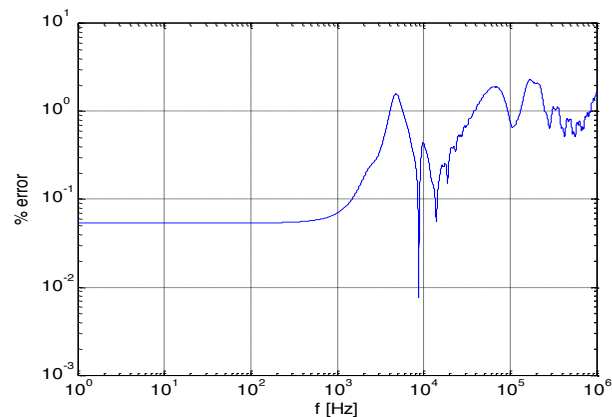

Fig. 7 Subsequent stroke lightning base current in frequency domain: percentage error

At very low frequency, a proper implementation is required in order to avoid round off errors in the exponentials. However this is just a problem in the computation of (9), the representation is formally correct and can be adopted for further computations.

At higher frequencies, the exponential functions in (9) can create numerical problems, it is necessary to reduce the approximation error in order get an accurate representation in a wider interval. It is found that reducing the approximation error by two orders of magnitude, the frequency representation is accurate in one more decade.

4. Conclusions

A simple procedure has been shown to represent the lightning base current in the frequency domain as a sum of elementary functions. The method has been successfully tested on a typical Heidler base current. However, it can be applied to any other current model with the same performance results.

The proposed method is fast and allows to choose the desired accuracy.

Then, differently from the FFT, the method doesn't impose a relationship between the time and frequency samplings. This is an advantage for the computational time.

References

1. Rakov, V.A. and Uman, M.A., "Review and evaluation of lightning return stroke models including some aspects of their application", *IEEE Trans. Electromagn. Compat.*, vol. 40, no. 4, pp. 403-426, Nov. 1998.
2. Izadi, M., Ab Kadir, M.Z.A., Gomes, C., Cooray, V. and Schoene, J., "Evaluation of lightning current and velocity profiles along lightning channel using measured magnetic flux density", *Progress in Electromagnetics Research*, Vol. 130, pp. 473-492, 2012.
3. Nucci, C.A, Diendorfer, G., Uman, M.A., Rachidi, F., Ianoz, M. and Mazzetti, C., "Lightning return stroke current models with specified channel-base current: A review and comparison", *J. Geophys. Res.*, vol. 95, no. D12, 1990.
4. Andreotti, A., Assante, D., Falco, S. and Verolino, L., "An improved procedure for the return stroke current identification", *IEEE Trans. on Mag.*, vol. 41, no. 5, 1872-1875, 2005.
5. Izadi, M., Ab Kadir, M.Z.A., Gomes, C. and Cooray, V., "Evaluation of lightning return stroke current using measured electromagnetic fields", *Progress In Electromagnetics Research*, Vol. 130, 581-600, 2012.
6. Ceclan, A., Topa, V., Micu, D. and Andreotti, A., "Lightning-Inverse Reconstruction by Remote Sensing and Numerical-Field Synthesis", *IEEE Trans. on Mag.*, vol. 49, no. 5, p. 1657, 2013.
7. Pavanello, D., Rubinstein, M. and Rachidi., F., "On the Determination of the Spatial-temporal Behavior of the Lightning Return Stroke Current by Multiple Field Measurements", *Progress In Electromagnetics Research Symposium*, Beijing, 2007.
8. Fisher, R.J., Schnetzer, G.H., Thottappillil, R., Rakov, V.A., Uman, M.A. and Goldberg, J.D. "Parameters of triggered-lightning flashes in Florida and Alabama", *Journal of Geophysical Research: Atmospheres*, pp. 22887-22902, Vol. 98, issue D12, 1993.
9. Heidler, F., "Travelling current source model for LEMP calculations", in *Proc. 6th Symposium Electromagn. Compat.*, pp. 5-7, Zurich, 1985.
10. Andreotti, A., Falco, S., and Verolino, L., "Some integrals involving Heidler's lightning return stroke current expression", *Electrical Engineering*, vol. 87, pp. 121-128, 2005.
11. Andreotti, A., Assante, D, Rakov, V.A. and Verolino, L., "Electromagnetic coupling of lightning to power lines: transmission-line approximation versus full-wave solution", *IEEE Trans. on Electromagn. Compat.*, vol. 53, no. 2, pp. 421-428, 2011.
12. Andreotti, A., Assante, D. and Verolino, L., "Fast and accurate evaluation of the underground lightning electromagnetic field", in *International Symposium Electromagn. Compat. - EMC Europe*, 2008.
13. Yazhou, C., Shanghe, L., Xiaorong, W., & Feizhou, Z., "A new kind of lightning channel-base current function", in 3rd International Symposium on Electromagnetic Compatibility, pp. 304-307, 2002.
14. Javor, V.; Rancic, P.D., "A Channel-Base Current Function for Lightning Return-Stroke Modeling", *IEEE Trans. on Electromagn. Compat.*, vol.53, no.1, pp.245 - 249, Feb. 2011
15. Zou, J., Li, M., Li, C., Lee, J.B. and Chang, S.H., "Fast Evaluation of the Horizontal Electric Field of Lightning With Cooray-

Rubinstein Formula in Time Domain Using a Piecewise Quadratic Convolution Technique", *IEEE Trans. on Electromagn. Compat.*, vol. 54, no. 5, pp. 1034-1041, 2012.

16. Lee, J.B., Zou, J., Li, M. and Chang, S.H., "Fast Evaluation of the lightning electromagnetic field in the time domain with an adaptive piecewise approximation of channel base current", *COMPEL*, vol. 32, no. 3, pp. 1030-1043, 2013.

17. Pavanello, D., Rachidi, F., Rubinstein, M., Bermudez, J.L. and Nucci, C.A., "Electromagnetic field radiated by lightning to tall towers: treatment of the discontinuity at the return stroke wave front", *Journal of Geophysical Research*, vol. 109, D06114, 2004.