AN ANALOG OF KARMARKAR'S ALGORITHM FOR INEQUALITY CONSTRAINED LINEAR PROGRAMS, WITH A "NEW" CLASS OF PROJECTIVE TRANSFORMATIONS FOR CENTERING A POLYTOPE

by

Robert M. Freund

Sloan W.P. No. 1921–87 August 1987

This research was done while author was a Visiting Scientist at Cornell University.

Research supported in part by ONR Contract N00014–87–K–0212.

Author's address:
Professor Robert M. Freund
M.I.T., Sloan School of Management
50 Memorial Drive
Cambridge, MA. 02139

ABSTRACT

We present an algorithm, analogous to Karmarkar's algorithm, for the linear programming problem: maximizing $c^T x$ subject to $Ax \leq b$, which works directly in the space of linear inequalities. The main new idea in this algorithm is the simple construction of a projective transformation of the feasible region that maps the current iterate $\bar{x}$ to the analytic center of the transformed feasible region.

Key words: Linear Program, Projective Transformation, Polytope, Center, Karmarkar's algorithm.

1. Introduction. Karmarkar's algorithm [4] is designed to work in the space of feasible solutions to the system $Ax = 0$, $e^T x = 1$, $x \geq 0$. Although any bounded system of linear inequalities $Ax \leq b$ can be linearly transformed into an instance of this particular system, it is more natural to work in the space of linear inequalities $Ax \leq b$ directly. From a researcher's point of view, the system $Ax \leq b$ is often easier to conceptualize and can be "seen" graphically, directly, if x is two- or three-dimensional, regardless of the number of constraints. Herein, we present an algorithm, analogous to Karmarkar's algorithm, that solves a linear program of the form: maximize $c^T x$, subject to $Ax \leq b$, under assumptions similar to those employed in Karmarkar's algorithm. Our algorithm performs a simple projective transformation of the feasible region that maps the current iterate $\bar{x}$ to the analytic center of the transformed feasible region.

2. Notation. Let e be the vector of ones, namely $e = (1,1,\ldots,1)^T$. If s is a vector in $\mathbb{R}^m$, then S refers to the diagonal matrix whose diagonal entries are the components of s, i.e.,

$$S = \begin{bmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_m \end{bmatrix}$$

If $\mathcal{X}$ is a subset of $\mathbb{R}^n$, then $\mathcal{X}^{++} = \{x \in \mathcal{X} \mid x > 0\}$.

3. <u>The Algorithm</u>. Our interest is in solving the linear program

$$P: \text{ maximize } c^T x = U$$
$$\text{subject to } Ax \leq b$$

where A is a matrix of size $m \times n$.

We assume that the feasible region $\mathcal{X} = \{x \in \mathbb{R}^n | Ax \leq b\}$ is bounded, has a nonempty interior, and that U, the optimal value of P, is known in advance. Furthermore, we assume that we have an initial point $x_0 \in \text{int } \mathcal{X}$ whose associated slack vector $s_0 = b - Ax_0$ satisfies:

$$e^T S_0^{-1} A = 0 \quad \text{and} \quad s_0 = e, \tag{1}$$

Condition (1) may appear to be restrictive, but we shall exhibit an elementary projective transformation that enables us to assume (1) holds, without any loss of generality. The first condition of (1) is simply the necessary and sufficient condition for x_0 to be the <u>center</u> of the system of linear inequalities $Ax \leq b$ (see, e.g., Sonnevend [5]), which is reviewed below. The second condition is that the rows of A have been scaled so that at x_0 the slack on each constraint is one. The algorithm is then stated as follows:

For $k = 0, 1, \ldots,$ do:

<u>Step k</u>: Define $s_k = b - Ax_k$, $v_k = U - c^T x_k$, and $\lambda_k = (1/m) S_k^{-1} e$, and $y_k = A^T \lambda_k$.

Set $A_k = S_k^{-1} A - e y_k^T$, $c_k = c - v_k y_k$, $b_k = S_k^{-1} b - e y_k^T x_k$, and $U_k = U - v_k y_k^T x_k$.

Define the search direction $d_k = \dfrac{(A_k^T A_k)^{-1} c_k}{\sqrt{c_k^T (A_k^T A_k)^{-1} c_k}}$

$$x_{k+1} = x_k + \frac{\alpha\, d_k}{1 + y^T(\alpha d_k)}$$, where $\alpha > 0$ is a steplength, e.g., $\alpha = 1/3$.

Note immediately that all of the effort lies in computing $(A_k^T A_k)^{-1} c_k$. Expanding, we see that

$$(A_k^T A_k) = [A^T S_k^{-2} A + y(-e^T S_k^{-1} A + e^T e y^T) - (A^T S_k^{-1} e)(y^T)]$$

is computed as two rank-1 updates of the matrix $A^T S_k^{-2} A$. Thus, as in Karmarkar's algorithm, the computational burden lies in solving $(A^T S_k^{-2} A)^{-1} c_k$ efficiently.

To measure the performance of the algorithm, we will use the potential function

$$F(x) = m\, \ell n\, (U - c^T x) - \sum_{i=1}^{m} \ell n\, (b - Ax)_i ,$$

which is defined for all x in the interior of the feasible region.

We will show below that this algorithm is an analog of Karmarkar's algorithm, by tracking how the algorithm performs in the slack space. Thus, let us rewrite P as

$$\begin{aligned} P:\ \underset{x,s}{\text{maximize}}\quad & c^T x = U \\ \text{subject to}\quad & Ax + s = b \\ & s \geq 0 , \end{aligned}$$

and define the primal feasible space as

$$(\mathcal{X};\mathcal{S}) = \{(x;s) \in \mathbb{R}^n \times \mathbb{R}^m \,|\, Ax + s = b,\ s \geq 0\}.$$

We also define the slack space alone to be

$$\mathcal{S} = \{s \in \mathbb{R}^m | s \geq 0, \; s = b - Ax \;\; \text{for some} \;\; x \in \mathbb{R}^n\}.$$

We first must develop some results regarding projective transformations of the spaces $\mathcal{X}$ and $\mathcal{S}$.

4. <u>A Class of Projective Transformations of $\mathcal{X}$ and $\mathcal{S}$</u>. Let $\bar{x}$ be a point in the interior of $\mathcal{X}$, so that $A\bar{x} + \bar{s} = b$ and $\bar{s} > 0$. Let $\bar{v} = U - c^T\bar{x}$. Suppose we have a given vector y that satisfies $y^T(x - \bar{x}) < 1$ for all $x \in \mathcal{X}$. Then the projective transformation $(z;r) = g(x;s)$ given by:

$$z = \bar{x} + \frac{(x - \bar{x})}{1 - y^T(x - \bar{x})} \qquad \text{and} \qquad r = \frac{\bar{S}^{-1}s}{1 - y^T(x - \bar{x})} \tag{2}$$

is well-defined for all $(x;s) \in (\mathcal{X};\mathcal{S})$. Furthermore, direct substitution shows that for $(x;s) \in (\mathcal{X};\mathcal{S})$, the z and r defined by $(z;r) = g(x;s)$ must satisfy:

$$(c - \bar{v}y)^T z \leq U - \bar{v}y^T\bar{x}$$
$$(\bar{S}^{-1}A - ey^T)z + r = (\bar{S}^{-1}b - ey^T\bar{x})$$
$$\text{and} \quad r \geq 0.$$

Thus, we can define the linear program

$$P_{y,\bar{x}}: \quad \text{maximize} \quad \bar{c}^T z = \bar{U}$$
$$\text{subject to} \quad \bar{A}z + r = \bar{b}$$
$$r \geq 0,$$

where

$$\bar{A} = (\bar{S}^{-1}A - ey^T) \quad ,$$
$$\bar{b} = (\bar{S}^{-1}b - ey^T\bar{x}) \quad ,$$
$$\bar{c} = (c - \bar{v}y) \quad , \qquad (3)$$
$$\bar{U} = (U - \bar{v}y^T\bar{x}).$$

Associated with this linear system we define

$$(\mathcal{Z};\mathfrak{R}) = \{(z;r) \in \mathbb{R}^n \times \mathbb{R}^m | \bar{A}z + r = \bar{b},\ r \geq 0\}$$

and define $\mathcal{Z}$ and $\mathfrak{R}$ analogously.

Note that the condition $y^T(x - \bar{x}) < 1$ holds for all $x \in \mathfrak{X}$ if and only if y lies in the interior of the polar of $(\mathfrak{X} - \bar{x})$, defined as

$$(\mathfrak{X} - \bar{x})^0 = \{y \in \mathbb{R}^n | y^Tx \leq 1 \quad \text{for all} \quad x \in (\mathfrak{X} - \bar{x})\}.$$

It can be shown that because $\mathfrak{X}$ is bounded and has a nonempty interior, that

$$(\mathfrak{X} - \bar{x})^0 = \{y \in \mathbb{R}^m | y = A^T\lambda \quad \text{for some} \quad \lambda \geq 0 \quad \text{satisfying} \quad \lambda^T\bar{s} = 1\},$$

and that $y \in \text{int}\ (\mathfrak{X} - \bar{x})^0$ (and hence $y^T(x - \bar{x}) < 1$ for all $x \in \mathfrak{X}$) if $y = A^T\lambda$ where $\lambda^T\bar{s} = 1$ and $\lambda > 0$. In this case we have the following:

<u>Lemma 1</u>. If $y = A^T\lambda$ where $\lambda > 0$ and $\lambda^T\bar{s} = 1$, then the transformation $(z;r) = g(x;s)$ is well-defined, and has an inverse, given by $(x;s) = h(z;r)$, where

$$x = \bar{x} + \frac{z - \bar{x}}{1 + y^T(z - \bar{x})} \quad , \quad s = \frac{\bar{S}r}{1 + y^T(z - \bar{x})} \quad . \qquad (4)$$

If $(z;r) = g(x;s)$, then $(x;s) \in (\mathfrak{X};\mathcal{S})$ if and only if $(z;r) \in (\mathfrak{Z};\mathfrak{R})$. Furthermore, any of the constraints in $c^T x \le U$, $Ax \le b$ is satisfied at equality if and only if the corresponding constraint of $\bar{c}^T z \le \bar{U}$, $\bar{A}z \le \bar{b}$ is satisfied at equality.

(The transformation $g(\cdot;\cdot)$ is a slight modification of a projective transformation presented as an exercise in Grünbaum [3] , on page 48. If $\mathfrak{X}$ is a polytope containing the origin in its interior, then its polar $\mathfrak{X}^0$ is also a polytope containing the origin in its interior. Grünbaum notes that a translation of $\mathfrak{X}^0$ by $y \in \text{int}\, \mathfrak{X}^0$ results in a projective transformation of $\mathfrak{X}$ given by

$$\mathfrak{Z} = \{z \in \mathbb{R}^n | z = \frac{x}{1 - y^T x} \text{ for some } x \in \mathfrak{X}\}.$$

Our transformation $g(\cdot;\cdot)$ is a translation of this transformation by $\bar{x}$.)

Recall that the <u>center</u> of the system of inequalities $Ax \le b$ is that point $x \in \mathfrak{X}$ that maximizes $\prod_{i=1}^{m} (b - Ax)_i$, or equivalently, $\sum_{i=1}^{m} \ln (b - Ax)_i$, see, e.g., Sonnevend [5]. Under our assumption of boundedness and full dimensionality, it is straightforward to show that $\bar{x}$ is the center of the system $Ax \le b$ if and only if,

$$e^T \bar{S}^{-1} A = 0 , \quad \text{where} \quad \bar{s} = b - A\bar{x} , \quad \text{and} \quad \bar{s} > 0.$$

We next construct a specific y that will ensure that $\bar{x}$ becomes the center of the projected polytope $\mathfrak{Z}$. If we define

$$\lambda = (1/m)\bar{S}^{-1} e \quad \text{and} \quad y = A^T \lambda,$$

then because $\lambda > 0$ and $\lambda^T\bar{s} = 1$, we know that y lies in the interior of $(\mathcal{X}-\bar{x})^0$, and $y^T(x-\bar{x}) < 1$ for all $x \in \mathcal{X}$. We have:

<u>Lemma 2</u>. Let $(\bar{x};\bar{s}) \in (\mathcal{X};\mathcal{S})$ be given, and suppose $\bar{s} > 0$. Let $\lambda = (1/m)\bar{S}^{-1}e$ be used to define $y = A^T\lambda$. Then $y \in \text{int}\,(\mathcal{X} - \bar{x})^0$, and by defining $g(\cdot;\cdot)$ by (2), we have:

(i) $(\bar{x},e) = g(\bar{x};\bar{s})$,

(ii) $\bar{x}$ is the center of the system $\bar{A}z \leq \bar{b}$, where $\bar{A}$, $\bar{b}$ are defined as in (3),

(iii) The set $\mathcal{R}$ is contained in the standard simplex in $\mathbb{R}^m$, namely $\Omega = \{r \in \mathbb{R}^m | e^Tr = m\ ,\ r \geq 0\}$.

Part (i) of Lemma 2 is obvious. To see (ii), note that e is the slack vector associated with $\bar{x}$ in $P_{y,\bar{x}}$, and so we must show that $e^T\bar{A} = 0$. This derivation is

$$e^T\bar{A} = e^T(\bar{S}^{-1}A - ey^T) = e^T\bar{S}^{-1}A - (1/m)e^Tee^T\bar{S}^{-1}A = 0.$$

For part (iii), note that for any $r \in \mathcal{R}$,

$$e^Tr = e^T(\bar{b} - \bar{A}\,z) = e^T\bar{b} = e^T(\bar{S}^{-1}b - ey^T\bar{x}) = e^T(\bar{S}^{-1}b - e^T(1/m)ee^T\bar{S}^{-1}A\bar{x})$$
$$= e^T\bar{S}^{-1}(b - A\bar{x}) = e^T\bar{S}^{-1}\bar{s} = e^Te = m.$$

Lemma 2 demonstrates that the projective transformation $g(\cdot;\cdot)$ transforms the slack space $\mathcal{S}$ to a subspace of the standard simplex, and transforms the current slack $\bar{s}$ to the center e of the standard simplex. Thus the projective transformation $g(\cdot;\cdot)$ corresponds to Karmarkar's projective transformation. We also have:

Lemma 3. The potential function of problem $P_{y,\bar{x}}$, defined as

$$G(z) = \ell n\ (\bar{U} - \bar{c}^T z) - \sum_{i=1}^{m} \ell n\ (\bar{b} - \bar{A}z)_i$$

differs from $F(x)$ by the constant $\sum_{i=1}^{m} \ell n\ (\bar{s}_i)$.

Thus, as in Karmarkar's algorithm, changes in the potential function are invariant under the projective transformation $g(\cdot;\cdot)$.

5. Analysis of the Algorithm. Here we examine a particular iterate of the algorithm. To ease the notational burden, the subscript k is suppressed. Let $\bar{x}$ be the current iterate. Let $\bar{s} = b - A\bar{x} > 0$ and $\bar{v} = U - c^T\bar{x}$. Then the vector $y = A^T\lambda$ is constructed, where $\lambda = (1/m)\bar{S}^{-1}e$, and the problem P is transformed to $P_{y,\bar{x}}$, where $\bar{A}$, $\bar{b}$, $\bar{c}$, $\bar{U}$ are given as in (3). The slack vector $\bar{s}$ is transformed by $g(\cdot;\cdot)$ to the center e of the standard simplex, as in Karmarkar's algorithm. Changes in the potential function are invariant under this transformation, by Lemma 3.

We now need to show that the direction $\bar{d}$ given by

$$\bar{d} = \frac{(\bar{A}^T\bar{A})^{-1}\bar{c}}{\sqrt{\bar{c}^T(\bar{A}^T\bar{A})^{-1}\bar{c}}}$$

corresponds to Karmarkar's search direction. Karmarkar's search direction is the projected gradient of the objective function in the transformed problem. Because the feasible region $\mathfrak{X}$ of P is bounded, A (and hence $\bar{A}$) has full column rank. Thus $(\bar{A}^T\bar{A})^{-1}$ exists, and so $\bar{A}z + r = \bar{b}$ is equivalent to $z = (\bar{A}^T\bar{A})^{-1}(\bar{b} - r)$. Substituting this last expression in $P_{y,\bar{x}}$, we form the equivalent linear program:

$$\text{minimize} \quad (\bar{A}(\bar{A}^T\bar{A})^{-1}\bar{c})^T r$$

$$\text{subject to} \quad (I - \bar{A}(\bar{A}^T\bar{A})^{-1}\bar{A}^T)r = (I - \bar{A}(\bar{A}^T\bar{A})^{-1}\bar{A}^T)\bar{b}$$

$$r \geq 0.$$

(This transformation is also used in Gay [2].) The objective function vector $\bar{A}(\bar{A}^T\bar{A})^{-1}\bar{c}$ of this program lies in the null space of $\mathfrak{R}$, so that the normed direction

$$\bar{p} = \frac{-\bar{A}(\bar{A}^T\bar{A})^{-1}\bar{c}}{\sqrt{\bar{c}^T(\bar{A}^T\bar{A})^{-1}\bar{c}}}$$

is Karmarkar's normed direction. In the space $\mathcal{Z}$, this direction $\bar{p}$ corresponds to

$$\bar{d} = \frac{(\bar{A}^T\bar{A})^{-1}\bar{c}}{\sqrt{\bar{c}^T(\bar{A}^T\bar{A})^{-1}\bar{c}}},$$

i.e., $\bar{d}$ is the unique direction d in $\mathbb{R}^n$ satisfying $\bar{A}d + \bar{p} = 0$.

Thus we see that the direction $\bar{d}$ given in the algorithm corresponds to Karmarkar's normed direction $\bar{p}$. Following Todd and Burrell [7], using a steplength of $\alpha = 1/3$ will guarantee a decrease in the potential function $F(x)$ of at least 1/5. Furthermore, as suggested in Todd and Burrell [7], performing a linesearch of $F(\cdot)$ on the line $\bar{x} + \alpha\bar{d}$, $\alpha \geq 0$, is advantageous.

The next iterate, in $\mathcal{Z}$ space, is the point $\bar{x} + \alpha\bar{d}$. We projectively transform back to $\mathcal{X}$ space using the inverse transformation $h(\cdot;\cdot)$ given by (4), to obtain the new iterate in $\mathcal{X}$ space, which is

$$\bar{x} + \frac{\alpha\bar{d}}{1 + y^T(\alpha\bar{d})}.$$

6. Remarks:

a. Getting Started.

We required, in order to start the algorithm, that the initial point x_0 must satisfy condition (1). This condition requires that x_0 be the center of the system of inequalities $Ax \leq b$ and that the rows of this system be scaled so that $b - Ax_0 = e$. If our initial point x_0 does not satisfy this condition, we simply perform one projective transformation to transform the linear inequality system into the required form. That is, we set $s_0 = b - Ax_0$, $v_0 = U - c^T x_0$, and define

$$y_0 = (1/m)A^T S_0^{-1} e$$

$$A_0 = S_0^{-1} A - e y^T$$

$$b_0 = S_0^{-1} b - e y^T x_0$$

$$c_0 = c - v_0 y$$

$$\text{and} \quad U_0 = U - v_0 y^T x_0 .$$

Then, exactly as in Lemma 2, x_0 is the center of the system $A_0 x \leq b_0$, and $e = b_0 - Ax_0$, and $e^T A_0 = 0$. Our initial linear program, of course, becomes

$$\text{maximize} \quad c_0^T x = U_0$$

$$\text{subject to} \quad A_0 x_0 \leq b_0$$

b. Complexity and Inscribed/Circumscribed Ellipsoids.

The algorithm retains the exact polynomial complexity as Karmarkar's principal algorithm, namely it solves P in $O(m^4 L)$ arithmetic operations. However, using Karmarkar's methodology for modifying

$(A_k^T S_k^{-2} A_k)^{-1}$, the modified algorithm should solve P in $O(m^{3.5}L)$ arithmetic operations. One of the constructions that Karmarkar's algorithm uses is that the ratio of the largest inscribed ball in the standard simplex, to the smallest circumscribed ball, is $1/(m - 1)$. In our system, this translates to the fact that if $\bar{x}$ lies in the interior of $\mathfrak{X} = \{x \in \mathbb{R}^n | Ax \leq b\}$, then there are ellipsoids E_{in} and E_{out}, each centered at $\bar{x}$, for which $E_{in} \subseteq \mathfrak{Z} \subseteq E_{out}$, where $\mathfrak{Z}$ is the transformed polytope $\{z \in \mathbb{R}^n | \bar{A}z \leq \bar{b}\}$, and $(E_{in} - \bar{x}) = (1/(m - 1))(E_{out} - \bar{x})$. This result was proven for the center of $\mathfrak{X}$ by Sonnevend [5]. We can explicitly characterize E_{in} and E_{out} by

$$E_{in} = \{z \in \mathbb{R}^n | (z - \bar{x})^T \bar{A}^T \bar{A}(z - \bar{x}) \leq 1/(m - 1)\}$$

and

$$E_{out} = \{z \in \mathbb{R}^n | (z - \bar{x})^T \bar{A}^T \bar{A}(z - \bar{x}) \leq m(m - 1)\},$$

where, of course, $\bar{A} = (\bar{S}^{-1}A - ey^T)$, and $y = (1/m)A^T\bar{S}^{-1}e$, $\bar{s} = \bar{b} - A\bar{x}$.

Using this ellipsoid construction, we could prove the complexity bound for the algorithm directly, without resorting to Karmarkar's results. But inasmuch as this algorithm was developed to be an analog of Karmarkar's for linear inequality systems, it is fitting to place it in this perspective.

c. <u>Other Results</u>.

The methodology presented herein can be used to translate other results regarding Karmarkar's algorithm to the space of linear inequality constraints. Although we assume that the objective function value is known in advance, this assumption can be relaxed. The results on objective function value bounding (see Todd and Burrell [7] and Anstreicher [1]), dual variables (see Todd and Burrell [7] and Ye [8]), fractional programming (Anstreicher [1]), and acceleration techniques (see Todd [6]), all should carry over to the space of linear inequality constraints.

Acknowledgement

I would like to acknowledge Michael Todd for stimulating discussions on this topic, and his reading of a draft of this paper.

REFERENCES

[1] K.M. Anstreicher. "A monotonic projective algorithm for fractional linear programming using dual variable." Algorithmica I (1986), 483-498.

[2] D. Gay, "A variant of Karmarkar's linear programming algorithm for problems in standard form," Mathematical Programming 37 (1987) 81-90.

[3] B. Grünbaum. Convex Polytopes. Wiley, New York (1967).

[4] N. Karmarkar. "A new polynomial time algorithm for linear programming." Combinatorica 4 (1984), 373-395.

[5] Gy. Sonnevend. "An 'analytical centre' for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming." Proc. 12th IFIP Conf. System Modelling, Budapest, 1985.

[6] M.J. Todd. "Improved bounds and containing ellipsoids in Karmarkar's linear programming algorithm." Manuscript, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, New York, (1986).

[7] M.J. Todd and B.P. Burrell. "An extension of Karmarkar's algorithm for linear programming using dual variables." Algorithmica I (1986), 409-424.

[8] Y. Ye. "Dual approach of Karmarkar's algorithm and the ellipsoid method." Manuscript, Engineering-Economic Systems Department, Stanford University, Stanford, California, (1987).