AN OPTIMIZATION APPROACH TO THE KANBAN SYSTEM

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WP #1635-85   March 1985

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ABSTRACT

In this paper we present an optimization model for the Kanban system in a multi-stage capacitated assembly-tree-structure production setting. We discuss solution procedures to the problem and address two special cases of practical interest.

1. INTRODUCTION

The Kanban system is a multi-stage production scheduling and inventory control system. It is motivated by the concept of just-in-time production and aims at reducing the level of inventory to a minimum. Briefly speaking, the concept of just-in-time production is that materials should flow through the entire production sequence without being stopped or accumulated in an intermediate stage. Under this concept, no inventory of any kind is viewed as an absolute necessity.

Obviously, in many instances inventories are justified because of the important role they play. For example, cycle stock is carried due to the trade-off that has to be made between setup cost and inventory holding cost; and safety stock is accumulated to protect against various uncertainties. Unfortunately, the basic concepts that justify the existence of inventories have been abused over the years. Managers very often accept the existence of setup work without looking into the possibility of reducing it, which could lead to a down-sized cycle stock. Similarly, instead of improving the accuracy of forecasts of demand and lead times and ameliorating preventive maintenance procedures, managers often choose to increase safety stock. In short, inventory has become more of a cover-up of production problems than of a solution to them.

The Kanban system, originally designed by Toyota to realize just-in-time production, is intended to keep a tight control over inventory and force the hidden problems to surface so that they can be identified and addressed directly.

1.1 Summary of the Operating Procedures of the Kanban System

For the purpose of this paper, we present a brief description of how the Kanban system operates. For more details, the reader is referred to [3, 5, 6, 7, 10]. "Kanban", in the Japanese language, refers to a card or tag. It can serve as a production, delivery, or purchase order. In the system, items are put into containers and different types of items are held in different containers. Once a container is full, a Kanban is attached to it. A Kanban usually carries the following information: (1) item name, (2) item number, (3) description of the item, (4) container type, (5) container capacity, (6) Kanban identification number, (7) preceding stage, and (8) succeeding stage.

In Figure 1, stage n represents an intermediate stage in a production setting. It encompases a production process P^n and a subsequent inventory point I^n. The type of production process involved can be fabrication, subassembly, delivery, or purchase. Using as inputs the items stored in the inventory point of the immediate predecessor, process P^n produces its own items to fill a container and then stores the full container in I^n with a Kanban attached to it. When the first piece of a full container in I^n is used by the production process of the immediate successor, the Kanban originally attached to the container is detached and kept aside. At the end of each time period (for example, at the end of every half-shift), all the Kanbans detached in I^n during the time period are collected and sent back to P^n. These Kanbans then serve as new production orders for P^n. Generally P^n uses a first-in-first-out rule to process these orders. Once P^n produces a full container (i.e., P^n fills an order), the Kanban which ordered the full container is attached to it and the container is sent to I^n.
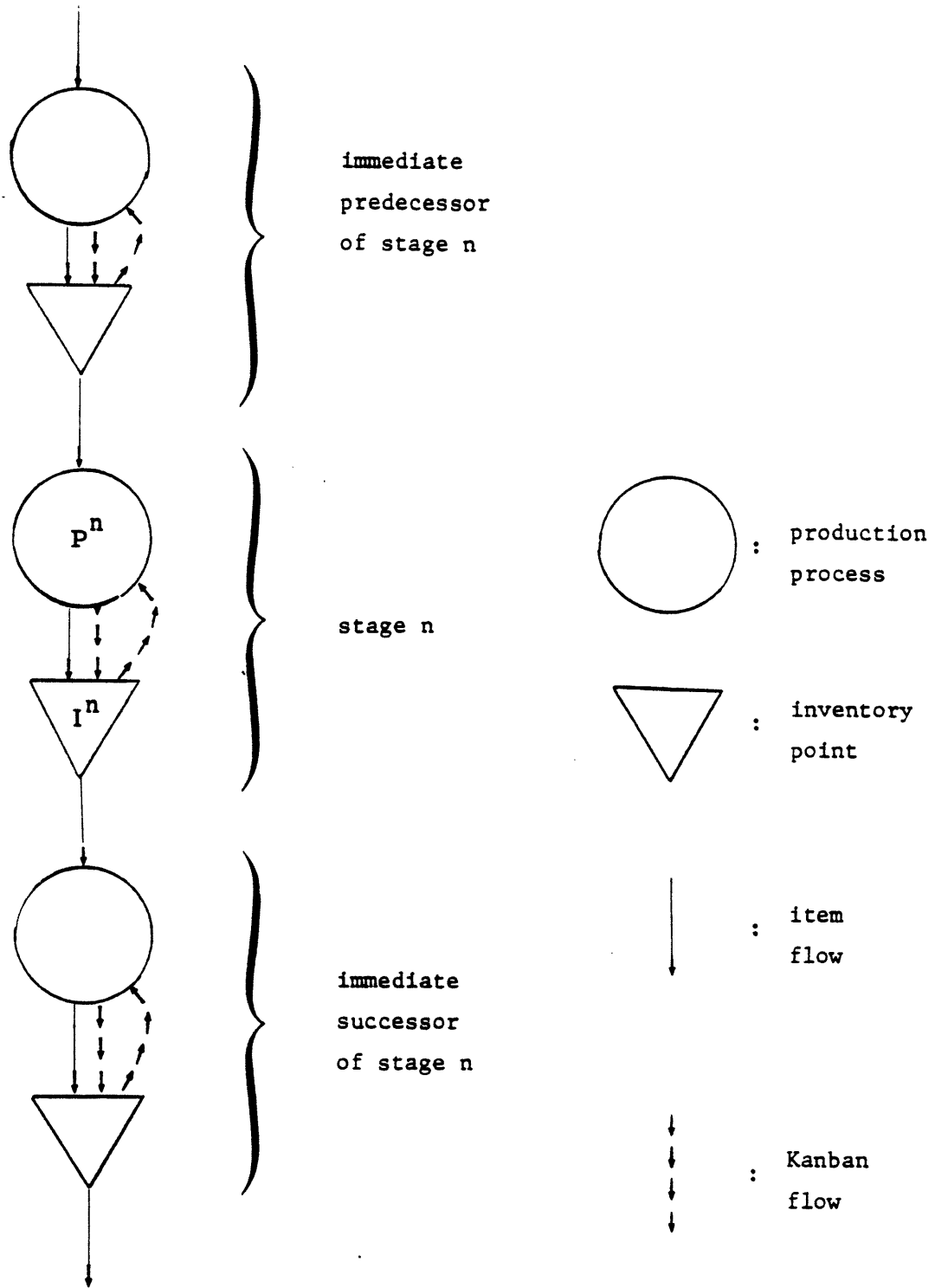
Figure 1. Flows of Items and Kanbans

Below, we outline four important observations regarding the system. First, the total number of Kanbans circulating between P^n and I^n is unchanged over time, unless management interferes to drain Kanbans from, or to inject more Kanbans into, stage n. Second, the maximum inventory buildup in I^n is limited by the number of Kanbans circulating between P^n and I^n. Consequently, by controlling the number of circulating Kanbans and requiring that every full container have a Kanban attached to it, managers can be assured that the inventory buildup will not exceed a certain limit. Third, the movement of Kanbans between P^n and I^n is triggered by the inventory withdrawal from I^n by the immediate successor. In other words, P^n will produce to replenish what has been withdrawn from I^n by the immediate successor. Fourth, by circulating Kanbans within every stage, all the stages in a production setting are chained together. Therefore, the production schedule of the final stage is transmitted back to all the upstream stages. Since a detached Kanban automatically becomes a new order, managers need not issue any other document to trigger an order in an upstream stage. The upstream stages can actually be self-operated.

Figure 1 depicts a serial production setting. Nevertheless, the reader can easily observe that the operating procedure, as described above, will also work with an assembly-type, a distribution-type, or a mixed-type production setting.

1.2 Purposes of the Paper

The Kanban system, enabling Toyota to drastically cut its inventory investment, has attracted much attention from production professionals worldwide. Most research efforts to date have focused on the comparison of the Kanban system, or the Japanese production methods in general, and the

Western production methods. Rice and Yoshikawa [8] contrasted Kanban with MRP (Material Requirements Planning). Schonberger [9] provided nine lessons on Japanese manufacturing techniques from which Western companies could learn in order to simplify their production problems. Most recently, Krajewski, King, Ritzman, and Wong [4] conducted a simulation experiment to identify the critical technical factors in the Japanese and U.S. manufacturing environments, represented by Kanban and MRP, respectively. With the exception of Kimura and Terada [3], efforts have not been made to develop mathematical models for the Kanban system. In [3], the authors provided several basic equations for the Kanban system in a multi-stage serial production setting to show how the fluctuation of final demand influences the fluctuation of production and inventory volumes at upstream stages. In their work, they assumed small container size and unlimited production capacity.

The purpose of this paper is two-fold. First, it provides an optimization model for the Kanban system in a multi-stage assembly production setting. The model assists managers in determining the number of circulating Kanbans, and hence the inventory level, at each stage. Contrasting with Kimura and Terada [3], we make no assumptions on the container size (except for two special cases in which we make assumption on the relative container size between stages); in addition, we allow limited production capacity. As a result, our model should be applicable in more general manufacturing situations. Second, the paper investigates solution procedures, for the resulting Kanban model, that will make it usable in practice. To this end, the initial model, which is nonlinear integer in nature, is transformed into an integer linear program. The integer linear program presents the following advantages: (1) it is more tractable than the nonlinear model; and (2) it provides the same set of feasible solutions and the same set of optimal solutions as the nonlinear model in terms of the decision variables controlled by managers.

To the same end, two special cases of practical interest are constructed on the basis of the relative container size between stages. In one case, the integer linear program is converted into a mixed integer linear program with the number of integer variables greatly reduced. In the other case, the linear programming (LP) technique is used and the relative error due to the LP approximation is shown to approach zero asymptotically.

2. MODEL DESCRIPTION

The model that we present deals with a multi-stage capacitated assembly-tree-structure production setting with each stage producing one type of item. There are $N+1$ stages in the setting. Let $n \in \{0,1,...,N\}$ index the stages with the understanding that $n_1 < n_2$ if stage n_1 succeeds stage n_2. We also denote an item by the index of the stage producing it. The final stage, stage 0, includes only the final assembly operation P^0, while every upstream stage $n \in \{1,2,...,N\}$ includes a production process P^n and an immediately succeeding inventory point I^n. An example of indexing is provided in Figure 2. Let $t \in \{0,1,...,T\}$ index the time periods with the understanding that the planning horizon starts at the beginning of period 1 and finishes at the end of period T. For the final stage, a time-phased production schedule is given and must be met. For each upstream stage, a production quota for the whole planning horizon is given and the quota is determined by the effective demand imposed upon the stage. Once an upstream P^n has reached its production quota, all the detached Kanbans remaining at P^n or to be sent to P^n in the future from I^n stop triggering any further production and are drained from the system by management at the end of the planning horizon.

Throughout the paper, we shall use $\lceil Z \rceil$ to denote the smallest integer which is larger than or equal to Z and $\lfloor Z \rfloor$ the largest integer which is smaller than or equal to Z. Proofs of propositions are omitted whenever they are not particularly difficult to reproduce.

Parameters

α^n = number of units of item n in a full container; $\alpha^n \in \{1,2,...\}$ $(n=0,1,...,N)$. These parameters represent container capacities.

β_t^n = production capacity, in terms of the number of full containers of item n, at P^n in period t; $\beta_t^n \in \{0,1,...\}$ $(n=1,...,N; t=1,...,T)$.

$s(n)$ = immediately succeeding stage of stage n $(n=1,...,N)$.

$P(n)$ = set of immediately preceding stages of stage n $(n=0,1,...,N)$.

$e^{n,s(n)}$ = number of units of item n which are required to make one unit of item $s(n)$; $e^{n,s(n)} \in \{1,2,...\}$ $(n=1,...,N)$.

V_0^n = number of full containers of item n available in I^n at the end of period 0; $V_0^n \in \{0,1,2,...\}$ $(n=1,...,N)$. Note that each of these full containers has a Kanban attached to it.

W_0^n = number of units of item n remaining in a partially filled container, whose Kanban has been detached, in I^n at the end of period 0; $W_0^n \in \{0,1,...,\alpha^n-1\}$ $(n=1,...,N)$.

X_t^0 = production requirement, in terms of the number of full containers of item 0 (i.e., the final product), at stage 0 in period t; $X_t^0 \in \{0,1,2,...\}$ $(t=1,...,T)$.
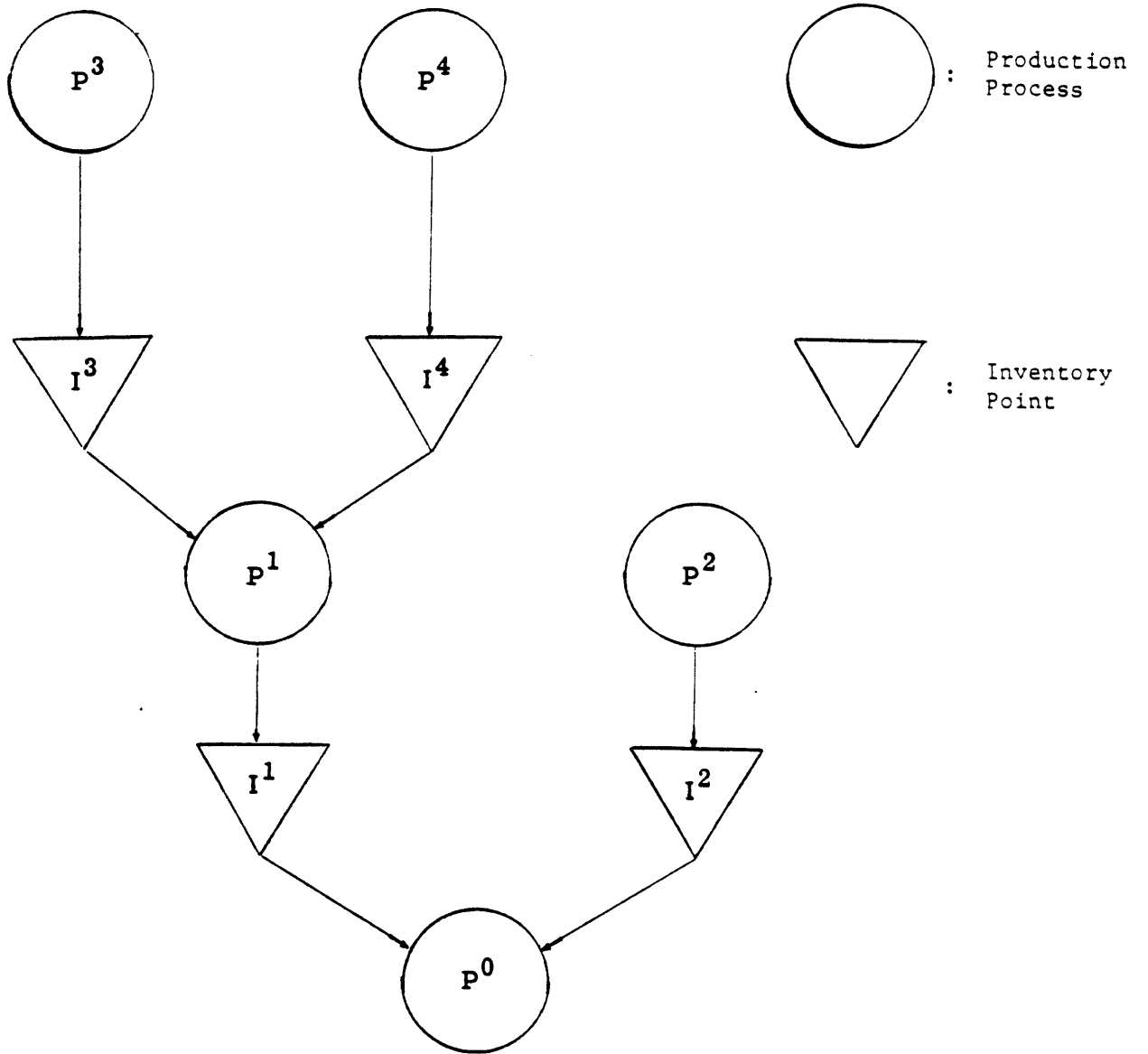
Figure 2. An Example of Indexing

$$Q^n = \max \left\{ 0, \left\lceil (e^{n,s(n)} \alpha^{s(n)} / \alpha^n) Q^{s(n)} - V_0^n - (W_0^n / \alpha^n) \right\rceil \right\}$$

= production quota or effective demand, in terms of the number of full containers of item n, imposed upon stage n for the whole planning horizon; $Q^n \in \{0,1,2,\ldots\}$ $(n=1,\ldots,N)$. Q^0 is defined as $\sum\limits_{t=1}^{T} X_t^0$ ..

We can define $\ell(n,1) \in \{0,1,2,\ldots\}$ and $\ell(n,2) \in \{1,2,\ldots\}$ as the production lead time and information transmission time, respectively, for stage n $(n=1,\ldots,N)$. The production lead time $\ell(n,1)$ has the interpretation that the full container of item n put into production in P^n in period t will be made available in I^n in period $t+\ell(n,1)$ for withdrawal by $P^{s(n)}$. Similarly, the information transmission time $\ell(n,2)$ has the interpretation that the Kanban detached in I^n in period t will begin to serve as a production order in P^n in period $t+\ell(n,2)$. Since the Kanbans detached in I^n in period t are collected at the end of that period and then sent back to P^n, the information transmission time $\ell(n,2)$, as used here, must be greater than or equal to one. To lessen the burden of notation, we shall let $\ell(n,1) = 0$ and $\ell(n,2) = 1$ for $n=1,\ldots,N$ from now on. This simplification will have no impact on the results of the paper. Note that at the beginning of the planning horizon, the initial inventory at stage n includes V_0^n full containers and W_0^n units of item n $(n=1,\ldots,N)$. Also note that both X_t^0 and β_t^n are allowed to vary from period to period in order to give management more flexibility in scheduling final assembly operations and shifting resources.

Variables

X_t^n = number of detached Kanbans of item n which respectively trigger the production of a full container in P^n in period t (n=1,...,N; t=1,...,T).

Y_t^n = number of Kanbans of item n which are detached from their associated containers in I^n in period t (n=1,...,N; t=1,...,T).

U_t^n = number of detached Kanbans of item n which are available in P^n at the end of period t and have not triggered any production yet (n=1,...,N; t=1,...,T).

V_t^n = number of full containers of item n which are available in I^n at the end of period t (n=1,..,N; t=1,...,T).

W_t^n = number of units of item n remaining in a partially filled container, whose Kanban has been detached, in I^n at the end of period t (n=1,...,N; t=1,...,T).

U_0^n = number of detached Kanbans of item n which are injected into P^n by management at the beginning of the planning horizon (n=1,...,N).

We shall use the following abbreviations for variables:

(1) $<U,V,W,X,Y>$ stands for all the variables involved, which includes N(T+1) U-type variables, NT V-type variables, NT W-type variables, NT X-type variables, and NT Y-type variables.

(2) $<U_0,X>$ stands for N U-type variables with t=0 and NT X-type variables.

(3) $<U_0>$ stands for $<U_0^1, U_0^2, \ldots, U_0^N>$.

(4) $<X^n>$ stands for $<X_1^n, X_2^n, \ldots, X_T^n>$.

We describe mathematically the Kanban system as follows:

$$(2.1) \quad U^n_{t-1} + Y^n_{t-1} - X^n_t - U^n_t = 0 \qquad n=1,\ldots,N; \ t=1,\ldots,T$$

$$(2.2) \quad V^n_{t-1} + X^n_t - Y^n_t - V^n_t = 0 \qquad n=1,\ldots,N; \ t=1,\ldots,T$$

$$(2.3) \qquad X^n_t = \min \begin{cases} U^n_{t-1} + Y^n_{t-1} \\[4pt] \beta^n_t \\[4pt] \left\lfloor (\alpha^k V^k_{t-1} + W^k_{t-1} + \alpha^k X^k_t)/(e^{k,n}\alpha^n) \right\rfloor \\[20pt] \hspace{4cm} \text{all } k \in P(n) \\[4pt] Q^n - \displaystyle\sum_{\tau=1}^{t-1} X^n_\tau \end{cases}$$

$$n=1,\ldots,N; \ t=1,\ldots,T$$

$$(2.4) \quad \alpha^k V^k_{t-1} + W^k_{t-1} + \alpha^k X^k_t \geq e^{k,0}\alpha^0 X^0_t$$

$$\text{all } k \in P(0); \ t=1,\ldots,T$$

$$(2.5) \quad \begin{cases} Y^n_t = \left\lceil (e^{n,s(n)}\alpha^{s(n)} X^{s(n)}_t - W^n_{t-1})/\alpha^n \right\rceil & n=1,\ldots,N; \ t=1,\ldots,T \\[6pt] Y^n_0 = 0 & n=1,\ldots,N \end{cases}$$

$$(2.6) \quad W^n_{t-1} + \alpha^n Y^n_t - e^{n,s(n)}\alpha^{s(n)} X^{s(n)}_t - W^n_t = 0 \qquad n=1,\ldots,N; \ t=1,\ldots,T$$

$$(2.7) \quad U^n_0 \ \text{nonnegative integer} \qquad n=1,\ldots,N$$

Constraints (2.1) and (2.2) describe the conservation of flow in P^n and I^n, respectively, in terms of Kanbans. Constraints (2.3) indicate that the number of full containers put into production in P^n in period t is determined by the available detached Kanbans, production capacity, available inventories in the previous stages, and remaining production quota. Constraints (2.4) ensure that the production schedule of the final stage can be carried out. Constraints (2.5) indicate the number of Kanbans which are detached from their associated containers in I^n in period t. Constraints (2.6) describe the conservation of flow in I^n for the number of units of item n remaining in a partially filled container. The nonnegative integrality of U^n_0 is enforced by (2.7). No setup is involved explicitly in (2.1)-(2.7). If an upstream stage P^n needs a setup in a particular period t (due,

for example, to the fact that the machinery in P^n is scheduled for other purposes in the previous period, i.e., $\beta^n_{t-1} = 0$), the setup is assumed to be executed externally to the model and the value of β^n_t is determined after making allowance for the setup.

Theorem 2.1   If $<U,V,W,X,Y>$ satisfies $(2.1)-(2.7)$, then

(a) U^n_t, V^n_t, W^n_t, X^n_t, and Y^n_t are nonnegative

      integers $n=1,\ldots,N;\ t=1,\ldots,T$

(b) $Y^n_t \leq V^n_{t-1} + X^n_t$ $n=1,\ldots,N;\ t=1,\ldots,T$

(c) $W^n_t \leq \alpha^n - 1$ $n=1,\ldots,N;\ t=1,\ldots,T$

(d) $U^n_0 + V^n_0 = U^n_t + V^n_t + Y^n_t$ $n=1,\ldots,N;\ t=1,\ldots,T$

(e) $\displaystyle\sum_{t=1}^{T} X^n_t = Q^n$ $n=1,\ldots,N$

The above theorem shows that $(2.1)-(2.7)$ implicitly enforce the following properties: (a) In addition to U^n_0 $(n=1,\ldots,N)$, the rest of the variables are also nonnegative integers. (b) The number of Kanbans detached from their associated containers in I^n in period t is smaller than or equal to the number of full containers available in I^n at the end of period t-1 plus the number of full containers received by I^n in period t. (c) The number of units of item n remaining in a partially filled container in I^n at the end of period t is smaller than the container size α^n. (d) The number of Kanbans circulating in each upstream stage is unchanged and equal to $U^n_0 + V^n_0$ during the planning horizon. (e) The production quota imposed on each upstream stage is met exactly.

We propose the following optimization model, henceforth referred to as model (M), for the Kanban system:

Minimize

$$(2.8) \qquad \sum_{n=1}^{N} C^n [U_0^n + V_0^n + 1 - (1/\alpha^n)]$$

s.t. (2.1) - (2.7),

where C^n = accumulated value of one full container of item n; in other words, C^n represents the sum of material, labor and all other manufacturing costs which have been accumulated by the system in a full container of item n. The cost objective (2.8) can be interpreted in two different ways. One interpretation relates to the value tied up in the inventory. Note that the expression $U_0^n + V_0^n$ represents the number of Kanbans circulating in stage n. By considering the possibility of at most $\alpha^n - 1$ units of item n remaining in a partially filled container in I^n, the expression $U_0^n + V_0^n + 1 - (1/\alpha^n)$ represents an upper bound for the inventory buildup, in terms of the number of full containers of item n, in stage n at any point in time. As a result, (2.8) represents an upper bound for the value tied up in the inventory of all upstream stages (i.e., stage 1 through stage N) in the system at any point in time. This bound is tight in the sense that it can be attained, in some period of the planning horizon, in some cases as the following example shows.

Example 1: Let N=1, T=3, C^1=6, α^1=3, $e^{1,0}$=1, α^0=1, $\beta_1^1 = \beta_2^1 = \beta_3^1 = \infty$, V_0^1=1, W_0^1=1, X_1^0=2, X_2^0=0, X_3^0=8, and Q^1=2.

Suppose that one new Kanban is distributed to P^1 at the start of the planning horizon, i.e., U_0^1=1. The value given by (2.8) is 16. The inventory buildup in I^1 at the end of period 2 includes two full containers and a partially filled container of two units of item 1, whose value is 6(2+2/3) = 16. This example shows the tightness of (2.8) as an upper bound. Note that in the example model (M) requires U_0^1 to be 1 in the optimal solution. □

Clearly, the multiplication of (2.8) by the inventory holding charge over the planning horizon results in an upper bound for the inventory holding cost over the same period. The other interpretation of (2.8) focuses on the number of Kanbans in circulation. A smaller number of Kanbans circulating at a stage reflects higher operating efficiency at that stage, and hence it is perceived as a desirable goal by workers and management. Now, consider $\sum_{n=1}^{N} C^n(U_0^n + V_0^n)$ instead of (2.8). Since $U_0^n + V_0^n$ represents the number of Kanbans circulating at stage n, the alternative objective function, which is to be minimized, reflects a weighted total number of Kanbans circulating in the upstream stages with C^n being used as the weight for stage n. The choice of C^n follows naturally from the fact that C^n is the sum of the costs which have been sunk into a full container up to the conclusion of manufacture at stage n. Note that the difference between the original objective function (2.8) and the alternative one is $\sum_{n=1}^{N} C^n[1 - (1/\alpha^n)]$, a constant term. Hence, the two objective functions are equivalent in terms of the sets of feasible and optimal solutions to model (M). For the rest of the paper, we shall continue to use (2.8) as the objective function.

3. MODEL SOLUTION

Model (M) is a complex integer problem. The nonlinear constraints (2.3) and linear constraints (2.5), when re-expressed in more operational forms, greatly increase the number of integer variables and the number of constraints.

3.1 Transformation

In this section, we shall transform model (M) into a simpler model such that both have the same set of feasible, and optimal, solutions in terms of $<U_0>$, and the same optimal value. The transformation is motivated by the observation that if $<U,V,W,X,Y>$ and $<\bar{U},\bar{V},\bar{W},\bar{X},\bar{Y}>$ are two feasible solutions to (2.1)-(2.7) and $<U_0> = <\bar{U}_0>$, then $<U,V,W,X,Y> = <\bar{U},\bar{V},\bar{W},\bar{X},\bar{Y}>$.

In other words, once U_0^n (n=1,...,N) assume their specific values, all other variables in (2.1)-(2.7) are uniquely determined. This observation corresponds to the characteristic of the Kanban system that it is self-operational once the Kanbans have been distributed to the stages. The key decision variables that need to be controlled by management are the U_0^n (n=1,...,N).

For future discussion, the following nomenclature will be adopted. A partial solution is said to satisfy, or to be feasible in, a set of constraints if there exists a complement to it such that the whole solution, i.e., the partial one together with its complement, satisfies all the constraints. For example, the partial solution $<X,Y>$ satisfies, or is feasible in, (2.1)-(2.7) if there exists $<U,V,W>$ such that the whole solution $<U,V,W,X,Y>$ satisfies (2.1)-(2.7). Similarly, a partial solution is said to be feasible (optimal) in an optimization model if there exists a complement to it such that the whole solution is feasible (optimal) in the model. Two optimization models are said to have the same feasible (optimal) partial solution if there exist two complements, which may or may not be different from each other, such that the two resulting whole solutions are feasible (optimal) in the two models, respectively.

Let $E^{n,s(n)} = e^{n,s(n)}\alpha^{s(n)}/\alpha^n$ for $n=1,\ldots,N$. The parameter $E^{n,s(n)}$ represents the number of full containers of item n required to make one full container of item $s(n)$. Depending upon the values of $e^{n,s(n)}$, $\alpha^{s(n)}$, and α^n, the value of $E^{n,s(n)}$ may or may not be integral. Also let $0 < \epsilon < \min\{1/\alpha^n \mid n=1,\ldots,N\} \leq 1$. Consider the following optimization model:

$$\text{minimize (2.8)}$$

s.t.

$$(3.1) \quad (W_0^n/\alpha^n) + V_0^n + \sum_{\tau=1}^{t} X_\tau^n - E^{n,s(n)} \sum_{\tau=1}^{t} X_\tau^{s(n)} \geq 0$$

$$n=1,\ldots,N; \ t=1,\ldots,T$$

$$(3.2) \quad U_0^n - \sum_{\tau=1}^{t} X_\tau^n + E^{n,s(n)} \sum_{\tau=1}^{t-1} X_\tau^{s(n)} - (W_0^n/\alpha^n) + 1 - \epsilon \geq 0$$

$$n=1,\ldots,N; \ t=1,\ldots,T$$

$$(3.3) \quad X_t^n \in \{0,1,\ldots,\beta_t^n\} \qquad n=1,\ldots,N; \ t=1,\ldots,T$$

$$(3.4) \quad U_0^n \in \{0,1,2,\ldots\} \qquad n=1,\ldots,N$$

We refer to the above model as model (MO). The relation between model (M) and model (MO) is summarized in the next theorem.

Theorem 3.1   (M) is feasible if and only if (MO) is feasible. The two models have the same set of feasible partial solutions $<U_0>$, the same set of optimal partial solutions $<U_0>$, and the same optimal value.

(MO) is an integer linear program which has 2NT constraints, excluding (3.3)-(3.4), and NT + N integral variables. The configuration of (MO) is computationally more favorable than that of the nonlinear integer problem (M). However, it should be pointed out that the constraints of (MO) do not describe the operating procedure of the Kanban system while those of (M) do. The link between (M) and (MO) hinges on U_0^n ($n=1,\ldots,N$), as shown in

Theorem 3.1. Since U_0^n ($n=1,\ldots,N$) are the only decision variables controlled by management, we can solve (MO) and still obtain a relevant feasible or optimal partial solution $\langle U_0\rangle$ to (M).

The proof of Theorem 3.1 follows directly from the next two lemmas.

Lemma 3.1. If $\langle U,V,W,X,Y\rangle$ satisfies (2.1)-(2.7), then it also satisfies

$$(2.3)' \quad \begin{cases} X_t^n \leq U_{t-1}^n + Y_{t-1}^n \\[2ex] X_t^n \leq \beta_t^n \\[2ex] X_t^n \leq \left\lfloor (\alpha^k V_{t-1}^k + W_{t-1}^k + \alpha^k X_t^k)/(e^{k,n}\alpha^n) \right\rfloor \\[1ex] \qquad\qquad\qquad\qquad \text{all } k \in P(n) \\[2ex] X_t^n \leq Q^n - \sum_{\tau=1}^{t-1} X_\tau^n \end{cases}$$

$$n=1,\ldots,N; \quad t=1,\ldots,T.$$

and

$(2.7)' \quad U_0^n, \ X_t^n$ nonnegative integers $\qquad n=1,\ldots,N; \ t=1,\ldots,T.$

Also, if $\langle U,V,W,X,Y\rangle$ satisfies (2.1)-(2.2), (2.3)', (2.4)-(2.6) and (2.7)', then $\langle U_0\rangle$ is a feasible partial solution to (2.1)-(2.7).

Lemma 3.2 If $\langle U,V,W,X,Y\rangle$ satisfies (2.1)-(2.2), (2.3)', (2.4)-(2.6), and (2.7)', then $\langle U_0,X\rangle$ satisfies (3.1)-(3.4). If $\langle U_0,X\rangle$ satisfies (3.1)-(3.4), then $\langle U_0\rangle$ is a feasible partial solution to (2.1)-(2.2), (2.3)', (2.4)-(2.6), and (2.7)'.

(The proof of Lemma 3.2 is provided in Appendix 1.)

Note that the constraints of model (MO), i.e., (3.1)-(3.4), do not

specifically involve the production quota Q^n. It can be easily shown that

$$\sum_{\tau=1}^{T} X_\tau^n \geq Q^n \quad \text{for } n=1,\ldots,N \quad \text{if } <X> \text{ satisfies (3.1) and (3.3).}$$

For any $<U_0,\bar{X}>$ satisfying (3.1) - (3.4), the proof of lemma 3.2 provides a way for constructing $< X >$ such that $\sum_{\tau=1}^{T} X_\tau^n \leq Q^n$ for $n=1, \ldots, N$ and

$<U_0,X>$ still satisfies (3.1) - (3.4). Obviously, the newly constructed

$<X>$ does satisfy the constraints $\sum_{\tau=1}^{T} X_\tau^n = Q^n$ for $n=1,\ldots,N$.

3.2 Feasibility Test

Before solving model (MO), it is important to know whether it is feasible or not. Let $\Omega = \{<X> \mid <X> \text{ satisfies (3.1) and (3.3)}\}$. It is obvious that $\Omega \neq \emptyset$ if and only if (MO) is feasible. Gabbay [2] devised a method to test if a multi-stage serial production setting is feasible. A similar feasibility test can be applied to Ω, which represents a multi-stage assembly production setting. The algorithm of the feasibility test is given below:

(Step 0) $\bar{X}_t^0 \leftarrow X_t^0$ for $t=1,\ldots,T$

$\qquad\quad n \leftarrow 1$

(Step 1) $Q_1^n \leftarrow \max \left\{ 0, \lceil E^{n,s(n)} \bar{X}_1^{s(n)} - v_0^n - (W_0^n/\alpha^n) \rceil \right\}$

$\qquad\quad Q_t^n \leftarrow \max \left\{ 0, \lceil E^{n,s(n)} \sum_{\tau=1}^{t} \bar{X}_\tau^{s(n)} - v_0^n - (W_0^n/\alpha^n) \rceil \right\} - \sum_{\tau=1}^{t-1} Q_\tau^n \quad \text{for } t=2,\ldots,T$

$\qquad\quad t \leftarrow T$

(Step 2) $\bar{X}_t^n \leftarrow \min \left\{ Q_t^n, \beta_t^n \right\}$

if t=1, then go to step 3; else,

$$Q^n_{t-1} \leftarrow Q^n_{t-1} + \max \{0, Q^n_t - \beta^n_t\}$$

$$t \leftarrow t-1$$

Return to the beginning of step 2.

(Step 3) If $\beta^n_1 < Q^n_1$, then $\Omega = \phi$ and exit; else, if n=N, then $\Omega \neq \phi$ and exit;

else, $n \leftarrow n+1$ and return to step 1.

Through steps 1 and 2, the algorithm computes $\langle \bar{X}^n \rangle \in \Omega$ (n; $\bar{X}^{s(n)}$)

$$= \{ \langle x^n \rangle \mid (w^n_0/\alpha^n) + v^n_0 + \sum_{\tau=1}^{t} X^n_\tau - E^{n,s(n)} \sum_{\tau=1}^{t} \bar{X}^{s(n)}_\tau \geq 0,$$

$X^n_t \in \{0,1,\ldots,\beta^n_t\}$ for t=1,...,T} such that $\sum_{\tau=1}^{t} \bar{X}^n_\tau \leq$

$\sum_{\tau=1}^{t} X^n_\tau$ (t=1,...,T) for every $\langle X^n \rangle \in \Omega$ (n; $\bar{X}^{s(n)}$). If $\beta^n_1 < Q^n_1$ in step 3,

then such $\langle \bar{X}^n \rangle$ does not exist and consequently $\Omega(n; \bar{X}^{s(n)}) = \Omega = \phi$.

4. MODEL SPECIALIZATION

In the previous discussion, we developed a general approach to the Kanban system. In practice, managers may find certain choices of container size desirable. In this section, we shall investigate the solution procedures to the Kanban model for two particular choices of container size.

4.1 A Container-For-Container Mode

By container-for-container mode we mean that exactly one full container of item n is required to make one full container of the subsequent item s(n) for all n $\in$ {1,...,N}. In other words, under this mode container sizes must be adjusted properly so that $e^{n,s(n)}\alpha^{s(n)} = \alpha^n$, or $E^{n,s(n)} = 1$, for all n $\in$ {1,...,N}. Note that $e^{n,s(n)}$ is given by the product

specification. The parameters to adjust are α^n for $n=1,\ldots,N$. The one-for-one scheme, whenever possible, is a convenient one and it is supported by the philosophy of just-in-time production. It does not need handling multiple containers from one stage to the next as would be the situation for the case where $E^{n,s(n)}$ is greater than one; neither does it tend to accumulate the inventory of work-in-process as would be the situation for the case where $E^{n,s(n)}$ is smaller than one.

Consider the following optimization model:

minimize (2.8)

s.t.

(4.1) $v_0^n + \sum_{\tau=1}^{t} x_\tau^n - \sum_{\tau=1}^{t} x_\tau^{s(n)} \geq 0$

$n=1,\ldots,N; \; t=1,\ldots,T$

(4.2) $U_0^n - \sum_{\tau=1}^{t} x_\tau^n + \sum_{\tau=1}^{t-1} x_\tau^{s(n)} \geq 0$

$n=1,\ldots,N; \; t=1,\ldots,T$

(4.3) $0 \leq x_t^n \leq \beta_t^n$ $n=1,\ldots,N; \; t=1,\ldots,T$

(4.4) $U_0^n \in \{0,1,2,\ldots\}$ $n=1,\ldots,N$

We refer to the above model as (M1). We summarize in the next theorem the relation between models (M0) and (M1).

Theorem 4.1   Assume that $E^{n,s(n)} = 1$ for all $n \in \{1,\ldots,N\}$. (M0) is feasible if and only if (M1) is feasible. The two models have the same set of feasible partial solutions $<U_0>$, the same set of optimal partial solutions $<U_0>$, and the same optimal value.

(The proof of Theorem 4.1 is provided in Appendix 2.)

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(M1) is a mixed integer linear progam which has 2NT constraints, excluding (4.3)-(4.4), and NT+N variables, of which N are required to be integral. Recall that NT+N variables are required to be integral in (MO). Therefore, it is easier to solve (M1) than to solve (MO); and Theorem 4.1 ensures that we can still obtain a relevant optimal partial solution $<U_0>$ for (MO), and hence for (M), by solving (M1). Note that for any feasible solution $<U_0, \bar{X}>$ to model (M1), the proof of Theorem 4.1 provides a way for constructing $<X>$ such that $<U_0, X>$ is a feasible solution to model (MO) under the container-for-container mode.

4.2 A One-Container-For-Multiple-Containers Mode

By one-container-for-multiple-container mode, we mean that one full container of item n is required to make an integral number of full containers of the subsequent item s(n) for all $n \in \{1,\ldots,N\}$. Under this mode, the container sizes must be adjusted properly so that the inverse of $E^{n,s(n)}$ is an integer for all $n \in \{1,\ldots,N\}$. This mode is not supported by the philosophy of just-in-time production because it encourages large-sized containers in the upstream stages and tends to create partially filled containers. Nevertheless, the specialization of the model to a one-container-for-multiple-containers mode is motivated by the following result regarding the lot-sizing problem for multi-stage assembly production. Under the assumptions of constant continuous final-product demand over an infinite planning horizon with a fixed setup cost per lot and a linear inventory holding cost on echelon inventory in each stage, Crowston, Wagner, and Williams [1] proved that the optimal lot sizes in minimizing average cost per time period have the following property: the optimal lot size of item n is an

integral multiple of the optimal lot size of item $s(n)$ for all $n \in \{1,\ldots,N\}$.

We impose the following conditions: (a) the inverse of $E^{n,s(n)}$ is an integer for all $n \in \{1,\ldots,N\}$; and (b) W_0^n is an integral multiple of $e^{n,s(n)}\alpha^{s(n)}$, or equivalently (W_0^n/α^n) is an integral multiple of $E^{n,s(n)}$, for all $n \in \{1,\ldots,N\}$. Condition (a) is the definition of one-container-for-multiple-containers mode. Condition (b) deals with the partially filled containers which exist at the beginning of the planning horizon. As shown in the next theorem, condition (b) can always be enforced under condition (a).

Theorem 4.2 Assume that condition (a) holds. Let $\bar{W}_0^n \in \{0,1,\ldots,\alpha^n-1\}$ and

$$\bar{\bar{W}}_0^n = e^{n,s(n)}\alpha^{s(n)} * \left\lfloor \bar{W}_0^n/(e^{n,s(n)}\alpha^{s(n)}) \right\rfloor \quad \text{for } n=1,\ldots,N.$$ Then, $\langle U_0,X\rangle$

satisfies (3.1)-(3.4) with $\langle W_0\rangle = \langle \bar{W}_0\rangle$ if and only if $\langle U_0,X\rangle$ satisfies

(3.1)-(3.4) with $\langle W_0\rangle = \langle \bar{\bar{W}}_0\rangle$.

(The proof of Theorem 4.2 is provided in Appendix 3.)

According to Theorem 4.2, we can always replace W_0^n by $e^{n,s(n)}\alpha^{s(n)} * \left\lfloor W_0^n/(e^{n,s(n)}\alpha^{s(n)}) \right\rfloor$ in model (MO) without altering the feasible region of the model under the one-container-for-multiple-containers mode. As a result, condition (b) does not really impose any additional restriction.

Consider the following optimization model:

minimize (2.8)

s.t. (3.1), (3.2)

(3.3)' $0 \le X_t^n \le \beta_t^n$ \hspace{2cm} $n=1,\ldots,N; \; t=1,\ldots,T$

$(3.4)'$ $U_0^n \geq 0$ $\qquad\qquad$ $n=1,\ldots,N$

We refer to the above model as model (M2). The model is a linear programming relaxation of (MO). We summarize their relation in the next theorem.

Theorem 4.3 Assume that conditions (a) and (b) hold. (MO) is feasible if and only if (M2) is feasible. If $<U_0,X>$ is feasible for (MO), it is also feasible for (M2). If $<U_0,X>$ is feasible for (M2), then $< \lceil U_0^1 \rceil + 1,\ldots, \lceil U_0^N \rceil + 1>$ is a feasible partial solution to (MO).

(The proof of Theorem 4.3 is provided in Appendix 4.)

It is easier to solve (M2), a linear program, than to solve (MO), an integer linear program. Theorem 4.3 provides a way to approximate the optimal partial solution $<U_0>$ of (MO) by solving (M2). We are therefore interested in knowing the performance of the approximation. Let Z_0 be the optimal objective value of (MO) and Z_2 the objective value given by $<\lceil U_0^1 \rceil + 1, \ldots, \lceil U_0^N \rceil + 1>$ where $<U_0>$ is an optimal partial solution to (M2). The next theorem shows that the relative error $(Z_2-Z_0)/Z_0$ caused by the LP approximation approaches zero asymptotically.

Theorem 4.4 Assume that conditions (a) and (b) hold. Also assume that (MO) and (M2) are feasible. Then, $(Z_2-Z_0)/Z_0 \to 0$ as $Q^0/T \to \infty$.

(The proof of Theorem 4.4 is provided in Appendix 5.)

In the above theorem, Q^0/T represents the average production requirement (in terms of the number of containers) per period at the final stage. When Q^0/T becomes very large, the relative error due to the approximation of (MO) by (M2) becomes negligible.

Example 2   Let $N=1$, $T=2$, $c^1=1$, $\alpha^1=5$, $e^{1,0}=1$, $\alpha^0=1$, $E^{1,0}=0.2$, $\beta_1^1=0$,

$\beta_2^1=\infty$, $\varepsilon=0.1$, $V_0^1=0$, $W_0^1=4$, $X_1^0=4$, and $X_2^0=26$. At optimality, $U_0^1=6$

for (MO) while $U_0^1=4.3$ for (M2). Since $\lceil 4.3 \rceil + 1 = 6$, the approximation method

actually finds the optimal U_0^1 for (MO) in this example.   □

Example 3   Let $N=1$, $T=2$, $c^1=1$, $\alpha^1=4$, $e^{1,0}=1$, $\alpha^0=1$, $E^{1,0}=0.25$, $\beta_1^1=\beta_2^1=\infty$,

$\varepsilon=0.1$, $V_0^1=0$, $W_0^1=0$, and $X_1^0=X_2^0=8$. At optimality, $U_0^1=2$ for (MO)

while $U_0^1=1.1$ for (M2). Since $\lceil 1.1 \rceil + 1 = 3 > 2$, the approximation

method finds a non-optimal feasible U_0^1 for (MO) in this case. With

$Q^0/T = 8$, the relative error $(Z_2-Z_0)/Z_0 = 4/11$.

Now let $X_1^0 = X_2^0 = 800$ while the other parameters remain

unchanged. At optimality, $U_0^1 = 200$ for (MO) while $U_0^1 = 199.1$ for

(M2). With $Q^0/T = 800$, the relative error $(Z_2-Z_0)/Z_0 = 4/803$.   □

5. CONCLUSION

In this paper we have presented an optimization model for the Kanban

system. The model is intended for a multi-stage capacitated assembly-type

production setting. We have also provided a general solution procedure to the

model and discussed two special cases of practical interest. Future research

topics would include the development of Kanban models for the distribution-

type and mixed-type production settings and the inclusion of independent

(external) demands for upstream items.

ACKNOWLEDGEMENT

APPENDIX 1: <u>Proof of Lemma 3.2</u>

[Proof] Let $<U,V,W,X,Y>$ satisfy (2.1)-(2.2), (2.3)', (2.4)-(2.6), and

(2.7)'. It follows from (2.2), (2.3)', (2.4), and (2.6) that $E^{n,s(n)}\alpha^{s(n)}X_t^{s(n)}$

$$\leq \alpha^n V_{t-1}^n + W_{t-1}^n + \alpha^n X_t^n = \alpha^n (V_0^n + \sum_{\tau=1}^{t-1} X_\tau^n - \sum_{\tau=1}^{t-1} Y_\tau^n) + (W_0^n + \alpha^n \sum_{\tau=1}^{t-1} Y_\tau^n$$

$$- E^{n,s(n)}\alpha^{s(n)} \sum_{\tau=1}^{t-1} X_\tau^{s(n)}) + \alpha^n X_t^n. \text{ Hence, } (W_0^n/\alpha^n) + V_0^n + \sum_{\tau=1}^{t} X_\tau^n$$

$$- E^{n,s(n)} \sum_{\tau=1}^{t} X_\tau^{s(n)} \geq 0. \text{ So, } <U_0,X> \text{ satisfies (3.1). It follows from}$$

(2.5) and (2.6) that $\displaystyle\sum_{\tau=1}^{t} Y_\tau^n = \left\lceil (E^{n,s(n)}\alpha^{s(n)} \sum_{\tau=1}^{t} X_\tau^{s(n)} - W_0^n)/\alpha^n \right\rceil$;

and it follows from (2.1) and (2.3)' that $\displaystyle U_0^n - \sum_{\tau=1}^{t} X_\tau^n + \sum_{\tau=1}^{t-1} Y_\tau^n \geq 0.$ Hence,

$$U_0^n - \sum_{\tau=1}^{t} X_\tau^n + E^{n,s(n)} \sum_{\tau=1}^{t-1} X_\tau^{s(n)} - (W_0^n/\alpha^n) + 1 - \varepsilon \geq$$

$$U_0^n - \sum_{\tau=1}^{t} X_\tau^n + \left\lceil (E^{n,s(n)}\alpha^{s(n)} \sum_{\tau=1}^{t-1} X_\tau^{s(n)} - W_0^n)/\alpha^n \right\rceil =$$

$$U_0^n - \sum_{\tau=1}^{t} X_\tau^n + \sum_{\tau=1}^{t-1} Y_\tau^n \geq 0. \text{ So, } <U_0,X> \text{ satisfies (3.2). The first half of the}$$

lemma is then proved.

Let $<U_0,\overline{X}>$ satisfy (3.1)-(3.4). Define $<X>$ as follows:

$$X_1^n = \min \{\overline{X}_1^n, Q^n\} \text{ and } X_t^n = \min \{\sum_{\tau=1}^{t} \overline{X}_\tau^n, Q^n\} - \min \{\sum_{\tau=1}^{t-1} \overline{X}_\tau^n, Q^n\} \text{ for } t=2,\ldots,T.$$

It follows from the definition of $<X>$ that $\displaystyle\sum_{\tau=1}^{t} X_\tau^n = \min \{\sum_{\tau=1}^{t} \overline{X}_\tau^n, Q^n\}$

for t=1,...,T and in particular, $\sum\limits_{\tau=1}^{T} X_\tau^n \leq Q^n$. We now show that $<U_0,\bar{X}>$ also

satisfies (3.1)-(3.4). It follows from the definition of Q^n that

$(W_0^n/\alpha^n) + V_0^n + Q^n - E^{n,s(n)}Q^{s(n)} \geq 0$. Since $<U_0,\bar{X}>$ satisfies (3.1),

$(W_0^n/\alpha^n) + V_0^n + \sum\limits_{\tau=1}^{t} X_\tau^n - E^{n,s(n)} \sum\limits_{\tau=1}^{t} X_\tau^{s(n)} = (W_0^n/\alpha^n) + V_0^n +$

$\min \{ \sum\limits_{\tau=1}^{t} \bar{X}_\tau^n, Q^n\} - E^{n,s(n)} * \min \{ \sum\limits_{\tau=1}^{t} \bar{X}_\tau^{s(n)}, Q^{s(n)}\} \geq 0$ and hence $<U_0,\bar{X}>$

satisfies (3.1). It also follows from the definition of Q^n and ϵ that

$-Q^n + E^{n,s(n)}Q^{s(n)} - (W_0^n/\alpha^n) + 1 - \epsilon > 0$. Since $<U_0,\bar{X}>$ satisfies (3.2) and

(3.4), $U_0^n - \sum\limits_{\tau=1}^{t} X_\tau^n + E^{n,s(n)} \sum\limits_{\tau=1}^{t-1} X_\tau^{s(n)} - (W_0^n/\alpha^n) + 1 - \epsilon = U_0^n -$

$\min \{ \sum\limits_{\tau=1}^{t} \bar{X}_\tau^n, Q^n\} + E^{n,s(n)} * \min \{ \sum\limits_{\tau=1}^{t-1} \bar{X}_\tau^{s(n)}, Q^{s(n)}\} - (W_0^n/\alpha^n) + 1 - \epsilon \geq 0$ and

hence $<U_0,\bar{X}>$ satisfies (3.2). By construction, $<\bar{X}>$ satisfies

(3.3). We have hence proved that $<U_0,\bar{X}>$ satisfies (3.1)-(3.4). We next

show that $<U_0,\bar{X}>$ is a feasible partial solution to (2.1)-(2.2), (2.3)',

(2.4)-(2.6), and (2.7)'. To construct a complement to $<U_0,\bar{X}>$, we let

U_t^n, V_t^n, Y_t^n, and W_t^n (n=1,...,N; t=1,...,T) be defined

according to (2.1)-(2.2) and (2.5)-(2.6). If $U_{t-1}^n + Y_{t-1}^n <$

X_t^n, then $U_{t-1}^n - X_t^n + Y_{t-1}^n \leq -1$ since all the variables

involved are integers. As a result, $U_0^n - \sum\limits_{\tau=1}^{t} X_\tau^n + E^{n,s(n)} \sum\limits_{\tau=1}^{t-1} X_\tau^{s(n)} -$

$(W_0^n/\alpha^n) + 1 - \epsilon \leq U_0^n - \sum\limits_{\tau=1}^{t} X_\tau^n + \left\lceil E^{n,s(n)} \sum\limits_{\tau=1}^{t-1} X_\tau^{s(n)} - (W_0^n/\alpha^n)\right\rceil + 1 - \epsilon =$

$U_{t-1}^n - X_t^n + Y_{t-1}^n + 1 - \epsilon \leq -1 + 1 - \epsilon = -\epsilon < 0$. A contradiction arises and

hence $U^n_{t-1} + Y^n_{t-1} \geq X^n_t$. By straightforward substitution, we can show that $\langle U,V,W,X,Y \rangle$ satisfies (2.3)' and (2.4). The second half of the lemma is then proved. $\square$

Appendix 2: <u>Proof of Theorem 4.1</u>

[Proof] Let $\langle U_0, X \rangle$ satisfy (3.1)-(3.4). Since all the variables involved are integral, $0 \leq (W^n_0/\alpha^n) < 1$, $0 < -(W^n_0/\alpha^n) + 1 - \epsilon < 1$ and $E^{n,s(n)} = 1$, we conclude that $\langle U_0, X \rangle$ also satisfies (4.1)-(4.4).

Let $\langle U_0, \bar{X} \rangle$ satisfy (4.1)-(4.4). We want to show $\langle U_0 \rangle$ is a feasible partial solution to (3.1)-(3.4). To construct a complement to $\langle U_0 \rangle$, we let $X^n_1 = \lceil \bar{X}^n_1 \rceil$

and $X^n_t = \lceil \sum_{\tau=1}^{t} \bar{X}^n_\tau \rceil - \lceil \sum_{\tau=1}^{t-1} \bar{X}^n_\tau \rceil$ for $t=2,\ldots,T$. It follows that $\sum_{\tau=1}^{t} X^n_\tau =$

$\lceil \sum_{\tau=1}^{t} \bar{X}^n_\tau \rceil$ and X^n_t is a nonnegative integer. If $V^n_0 + \sum_{\tau=1}^{t} X^n_\tau - \sum_{\tau=1}^{t} X^{s(n)}_\tau < 0$, then

$V^n_0 + \sum_{\tau=1}^{t} X^n_\tau - \sum_{\tau=1}^{t} X^{s(n)}_\tau \leq -1$ since all the variables involved are integral. Then,

$V^n_0 + \sum_{\tau=1}^{t} \bar{X}^n_\tau - \sum_{\tau=1}^{t} \bar{X}^{s(n)}_\tau \leq (V^n_0 + \sum_{\tau=1}^{t} X^n_\tau - \sum_{\tau=1}^{t} X^{s(n)}_\tau) + (\sum_{\tau=1}^{t} X^{s(n)}_\tau - \sum_{\tau=1}^{t} \bar{X}^{s(n)}_\tau) \leq$

$-1 + (\lceil \sum_{\tau=1}^{t} \bar{X}^{s(n)}_\tau \rceil - \sum_{\tau=1}^{t} \bar{X}^{s(n)}_\tau) < 0$, which contradicts (4.1) for $\langle U_0, \bar{X} \rangle$.

Hence, we conclude that $V^n_0 + \sum_{\tau=1}^{t} X^n_\tau - \sum_{\tau=1}^{t} X^{s(n)}_\tau \geq 0$ and $\langle U_0, X \rangle$ satisfies (4.1).

Similarly, we can show that $\langle U_0, X \rangle$ satisfies (4.2). It follows from the

construction of $<X>$ that $X_t^n = \left\lceil \sum_{\tau=1}^{t} \bar{X}_\tau^n - \lceil \sum_{\tau=1}^{t-1} \bar{X}_\tau^n \rceil \right\rceil = \left\lceil \bar{X}_t^n - (\lceil \sum_{\tau=1}^{t-1} \bar{X}_\tau^n \rceil \right.$ —

$\left. \sum_{\tau=1}^{t-1} \bar{X}_\tau^n) \right\rceil \leq \lceil \bar{X}_t^n \rceil \leq \beta_t^n$; hence, $<U_0,X>$ satisfies (4.3). Noting that all the

variables in $<U_0,X>$ are integral, $0 \leq (W_0^n/\alpha^n) < 1$, $0 < - (W_0^n/\alpha^n) + 1$

$- \epsilon < 1$ and $E^{n,s(n)} = 1$, we conclude that $<U_0,X>$ also satisfies (3.1)–(3.4).

$\square$

Appendix 3: <u>Proof of Theorem 4.2</u>

[Proof] Note that $0 \leq \bar{W}_0^n - \bar{\bar{W}}_0^n \leq e^{n,s(n)}\alpha^{s(n)} - 1$, or equivalently,

$0 \leq (\bar{W}_0^n/\alpha^n) - (\bar{\bar{W}}_0^n/\alpha^n) \leq E^{n,s(n)} - (1/\alpha^n)$. Also note that $0 < \epsilon < (1/\alpha^n) \leq E^{n,s(n)}$.

Let $<U_0,X>$ satisfy (3.1)–(3.4) with $<W_0> = <\bar{W}_0>$. It follows from

(3.1) that $(\bar{\bar{W}}_0^n/\alpha^n) + V_0^n + \sum_{\tau=1}^{t} X_\tau^n - E^{n,s(n)} \sum_{\tau=1}^{t} X_\tau^{s(n)} = (\bar{\bar{W}}_0^n/\alpha^n) + V_0^n +$

$\sum_{\tau=1}^{t} X_\tau^n - E^{n,s(n)} \sum_{\tau=1}^{t} X_\tau^{s(n)} - [(\bar{W}_0^n/\alpha^n) - (\bar{\bar{W}}_0^n/\alpha^n)] \geq - [(\bar{W}_0^n/\alpha^n) - (\bar{\bar{W}}_0^n/\alpha^n)] \geq -$

$[E^{n,s(n)} - (1/\alpha^n)]$. Since $(\bar{\bar{W}}_0^n/\alpha^n)$, V_0^n, X_τ^n (for $\tau=1,\ldots,t$) and $E^{n,s(n)}X_\tau^{s(n)}$

(for $\tau=1,\ldots,t$) are all integral multiples of $E^{n,s(n)}$, we conclude that $(\bar{\bar{W}}_0^n/\alpha^n) +$

$V_0^n + \sum_{\tau=1}^{t} X_\tau^n - E^{n,s(n)} \sum_{\tau=1}^{t} X_\tau^{s(n)} \geq 0$. It is straightforward to show that

$U_0^n - \sum_{\tau=1}^{t} X_\tau^n + E^{n,s(n)} \sum_{\tau=1}^{t-1} X_\tau^{s(n)} - (\bar{\bar{W}}_0^n/\alpha^n) + 1 - \epsilon \geq 0$.

Let $<U_0,X>$ satisfy (3.1)–(3.4) with $<W_0> = <\bar{\bar{W}}_0>$. It follows from (3.2)

that $U_0^n - \sum_{\tau=1}^{t} X_\tau^n + E^{n,s(n)} \sum_{\tau=1}^{t-1} X_\tau^{s(n)} - (\bar{W}_0^n/\alpha^n) + 1 \geq E^{n,s(n)}$ since U_0^n, X_τ^n (for

$\tau=1,\ldots,t)$, $E^{n,s(n)} X_\tau^{s(n)}$ (for $\tau=1,\ldots,t-1)$, $(\bar{W}_0^n/\alpha^n)$ and 1 are all integral

multiples of $E^{n,s(n)}$ while $0 < \varepsilon < (1/\alpha^n) \leq E^{n,s(n)}$. Consequently, $U_0^n - \sum_{\tau=1}^{t} X_\tau^n +$

$E^{n,s(n)} \sum_{\tau=1}^{t-1} X_\tau^{s(n)} - (\bar{W}_0^n/\alpha^n) + 1 - \varepsilon = U_0^n - \sum_{\tau=1}^{t} X_\tau^n + E^{n,s(n)} \sum_{\tau=1}^{t-1} X_\tau^{s(n)} -$

$(\bar{W}_0^n/\alpha^n) + 1 - \varepsilon + (\bar{W}_0^n/\alpha^n) - (\bar{W}_0^n/\alpha^n) \geq E^{n,s(n)} - \varepsilon - E^{n,s(n)} + (1/\alpha^n) > 0$. It

is straightforward to show that $(\bar{W}_0^n/\alpha^n) + V_0^n + \sum_{\tau=1}^{t} X_\tau^n - E^{n,s(n)} \sum_{\tau=1}^{t} X_\tau^{s(n)} \geq 0$.

$\square$

Appendix 4: <u>Proof of Theorem 4.3</u>

[Proof] We shall prove only the third part of the theorem. Let $\langle \bar{U}_0, \bar{X} \rangle$

satisfy (3.1)-(3.2), (3.3)' and (3.4)'. Define $U_0^n = \lceil \bar{U}_0^n \rceil + 1$, $X_1^n = \lceil \bar{X}_1^n \rceil$ and

$X_t^n = \lceil \sum_{\tau=1}^{t} \bar{X}_\tau^n \rceil - \lceil \sum_{\tau=1}^{t-1} \bar{X}_\tau^n \rceil$ for $t=2,\ldots,T$. We want to show that $\langle U_0, X \rangle$

satisfies (3.1)-(3.4). If $(W_0^n/\alpha^n) + V_0^n + \sum_{\tau=1}^{t} X_\tau^n - E^{n,s(n)} \sum_{\tau=1}^{t} X_\tau^{s(n)} < 0$, then

$(W_0^n/\alpha^n) + V_0^n + \sum_{\tau=1}^{t} X_\tau^n - E^{n,s(n)} \sum_{\tau=1}^{t} X_\tau^{s(n)} \leq - E^{n,s(n)}$. Hence, $(W_0^n/\alpha^n) + V_0^n +$

$\sum_{\tau=1}^{t} \bar{X}_\tau^n - E^{n,s(n)} \sum_{\tau=1}^{t} \bar{X}_\tau^{s(n)} \leq - E^{n,s(n)} + E^{n,s(n)} (\lceil \sum_{\tau=1}^{t} \bar{X}_\tau^{s(n)} \rceil - \sum_{\tau=1}^{t} \bar{X}_\tau^{s(n)})$

< 0, which contradicts (3.1) for $\langle \bar{U}_0, \bar{X} \rangle$. Thus, we conclude that $\langle U_0, X \rangle$

satisfies (3.1). Since $\langle \bar{U}_0, \bar{X} \rangle$ satisfies (3.2), it follows that $U_0^n - \sum_{\tau=1}^{t} X_\tau^n +$

$E^{n,s(n)} \sum_{\tau=1}^{t-1} X_\tau^{s(n)} - (W_0^n/\alpha^n) + 1 - \epsilon \geq 1 - (\left\lceil \sum_{\tau=1}^{t} \bar{X}_\tau^n \right\rceil - \sum_{\tau=1}^{t} \bar{X}_\tau^n) > 0$ and we

conclude that $\langle U_0, \bar{X} \rangle$ satisfies (3.2). $\qquad\qquad\qquad\square$

Appendix 5: Proof of Theorem 4.4

[**Proof**] Let $\langle U_0, \bar{X} \rangle$ and $\langle \bar{U}_0, \bar{X}_0 \rangle$ be the optimal solutions to (MO) and (M2),

respectively. Since $\sum_{n=1}^{N} c^n \bar{U}_0^n \leq \sum_{n=1}^{N} c^n U_0^n \leq \sum_{n=1}^{N} c^n (\left\lceil \bar{U}_0^n \right\rceil + 1) < \sum_{n=1}^{N} c^n (\bar{U}_0^n + 2)$,

we conclude that $Z_2 - Z_0 < 2 \sum_{n=1}^{N} c^n$. Since $\langle U_0, \bar{X} \rangle$ satisfies (3.1) and (3.2),

$(W_0^n/\alpha^n) + V_0^n + \sum_{\tau=1}^{t-1} X_\tau^n - E^{n,s(n)} \sum_{\tau=1}^{t-1} X_\tau^{s(n)} \geq 0$ and $U_0^n - \sum_{\tau=1}^{t} X_\tau^n$

$+ E^{n,s(n)} \sum_{\tau=1}^{t-1} X_\tau^{s(n)} - (W_0^n/\alpha^n) + 1 - \epsilon \geq 0$. By adding the previous two

inequalities together, we have $U_0^n + V_0^n - X_t^n + 1-\epsilon \geq 0$. Since U_0^n, V_0^n and X_t^n are

all integral and $0 < 1-\epsilon < 1$, we have $U_0^n + V_0^n \geq X_t^n$. Since $U_0^n + V_0^n \geq X_t^n$ holds

for all $t=1, \ldots, T$, we have $U_0^n + V_0^n \geq (\sum_{\tau=1}^{T} X_\tau^n)/T \geq Q^n/T$. It follows that

$Z_0 \geq \sum_{n=1}^{N} c^n Q^n/T$ and $(Z_2 - Z_0)/Z_0 < 2 \sum_{n=1}^{N} c^n / (\sum_{n=1}^{N} c^n Q^n/T)$. By the definition

of Q^n, $Q^n/T \to \infty$ as $Q^0/T \to \infty$. Hence, $(Z_2 - Z_0)/Z_0 \to 0$ as $Q^0/T \to \infty$. $\qquad\square$

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