COMBINATORIAL THEOREMS ON THE SIMPLOTOPE THAT GENERALIZE RESULTS ON THE SIMPLEX AND CUBE

by

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Abstract

In the context of the theory and computation of fixed points of continuous mappings, researchers have developed combinatorial analogs of Brouwer's fixed-point theorem on the simplex and on the n-cube. Although the simplex and the cube have different combinatorial properties regarding their boundaries, they are both instances of a simplotope, which is the cross-product of simplices. This paper presents three combinatorial theorems on the simplotope, and shows how each translates into some known and new results on the simplex and cube, including various forms of Sperner's lemma. Each combinatorial theorem also implies set covering lemmas on the simplotope, the simplex, and the cube, including the Generalized Covering lemma, the Knaster-Kuratowski-Mazurkiewicz lemma, and a lemma of Freidenfelds.

Key Words: simplotope, simplex, cube, fixed-point, V-complex, combinatorial lemma, set covering.

1. Introduction

In the context of the theory and computation of fixed-points of continuous mappings, researchers have developed combinatorial analogs of Brouwer's fixed-point theorem on the simplex (see [1], [3], [5], [9], and [10]) and the n-cube (see [3], [5], [7]). Although the simplex and the cube have different combinatorial properties regarding their boundaries, they are both instances of a *simplotope*, which is the cross-product of simplices. This paper presents three combinatorial theorems on the simplotope, and shows how each translates into some known and new results on the simplex and cube. It is shown that these three theorems are each equivalent to Brouwer's fixed-point theorem, in the sense that each yields a relatively short proof of Brouwer's theorem, and *vice versa*. Furthermore, each combinatorial theorem implies a set covering lemma on the simplotope, that in turn implies set covering lemmas on the cube and simplex, including the Generalized Covering lemma [5], the Knaster-Kuratowski-Mazurkiewicz lemma [6] and a lemma of Freidenfelds [2] on the simplex.

Sperner's lemma [10] and Scarf's dual Sperner lemma [9] rely on a "proper" labelling and a "dual proper" labelling of the vertices of a triangulation of the simplex, where the labelling is restricted on the boundary in each instance. On the other hand, the Generalized Sperner Lemma ([1] or [3]) relies on no restriction on the labelling used. Generalizing the above, the first of the three combinatorial theorems on the simplotope presented herein does not depend on any restrictions on the labelling on the boundary. The second and third combinatorial theorems on the simplotope depend on a *proper* and *dual proper* labelling of the simplotope, where these terms are defined precisely in section 4.

The notation used is presented in section 2. In section 3, we give a synopsis of the terminology and theory of V-complexes, as presented in [4]. This theory is central to the proofs of the combinatorial theorems to follow. In section 4, the three combinatorial theorems are proved, and their equivalence to Brouwer's theorem is demonstrated. Furthermore, three set covering lemmas on the simplotope are also presented. In section 5, the results of sections 4 are applied to the simplex. These results include Sperner's lemma [10], Scarf's dual Sperner lemma [9], the Generalized Sperner lemma ([1] a [3]), The Generalized Covering lemma of [5], the Knaster-Kuratowski-Mazurkiewicz lemma [6], and a lemma of Freidenfelds [2]. In section 6, the results of section 4 are applied to the cube. These results include lemmas 1 and 2 of [5], and new results as well. Section 7 contains concluding remarks.

2. Notation

Let $\mathbb{R}^n$ denote real n-dimensional space, and define e to be the vector of 1's, namely $e = (1,\ldots, 1)$. Let ϕ denote the empty set, and let $|S|$ denote the cardinality of a set S. For two sets S, T, let $S \setminus T = \{x \mid x \in S, x \notin T\}$.

Let $v^0,\ldots, v^m$ be vectors in $\mathbb{R}^n$. If the matrix

$$\begin{bmatrix} v^0 & \ldots & v^m \\ 1 & \ldots & 1 \end{bmatrix}$$

has rank $(m + 1)$, then the convex hull of $v^0,\ldots, v^m$, denoted $\langle v^0,\ldots, v^m\rangle$ is said to be a real m-dimensional simplex, or more simply an m-simplex. If $\sigma = \langle v^0,\ldots, v^m\rangle$ is an m-simplex and $\{v^{j_0},\ldots, v^{j_k}\}$ is a nonempty subset of $\{v^0,\ldots, v^m\}$, then $\tau = \langle v^{j_0},\ldots, v^{j_k}\rangle$ is a k-face or face of σ.

Let H be an m-dimensional convex set in $\mathbb{R}^n$. Let C be a collection of m-simplices σ together with all of their faces. C is a triangulation of H if

i) $H = \bigcup_{\sigma \in C} \sigma$,

ii) $\sigma, \tau \in C$ imply $\sigma \cap \tau \in C$, and

iii) If σ is an (m-1)-simplex of C, σ is a face of at most two m-simplices of C.

C is said to be locally finite if for each vertex $v \in H$, the set of simplices $\sigma \in C$ that contain v is a finite set.

If $S_1,\ldots, S_n$ are n simplices in $\mathbb{R}^{m_1},\ldots, \mathbb{R}^{m_n}$, respectively, the set $S = S_1 \times \ldots \times S_n$ in $\mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$ is called a simplotope. Thus a simplotope is the cross product of n simplices, for $n \geq 1$. Note that any simplex is itself a simplotope (by setting n = 1), and the n-cube $\{x \in \mathbb{R}^n \mid 0 \leq x \leq e\}$ is just the cross product of the n 1-simplices $\{x_j \in \mathbb{R}^1 \mid 0 \leq x_j \leq 1\}$, $j = 1,\ldots, n$.

3. Review of V-Complex Terminology and Results

This section presents a condensation of the terminology and major results concerning the theory of V-complexes, as presented in [4]. This material is central to proofs of the combinatorial theorems in Section 4.

An abstract complex consists of a set of vertices K^0 and a set of finite nonempty subsets of K^0, denoted K, such that

i) $v \in K^0$ implies $\{v\} \in K$

ii) $\phi \neq x \subset y \in K$ implies $x \in K$.

An element x of K is called in abstract simplex, or more simply a simplex. If $x \in K$ and $|x| = n + 1$, then x is called an n-simplex, where $|\cdot|$ denotes cardinality. Technically, an abstract complex is defined by the pair (K^0, K). However, since the set K^0 is implied by K, it is convenient to denote the complex by K alone. An abstract complex K is said to be finite if K^0 is finite, and is locally finite if for each $v \in K^0$, the set of simplices $x \in K$ for which $v \in x$ is a finite set.

An n-dimensional pseudomanifold, or more simply an n-pseudomanifold, where $n \geq 1$, is a complex K such that

i) $x \in K$ implies there exists $y \in K$ with $|y| = n + 1$ and $x \subset y$.

ii) If $x \in K$ and $|x| = n$, then there are at most two n-simplices of K that contain x.

Let K be an n-pseudomanifold, where $n \geq 1$. The boundary of K, denoted ∂K, is defined to be the set of simplices $x \in K$ such that x is contained in an $(n - 1)$-simplex $y \in K$, and y is a subset of exactly one n-simplex of K.

A 0-dimensional pseudomanifold K is defined to be a set of one of the following two forms:

i) $K = \{\phi, \{v\}\}$, where $K^0 = \{v\}$, or

ii) $K = \{\phi, \{u\}, \{v\}\}$, where $K^0 = \{u, v\}$.

Because K contains ϕ, the empty set, as a member, K is not properly a complex by the usual definition. Here, however, ϕ is defined as a -1-simplex. If K is of type (i) above, we denote $\partial K = \{\phi\}$. If K is of type (ii) above, then $\partial K = \phi$, i.e. K has no boundary.

If C is a triangulation of a set H in $\mathbb{R}^n$ with vertex set K^0, then corresponding to each simplex σ in C is its set of vertices $\{v^0,\ldots, v^k\}$. Let K be collection of these sets of vertices together with their nonempty subsets. Then K is a pseudomanifold and K is defined to be the pseudomanifold <u>corresponding</u> to C.

Let K be a locally-finite abstract complex with vertices K^0. Let N be a fixed finite nonempty set, called the <u>label set</u>. Let $\mathcal{T}$ denote a collection of subsets of N, denoted the <u>admissible</u> subsets of N. Let $A(\cdot)$ be a map $A(\cdot) : \mathcal{T} \to 2^K \setminus \phi$, where 2^S denotes the collection of subsets of a set S. K, N, $\mathcal{T}$, $A(\cdot)$ are said to constitute a <u>V-complex</u> with operator $A(\cdot)$ and admissible sets $\mathcal{T}$, if the following eight conditions are met:

i) K is a locally finite complex with vertices K^0

ii) $\mathcal{T} \subset 2^N$

iii) $T \in \mathcal{T}$, $S \in \mathcal{T}$ implies $S \cap T \in \mathcal{T}$

iv) $A(\cdot) : \mathcal{T} \to 2^K \setminus \phi$

v) For any $x \in K$, there is a $T \in \mathcal{T}$ such that $x \in A(T)$

vi) For any $S, T \in \mathcal{T}$, $A(S \cap T) = A(S) \cap A(T)$

vii) For $T \in \mathcal{T}$, $A(T)$ is a pseudomanifold of dimension $|T|$

viii) If $T \in \mathcal{T}$ and $T \cup \{j\} \in \mathcal{T}$ but $j \notin T$, then $A(T) \subset \partial A(T \cup \{j\})$.

The nomenclature "V-complex" is short for variable-dimension complex, and derives from property (vii) above, where the dimension of the pseudomanifolds A(T) varies over the range of $T \in \mathcal{T}$.

If K is a V-complex, for each $x \in K$, we define:

$$T_x = \bigcap_{\substack{T \in \mathcal{T} \\ x \in A(T)}} T$$

x is a full simplex if $|x| = |T_x| + 1$. For each $T \in \mathcal{T}$, we also define $\partial' A(T)$ as $\partial' A(T) = \{x \in \partial A(T) \mid T_x = T\}$. If $\phi \in \mathcal{T}$ and $A(\phi) = \{\phi, \{v\}\}$, then $\partial' A(\phi) = \{\phi\}$. If $A(\phi) = \{\phi, \{u\}, \{v\}\}$, then $\partial' A(\phi) = \phi$.

Let K be a V-complex with label set N. A function $L(\cdot) : K^0 \to N$ that assigns an element of N to each vertex of K is said to be a labelling function. If $L(\cdot)$ is a labelling function, for each $x \in K$, we define $L(x) = \bigcup_{v \in x} L(v)$. Two distinct simplices $x, y \in K$ are defined to be adjacent, written $x \sim y$, if

i) x and y are full, and

ii) $L(x \cap y) = T_x \cup T_y$.

Note that if $x \sim y$ for some y, $L(x) \supset T_x$. To see this, observe that if $x \sim y$, $L(x) \supset L(x \cap y) = T_x \cup T_y \supset T_x$.

For a given V-complex K and labelling function $L(\cdot)$, we define the two sets:

$G = \{x \in K \mid x \text{ is full and } L(x) \supset T_x, \text{ and } L(x) \notin \mathcal{T}\}$, and

$B = \{x \in K \mid x \in \partial' A(T_x) \text{ and } L(x) = T_x\}$.

G and B are short for "good" and "bad", for in most applications of V-complexes, a path-following scheme will terminate with an element of G or B. G typically contains those simplices with pre-specified desirable properties, whereas

B does not. G can also be thought of as the "goal" set. Note that $B \cap G = \phi$. The following result is proved in [4]:

Lemma 3.1 ([4], Lemma 11). If $x \in K$, then x is adjacent to at most two other simplices in K.

With the above lemma in mind, we can construct paths of simplices in K. Let $\langle x_i \rangle_i$ be a maximal sequence of simplices in K such that $x_i \sim x_{i+1}$, and $x_{i-1} \neq x_{i+1}$. If x_i is left endpoint of this sequence, and $x \notin G$, then there exists a unique simplex $x_{i-1} \subset x_i$, such that $x_{i-1} \in B$, and we append x_{i-1} to the sequence. Likewise, if x_i is a right endpoint of the sequence, and $x_i \notin G$, then there exists a unique simplex $x_{i+1} \subset x_i$, such that $x_{i+1} \in B$, and we append x_{i+1} to the sequence. The new sequence, with possible endpoints added, is a path on K.

We have the following results, which are central to the proofs the combinatorial theorems in the next section:

Lemma 3.2 ([4], Lemma 12). Let $x \in K$. If x is an endpoint of a path on K, then $x \in B \cup G$.

Lemma 3.3 ([4], Lemma 13). If K is finite, then B and G have the same parity.

4. Three Combinatorial Theorems on the Simplotope, and Extensions

In this section, we present three combinatorial theorems on the simplotope, each of which is a generalization of related results on the simplex and the cube. We also present three covering lemmas related to these theorems, and show the equivalence of these results with Brouwer's fixed point theorem.

Define the standard $(m - 1)$-simplex in $\mathbb{R}^m$ to be the set $S^{m-1} = \{x \in \mathbb{R}^m \mid e \cdot x = 1, x \geq 0\}$. Our concern centers on the simplotope formed by taking the product of n standard simplices, namely $S = S^{m_1-1} \times \ldots \times S^{m_n-1}$, where we presume $m_j > 1$, $j = 1,\ldots, n$, to avoid trivialities.

If v is an element of S, let v^j denote the $j^{\underline{th}}$ concatenated vector of v, $j = 1,\ldots, n$, and let v^j_k denote the $k^{\underline{th}}$ component of v^j, $k = 1,\ldots, m_j$, $j = 1,\ldots, n$. Furthermore, define $F^j(v) = \{(j, k) \mid v^j_k > 0\}$, $j = 1,\ldots, n$, i.e. $F^j(v)$ is the carrier of v with respect to the j coordinates of v. If x is a set of vectors $v \in S$, then define $F^j(x) = \bigcup_{v\in x} F^j(v)$. Define e^{jk} to be the $k^{\underline{th}}$ unit vector in $\mathbb{R}^{m_j}$, and define $E = (e^{11}; e^{21};\ldots; e^{n1})$, and define $M = \sum_{j=1}^{n} (m_j - 1)$; i.e. M is the dimension of S.

In the context of a simplotope $S = S^{m_1-1} \times \ldots \times S^{m_n-1}$, define the label set N by $N = \{(j, k) \mid j \in \{1,\ldots, n\}, k \in \{1,\ldots, m_j\}\}$, and define $N^j = \{(j, 1),\ldots, (j, m_j)\}$, $j = 1,\ldots, n$. If $T \subset N$, denote $T^j = \{(j, k) \mid (j, k) \in T\}$, $j = 1,\ldots, n$.

Let C be a triangulation of S with vertex set K^o, and let K be the pseudomanifold corresponding to C. Let $L(\cdot) = K^o \to N$ be a labelling function on K^o. Then for $v \in K^o$, define $L^j(v) = \{(j, k) \mid (j, k) \in L(v)\}$, $j = 1,\ldots, n$, and for x a subset of K^o, define $L(x) = \bigcup_{v\in x} L(v)$ and $L^j(x) = \bigcup_{v\in x} L^j(v)$.

The following elementary lemma will be used in the analysis presented in the remainder of this section:

Lemma 0. Let $y^1, \ldots, y^n$ be elements of S^{n-1}, not necessarily distinct. Then there exists a nonempty set $S \subset \{1, \ldots, n\}$ with the property that

$$S = \{j \mid (y^i)_j > 0 \text{ for some } i \in S\}.$$

PROOF: Let Y be the matrix whose $i^{\underline{th}}$ column is y^i, $i = 1, \ldots, n$, and consider the system of equations:

$$\left.\begin{aligned} [I - Y]\lambda &= 0 \\ e \cdot \lambda &= 1 \\ \lambda &\geq 0 \end{aligned}\right\} (*) ,$$

where I is the identity matrix. If this system has no solution, then by a theorem of the alternative, there exists π, α, such that

$$\pi I - \pi Y + \alpha e \geq 0 , \alpha < 0.$$

Let π_i denote the smallest component of π, i.e. $\pi_i \leq \pi_j$, for $j = 1, \ldots, n$. Then we can write $\pi = \pi_i e + \beta$, $\beta \geq 0$, and $\beta_i = 0$. From the above, we have that $\pi_i - \pi y^i + \alpha \geq 0$, and $\alpha < 0$. However, $\pi y^i = (\pi_i e + \beta)y^i = \pi_i + \beta y^i$, since $e \cdot y^i = 1$, whereby $\pi_i - \pi y^i + \alpha \geq 0$ means $-\beta y^i + \alpha \geq 0$. But $\beta y^i \geq 0$ (since $\beta \geq 0$ and $y^i \geq 0$) and $\alpha < 0$, which is a contradiction. Therefore, the system (*) has a solution $\bar{\lambda}$.

Let $S = \{j \mid \bar{\lambda}_j > 0\}$. Because $e \cdot \bar{\lambda} = 1$, $S \neq \phi$. For each $j \in S$, $\bar{\lambda}_j > 0$. From (*), we have $\bar{\lambda}_j = \sum_{i=1}^{n} \bar{\lambda}_i (y^i)_j$, whereby if $\bar{\lambda}_j > 0$, there exists some i with $\bar{\lambda}_i > 0$ and $(y^i)_j > 0$; i.e. if $j \in S$, there exists $i \in S$ with $(y^i)_j > 0$. Thus $S \subset \{j \mid (y^i)_j > 0 \text{ for some } i \in S\}$.

Now suppose $j \in \{j \mid (y^i)_j > 0 \text{ for some } i \in S\}$. Thus there exists $i \in S$ with $\bar{\lambda}_i > 0$ and $(y^i)_j > 0$, whereby from (*) we must have $\bar{\lambda}_j > 0$, i.e. $j \in S$. Thus $S \supset \{j \mid (y^i)_j > 0 \text{ for some } i \in S\}$, from which it follows that $S = \{j \mid (y^i)_j > 0 \text{ for some } i \in S\}$. ⊠

Corollary 0.1 Let $T \subset \{1,\ldots, n\}$, $T \neq \phi$, and define $S^T = \{x \in S^{n-1} \mid x_j = 0$ for $j \in \{1,\ldots, n\} \setminus T\}$, i.e. S^T is the face of S^{n-1} whose carrier is T. For each $i \in T$, let y^i be a given element of S^T. Then there exists $S \subset T$, $S \neq \phi$, such that $S = \{j \mid (y^i)_j > 0$ for some $i \in S\}$.

Corollary 0.2 Let $S = S^{m_1-1} \times \ldots \times S^{m_n-1}$. Let C be a triangulation of S, with vertex set K^o, and let K be the pseudomanifold corresponding to C. Suppose there exists $x \in K$ and $j \in \{1,\ldots, n\}$ such that $L^j(x) = F^j(x)$. Then there exists $z \subset x$, $z \neq \phi$, such that $L(z) = F^j(z)$.

PROOF: Let $T = \{k \mid (j, k) \in L^j(x)\}$. Because $x^j \in S^{m_j-1}$, $L^j(x) = F^j(x) \neq \phi$, whereby $T \neq \phi$. For each $k \in T$, there exists some vector $\overset{k}{v} \in x$, such that $L(\overset{k}{v}) = (j, k)$. Let y^k be the $j^{\underline{th}}$ concatenated vector of v^k, i.e. $y^k = (\overset{k}{v})^j$. Note that since $F^j(x) = \{(j, k) \mid k \in T\}$, then $y^k \in S^T$ for each $k \in T$. By corollary 0.1, there exists a nonempty subset $S \subset T$, $S \neq \phi$, such that $S = \{\ell \mid (y^k)_\ell > 0$ for some $k \in S\}$. Let $z = \{\overset{k}{v} \mid k \in S\}$. Then $F^j(z) = \{(j, k) \mid k \in S\} = L(z)$. ⊠

With the above material and notation as background, we present our first result.

Theorem 1. Let $L(\cdot) : K^o \to N$ be a labelling function on the vertices of a triangulation C of $S = S^{m_1-1} \times \ldots \times S^{m_n-1}$ and let K be the pseudomanifold corresponding to C. Then there exists a simplex $x \in K$ and an index $j \in \{1,\ldots, n\}$ such that $L(x) = F^j(x)$.

The proof of Theorem 1 appears below, and proceeds by first defining a V-complex associated with K. Next, the special sets B and G are examined, and it is shown that if $x \in G$ or $x \in B \setminus \phi$, then there exists a subset y of x such that $L(y) = F^j(y)$ for some $j \in \{1,\ldots, n\}$. Because $\phi \in B$, and B and G have the same parity by lemma 3.3 there must exist some x such that $x \in G \cup B \setminus \phi$.

Proof of Theorem 1: We first construct a V-complex on K. Let $\mathcal{T} = \{T \subset N \mid (j, 1) \notin T$ for $j = 1,\ldots, n\}$, and let $A(\phi) = \{\{E\}, \phi\}$. For $T \in \mathcal{T}$, $T \neq \phi$, let $A(T)$ be the pseudomanifold corresponding to the restriction of C to $\{v \in S \mid v_k^j = 0$ for any $(j, k) \notin T^j \cup \{(j, 1)\}, j = 1,\ldots, n\}$. It is simple to verify that K, $A(\cdot)$, N, $\mathcal{T}$ constitute a V-complex.

Note that if $x \in A(T)$, then $F^j(x) \subset T^j \cup \{(j, 1)\}$, $j = 1,\ldots, n$, and if x is full, then $T_x^j \cup \{(j, 1)\} = F^j(x)$ for $j = 1,\ldots, n$. Since $A(\phi) = \{\{E\}, \phi\}$, $\phi \in B$. Suppose $\phi \neq x \in B$. Then $x \in \partial' A(T_x)$ and $L(x) = T_x$. Because $x \in \partial' A(T_x)$, there exists some j for which $v_1^j = 0$ for all $v \in x$, whereby $F^j(x) = T_x^j$, because x is a $(|T_x| - 1)$-simplex. But because $L(x) = T_x$, $L^j(x) = T_x^j = F^j(x)$. Applying Corollary 0.2, there exists a nonempty subset z of x such that $L(z) = F^j(z)$.

Now suppose $x \in G$. Then $T_x \subset L(x) \notin \mathcal{T}$. Thus there exists some $j \in \{1,\ldots, n\}$ such that $L(x) = T_x \cup \{(j, 1)\}$. Because x is full $F^j(x) = T_x^j \cup \{(j, 1)\} = L^j(x)$. Applying Corollary 0.2, there exists a nonempty subset z of x such that $L(z) = F^j(z)$.

Since K is finite, by lemma 3.3, B and G have the same parity, whereby there exixts some $x \in G \cup B \setminus \phi$. From the above remarks, there exixts $z \subset x$, $z \neq \phi$, with $L(z) = F^j(z)$ for some $j \in \{1,\ldots, n\}$. ☒

Note that the proof of theorem 1 is constructive. For a given triangulation C of S, the algorithm for finding an element x of $G \cup B \setminus \phi$ consists of starting at the endpoint $\phi \in B$ and following the unique path of adjacent simplices until the path terminates with an element x of $G \cup B \setminus \phi$. One of the finitely many subsets z of x will satisfy $L(z) = F^j(z)$ for some j.

Now suppose $L(\cdot) : K^0 \to N$ is a labelling function. $L(\cdot)$ is called a proper labelling if $v_k^j = 0$ implies $L(v) \neq (j, k)$, $j = 1,\ldots, n$, $k = 1,\ldots, m_j$.

$L(\cdot)$ is called dual proper if whenever $v \in K^o$ and $v \in \partial S$, then $v^j_k > 0$ implies $L(v) \neq (j, k)$, $j = 1, \ldots, n$, $k = 1, \ldots, m_j$.

Our next result is:

Theorem 2: Let $L(\cdot) : K^o \to N$ be a dual proper labelling function on the vertices of a triangulation C of $S = S^{m_1 - 1} \times \ldots \times S^{m_n - 1}$ and let K be the pseudomanifold corresponding to C. Suppose furthermore that for any $x \in K$, and any $j \in \{1, \ldots, n\}$, $\{k \mid v^j_k = 0 \text{ for some } v \in x\} \neq \{1, \ldots, m_j\}$. Then there exists $x \in K$ and $j \in \{1, \ldots, n\}$, such that $L(x) = N^j$.

Proof of Theorem 2 From theorem 1, there exists $x \in K$ and $j \in \{1, \ldots, n\}$ such that $L(x) = F^j(x)$. Suppose $F^j(x) \neq N^j$. Then for all k such that $(j, k) \in N^j \setminus F^j(x)$, $v^j_k = 0$ for all $v \in x$. Thus $v \in \partial S$ for all $v \in x$, and since $L(\cdot)$ is dual proper, for all $(j, k) \in F^j(x) = L(x)$, there exists $v \in x$ with $v^j_k = 0$. Thus $\{k \mid v^j_k = 0 \text{ for some } v \in x\} = \{1, \ldots, m_j\}$, a contradiction. Therefore $L(x) = F^j(x)$. ☒

Note that an algorithm for finding a simplex $x \in K$ such that $L(x) = F^j(x)$ for some j is just the algorithm suggested for theorem 1. Our third combinatorial theorem is:

Theorem 3 (van der Laan and Talman [8]) Let $L(\cdot) : K^o \to N$ be a proper labelling function on the vertices of a triangulation C of $S = S^{m_1 - 1} \times \ldots \times S^{m_n - 1}$, and let K be the pseudomanifold corresponding to C. Then there exists $x \in K$ and $j \in \{1, \ldots, n\}$ such that $L(x) = N^j$.

This theorem was first proved in van der Laan and Talman [8]. In their paper, a simplex x for which $L(x) = N^j$ is called a "j-stopping face", and their proof of the existence of a j-stopping face is for a special triangulation of S developed specifically for computational capabilities. The proof given below is more general, as it does not depend on any particular triangulation of S.

Proof of Theorem 3 We first construct a V-complex. Let $\mathcal{T} = \{T \subset N \mid (j, m_j) \notin T$, and $(j, k) \in T$ and $k > 1$ implies $(j, k - 1) \in T$, for all $j = 1,\ldots, n\}$. Define $A(\phi) = \{\{E\}, \phi\}$, and for $\phi \neq T \in \mathcal{T}$, define the region $R_j(T)$ as follows:

$$R_j(T) = \begin{cases} \{e^{j1}\} & \text{if } T^j = \phi \\ \\ \langle e^{j1},\ldots, e^{j,\,k+1}\rangle & \text{if } T^j = \{(j, 1), (j, 2),\ldots, (j, k)\}, \end{cases}$$

where $\langle\cdot\rangle$ denotes convex hull. Then define $A(T)$ to be the pseudomanifold corresponding to the restriction of C to $R_1(T) \times R_2(T) \times \ldots \times R_n(T)$. Note that if $v \in A(T)$, then $v^j_k = 0$ for $k > |T^j| + 1$. Also, note that if $|T^j| = t$, $R_j(T) = \langle e^{j1},\ldots, e^{j,\,t+1}\rangle$. If $x \in \partial' A(T)$ and $T \neq \phi$, then there exists $j \in \{1,\ldots, n\}$ such that $t = |T^j| > 0$ and for some $k \le t$ $v^j_k = 0$ for all $v \in x$. It is simple to verify that K, $A(\cdot)$, $\mathcal{T}$, N constitute a V-complex.

First examing the set B, note that $\phi \in B$, since $A(\phi) = \{\{E\}, \phi\}$. Suppose $x \in B$, $x \neq \phi$. Then $x \in \partial' A(T_x)$ and $L(x) = T_x$. Thus there exists $j \in \{1,\ldots, n\}$ such that $t = |T_x{}^j| > 0$ and for some $k \le t$, $v^j_k = 0$ for all $v \in x$. But since $L(\cdot)$ is proper, $(j, k) \notin L(x)$, whereby $(j, k) \notin T_x$, a contradiction. Thus $x \notin B$, and so $B = \{\phi\}$.

Now suppose $x \in G$. Then $T_x = L(x) \notin \mathcal{T}$; thus there exists $(j, k) \in N$ such that $(j, k) \notin T_x$, $L(x) = T_x \cup \{(j, k)\}$, and $L(x) \notin \mathcal{T}$. Let $t = |T^j_x|$. Because $L(\cdot)$ is proper, $k \le t + 1$ and since $L(x) \notin \mathcal{T}$, $k = m_j$, whereby $t = m_j - 1$ and so $L^j(x) = \{(j, 1), (j, 2),\ldots, (j, m_j)\} = N^j$. From lemma 3.3, G must have an odd number of elements, hence at least one, say x. Let $y = \{v \in x \mid L(v) \in L^j(x)\}$. Then $L(y) = N^j$. ⊠

Theorems 1, 2, and 3 are each "equivalent" to Brouwer's fixed-point theorem on the simplotope, stated below:

Brouwer's Theorem on the Simplotope: Let $f(\cdot) : S \to S$ be a continuous mapping, where $S = S^{m_1-1} \times \ldots \times S^{m_n-1}$. Then there exists $v \in S$ such that $f(v) = v$.

The combinatorial theorems are equivalent to Brouwer's theorem in that each theorem provides a relatively straightforward proof of Brouwer's theorem, and *vice versa*. This is shown as follows:

Proof of Equivalence of Theorems 1, 2, and 3 with Brouwer's theorem: Consider Theorem 1 first. Suppose $f(\cdot) : S \to S$ is given and let C be a triangulation of S with vertex set $K°$. For each $v \in K°$, assign the label $L(v) = (j, k) \in N$ such that (j, k) is any element of N that satisfies $f^j_k(v) - v^j_k \geq f^\ell_m(v) - v^\ell_m$ for any $(\ell, m) \in N$, $m \neq j$. By theorem 1 there exists a simplex x and an index j such that $L(x) = F^j(x)$. Let v^* be any limit point of such a sequence of simplices x for a sequence of triangulations whose meshes approach zero. Then a continuity argument implies that $f(v^*) = v^*$. This shows that theorem 1 implies Brouwer's theorem.

The proof that theorem 2 implies Brouwer's theorem follows along similar lines, where the labelling rule is such that for $v \notin \partial S$, $L(v) = (j, k) \in N$ is any element of N that satisfies $f^j_k(v) - v^j_k \geq f_m(v) - v_m$ for any $(\ell, m) \in N$, $m \neq j$; and if $v \in \partial S$, $L(v)$ is any element (j, k) of N for which $v^j_k = 0$. The proof that theorem 3 implies Brouwer's theorem follows by defining $L(v) = (j, k)$ if $v^j_k > 0$ and $f^j_k(v) - v^j_k \leq f^\ell_m(v) - v^\ell_m$ for any $(\ell, m) \in N$, $m \neq j$.

To see that Brouwer's theorem implies theorems 1, 2, and 3, again consider theorem 1 first. Suppose $L(\cdot)$ is given. Then define, for each vertex $v \in K°$, $f(v)$ as follows:

$$f^{\ell}(v) = \begin{cases} e^{jk} & \text{if } L(v) = (j,\ k) \text{ and } \ell = k \\ v^{\ell} & \text{if } L(v) = (j,\ k) \text{ and } \ell \neq k\ , \end{cases}$$

and extend $f(\cdot)$ in a PL manner over all of S. Then $f(\cdot) : S \to S$ is continuous and so has a fixed point v^*. Let $x = \{v^o,\ldots,\ v^m\}$ be the vertices of the smallest simplex containing v^*. Let $(j,\ k) = L(v^o)$. Then because $f(v^*) = v^*$, we must have $L^j(x) = F^j(x)$, and by corollary 0.2, there exists $z \subset x$, $z \neq \phi$, such that $L(z) = F^j(z)$.

The proof that theorem 2 is implied by Brouwer's theorem follows from an identical argument to that above, except that since $L(\cdot)$ is dual proper, this means $L^j(x) = F^j(x) = N^j$.

To prove that theorem 3 is implied by Brouwer's theorem, let $L(\cdot)$ be a proper labelling of K^o, and for each $v \in K^o$, define $f(v)$ as follows:

$$f^{\ell}(v) = \begin{cases} v^{\ell} & \text{if } L(v) = (j,\ k) \text{ and } \ell \neq k \\ e^{j,k+1} & \text{if } L(v) = (j,\ k) \text{ and } k < m_j \\ e^{j,1} & \text{if } L(v) = (j,\ m_j) \end{cases}\ ,$$

and extend $f(\cdot)$ in a PL manner over all of S. Then $f(\cdot) : S \to S$ is continuous, and so by Brouwer's theorem has a fixed point v^*. Let $x = \{v^o,\ldots,\ v^m\}$ be the vertices of the smallest simplex of C containing v^*, and let $(j,\ k) = L(v^o)$. Then it is simple to show that $L(x) = N^j$, proving theorem 3. ☒

Analogous to the Knaster-Kuratowski-Mazurkiewicz covering lemma [6] on the simplex, theorems 1, 2, and 3 also imply covering lemmas on the simplotope. Again, let $S = S^{m_1-1} \times\ldots\times S^{m_n-1}$, and let $N = \{(j,\ k) \mid j \in \{1,\ldots,\ n\},\ k \in \{1,\ldots,\ m_j\}\}$. We have the following:

<u>Covering Lemma 1</u> Let C^{jk}, $(j,\ k) \in N$, be a family of closed sets such that $\bigcup_{(j,k)\in N} C^{jk} = S$. Then there exists $j \in \{1,\ldots,\ n\}$, and $v \in S$ such that $v \in \bigcap_{(j,k)\in F^j(v)} C^{jk}$, i.e. $F^j(v) \subset \{(j,\ k) \mid v \in C^{jk}\}$.

Covering Lemma 2 Let C^{jk}, $(j, k) \in N$, be a family of closed sets such that $\bigcup_{(j,k)\in N} C^{jk} = S$, and $C^{jk} \supset \{v \in S \mid v^j_k = 0\}$ for each $(j, k) \in N$. Then there exists $j \in \{1,\dots, n\}$ such that $\bigcap_{k=1,\dots,m_j} C^{jk} \neq \phi$.

Covering Lemma 3 Let C^{jk}, $(j, k) \in N$ be a family of closed sets such that $\bigcup_{(j,k)\in N} C^{jk} = S$, and for each $T \subset N$ with $T^j \neq \phi$ for $j = 1,\dots, n$, $\bigcup_{(j,k)\in T} C^{jk} \supset \{v \in S \mid v^\ell_m = 0 \text{ for } (\ell, m) \notin T\}$. Then there exists $j \in \{1,\dots, n\}$ such that $\bigcap_{k=1,\dots,m_j} C^{jk} \neq \phi$.

The proofs of each of these covering lemmas is similar. We will prove covering lemma 1; the other two are proved in a parallel manner. Suppose the family of closed sets C^{jk} is given. Then let C be a triangulation of S, and for any $v \in K^o$, let $L(v)$ = any (j, k) such that $v \in C^{jk}$. By theorem 1, there exists j and x such that $L(x) = F^j(x)$. Taking a sequence of triangulations whose mesh goes to zero, and enumerating an infinite sub-sequence of x's and j's, we have in the limit a point v^* such that $\{(j, k) \mid v^* \in C^{jk}\} \supset F^j(v^*)$ for some j. ⊠

Finally, note that each of the covering lemmas implies Brouwer's fixed point theorem on the simplotope. To see this for covering lemma 1, let $f(\cdot) : S \to S$ be a given continuous function, and define $C^{jk} = \{v \in S \mid f^j_k(v) - v^j_k \geq f^\ell_m(v) - v^\ell_m \text{ for any } (\ell, m) \in N\}$. Any v for which $F^j(v) \subset \{(j, k) \mid v \in C^{jk}\}$ must be a fixed point of $f(\cdot)$. The derivations of Brouwer's theorem from covering lemmas 2 and 3 follows along similar lines.

5. Applications to the Simplex

This section states the results of section 4 for the case when S is the (trivial) cross-product of one simplex, i.e. $S = S^{m_1-1} = S^{m-1}$. For $x \subset S^{m-1}$, let $F(x) = \{j \mid v_j > 0 \text{ for some } v \in x\}$, i.e. F(x) is the carrier of x. Then theorems 1, 2, and 3, and covering lemmas 1, 2, and 3 become previously known results related to the simplex:

Theorem 1 on the Simplex (Generalized Sperner Lemma of [1] or [3]): Let S^{m-1} be given and let C be a triangulation of S^{m-1} with vertex set K^o, and let K be the pseudomanifold corresponding to C. Let $L(\cdot) : K^o \to \{1,\ldots, m\}$ be given. Then there exists an odd number of $x \in K$ such that $L(x) = F(x)$.

Theorem 2 on the Simplex (Dual Sperner Lemma of Scarf [9]): Let S^{m-1} be given and let C be a triangulation of S^{m-1} with vertex set K^o, and let K be the pseudomanifold corresponding to C, and suppose no simplex of K meets every facet of S^{m-1}. Let $L(\cdot) : K^o \to \{1,\ldots, m\}$ be given, such that for $v \in \partial S$, $L(v) = j$ implies $v_j = 0$. Then there exists an odd number of $x \in K$ such that $L(x) = \{1,\ldots, m\}$.

Theorem 3 on the Simplex (Sperner's Lemma [10]): Let S^{m-1} be given and let C be a triangulation of S^{m-1} with vertex set K^o, and let K be the pseudomanifold corresponding to C. Let $L(\cdot) : K^o \to \{1,\ldots, m\}$ be a labelling function with the property that $L(v) = j$ implies $v_j > 0$. Then there exists an odd number of $x \in K$ such that $L(x) = \{1,\ldots, m\}$.

Note that the above three results are, respectively, instances of theorem 1, 2, and 3, with the stronger conclusion that there are an odd number of simplices x with the respective required lables. The conclusions that the number of simplices is odd follows from the uniqueness of the cross-product. For a complete proof, refer to [5].

We also have:

<u>Covering Lemma 1 on the Simplex (Generalized Covering Lemma of [4])</u>: Let S^{m-1} be given and let $C^1,\ldots, C^m$ be m closed sets such that $\bigcup_{k=1}^{m} C^k = S^{m-1}$. Then there exists $v \in S^{m-1}$ such that $v \in \bigcap_{k\in F(v)} C^k$, i.e. $F(v) \subset \{k \mid v \in C^k\}$.

<u>Covering Lemma 2 on the Simplex (Freidenfelds [2])</u>: Let S^{m-1} be given and let $C^1,\ldots, C^m$ be m closed sets such that $\bigcup_{k=1}^{m} C^k = S^{m-1}$, and $C^k \supset \{v \in S^{m-1} \mid v_k = 0\}$, $k = 1,\ldots, m$. Then $\bigcap_{k=1}^{m} C^k \neq \phi$.

<u>Covering Lemma 3 on the Simplex (Knaster-Kuratowski-Mazurkiewicz Lemma [6])</u>: Let S^{m-1} be given, and let $C^1,\ldots, C^m$ be m closed sets such that $\bigcup_{k=1}^{m} C^k = S^{m-1}$, and for any $T \subset \{1,\ldots, m\}$, $T \neq \phi$, $\bigcup_{k\in T} C^k \supset \{v \in S^{m-1} \mid v_j = 0 \text{ for } j \notin T\}$. Then $\bigcap_{k=1}^{m} C^k \neq \phi$.

Note that Freidenfelds' covering lemma is a direct consequence of the Generalized Covering Lemma on the simplex. From the latter lemma, we have that there exists $v \in S^{m-1}$ such that $v \in \bigcap_{k\in F(v)} C^k$. But if the conditions of the former lemma are met, then $v \in \bigcap_{k\notin F(v)} C^k$, whereby $v \in \bigcap_{k=1}^{m} C^k$.

6. Applications to the Cube

In applying the results of section 4 to the cube, note that when $m_j = 2$, $j = 1, \ldots, n$, then $S = S^{m_1-1} \times \ldots \times S^{m_n-1}$ is isomorphic to the n-cube, defined to be $C^n = \{v \in \mathbb{R}^n \mid 0 \le v \le e\}$. Because $C^n \subset \mathbb{R}^n$, whereas $S = S^{2-1} \times \ldots \times S^{2-1} \subset \mathbb{R}^{2n}$, it is more convenient to state our results on C^n rather than on S. For $w \in S$, its corresponding element in C^n is given by v, where $v_j = w_1^j$, $j = 1, \ldots n$; and for $v \in C^n$, its corresponding element in S is given by w, where $w_1^j = v_j$, $w_2^j = 1 - v_j$, $j = 1, \ldots, n$. In the context of S, we defined $N = \{(j, k) \mid j \in \{1, \ldots, n\}, k \in \{1, \ldots, m_j\}\} = \{(1, 1), (1, 2), \ldots, (n, 1), (n, 2)\}$. Regarding the cube C^n, we define $\bar{N} = \{1, -1, \ldots, n, -n\}$, where we have the correspondence $(j, 1) \leftrightarrow j$ and $(j, 2) \leftrightarrow -j$ between N and $\bar{N}$.

Finally, for $v \in C^n$, we define its $j^{\underline{th}}$ carrier function $\bar{F}^j(v)$, by

$$\bar{F}^j(v) = \begin{cases} \{j\} & \text{if } v_j = 1 \\ \{-j\} & \text{if } v_j = 0 \\ \{j, -j\} & \text{if } 0 < v_j < 1 \end{cases}, \quad j = 1, \ldots, n.$$

Note we have the following correspondence between $F^j(w)$ and $\bar{F}^j(v)$:

$$F^j(w) = \{(j, 1)\} \quad \leftrightarrow \quad \{j\} \quad = \bar{F}^j(v)$$
$$F^j(w) = \{(j, 2)\} \quad \leftrightarrow \quad \{-j\} \quad = \bar{F}^j(v)$$
$$F^j(w) = \{(j, 1), (j, 2)\} \leftrightarrow \{j, -j\} = \bar{F}^j(v)$$

With the above notation, we can now state Theorems 1, 2, and 3, and covering lemma 1, 2, and 3, in the context of the n-cube C^n.

Theorem 1 on the Cube: Let C be a triangulation of C^n, with vertex set K^o, and let K be the pseudomanifold corresponding to C. Let $L(\cdot) : K^o \to \bar{N}$ be given. Then there exists $x \in K$ and $j \in \{1, \ldots, n\}$ such that $L(x) = \bar{F}^j(x)$.

Theorem 2 on the Cube (Freund, [5]): Let C be a triangulation of C^n, with vertex set K^o, and let K be the pseudomanifold corresponding to C. Let $L(\cdot) : K^o \to \bar{N}$ be a given labelling function such that if $v \in \partial C^n$, then $L(v) = j$ implies $v_j = 0$ and $L(v) = -j$ implies $v_j = 1$, $j = 1, \ldots, n$. Then there exists $x \in K$ and $j \in \{1, \ldots, n\}$ such that $L(x) = \{-j, j\}$.

Theorem 3 on the Cube (Freund [5], van der Laan and Talman [8]): Let C be a triangulation of C^n with vertex set K^o and let K be the pseudomanifold corresponding to C. Let $L(\cdot) : K^o \to \bar{N}$ be a given labelling function such that $L(v) = j$ implies $v_j > 0$ and $L(v) = -j$ implies $v_j < 1$, $j = 1, \ldots, n$. Then there exists $x \in K$ and $j \in \{1, \ldots, n\}$ such that $L(x) = \{-j, j\}$.

The above three results are, respectively, direct instances of theorems 1, 2, and 3, except that in the case of theorem 2, we no longer need the hypothesis that no simplex x of C meets the two facets $\{v \in C^n \mid v_j = 0\}$ and $\{v \in C^n \mid v_j = 1\}$ for any $j = 1, \ldots, n$. For the details of the proof, refer to [5].

We also have:

Covering lemma 1 on the Cube: Let $D^1, \ldots, D^n, D^{-1}, \ldots, D^{-n}$ be 2n closed sets such that $\bigcup_{k=1}^{n} (D^k \cup D^{-k}) = C^n$. Then there exists $v \in C^n$ and $j \in \{1, \ldots, n\}$ such that $v \in \bigcap_{k \in F^j(v)} D^k$, i.e. $F^j(v) \subset \{k \mid v \in D^k\}$.

Covering lemma 2 on the Cube: Let $D^1, \ldots, D^n, D^{-1}, \ldots, D^{-n}$ be 2n closed sets such that $\bigcup_{k=1}^{n} (D^k \cup D^{-k}) = C^n$, and such that $D^k \supset \{v \in C^n \mid v_k = 0\}$ and $D^{-k} \supset \{v \in C^n \mid v_k = 1\}$, $k = 1, \ldots, n$. Then there exists $j \in \{1, \ldots, n\}$ such that $D^j \cap D^{-j} \neq \phi$.

Covering lemma 3 on the Cube: Let $D^1, \ldots, D^n, D^{-1}, \ldots, D^{-n}$ be 2n closed sets such that $\bigcup_{k=1}^{n} (D^k \cup D^{-k}) = C^n$, and for each $T \subset \bar{N}$, with $T \cap \{-j, j\} \neq \phi$ for

each $j = 1, \ldots, n$, $\bigcup_{k \in T} D^k \supset \{v \in C^n \mid v_j = 0 \text{ if } j \notin T,\ v_j = 1 \text{ if } -j \notin T\}$.
Then there exists $j \in \{1, \ldots, n\}$ such that $D^j \cap D^{-j} \neq \phi$.

Covering lemmas 2 and 3 are illustrated in Figures 1 and 2.

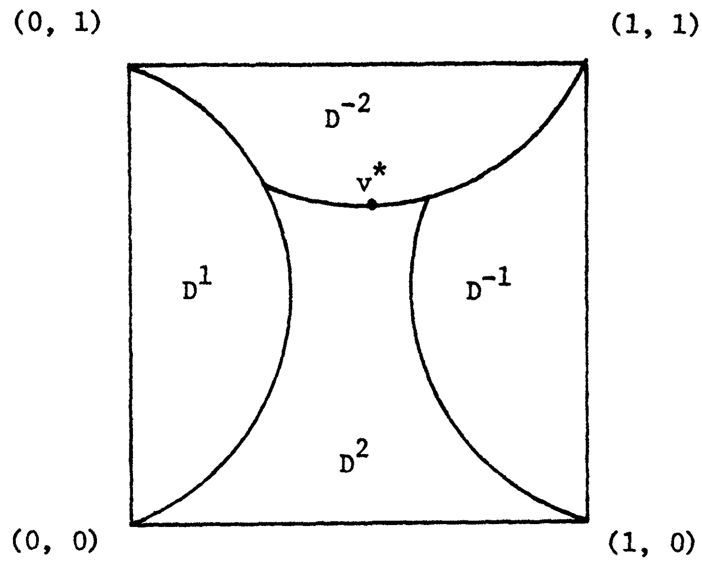

Illustration of Covering lemma 2 on the Cube, n = 2.

Figure 1.

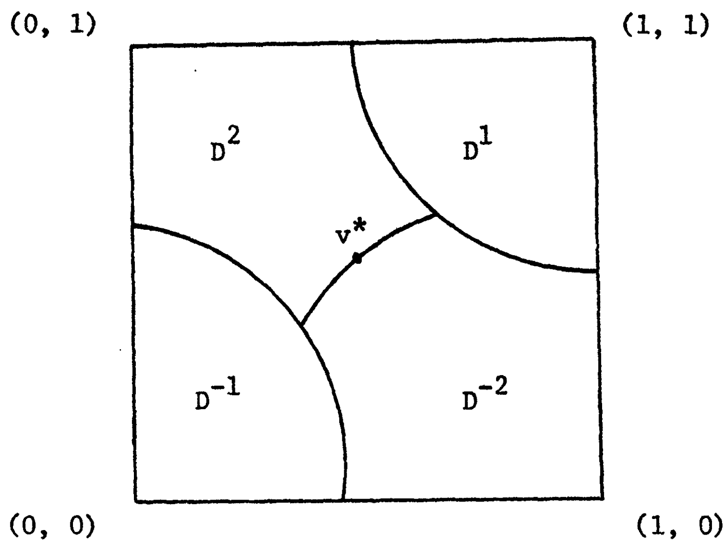

Illustration of Covering lemma 3 on the Cube, n = 2.

Figure 2.

7. Concluding Remarks

In this study, we have presented three combinatorial theorems on the simplotope, each of which is equivalent to Brouwer's fixed point theorem, and three covering lemmas on the simplotope that derive from the combinatorial theorems and which imply Brouwer's theorem. The first theorem implies the Generalized Sperner Lemma on the simplex [3] and the Generalized Covering theorem on the simplex [5], and implies two new results on the cube. The second theorem implies Scarf's dual Sperner Lemma [9] on the simplex, Freidenfeld's covering lemma on the simplex [2], lemma 2 in [5], and a new covering lemma on the cube. The third combinatorial theorem on the simplotope, originally due to van der Laan and Talman [8], implies Sperner's lemma on the simplex [10], the Knaster-Kuratowski-Mazurkiewicz Covering lemma [6] on the simplex, lemma 1 in [5], and a new covering lemma on the cube.

One combinatorial result that has not been mentioned up to this point is Kuhn's Strong Cubical lemma, presented in [7]. This lemma is different in many ways from other combinatorial results related to Brouwer's theorem. Kuhn's strong cubical lemma starts with a vector labelling that is "proper", which is then condensed into a reduced integer labelling, unlike other results discussed herein. Furthermore, the lemma asserts the existence on the n-cube of an n-simplex with (n + 1) distinct labels, whereas the results herein pertaining to the n-cube assert the existence on the n-cube of a 1-simplex with two complementary labels. In a forthcoming paper, I hope to report on a generalization of Kuhn's Strong Cubical lemma to the simplotope.

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