# CONSECUTIVE OPTIMIZORS FOR A PARTITIONING PROBLEM WITH APPLICATIONS TO OPTIMAL INVENTORY GROUPINGS FOR JOINT REPLENISHMENT*

A. K. Chakravarty
J. B. Orlin
and
U. G. Rothblum

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A. K. Chakravarty, Washington State University, Department of Management, Pullman, Washington 99163

J. B. Orlin, Massachusetts Institute of Technology, Sloan School of Management, Cambridge, Massachusetts 02139.

U. G. Rothblum, Yale University, School of Organization and Management, New Haven, Connecticut 06520

CONSECUTIVE OPTIMIZORS FOR A PARTITIONING PROBLEM

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## Abstract

We consider several subclasses of the problem of grouping $n$ items (indexed $1, \ldots, n$) into $m$ subsets so as to minimize the function $g(S_1, \ldots, S_m)$. In general, these problems are very difficult to solve to optimality, even for the case $m = 2$. Here we provide several sufficient conditions on $g(\cdot)$ which guarantee that there is an optimum partition in which each subset consists of consecutive integers (or else the partition $S_1, \ldots, S_m$ satisfies a more general condition called "semi-consecutiveness"). Moreover, by restricting attention to "consecutive" (or "semi-consecutive") partitions we can solve the partition problem efficiently for small values of $m$. If in addition, $g$ is symmetric then the partition problem reduces to a shortest path problem, solvable in $O(n^2m)$ steps.

We apply the above results to the problem of grouping inventory items into subgroups with a common order cycle per subgroup so as to minimize the resulting economic order quantity costs. Under relatively minor assumptions on the cost structure and on the set of feasible policies, we may reindex the items a priori so as to guarantee the optimality of a "consecutive" partition.

## 1. Introduction

Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers ordered so that for some integer $0 \le r \le n$, $b_1, \ldots, b_r$ are negative, $b_{r+1}, \ldots, b_n$ are non-negative and

$$a_1 / b_1 \le \ldots \le a_r / b_r \quad \text{and} \quad a_{r+1} / b_{r+1} \le \ldots \le a_n / b_n \,, \tag{1}$$

where for $b_i = 0$ we consider $a_i / b_i$ to be $+\infty$ or $-\infty$ according as $a_i > 0$ or $a_i < 0$ . If $a_i = b_i = 0$ , $a_i / b_i$ is defined arbitrarily so that inequality (1) holds. As usual, we let $a$ and $b$ denote the vectors whose coordinates are $a_i$ and $b_i$ , respectively.

Let $N = \{1, \ldots, n\}$ . For each subset $S$ of $N$ let $a_S = \sum_{i\varepsilon S} a_i$ and let $b_S = \sum_{i\varepsilon S} b_i$ . We consider the problem of partitioning the set $N$ into $m$ sets, say $S_1, \ldots, S_m$ , including possibly some empty sets, so as to minimize a real-valued function $g_m(S_1, \ldots, S_m)$, where $g_m(\cdot)$ is assumed to have the form

$$g_m(S_1, \ldots, S_m) = h_m(a_{S_1}, b_{S_1}, \ldots, a_{S_m}, b_{S_m}) \,, \tag{2}$$

where $h_m(\cdot)$ is a real valued function of $2m$ variables. We allow that the function $h_m(\cdot)$ be nonsymmetric, in which case the cost of a partition depends on the order in which the sets are selected.

A subset $S \subseteq N$ is called <u>consecutive</u> if its elements are consecutive integers, e.g., $\{4,5,6\}$. In particular, the empty set is considered to be consecutive. A partition $P = \{S_1, \ldots, S_m\}$ is called <u>consecutive</u> if $S_i$ is consecutive for each $i$ . A partition $P = \{S_1, \ldots, S_m\}$ is called <u>semi-consecutive</u> if there is a permutation $\pi$ of the set of integers $\{1, \ldots, m\}$ such that the sets $S_{\pi(1)} \cup \ldots \cup S_{\pi(j)}$ are consecutive for $j = 1, \ldots, m$ . For example, the partition $\{S_1,S_2,S_3\} = \{\{3,7,8\}, \{1,2,9\} \; \{4,5,6\}\}$ is semi-consecutive because the three subsets $S_3$, $S_3 \cup S_1$, $S_3 \cup S_1 \cup S_2$ are all consecutive.

The purpose of this paper is to provide sufficient conditions that the optimization problem defined above has an optimal consecutive or semi-consecutive partition. Specifically, we show that if $h_m$ is concave in its $2m$ variables then there is an optimal semi-consecutive partition. Moreover, if in addition, $b \geq 0$ or $a \geq 0$ , then there is an optimal consecutive partition. Also, we show that if $b_i = 1$ for each $i$ and , in addition, $h_m$ is concave in the "a variables" for each fixed value of the "b variables" then a consecutive optimal partition exists. Of course, when $b_i = 1$ for each $i$ we have that for a set $S \subseteq \{1, \ldots, n\}$ $b_S$ equals $|S|$, the cardinality of $S$ . In this case, for a given partition $P = \{S_1, \ldots, S_m\}$ , $(b_{S_1}, \ldots, b_{S_m})$ is called the <u>shape</u> of the partition $P$ (e.g., Hwang [1981]).

When we wish to specify the number $m$ of sets in a partition $P$ we call $P$ an <u>m-partition</u>. We observe that the number of ordered m-partitions that are consecutive is $O(m!n^{m-1})$ and the number of semi-consecutive partitions is $O(m!n^{2m-2})$. Thus for a fixed value of $m$ we can determine an optimal consecutive or semi-consecutive m-partition in polynomial time via complete enumeration. Although complete enumeration is quite impractical even for moderate sizes of $m$ , the above fact contrasts sharply with the NP-completeness of the optimal partition problem for any fixed $m \geq 2$ , as the knapsack problem is a special case of the partition problem for $m = 2$ . (Of course, the total number of ordered m-partitions is $m^n$ .) We note that the sensitivity of optimal consecutive partitions to changes in $m$ can be studied by using the results of Denardo, Huberman and Rothblum [1982].

We formally state our main results concerning the existence of consecutive and semi-consecutive optimal partitions in Section 2, where we also survey a number of places where special cases of our results are used. In Section 3, we show how our results apply to problems that involve optimal groupings for joint replenishment of inventories. Next, in Section 4, we discuss the special case where $g_m(\cdot)$ is separable and symmetric, in which case optimal consecutive partitions can be determined in $O(mn^2)$ steps and optimal semi-consecutive partitions can be determined in $O(mn^4)$ steps. In Section 5, we discuss issues of NP-completeness which apply to our results. Finally, proofs of our results are given in Section 6.

## 2. The Main Results

We next state our main results concerning optimal partitions that are either consecutive or semi-consecutive. The proofs are deferred to Section 6.

Theorem 1: Suppose $h_m$ is concave in its 2m variables and in addition $b \geq 0$ or $a \geq 0$. Then there exists a consecutive optimal partition.

Theorem 2: Suppose that $h_m$ is concave in its 2m variables. Then there exists a semi-consecutive optimal partition.

Theorem 3: Suppose $b_i = 1$ for $i = 1, \ldots, n$ and $h_m(\cdot)$ is concave in the "a variables" for each fixed value of the "b variables." Then there exists a consecutive optimal partition.

Although the partition problems considered in the above Theorems are quite restricted, there have been several realms where special cases of these results have been used. We next give a brief survey of such applications.

One classical partitioning problem occurs in Hypothesis Testing in Statistics. Outcomes of a random phenomenon are partitioned into two sets, one of which corresponds to the region where the null hypothesis is accepted and the other to the region where it is rejected. In this case the "a variables" correspond to probability of type 1 error, the "b variables" correspond to probability of type 2 error and the corresponding objective function is linear. It follows that Theorem 1 applies and an optimal consecutive partition exists. This fact is the well-known Neyman-Pearson Lemma. For more details on the applicaton to statical testing see DeGroot [1975, p. 374].

We next give a number of examples of studies where special cases of the results of Theorem 3 were obtained. In particular, the special case of Theorem 3 where h is separable and symmetric is discussed in Chakravarty, Orlin and Rothblum [1982]. They apply the corresponding result to a problem of inventory grouping with joint replenishment (see Section 4 for more complex such problems). Also, Hwang [1981] and Hwang, Sun and Yao [1982] consider restricted classes of functions for which a corresponding optimization problem have consecutive optimizers. Their conditions entail monotonicity and additivity requirements, and some of their results are special cases of Theorem 3. The above papers provide applications of their results to problems in storage and problems in group testing. Finally, Barnes and Hoffman [1982] considered a special instance of the partitioning problem studied in Theorem 3. The motivation for their work arose in connection with the partitioning of eigenvalues of a matrix so as to derive estimates on a graph theoretic partitioning problem. They used the theory of submodular set functions to establish the existence of consecutive optimal partitions. Their results were developed independently of those in Chakravarty, Orlin and Rothblum [1982], and their proof is an interesting application of the theory of submodular set functions. In Section 5, we show that the Barnes-Hoffman partitioning problem is NP-complete.

## 3. An Application to Inventory Grouping

Consider an economic order quantity model involving $n$ items, where the i-th item has (deterministic) demand rate $D_i$, a unit inventory holding cost $h_i$ per unit time and a fixed cost $K_i$ for placing an order. The problem is to partition the $n$ items into $m$ subgroups and choose order cycles for the groups out of a given set of allowable (joint) order cycles, so as to minimize the net average cost per unit time. Motivation and more detailed explanation of special cases of this model are given in Chakravarty [1982a, 1982b]. This model is a generalization of the joint replenishment models considered by Goyal [1974], Silver [1975], and Nocturne [1973] who study the problem in which a unit of order time $\lambda$ is determined for the group of $n$ items, and the order cycles for each item is an integer multiple of $\lambda$. They give heuristic solutions to this problem. A related model was studied by Chakravarty, Orlin and Rothblum [1982].

Let $a_i = 2^{-1}h_iD_i$ and $b_i = K_i$ where the items are labeled so that (1) holds. As in the ordinary EOQ model (e.g., Wagner [1969, pp. 18-19]), if item $i$ has order cycle $\tau$ then its order quantity is $\tau D_i$, and the average net cost per unit time of a group $S$ of items having the same order cycle $\tau$ is:

$$c(S,\tau) = \sum_{i\varepsilon S} 2^{-1}\tau D_i h_i + \sum_{i\varepsilon S} K_i\tau^{-1} = \tau a_S + \tau^{-1}b_S . \qquad (3)$$

So, if the items are partitioned into groups $S_1, \ldots, S_m$ having order cycles $t_1, \ldots, t_m$, respectively, the total average cost per unit time is

$$c_m(S_1, \ldots, S_m, t_1, \ldots, t_m) = \sum_{j=1}^{m} c(S_j, t_j) . \qquad (4)$$

The m-vector of order cycles is assumed to be taken out of a set $T$ of allowable m-vectors of order cycles. This formulation allows one to impose joint restrictions on the order cycles, e.g., the requirement that all order cycles are integer multiples of the smallest one, or individual restrictions, e.g., that each order cycle be an integer in the set $\{1,7,30\}$. It follows that for a fixed partition $S_1, \ldots, S_m$ of the items, the minimum average cost per unit time is:

$$g_m(S_1, \ldots, S_m) = \inf_{t \in T} c_m(S_1, \ldots, S_m, t_1, \ldots, t_m)$$
$$= \inf_{t \in T} \sum_{j=1}^{m} t_j a_{S_j} + t_j^{-1} b_{S_j} \quad . \tag{5}$$

Evidently, as $c_m(S_1, \ldots, S_m, t_1, \ldots, t_m) \geq 0$ , the infimum defining $g$ is finite for each partition. The problem is to find a partition of the items into subsets so as to minimize $g_m(\cdot)$ , where $g_m$ is given above. Now, as $c(S,t)$ is linear in $a_S$ and $b_S$ (for fixed $t$ ), we conclude that $c_m(S_1, \ldots, S_m, t_1, \ldots, t_m)$ is linear in $a_{S_1}, b_{S_1}, \ldots, a_{S_m}, b_{S_m}$ and therefore $g_m(S_1, \ldots, S_m)$ , as the infimum of linear functions, is concave in $a_{S_1}, b_{S_1}, \ldots, a_{S_m}, b_{S_m}$ , i.e., the assumptions of Theorem 1 are satisfied. Hence, our results apply and there exists an optimal partition consisting of consecutive sets.

We next consider a modified version of the model described above where the terms $\tau a_S$ and $\tau^{-1} b_S$ in (3) are replaced, respectively, by expressions $e_\tau(\tau a_S)$ and $f_\tau(\tau^{-1} b_S)$ , where $e_\tau(\cdot)$ and $f_\tau(\cdot)$ are real valued functions. This formulation allows one to introduce discounting on the holding cost as well as the setup cost, where the value of the discount depends on the monetary volume. Under the above modification (5) has to be replaced by:

$$g_m(S_1, \ldots, S_m) = \inf_{t \in T} \sum_{j=1}^{m} e_{t_j}(t_j a_{S_j}) + f_{t_j}(t_j^{-1} b_{S_j}) \quad . \tag{6}$$

It is easily seen that if the function $e_\tau$ and $f_\tau$ are concave then the assumptions of Theorem 1 hold and if $K_i$ is independent of $i$ and the functions $e$ are concave then the assumptions of Theorem 3 hold. In either case we conclude that there exists an optimal grouping of the products where each set is consecutive. We emphasize that concavity of the functions $e_\tau$ and $f_\tau$ is reasonable as it represents higher discounting as the volume increases.

A special case of the general model allows the decision maker to choose the order cycles of the groups independently out of a given set (which is independent of the enumeration of the group), i.e., $T = U \times U \times \ldots \times U$ for some set $U \subseteq R$. In this case $g_m$ is symmetric and separable and the results of (the forthcoming) Section 4 apply. In particular, the problem of finding an optimal grouping can be reduced to a shortest path problem.

Finally, we consider the case in which the cost of ordering in period P is concave in the total quantity ordered in that period, where we might have a simultaneous reorder from different groups. If we let P be the least common multiple of the order cycles, then the total cost of ordering is:

$$\sum_{p=0}^{P-1} e\left( \sum_{t_j | p} t_j a_{S_j} \right) \tag{7}$$

where $t_j | p$ means that $t_j$ is an integral divisor of $p$. As before, the infimum of concave functions is concave, and thus there is a consecutive optimal partition.

## 4. The Symmetric Separable Case

We next consider the case where $g$ is symmetric and separable, i.e., for some function $h : R^2 \to R$

$$g_m(S_1, \ldots, S_m) = \sum_{j=1}^{m} h(a_{S_j}, b_{S_j}) .$$

In this case the order of the subsets in a consecutive partition does not matter. In particular, without loss of generality, one can assume that in an (optimal) consecutive partition the indices in $S_i$ precede the indices in $S_j$ for $i < j$. Chakravarty, Orlin and Rothblum [1982] showed that the problem of determining an optimal consecutive partition reduces to the problem of finding the shortest path between two nodes in a graph where the number of edges in the path is at most $m$. This problem can be solved in $O(mn^2)$ steps using a standard dynamic programming recursion.

In the symmetric and separable case, one can determine an optimal semi-consecutive partition as follows. Let $S_{ij} = \{i+1, \ldots, j\}$ for all $i,j$ with $0 \le i \le j \le n$. Let $f^t_{ij}$ be the optimal value of a semi-consecutive partitioning of $S_{ij}$ into $t$ subsets. Finally, let $h'(S) = h(a_S, b_S)$ for each $S \subseteq N$. Then $f^1_{ij} = h'(S_{ij})$ for $0 \le i \le j \le n$.

$$f^t_{ij} = \min_{i \le k \le q \le j} h'(S_{ik} \cup S_{qj}) + f^{t-1}(S_{kq}) . \tag{8}$$

It is easy to see that the above recursions can be computed in $O(n^4)$ steps for each value of $t$, and thus the total number of computations is $O(mn^4)$.

## 5. The NP-Completeness of an Optimal Partition Problem

Barnes and Hoffman [1982] consider the following partition problem. Given a set of $n$ integers $a_1, \ldots, a_n$ and $m$ non-negative integers $d_1, \ldots, d_m$, where $\sum_{i=1}^{m} d_i = N$, partition $N$ into $m$ subsets $S_1, \ldots, S_m$ such that $|S_i| = d_i$ for $i = 1, \ldots, m$ and so as to maximize

$$\sum_{j=1}^{m} d_j^{-1} \left( \sum_{i \in S_j} a_i \right)^2 . \tag{9}$$

Barnes and Hoffman demonstrated that there is always an optimal partition consisting of consecutive subsets. Moreover, since they were interested in problems in which $m$ was quite small (e.g., $m = 3$), implicit enumeration of all feasible consecutive partitions was quite practical.

The Barnes-Hoffman partition problem satisfies the conditions of Theorem 3 (except for the fact that they maximize a convex function rather than minimize a concave function). Thus the optimality of consecutive partitions is a special case of Theorem 3.

Although the Barnes-Hoffman problem is polynomially solvable for fixed $m$, the corresponding recognition problem is NP-complete if $m$ is allowed to vary jointly with $n$. We next prove the NP-completeness via a transformation from 3-partition.

3-PARTITION

INPUT: non-negative integers $b_1, \ldots, b_{3k}$ such that $b_i \le k^2$ for $i = 1, \ldots, 3k$.

QUESTION: Is it possible to partition $\{1, \ldots, 3k\}$ into $k$ subsets $S_1, \ldots, S_k$ such that

$$\sum_{i \in S_j} b_i = (1/k) \sum_{i=1}^{3k} b_i \quad \text{for} \quad j = 1, \ldots, k \quad ? \tag{10}$$

We consider the version of 3-Partition in which the subsets are not specified to have three elements each. Garey and Johnson [1978] showed that the above variant of the 3-Partition problem is NP-complete.

<u>Theorem 4</u>. The recognition version of the Barnes-Hoffman partition problem is NP-complete.

## 6. Proofs

Proof of Theorem 1.

We consider only the case where $b \geq 0$ as the case where $a \geq 0$ follows from analogous arguments. Our proof follows by induction on the number $m$ of subsets in the partitions. The case $m = 1$ is trivial. We next consider the case $m = 2$. We first remark that the case where $a_i = b_i = 0$ for some $i$ can be ignored. Noting that each 2-partition may be written as $\{S, N \setminus S\}$ for some set $S \subseteq N$ and that for such $S$, $a_{N\setminus S} = a_N - a_S$ and $b_{N\setminus S} = b_N - b_S$, we can write the optimal partitioning problem with $m = 2$ as the following optimization problem:

$$\min_{S\subseteq N} g_2(S, N\setminus S) = \min_{S\subseteq N} h_2(a_S, b_S, a_{N\setminus S}, b_{N\setminus S}) \tag{11}$$

$$= \min_{S\subseteq N} h_2(a_S, b_S, a_N - a_S, b_N - b_S) .$$

Since the set of ordered pairs $\{(a_S, b_S) : S \subseteq N\}$ is finite, its convex hull, which will be denoted by $C$, is a convex polyhedron. It now follows from (11) that

$$\min_{S\subseteq N} g_2(S, N\setminus S) = \min_{S\subseteq N} h_2(a_S, b_S, a_N - a_S, b_N - b_S) \tag{12}$$

$$\geq \min_{(x,y)\in C} h_2(x, y, a_N - x, b_N - y) .$$

Since the function $h^2(x, y, a_N - x, b_N - y)$ is concave in the variables $(x, y)$, the latter minimization problem attains a minimum at an extreme point of $C$. It therefore suffices to show that for each extreme point $(x^*, y^*)$ of $C$ there exists a set $S^* \subseteq N$ such that $(x^*, y^*) = (a_{S*}, b_{S*})$ and $\{S^*, N \setminus S^*\}$ is a consecutive partition.

Let $(x^*, y^*)$ be an extreme point of $C$. Since $C$ is a polyhedron, it is well known that there exists a linear function $f$ on $C$ that attains its unique minimum over $C$ at $(x^*, y^*)$. Since $f$ is linear it has a representation $f(x, y) = \alpha x + \beta y$ where $\alpha$ and $\beta$ are real numbers. The uniqueness of $(x^*, y^*)$ as the minimizer of $f$ over $C$ assures that we do not have $\alpha = \beta = 0$. Next observe that as $C$ is the convex hull of $\{(a_S, b_S) : S \subseteq N\}$,

$$\alpha x^* + \beta y^* = \min_{(x,y)\in C} \alpha x + \beta y = \min_{S\subseteq N} \alpha a_S + \beta b_S = \min_{S\subseteq N} \sum_{i\in S} (\alpha a_i + \beta b_i) \ . \quad (13)$$

The set $S^* = \{i : \alpha a_i + \beta b_i < 0\}$ is clearly optimal for the last optimization problem. Hence, we conclude from the uniqueness of $(x^*, y^*)$ as the minimizer of $f$ over $C$ that $(x^*, y^*) = (a_{S*}, b_{S*})$.

We next show that the fact that $b \geq 0$ assures that both $S^*$ and $N \setminus S^*$ are consecutive. We consider a number of cases separately. If $\alpha > 0$ then $S^* = \{i : a_i / b_i < -\beta / \alpha\}$ and if $\alpha < 0$ then $S^* = \{i : a_i / b_i > -\beta / \alpha\}$. In either case (1) assures that both $S^*$ and $N \setminus S^*$ are consecutive. Finally, if $\alpha = 0$ then $\beta \neq 0$, and $S^* = \{i : b_i < 0\} = \phi$ if $\beta > 0$ and $S^* = \{i : b_i > 0\}$ if $\beta < 0$. In either case we trivially have that both $S^*$ and $N \setminus S^*$ are consecutive. This completes our proof of Theorem 1 when $m = 2$.

We next consider the case where $m \geq 3$. For each subset $S$ of $N$, let $\max S = \max \{i : i \in S\}$, $\min S = \min \{i : i \in S\}$ and $d(S) = \max S - \min S$. Let $P = \{S_1, \ldots, S_m\}$ be an optimal m-partition of $N$ that minimizes $\sum_{j=1}^{m} d(S_j)$ with repsect to all such (optimal) partitions. We claim that for each $j = 1, \ldots, m$, $S_j$ is consecutive. We see this as follows. Suppose that not all subsets are consecutive. Then we may determine two sets of $P$, say $S_1$ and $S_2$, such that for some $i \in S_2$, $\min S_1 < i < \max S_1$. For $j = 1, 2,$ let $m_j = \min S_j$ and $M_j = \max S_j$. Observe that $\min \{M_1, M_2\} \geq i$ and $\max \{m_1, m_2\} \leq i$. Hence,

$$\min\{M_1, M_2\} - \max\{m_1, m_2\} \geq 0 . \quad (14)$$

Let $\{S_1', S_2'\}$ be an optimal parition of $S_1 \cup S_2$ into two subsets which are consecutive with respect to $S_1 \cup S_2$ where $S_3, \ldots, S_m$ remain fixed (as is possible by the established result of Theorem 1 when $m = 2$ and the observation that $h_m$ is concave in the variables corresponding to $S_1$ and $S_2$ when the remaining variables are fixed). Then

$$d(S_1') + d(S_2') \leq \max\{M_1, M_2\} - \min\{m_1, m_2\} - 1 \quad (15)$$

and therefore, using (14), we conclude that

$$\begin{aligned} d(S_1) + d(S_2) &= M_1 - m_1 + M_2 - m_2 \quad (16)\\ &= \max\{M_1, M_2\} + \min\{M_1, M_2\} - \min\{m_1, m_2\} - \max\{m_1, m_2\}\\ &\geq d(S_1') + d(S_2') + 1 . \end{aligned}$$

This contradicts the choice of $S_1, \ldots, S_m$ , completing our proof. □

Proof of Theorem 2.

Our proof follows by induction on the number $m$ of subsets in the partitions. The case $m = 1$ is trivial. We next consider the case $m = 2$ . The argument used in the proof of Theorem 1 shows that it suffices to prove that for every pair of real numbers $\alpha$ and $\beta$ , the set $S^* = \{i : \alpha a_i + \beta b_i < 0\}$ has the property that $\{S^*, N \setminus S^*\}$ is a semi-consecutive partition of $N$ , or equivalently, that either $S^*$ or $N \setminus S^*$ is a consecutive set. We consider a number of cases separately. If $\alpha > 0$ then $S^* = \{i = 1, \ldots, r : a_i / b_i > -\beta / \alpha\} \cup \{i = r+1, \ldots, n : a_i / b_i < -\beta / \alpha\}$ and if $\alpha < 0$ then $S^* = \{i = 1, \ldots, r : a_i / b_i < -\beta / \alpha\} \cup \{i = r+1, \ldots, n : a_i / b_i > -\beta / \alpha\}$ . In the former case

(1) assures that $S^*$ is consecutive and in the latter case (1) assures that $N \setminus S^*$ is consecutive. Hence, in either case the partition $\{S^*, N \setminus S^*\}$ is semi-consecutive. Finally, if $\alpha = 0$ then $\beta \neq 0$ and $S^* = \{i : b_i < 0\} = \{1, \ldots, r\}$ if $\beta > 0$ and $S^* = \{i : b_i > 0\}$ if $\beta < 0$. In either case we have that $S^*$ is consecutive (though in the latter case $N \setminus S^*$ is not consecutive if $r > 1$ and for some $i$, $a_i > 0 = b_i$) and therefore the partition $\{S^*, N \setminus S^*\}$ is semi-consecutive.

Next assume that the conclusions of Theorem 2 hold whenever the number of sets in the partitioning problem is less than $m$ ($\geq 3$), and consider partitioning problems where the number of sets is $m$. We prove the existence of a semi-consecutive optimal partition for the corresponding partitioning problems by induction on the number of elements in the partitioned set $N$. The cases where the number of elements in $N$ is 1, 2 or 3 are straight forward. Next assume that each partitioning problem of a set consisting of less than $n$ elements into $m$ sets, where the assumptions of Theorem 2 are satisfied, has a semi-consecutive optimal partition. We next consider such partitioning problems of a set $N$ where the number of elements in $N$ is $n$. For each subset $S \subseteq N$, let $S^- = S \cap \{1, \ldots, r\}$ and $S^+ = S \cap \{r+1, \ldots, n\}$, where $r$ is defined through (1). Given any optimal partition, say $\{S_1, \ldots, S_m\}$, one can hold $S_1^-, \ldots, S_m^-$ fixed and, by applying Theorem 1, repartition $S_1^+ \cup \ldots \cup S_m^+ = N^+$ such that the sets in the new (optimal) partition of $N^+$ are all consecutive. Similarly, by applying an appropriate modification of Theorem 1, one can hold the (new) sets $S_1^+, \ldots, S_m^+$ fixed and repartition $S_1^- \cup \ldots \cup S_m^- = N^-$ such that the sets in the new (optimal) partition of $N^-$ are all consecutive. The above arguments establish the existence of an optimal partition $\{S_1, \ldots, S_m\}$ for which $S_1^-, \ldots, S_m^-, S_1^+, \ldots, S_m^+$ are all consecutive. We remark that if $r = 0$ or $r = n$ then the partition $\{S_1, \ldots, S_m\}$ is consecutive and therefore semi-consecutive. Henceforth, we assume through the end of this proof that $1 \leq r < n$.

We continue our proof by assuming that the partitioning problem of $N$ into $m$ sets has an optimal partition $\{S_1, \ldots, S_m\}$ for which the sets $S_1^-, \ldots, S_m^-$, $S_1^+, \ldots, S_m^+$ are all consecutive and in addition there is an index $j \in \{1, \ldots, m\}$ with either $|S_j^-| > 1$ or $|S_j^+| > 1$. We consider only the case where $|S_j^+| > 1$ as the alternative case follows from similar arguments. Let $T = S_j^+$. One can combine all the elements in $T$ into a single element and attach an "a" value $a_T$ and a "b" value $b_T$ to this combined element. Evidently, as $T \subseteq N^+$ we have that if $\alpha \leq a_i / b_i \leq \beta$ for each $i \in T$ then $\alpha \leq a_T / b_T \leq \beta$. It follows that the corresponding partitioning problem where the elements in $T$ are combined into a single element satisfies (1) and the other assumptions of Theorem 2. As the number of elements in the partitioned set of this modified problem is less than $n$, we conclude from the induction assumption that a semi-consecutive optimal partition exists. It immediately follows that the original problem too has a semi-consecutive optimal solution.

It remains to consider the case where any optimal partition $\{S_1, \ldots, S_m\}$ for which $S_1^-, \ldots, S_m^-, S_1^+, \ldots, S_m^+$ are all consecutive has the property that each of the sets $S_1^-, \ldots, S_m^-, S_1^+, \ldots, S_m^+$ has at most a single element. Let $\{S_1, \ldots, S_m\}$ be such an optimal partition (whose existence follows from our earlier arguments). Let $S_p$ and $S_q$ be the sets in $\{S_1, \ldots, S_m\}$ which contain $r$ and $r+1$, respectively (recall that $1 \leq r < n$). We establish existence of a semi-consecutive optimal partition by considering two cases.

First assume that either $S_p$ or $S_q$ is consecutive. We consider only the former case as the latter case follows from identical arguments. By using the induction assumption concerning partitioning problems of a set into $m - 1$ sets, one can fix $S_p$ and repartition $\cup \{S_t : t = 1, \ldots, m, t \neq p\}$ such that the resulting partition of this set, say $\{\tilde{S}_t : t = 1, \ldots, m, t \neq p\}$ will be semi-consecutive and will not reduce the value of the objective function. It easily follows that, with $\tilde{S}_p = S_p$, $\{\tilde{S}_1, \ldots, \tilde{S}_m\}$ is a semi-consecutive optimal partition of $N$.

We next consider the case where neither $S_p$ nor $S_q$ is consecutive. As $|S_p^-| \le 1$ and $|S_q^+| \le 1$ we have that $S_p^- = \{r\}$ and $S_q^+ = \{r+1\}$. Also observe that as $|S_p^-| \le 1$, $|S_p^+| \le 1$ and $S_p$ is not consecutive, we have that $r+1 \notin S_p^+$ and for some $r+1 < j \le n$, $S_p^+ = \{j\}$. In particular, $p \ne q$. A similar argument shows that for some $1 \le i < r$, $S_q^- = \{i\}$. We can next hold $\{S_t : t = 1, \ldots, m, t \ne p, t \ne q\}$ fixed and apply the (established) conclusions of Theorem 2 for the case $m = 2$, to repartition $S_p \cup S_q = \{i, r, r+1, j\}$ into sets $\tilde{S}_p$ and $\tilde{S}_q$ where $\{\tilde{S}_p, \tilde{S}_q\}$ is a semi-consecutive partition with respect to $\{i, r, r+1, j\}$ and where the value of the objective is not reduced. Now, if either of the sets $\tilde{S}_p^-$, $\tilde{S}_q^-$, $\tilde{S}_p^+$, $\tilde{S}_q^+$ consists of more than a single element, one can apply a combination of the arguments used earlier to establish the existence of a semi-consecutive optimal partition. Specifically, if for example $|\tilde{S}_p^-| > 1$, one can repartition $N^-$ into $k-1$ sets where $k$ is the number of nonempty sets among $\{S_1^-, \ldots, S_m^-\}$ where each of the sets in the new partition is consecutive. As $|N^-| = k$ we have that one of these sets contains at least two elements. By combining the elements in that set and applying the induction assumption concerning partitioning problems of sets consisting of less than $n$ elements into $m$ sets, we conclude the existence of a semi-consecutive optimal partition. We continue by assuming that the sets $\tilde{S}_p^-$, $\tilde{S}_q^-$, $\tilde{S}_p^+$, $\tilde{S}_q^+$ each contains at most a single element, in which case each contains precisely one element. It follows from the semi-consecutiveness of the partition $\{\tilde{S}_p, \tilde{S}_q\}$ with respect to $\{i, r, r+1, j\}$ that either $\tilde{S}_p = \{r, r+1\}$ or $\tilde{S}_q = \{r, r+1\}$. In either case, the set containing $r$ (which also contains $r+1$) is consecutive and our earlier arguments assure the existence of a semi-consecutive optimal partition. □

Proof of Theorem 3:

Chakravarty, Orlin and Rothblum established a special case of Theorem 3 where $g_m(\cdot)$ is symmetric and separable. Their arguments apply to the general case considered in Theorem 3 and are omitted. □

Proof of Theorem 4:

Let $b_1, \ldots, b_{3k}$ be the input for a given 3-Partition problem. We will show below how to transform the 3-Partition problem into the Barnes-Hoffman partition problem. The transformation is based on the following elementary lemma.

Lemma. Let $a_1, \ldots, a_n$ and $d_1, \ldots, d_m$ be the data for the Barnes-Hoffman partition problem. Then the maximum cost partition has value at most $\sum_{i=1}^n a_i^2$. Moreover, it is possible to achieve this bound if and only if it is possible to determine a feasible partition $S_1, \ldots, S_m$ such that $a_i = a_t$ for every two indices $i,t$ in the same subset $S'$.

Proof of Lemma. By the Cauchy-Schwarz inequality

$$\left(\sum_{i \varepsilon S_j} a_i\right)^2 \leq |S_j| \sum_{i \varepsilon S_j} a_i^2$$ and equality holds if and only if

$a_i = a_t$ for all $i,t \varepsilon S_j$. Thus:

$$\sum_{j=1}^m |S_j|^{-1} \left(\sum_{i \varepsilon S_j} a_i\right)^2 \leq \sum_{i=1}^n a_i^2$$

and equality holds if and only if $a_i = a_t$ for every two indices $i,t$ in the same subset.

To complete the proof of the theorem, let $\bar{b} = k^{-1} (\sum_{i=1}^{3k} b_i)$. ($\bar{b}$ is the subset sum for each subset of the 3-partition). Let $n = \sum_{i=1}^{3k} b_i$, let $m = 3k$ and let $d_i = b_i$ for $i = 1, \ldots, 3k$. Finally, let

$$a_{i+j\bar{b}} = j \quad \text{for} \quad 1 \leq i \leq \bar{b} \quad \text{and} \quad 0 \leq j \leq m-1 .$$

Thus there are exactly $\bar{b}$ indices $i$ for which $a_i = j$ for any $j = 0, \ldots, m-1$. By the previous lemma, we can achieve a value of $\sum_{i=1}^{n} a_i^2$ if only if we can determine a feasible partition $S_1, \ldots, S_m$ such that $a_i = a_t$ for all $i,t$ in the same subset. Suppose we can achieve this latter result. Then let $\{P_0, \ldots, P_{k-1}\}$ be a partition of $\{b_1, \ldots, b_{3k-1}\}$ such that $P_i$ consists of those elements $b_j$ such that $a_t = i$ for $t \in S_j$. Then $\{P_0, \ldots, P_{k-1}\}$ is a 3-partition of $b_1, \ldots, b_{3k}$. Conversely, if $\{P_0, \ldots, P_{k-1}\}$ is a 3-partition of $b_1, \ldots, b_{3k}$, then let $S_j$ consist of $b_j$ elements whose corresponding "a" value is $t$, where $b_j \in P_t$. We have thus seen that there is a feasible partition $\{S_1, \ldots, S_m\}$ such that the "a" value for each $S_i$ is constant if and only if there is a 3-partition for the b's. Thus the Barnes-Hoffman partition problem is NP-hard. Since the recognition variant is clearly in the class NP, the latter problem is NP-complete. □

## References


Barnes, E.R., and A.J. Hoffman (1982), "Partitioning, spectra, and linear programming". To appear in the Proceedings of the Silver Jubilee Conference on Combinatorics, University of Waterloo, Ontario, Canada.

Chakravarty, A.K. (1982a), "Multi-item inventory aggregation into groups", Journal of Operational Research Society (U.K.) 32, 19-26.

Chakravarty, A.K. (1982b), "Consecutiveness rule for inventory grouping with integer multiple constrained group-review periods", unpublished manuscript.

Chakravarty, A.K., J.B. Orlin and U.G. Rothblum (1982), "A partitioning problem with additive objective with an application to optimal inventory groupings for joint replenishment", Operations Research 30, pp. 1018-1022.

DeGroot, M.H. (1975), Probability and Statistics, Addison-Wesley.

Denardo, E.V., G. Huberman and U.G. Rothblum (1982), "Optimal locations on a line are interleaved", Operations Research 30, pp. 745-759.

Garey, M.R. and D.S. Johnson (1978), "Strong NP-completeness results: motivation, examples, and implication," S, Assoc. Comput. Mach. 25, 499-508.

Goyal, S.K. (1974), "Determination of Optimum packing frequence of items jointly replenished", Management Science 21, pp. 436-443.

Hwang, F.K. (1981), "Optimal partitions," Journal of Optimization Theory and Applications 34, pp. 1-10.

Hwang, F.K., J. Sun and Y. Yao (1982), "Optimal set partitioning," an unpublished manuscript.

Nocturne, D. (1973), "Economic order frequency for several items jointly replenished", Management Science 19, 1093.

Silver, E.A. (1976), "A simple method of determining order quantities in joint replenishments under deterministic demands", Management Science 22, 1351.

Wagner, H.M. (1969), Principles of Operations Research, Prentice-Hall.