Preferential Defense Strategies. Part I: The Static Case *

Patrick A Hosein[†] Michael Athans[‡]

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Abstract

In a military conflict, the defense tries to save as many of its valuable assets as possible. For any given offensive strategy the defense must decide which of its assets should be defended and which of them it should leave completely undefended. This is called a preferential defense strategy. We will present an algorithm for obtaining a near-optimal defense strategy under the assumption that all defensive weapons are identical.

1 Introduction

We consider the following problem faced by the defense in a military conflict. The offense (the enemy) launches a number of offensive weapons which are aimed at valuable assets (military installations, population centers, Command and Control (C^2) nodes, weapon farms, harbors etc.) of the defense. Since these weapons will be the targets of the defense's weapons, henceforth we will call them targets. Each of these targets is aimed at exactly one of the defense's assets and, if it is not intercepted, it destroys the asset with some lethality probability. This probability will depend on the accuracy of the targets as well as the nature (i.e. hardness) of the asset. We will assume that the impact of a target on an asset is independent of all other targets and assets. The defense has a number of defensive weapons with which to engage these incoming targets. The engagement of a target by a weapon will be modeled as a stochastic event. A probability, called a "kill probability", will be assigned to each weapon-target pair. This will be the probability that the weapon destroys the target if it is assigned to it and reflects the characteristics of the engagement of the specific weapon-target pair. We will assume that the engagement of a weapon-target pair is independent of all other weapons and targets. Values are assigned to the defended assets and the objective

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[†]AT&T Bell Laboratories, Holmdel, New Jersey 07733.

[‡]Massachusetts Institute of Technology, Cambridge, Massachusetts 02139.

of the defense is to assign weapons to targets so as to maximize the expected total value of the assets which survive after all weapon-target engagements and all target impacts. We will call this the Asset-Based Weapon-Target Allocation (WTA) problem. In this paper we will assume that all weapons are fired simultaneously. In part II of our report we will consider the dynamic case in which weapons are fired in stages and the outcomes of engagements of a stage are observed before assignments for the following stage are made.

Note that a particular target may be engaged by more than one weapon (Salvo attacks).[1] Also note that, in order to save an asset, the defense must, with high probability, destroy all of the targets aimed for it. Each of these targets must be attacked with enough weapons so as to make the probability that one or more of them survives sufficiently small. However, if this is done, the defense may not have enough weapons to defend all of the assets. Therefore the defense must decide which of the assets should be defended and assign all of its weapons to the defense of these assets. No weapons should be assigned to the targets aimed for the other not-to-be-defended assets. This is known as a preferential defense strategy (see for example Bracken et al. [2]).

Another frequently used model for weapon-target allocation is the following. Assign values to the offensive targets (based on target type, probable point of impact etc.) and assign weapons to targets so as to minimize the total expected surviving value of the targets which survive all engagements. In this model the assets are taken into account only through the assignment of target values. We will call this the Target-Based WTA problem. This problem can be shown to be a special case of the Asset-Based problem.

Target-Based objectives lead to *subtractive* defense strategies. In other words the defense tries to destroy as many of the most lethal targets as possible, or at least the most valuable ones. On the other hand Asset-Based objectives lead to *preferential* defense strategies. Note that, by directing multiple targets at an asset, the offense is in effect trying to make a subtractive defense useless. This is because in a subtractive defense it is likely that at least one of the targets aimed at an asset will get through, and that the asset will almost surely be destroyed. On the other hand, a preferential defense requires more information because the defense has to know the point of impact of each target. If this information is not available, then the best that the defense can do is to

[1]The weapon-target allocation problem is but one of the many problems that need to be addressed in the field of Command and Control (C^2) theory. The perspectives paper by Athans [1] presents some of the other basic problems in the theory of C^2 systems.

use a subtractive defense. Therefore an understanding of both the Target-Based and Asset-Based problems is needed in order to produce the best defense possible. In this paper we will only consider the Asset-Based problem (see [7] for details of the Target-Based problem).

Some important properties of the Asset-Based WTA problem are that it is (a) *NP-Complete* (i.e. one must essentially resort to complete enumeration to find the optimal solution), (b) *Discrete* (fractional weapon assignments are not allowed), (c) *Nonlinear* (the objective function is non-linear; it is also neither convex nor concave), (d) *Stochastic* (weapon-target engagements and target impacts are modeled as stochastic events) and (e) *Large-Scale* (the number of weapons and targets is large, making enumeration techniques impractical). These properties of the problem rule out any hope of obtaining efficient optimal algorithms. In this paper we will study the special case of the Asset-Based problem in which the kill and lethality probabilities are solely target dependent. We will present an algorithm for obtaining a sub-optimal solution to the problem and show that the solution is in fact near-optimal.

Most of the literature on weapon-target allocation considers the Target-Based problem. Matlin [9] provides a review of the literature on weapon-target allocation problems. Several references are given and are classified by the model under consideration. Eckler and Burr [6] also give a review of the material on weapons allocation problems. Besides giving references, they summarize different mathematical models and provide some analysis.

A major result, obtained by Lloyd and Witsenhausen [8], is that the Target-Based problem is NP-Complete. What this means is that the computation time of any optimal algorithm for the problem will grow exponentially with the size of the problem. Since this is a special case of the Asset-Based problem then we can conclude that the Asset-Based problem is also NP-Complete.

In [3], Burr et al. take a different approach to the weapon-target allocation problem. Instead of fixing the number of defensive weapons and minimizing the amount of damage caused by the offense's weapons, they minimize the number of defensive weapons needed by the defense to provide a given level of defense (i.e. an upper bound on the damage caused by the offensive weapons).

A group at Alphatech Inc., under the leadership of Dr. D. A. Castañon, has examined both Target-Based and Asset-Based problems in the context of the Strategic Defense System. Their recent reports, although unclassified, are restricted and the first author did not have access to these documents. On the other hand, personal communication with Dr. Castañon [4] ensured that no significant duplication of effort and results (unclassified and/or unrestricted) occurred.

2 Problem Definition

In this section we will define the Asset-Based problem. We will assume that all weapons are fired simultaneously (i.e. no feedback of information) and that damage assessment is performed after all target inpacts. In part II of this report we will consider the case in which weapons are assigned dynamically. There we will assume that the results of engagements of previous assignments are observed (i.e. information is fed back) before assignments for the present stage are made.

We will assume that the engagement of a target by a weapon is independent of all other weapons and targets and that the impact of a target on an asset is independent of all other targets and assets. The following notation will be used.

$K \stackrel{\text{def}}{=}$ the number of assets of the defense,

$N \stackrel{\text{def}}{=}$ the number of targets (offense weapons),

$M \stackrel{\text{def}}{=}$ the number of defense weapons,

$G_k \stackrel{\text{def}}{=}$ the set of targets aimed for asset k, $\quad k = 1, 2, \ldots, K$,

$n_k \stackrel{\text{def}}{=}$ the number of targets aimed for asset k, (i.e. $|G_k|$), $k = 1, 2, \ldots, K$,

$W_k \stackrel{\text{def}}{=}$ the value of asset k, $\quad k = 1, 2, \ldots, K$,

$p_{ij} \stackrel{\text{def}}{=}$ the probability that weapon j destroys target i if assigned to it,
$i = 1, 2, \ldots, N; \quad j = 1, 2, \ldots, M$,

$\pi_i \stackrel{\text{def}}{=}$ the probability that target i destroys the asset to which it is aimed, $i = 1, 2, \ldots, N$,

$Z_+^n \stackrel{\text{def}}{=}$ The set of ordered n-tuples of non-negative integers.

The decision variables will be denoted by:

$$x_{ij} = \begin{cases} 1 & \text{if weapon } j \text{ is assigned to target } i \\ 0 & \text{otherwise} \end{cases}$$

The probability that target i is destroyed is given by $1 - \prod_{j=1}^{M}(1 - p_{ij})^{x_{ij}}$. Therefore the probability that asset k survives all targets aimed for it is given by $\prod_{i \in G_k}[1 - \pi \prod_{j=1}^{M}(1 - p_{ij})^{x_{ij}}]$. Hence we can state the problem as:

Problem 2.1 *The Asset-Based WTA problem can be stated as:*

$$\max_{\{x_{ij} \in \{0,1\}\}} J = \sum_{k=1}^{K} W_k \prod_{i \in G_k} (1 - \pi_i \prod_{j=1}^{M} (1 - p_{ij})^{x_{ij}}),$$

$$\text{subject to} \qquad \sum_{i=1}^{N} x_{ij} = 1, \quad j = 1, 2, \ldots, M.$$

The objective function is the sum over all assets of the value of the asset times the probability of survival of the asset. The constraint is due to the fact that each weapon can be assigned to only one target.

The solution to problem 2.1 provides us with an assignment of weapons to targets. However, recall that it may be optimal to use a preferential defense strategy, i.e. defend some of the assets and leave the others undefended. From the solution of problem 2.1 one can tell which of the assets should be defended and how each of the defended assets should be defended. We will find that the assets which are defended have large values and/or have few targets aimed for them. The assets which have few targets aimed for them will be assigned a small number of weapons per target. If an asset has many targets aimed for it but has such a large value that it is optimal to defend it, then, we will find that many weapons per target will be assigned to defend it. This must be done to ensure that the probability, that one or more of the targets aimed for it get through, is small.

The Asset-Based problem has proven to be significantly more difficult than the Target-Based problem. The difficulty stems from the fact that, unlike the Target-Based problem which has a convex objective function, the objective function of problem 2.1 is neither convex nor concave. Even if we assume that the kill probabilities are independent of the weapons, the problem is still difficult. However, it has not yet been proven whether the problem, under the assumption of weapon independent kill probabilities, is NP-Complete or polynomial time solvable.

3 Asset Dependent Kill and Lethality Probabilities

In this section we will assume that the kill probability of a weapon-target pair is dependent solely on the asset to which the target is aimed. We will also assume that the lethality probability π_k of a target is dependent solely on the asset to which the target is aimed. We will denote the kill probability of a weapon on target i by p_k, where k is the asset to which target i is aimed. Let us denote the number of weapons that are assigned to target i by x_i.

Problem 3.1 *The Asset-Based WTA Problem with Asset dependent probabilities can be stated as:*

$$\max_{\{x_i \in Z_+\}} J = \sum_{k=1}^{K} W_k \prod_{i \in G_k} (1 - \pi_k (1 - p_k)^{x_i}),$$

$$\text{subject to} \qquad \sum_{i=1}^{N} x_i = M.$$

The optimal assignment of problem 3.1 has some important properties which we give in the following theorems.

Theorem 3.1 *If $\vec{x}$ is an optimal assignment for problem 3.1 then*

$$|x_a - x_b| \leq 1, \qquad \forall\, a, b \in G_k, \quad k = 1, \ldots, K.$$

Proof: Pick any asset k and assume that $x_a > x_b + 1$ for some pair of targets $a, b \in G_k$. Let $J(\vec{x})$ denote the value of this assignment. Now consider the assignment which is the same as $\vec{x}$ except that a single weapon is removed from target a and assigned to target b. If we use the notation $e_i^T = (0, \ldots, 0, 1, 0, \ldots, 0)$, then this assignment can be written as $\vec{x} - e_a + e_b$. We will denote the value of this assignment by $J(\vec{x} - e_a + e_b)$. We have:

$$J(\vec{x}) - J(\vec{x} - e_a + e_b) = W_k \pi_k p_k [(1 - p_k)^{x_a - 1} - (1 - p_k)^{x_b}] \prod_{\substack{i \in G_k \\ i \neq a, b}} (1 - (1 - p_k)^{x_i}) \qquad (1)$$

Since $x_a > x_b + 1$ then the right hand side of 1 is negative. Therefore,

$$J(\vec{x}) - J(\vec{x} - e_a + e_b) < 0.$$

This is a contradiction since the assignment $\vec{x}$ was assumed to be optimal. ■

This theorem states that, in the optimal assignment, the numbers of weapons assigned to any two targets aimed for the same asset are either equal or differ by one. Theorem 3.1 can be used to simplify problem 3.1 by introducing a new decision variable, X_k, which will be used to denote the number of weapons assigned to all targets aimed for asset k. Given X_k one can obtain an optimal assignment by spreading the weapons evenly among the targets. We will let $\vec{X}$ be the K-dimensional vector with elements X_k. Let us define $J_k(X_k)$ to be the expected surviving value of asset k if X_k weapons are assigned to its targets and these weapons are spread as evenly as possible among the targets. We then have[2]

$$J_k(X) = W_k(1 - \pi_k(1 - p_k)^{\lceil \frac{X}{n_k} \rceil})^{X - n_k \lfloor \frac{X}{n_k} \rfloor}(1 - \pi_k(1 - p_k)^{\lfloor \frac{X}{n_k} \rfloor})^{n_k(1 + \lfloor \frac{X}{n_k} \rfloor) - X} \tag{2}$$

We can now use equation 2 to simplify problem 3.1 to

Problem 3.2

$$\max_{\vec{X} \in Z_+^K} J = \sum_{k=1}^{K} W_k J_k(X_k),$$

$$\text{subject to} \quad \sum_{k=1}^{K} X_k = M.$$

Note that the objective function in problem 3.2 is separable. However, the functions J_k are not concave which means that multiple maxima may exist for the problem. This makes the problem difficult. Our approach is to find concave approximations to each of the functions J_k and then solve the resulting concave optimization problem. This provides us with a sub-optimal solution which we will show is near-optimal.

In figure 1 we have plotted an example of the function J_k for the case $n_k = 10, p_k = 0.8, \pi_k = 0.9$ and $W_k = 1$. Clearly it is neither convex nor concave. Note that between multiples of n_k the function is convex. This is due to the fact that the weakest link in the defense of the asset is the target to which the least number of weapons is assigned. As a function of multiples of n_k, the function is convex and then becomes concave. This change occurs roughly at the point where the expected number of surviving targets is one i.e. the value of X for which

$$n_k \pi_k (1 - p_k)^{\frac{X}{n_k}} = 1.$$

These two properties can be stated formally as follows.

Property 1: If $\lfloor \frac{X}{n_k} \rfloor < X < \lceil \frac{X}{n_k} \rceil$ then

$$J_k(X-1) - 2J_k(X) + J_k(X+1) \geq 0.$$

[2]For ane real variable x, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x while $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

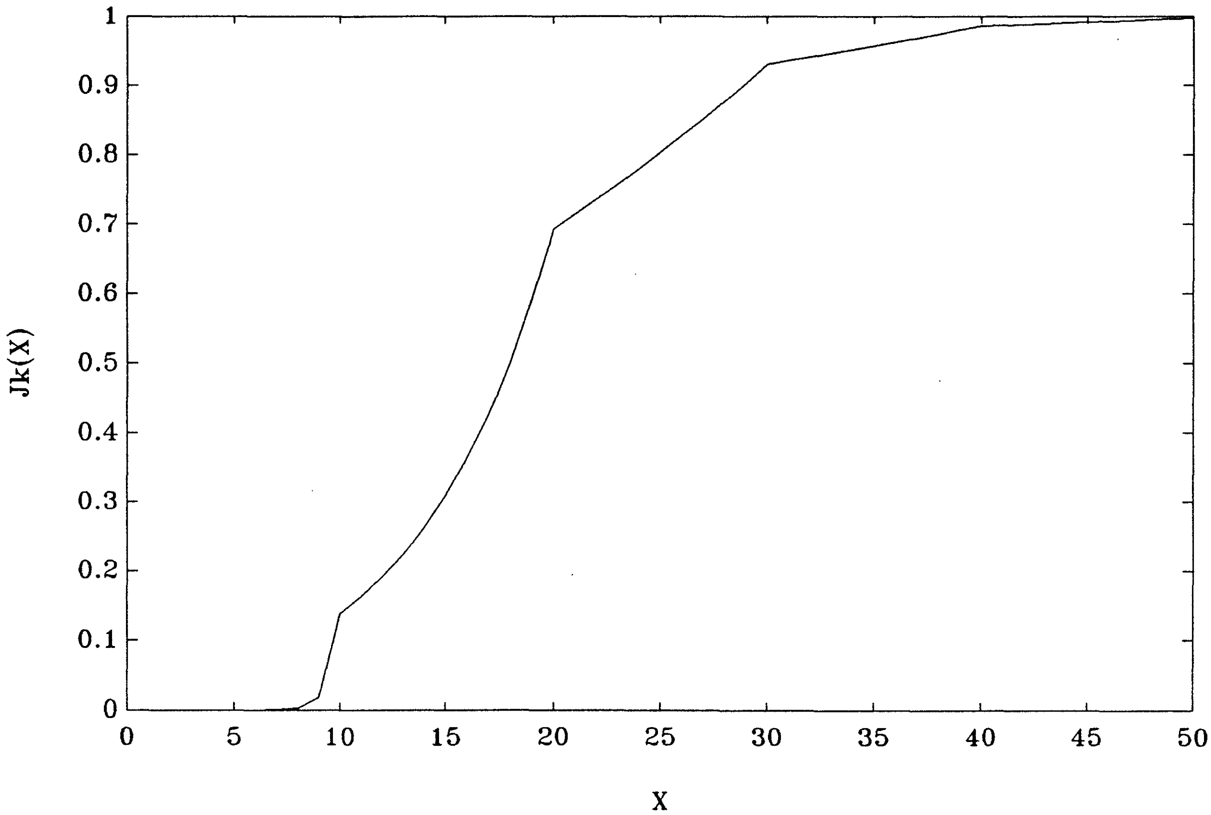

Figure 1: An example of the function J_k, the expected surviving value of asset k, for $n_k = 10, p_k = 0.8, \pi_k = 0.9$ and $W_k = 1$, vs. the total number of weapons assigned to defend it, X.

Proof: Let us define

$$\alpha \equiv 1 - \pi_k(1 - p_k)^{\lceil \frac{X}{n_k} \rceil},$$

and

$$\beta \equiv 1 - \pi_k(1 - p_k)^{\lfloor \frac{X}{n_k} \rfloor},$$

Note that

$$J_k(X + 1) = J_k(X)\frac{\alpha}{\beta},$$

and

$$J_k(X - 1) = J_k(X)\frac{\beta}{\alpha}.$$

Therefore,

$$\begin{aligned} J_k(X-1) - 2J_k(X) + J_k(X+1) &= J_k(X)[\frac{\alpha}{\beta} + \frac{\beta}{\alpha} - 2] \\ &= J_k(X)[\alpha - \beta]^2/(\alpha\beta) \\ &\geq 0, \end{aligned}$$

which completes the proof. ■

Property 2: There exists an $r_k \in Z_+$ such that for all $r \geq r_k$,

$$J_k(n_k r) - 2J_k(n_k(r+1)) + J_k(n_k(r+2)) \leq 0.$$

Furthermore, if $r_k > 0$ then for all $0 < r \leq r_k$,

$$J_k(n_k(r-1)) - 2J_k(n_k r) + J_k(n_k(r+1)) \geq 0.$$

Proof: Note that

$$J_k(n_k r) = W_k(1 - \pi_k(1-p_k)^r)^{n_k}.$$

Let $J_k''(n_k r)$ denote the second derivative of $J_k(n_k r)$ with respect to r. We have

$$J_k''(n_k r) = W_k n_k \pi_k (1-p_k)^r \log^2(1-p_k)(1 - \pi_k(1-p_k)^r)^{n_k - 2}[n_k \pi_k (1-p_k)^r - 1].$$

Therefore if $r \leq \left\lfloor \frac{-\log(n_k \pi_k)}{\log(1-p_k)} \right\rfloor$, then $J_k''(n_k r) \geq 0$ and the function is convex. Otherwise if $r \geq \left\lceil \frac{-\log(n_k \pi_k)}{\log(1-p_k)} \right\rceil$ then $J_k''(n_k r) \leq 0$ and the function is concave. ■

The first property says that the function J_k is convex between multiples of n_k. The second property says that the function $J_k(rn_k)$ is first convex for small values of r and concave for larger values.

We will approximate each of the functions J_k by its concave hull which we will denote by $\bar{J}_k$. Note that, because of property 1, the line through the origin which is tangent to the function J_k will touch at a point where X is a multiple of n_k. Define $\ell_k \in Z_+$ to be such that $X = \ell_k n_k$ is the point at which the tangent through the origin is tangent to J_k. Next note that, because of property 2, $\ell_k \geq r_k$ where r_k is the point at which the function $J_k(rn_k)$ changes from being convex to concave. This implies that the function $J_k(rn_k)$ is concave for $r \geq \ell_k$. These facts can now be used to obtain the concave hull $\bar{J}_k$ as

$$\bar{J}_k(X) = \begin{cases} \frac{X}{n_k \ell_k} J_k(n_k \ell_k) & \text{if } X \leq n_k \ell_k \\ J_k(n_k \lfloor \frac{X}{n_k} \rfloor) + (\frac{X}{n_k} - \lfloor \frac{X}{n_k} \rfloor)(J_k(n_k \lceil \frac{X}{n_k} \rceil) - J_k(n_k \lfloor \frac{X}{n_k} \rfloor)) & \text{if } X > n_k \ell_k \end{cases} \quad (3)$$

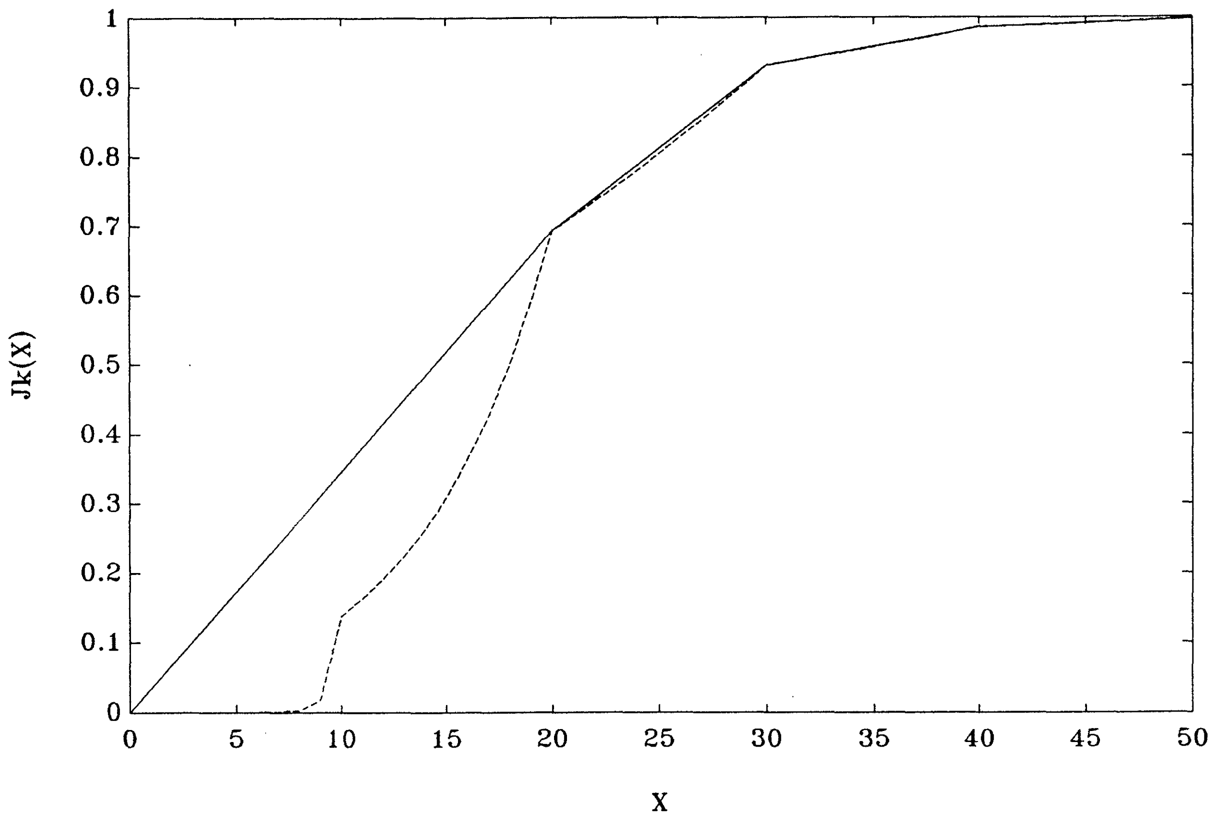

Figure 2: The hull $\bar{J}_k(X)$ of the function $J_k(X)$ shown in figure 1. The dashed line is $J_k(X)$ and the solid line is $\bar{J}_k(X)$.

In figure 2 we have drawn the hull for the function that was plotted in figure 1. The dotted line is the function J_k while the solid line is the function $\bar{J}_k$. Note that in this example $\ell_k = 2$ and $r_k = 1$. Secondly note that $\bar{J}_k(X)$ is a good approximation of $J_k(X)$ for $X \geq n_k \ell_k$. This fact will be used in obtaining bounds on the optimal value for the problem.

Consider, now the problem in which we replace the objective function in problem 3.2 by its concave hull.

Problem 3.3 *An approximation to problem 3.2 is given by:*

$$\max_{\vec{X} \in Z_+^K} \bar{J} = \sum_{k=1}^{K} W_k \bar{J}_k(X_k),$$

$$\text{subject to} \quad \sum_{k=1}^{K} X_k = M.$$

This is a separable concave maximization problem. Before we solve this problem we need to present the following result.

Let $F : \Re^n \longrightarrow \Re$ be defined as:

$$F(\vec{x}) = \sum_{i=1}^{n} f_i(x_i) \qquad x_i \in Z_+, \quad i = 1, 2, \ldots, n$$

where each of the functions $f_i(x)$ has the property:

$$f_i(x-1) - f_i(x) \leq f_i(x) - f_i(x+1).$$

For any $m \in Z_+$ consider the following optimization problem:

$$\max_{\vec{x} \in Z_+^n} F(\vec{x}) \tag{4}$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_i = m.$$

The following algorithm, called the Maximum Marginal Return (MMR) algorithm, is optimal for this problem.

```
procedure MMR
begin
        x⃗ := [0,...,0]^T
        for i := 1:n do Δ_i := f_i(x_i + 1) − f_i(x_i)
        for j := 1:m do
        begin
                Let k be such that Δ_k = max_i{Δ_i};
                x_k := x_k + 1;
                Δ_k := f_k(x_k + 1) − f_k(x_k);
        end
end
```

This is basically a greedy algorithm. In each iteration the index k is found for which an increase in x_k by unity produces the maximum increase in the objective function. The value of x_k is increased by one and the process is repeated until the constraint is satisfied. If the marginal return data is stored in heaps then the initial heap data structure requires $O(n)$ operations to construct. In each iteration the maximum marginal return item is removed and the heap must be reorganized. This requires $O(\log n)$ operations. Since there are m iterations, the worst case complexity of the algorithm is $O(n + m \log n)$. We will next prove the optimality of the algorithm.

Theorem 3.2 *The solution produced by the MMR algorithm is optimal for problem 4.*

Proof The proof of this theorem for the special case in which the functions f_i have the form $f(x) = V(1-p)^x$ is given in [5]. We have generalized their proof. The proof of the theorem will be by induction. Note that the theorem is trivially true for the case $m = 1$. Assume that it is true for $m = \bar{m}$. Denote the optimal solution for this case by $\vec{\bar{x}}$. Now suppose that:

$$f_k(\bar{x}_k + 1) - f_k(\bar{x}_k) = \max_i\{f_i(\bar{x}_i + 1) - f_i(\bar{x}_i)\} \tag{5}$$

Let us denote the solution produced by the algorithm for the case $m = \bar{m} + 1$ by $\vec{x}^*$. Note that $\vec{x}^* = \vec{\bar{x}} + e_k$. Let $\vec{z}$ be any feasible solution to the problem with $m = \bar{m} + 1$ other than $\vec{x}^*$. There must exist some j such that $z_j > x_j^* \geq \bar{x}_j$. Let $\vec{\bar{\bar{x}}} = \vec{\bar{x}} + e_j$. We have

$$F(\vec{z}) = F(\vec{z} - e_j) - [f_j(z_j - 1) - f_j(z_j)]. \tag{6}$$

We also have that

$$F(\vec{\bar{\bar{x}}}) = F(\vec{\bar{x}}) - [f_j(\bar{x}_j) - f_j(\bar{x}_j + 1)]. \tag{7}$$

Since $\vec{\bar{x}}$ is optimal for the case $m = \bar{m}$ we have

$$F(\vec{z} - e_j) \leq F(\vec{\bar{x}}) \tag{8}$$

and by the assumptions on the functions f_i stated in the problem we have

$$f_j(\bar{x}_j) - f_j(\bar{x}_j + 1) \leq f_j(z_j - 1) - f_j(z_j) \tag{9}$$

since $z_j > \bar{x}_j$. If we subtract 7 from 6 and use the inequalities 8 and 9 then we can show that $F(\vec{z}) \leq F(\vec{\bar{\bar{x}}})$. Furthermore one can use 5 to show that $F(\vec{x}^*) \geq F(\vec{\bar{\bar{x}}})$. We therefore have that

$$F(\vec{z}) \leq F(\vec{x}^*).$$

This implies that the solution $\vec{z}$ is no better than the solution $\vec{x}^*$ obtained by the algorithm in the theorem. Since $\vec{z}$ can be any feasible solution we conclude that the solution obtained by the algorithm for the case $m = \bar{m}+1$ is optimal. Therefore, by induction, the theorem is true for all $m \geq 0$. ■

Note that problem 3.3 has the form required to apply theorem 3.2. We can therefore conclude that the MMR algorithm produces the optimal solution to problem 3.3. Denote the optimal solution to 3.3, if the MMR algorithm is used, by $\vec{\vec{X}}^*$. The assignment $\vec{\vec{X}}^*$ has the following important property.

Property 3: For all but one of the assets k, $\bar{X}_k^*$ is a multiple of n_k.

Proof: The proof will be by contradiction. Assume that the property does not hold. This means that there exist at least two assets with the property that the total number of weapons assigned to the targets aimed for them is not a multiple of the number of targets aimed for them. For simplicity let us assume that two of these assets are assets 1 and 2. Note that the function $\bar{J}_k$ is linear between multiples of n_k. Therefore, the marginal return of asset k is constant between multiples of n_k. If the marginal return for asset 1 on termination of the algorithm is greater than that of asset 2 then the weapons that were assigned to asset 2 would have been assigned to asset 1 leading to a contradiction. Therefore the marginal return of asset 2 on termination of the algorithm must be greater than or equal to that of asset 1. If this is the case, then, since the algorithm started assigning weapons to asset 2 then it would have continued doing so until the number of weapons assigned was a multiple of n_2 (i.e. until the marginal return for asset 2 changed value). This is a contradiction since we assumed that the number of weapons assigned to asset 2 was not a multiple of n_2. ■

This property states the following. If an asset is defended, then the same number of weapons is assigned to each of the targets aimed for the asset. Because the total number of weapons is arbitrary, then it may not be possible to do this for all of the defended assets. Therefore, the property may not hold for one of the defended assets. Let target v be the target for which $\bar{X}_v^*$ is *not* a multiple of n_v, i.e the property is *violated.*

By examining equation 3 one can see that if X is a multiple of n_k then $\bar{J}_k(X) = J_k(X)$. Since $\bar{X}_k^*$ is a multiple of n_k for all assets k except asset v then:

$$\bar{J}(\vec{\bar{X}}^*) = J(\vec{\bar{X}}^*) - J(\bar{X}_v^*) + \bar{J}(\bar{X}_v^*). \tag{10}$$

Finally note that $\bar{J}(X)$ is an upper bound to $J(X)$. Therefore, if we denote the optimal solution to the original problem (i.e. 3.1) by $\vec{X}^*$ then

$$\bar{J}(\vec{\bar{X}}^*) \geq J(\vec{X}^*). \tag{11}$$

Furthermore since $\vec{X}^*$ is optimal for Problem 3.1 then

$$J(\vec{X}^*) \geq J(\vec{\bar{X}}^*). \tag{12}$$

Combining equations 10, 11 and 12 we obtain

$$J(\vec{\bar{X}}^*) + \bar{J}(\bar{X}_v^*) - J(\bar{X}_v^*) \geq J(\vec{X}^*) \geq J(\vec{\bar{X}}^*) \tag{13}$$

Therefore the optimal value of problem 3.3, the approximate problem, can be used to obtain upper and lower bounds on the optimal value of problem 3.2.

Notice that the solution to the approximate problem 3.3 is a suboptimal solution to the original problem 3.2. The difference in value between the optimal and suboptimal solutions is bounded by

$$J(\vec{X}^*) - J(\vec{\bar{X}}^*) \leq \bar{J}(\bar{X}_v^*) - J(\bar{X}_v^*) \tag{14}$$

Note that if $\bar{J}(\bar{X}_v^*) = J(\bar{X}_v^*)$, which would be the case if $\bar{X}_v^*$ is also a multiple of n_v then we obtain

$$J(\vec{X}^*) = J(\vec{\bar{X}}^*)$$

which implies that $\vec{\bar{X}}^*$ is an optimal solution to problem 3.2. In other words, if the total number of weapons is such that for *each* defended asset the same number of weapons is assigned to each of the targets aimed for the asset, then the algorithm produces the optimal solution.

Let us now consider the case in which $\bar{J}(\bar{X}_v^*) > J(\bar{X}_v^*)$. In this case $\bar{X}_v^*$ is not a multiple of n_v. By the nature of the algorithm used, if the number of weapons is increased by $n_v\lceil\frac{\bar{X}_v^*}{n_v}\rceil - \bar{X}_v^*$ weapons, then the optimal solution to problem 3.3 will be the same except that $\bar{X}_v^*$ will be increased by the number of additional weapons making it a multiple of n_v. The analysis in the previous paragraph can then be used to show that the optimal solution for the approximate problem is also optimal for the original problem 3.2. Similarly if the number of weapons is *decreased* by $\bar{X}_v^* - n_v\lfloor\frac{\bar{X}_v^*}{n_v}\rfloor$ then the resulting optimal solution of the approximate problem is optimal for the original problem 3.2 with the decreased number of weapons. These results suggest that the optimal solution obtained for the approximate problem is close to being optimal for the true problem (i.e. 3.2). We will now state our result as a theorem.

Theorem 3.3 *Consider the Asset-Based WTA problem in which the kill probability of a weapon-target pair and the lethality probability of a target-asset pair is dependent solely on the asset to which the target is aimed. Let* $\vec{\bar{X}}^*$ *be the optimal solution to the approximate problem defined in 3.3 obtained by the use of the MMR algorithm. Let* $\vec{X}^*$ *denote the optimal solution of the true problem (i.e. problem 3.2) then*

$$J(\vec{\bar{X}}^*) + \bar{J}(\bar{X}_v^*) - J(\bar{X}_v^*) \geq J(\vec{X}^*) \geq J(\vec{\bar{X}}^*).$$

Furthermore if we let

$$\varepsilon = \max_{k} \max_{\{0 \le X_k \le \ell_k n_k\}} \bar{J}_k(X_k) - J_k(X_k)$$

then

$$J(\vec{X}^*) - J(\vec{\bar{X}}^*) \le \varepsilon.$$

Proof: The first part of the theorem has already been proved. The second part is obtained by upper bounding the difference $\bar{J}(\bar{X}_v^*) - J(\bar{X}_v^*)$ by its maximum possible value. ■

Note that ε is the maximum, over all assets, of the maximum difference between the function J_k and its concave hull $\bar{J}_k$. Therefore ε is dependent solely on the shape of the functions J_k. If we increase the size of the problem by increasing the number of assets, targets and weapons but do not change the types of assets so that ε does not change then we find that as the problem size increases, the percentage error of the suboptimal solution decreases because ε remains constant. Therefore, for large-scale problems we expect that the algorithm will perform well.

Note that the bound ε can be computed even before problem 3.2 is solved. This provides an upper bound on the error of the suboptimal solution that is obtained by the algorithm. However, after the approximate problem 3.3 has been solved a much tighter bound is obtained by the difference $\bar{J}(\bar{X}_v^*) - J(\bar{X}_v^*)$. Furthermore, as we have shown, the solution can be made optimal by slightly decreasing or increasing the number of weapons.

4 Target Dependent Kill and Lethality Probabilities

In the previous section we assumed that the kill and lethality probabilities were solely asset dependent. In this section we will assume that the kill probabilities are weapon independent and that the lethality probabilities are solely target dependent. These assumptions are valid for the case of a single cluster of weapons and a single type of assets. Note that this is a more general problem than that of the previous section. We will present an algorithm, similar to the one in the previous section, which yields a suboptimal solution for this problem. The following notation will be used.

$K \stackrel{\text{def}}{=}$ the number of assets of the defense,

$N \stackrel{\text{def}}{=}$ the number of targets (offense weapons),

$M \stackrel{\text{def}}{=}$ the number of defense weapons,

$W_k \stackrel{\text{def}}{=}$ the value of asset k to the defense, $k = 1, 2, \ldots, K$,

$G_k \stackrel{\text{def}}{=}$ the set of targets aimed for asset k, $\quad k = 1, 2, \ldots, K$,

$n_k \stackrel{\text{def}}{=}$ the number of targets aimed for asset k, ($|G_k|$), $\quad k = 1, 2, \ldots, K$,

$\pi_i \stackrel{\text{def}}{=}$ the probability that target i destroys the asset to which it is aimed, $i = 1, 2, \ldots, N$,

$p_i \stackrel{\text{def}}{=}$ the probability that a weapon destroys target i if it is fired at it, $i = 1, 2, \ldots, N$.

$x_i \stackrel{\text{def}}{=}$ the number of weapons assigned to target i, $\quad i = 1, 2, \ldots, N$,

$\vec{x} \stackrel{\text{def}}{=}$ the N-dimensional vector $[x_1, \ldots, x_N]^T$,

$X_k \stackrel{\text{def}}{=}$ the number of weapons assigned to the defense of asset k, (i.e. $\sum_{i \in G_k} x_i$),

$\vec{X} \stackrel{\text{def}}{=}$ the K-dimensional vector $[X_1, \ldots, X_K]^T$,

$Z_+^N \stackrel{\text{def}}{=}$ The set of ordered n-tuples of non-negative integers.

Under the assumptions we have made, the probability that an asset survives is the product over all targets of the probability that the target is destroyed. We therefore have:

Problem 4.1 *The Asset-Based WTA problem with Target dependent probabilities can be stated as:*

$$\max_{\vec{x} \in Z_+^N} J(\vec{x}) = \sum_{k=1}^{K} W_k \prod_{i \in G_k} (1 - \pi_i(1 - p_i)^{x_i}),$$

$$\text{subject to} \quad \sum_{i=1}^{N} x_i = M.$$

The objective function is the total expected surviving asset value and the constraint is due to the fact that the total number of weapons fired must equal the number of weapons available.

Because problem 4.1 is separable with respect to the assets, it can be re-formulated as follows. Let $J_k(X)$ denote the maximum expected surviving value of asset k given that X weapons are used to defend it. $J_k(X)$ can be obtained by solving the following problem.

Problem 4.2 *The subproblem (SUB) is defined by:*

$$J_k(X_k) = \max_{\vec{x} \in Z_+^{n_k}} W_k \prod_{i \in G_k} (1 - \pi_i(1 - p_i)^{x_i}),$$

$$\text{subject to} \quad \sum_{i \in G_k} x_i = X_k.$$

We can now restate the original problem.

Problem 4.3 *Problem 4.2 can be restated as (MAIN):*

$$\max_{\vec{X} \in Z_+^K} J(\vec{X}) = \sum_{k=1}^{K} J_k(X_k)$$

$$\text{subject to} \qquad \sum_{k=1}^{K} X_k = M.$$

We will first consider the subproblems 4.2. The approach will then be the same as in the previous section. We will find the hull of J_k and then use an MMR algorithm on the approximate problem in which J_k is replaced by its hull in problem 4.3. We will then show that the solution of this approximate problem is a near-optimal solution to the true problem 4.3. Since the approach is identical to that used in the previous section, some of the details will be omitted.

4.1 Solution of the Subproblem

Since the logarithm function is monotonic, if we replace the objective function of problem 4.2 by its logarithm then the optimal assignment of the resulting problem will also be optimal for the original problem. If we take the logarithm of the objective function of SUB we obtain

$$\ln W_k + \sum_{i \in G_k} \ln[1 - \pi_i(1 - p_i)^{x_i}].$$

The first term is constant so we can remove it and optimize the second term. The optimization problem is:

$$\max_{\vec{x} \in Z_+^{n_k}} \mathcal{F}(\vec{x}_k) = \sum_{i \in G_k} \ln[1 - \pi_i(1 - p_i)^{x_i}], \tag{15}$$

$$\text{subject to} \qquad \sum_{i \in G_k} x_i = X_k.$$

Note that the function $\mathcal{F}(\vec{x}_k)$ is separable with respect to the target index i. Next note that each of the functions $\ln[1 - \pi_i(1 - p_i)^{x_i}]$ is concave. This can be verified by showing that the second derivative of this function (with respect to x_i) is non-positive. Therefore, the objective function $\mathcal{F}$ satisifies the conditions required to apply theorem 3.2. We can therefore conclude that the MMR algorithm can be used to obtain an optimal solution to this problem. This solution will also be optimal for problem 4.2.

We next need to obtain the concave hull of the function J_k. In the previous section this task was easy because in that case the functions J_k had two special properties which could be exploited,

(a) J_k is convex between multiples of n_k and (b) as a function of multiples of n_k the function is first convex and then concave. In this case however, the functions J_k do not have these special properties. The functions J_k in this section differ from those of the previous section because the kill probabilities and lethality probabilities are target dependent instead of asset dependent. The effect of these differences is that the function J_k is "smoother" for target dependent probabilities compared to the case for asset dependent probabilities.

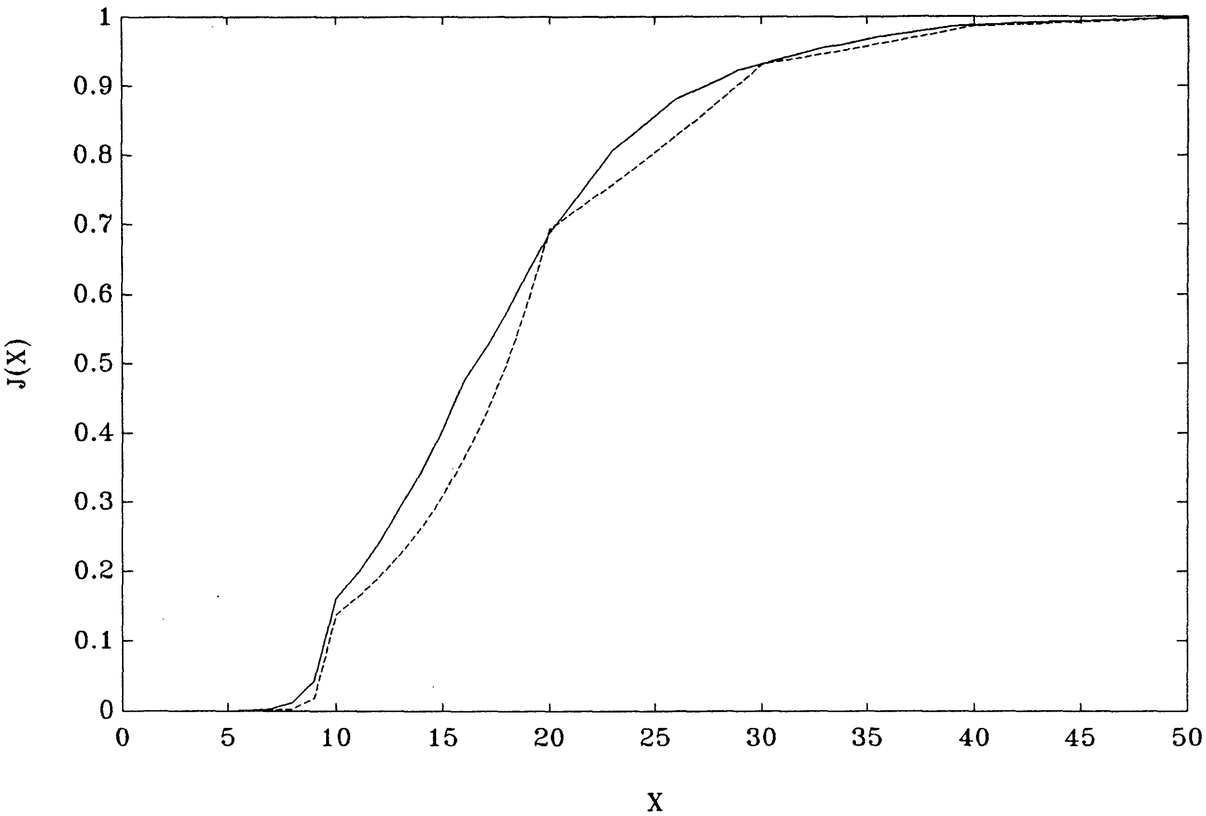

Figure 3: Maximum expected surviving value of a unit valued asset versus the number of weapons assigned to its targets for (a) target dependent parameters (solid curve) and (b) target independent parameters (dashed curve).

Figure 3 illustrates the effect of having target dependent kill and lethality probabilities. We considered the problem of a single asset with $W = 1$ and $n = 10$. The dashed curve is for the case of target independent parameters, $p_i = 0.8, \pi_i = 0.9$. The solid curve is for the case of target dependent kill probabilities, $\{p_i\} = [.7,.7,.7,.8,.8,.8,.9,.9,.9,.9]$, and target dependent lethality probabilities $\{\pi_i\} = [.8,.8,.8,.9,.9,.9,1,1,1,1]$. One can see that unlike the dashed curve, the solid curve is almost concave. For all practical purposes, the solid curve is concave in the region of a heavy defense.

4.2 Solution of the Main Problem

Let us define $\bar{J}_k$ to be the concave hull of the function J_k (defined in problem 4.2). We will approximate problem 4.3 by its concave hull. This approximate problem can then be solved to obtain a sub-optimal solution.

Problem 4.4 *The approximate problem to problem 4.3 is given by:*

$$\max_{\vec{X} \in Z_+^K} \bar{J}(\vec{X}) = \sum_{k=1}^{K} \bar{J}_k(X_k)$$

$$\text{subject to} \qquad \sum_{k=1}^{K} X_k = M.$$

Theorem 3.2 can be applied to show that the MMR algorithm produces the optimal solution to this problem as well. Let $\bar{\vec{X}}^*$ denote the optimal solution of the approximate problem 4.4. By the nature of the algorithm, we can show that for all but one of the assets

$$J_k(\bar{X}_k^*) = \bar{J}_k(\bar{X}_k^*).$$

Let the asset for which this equality does not hold be asset v. Also let $\vec{X}^*$ denote the optimal solution to the true problem 4.3. Using the same analysis as in the previous section we can then show that

$$J(\bar{\vec{X}}^*) + \bar{J}(\bar{X}_v^*) - J(\bar{X}_v^*) \geq J(\vec{X}^*) \geq J(\bar{\vec{X}}^*). \tag{16}$$

Therefore the optimal solution to the approximate problem can be used to obtain upper and lower bounds on the optimal value of true problem (i.e. problem 4.3).

Notice that the solution to the approximate problem 4.4 is a near optimal solution to the true problem (4.3). The difference in value of these two solutions is bounded by:

$$J(\vec{X}^*) - J(\bar{\vec{X}}^*) \leq \bar{J}(\bar{X}_v^*) - J(\bar{X}_v^*). \tag{17}$$

If $\bar{J}(\bar{X}_v^*) = J(\bar{X}_v^*)$ then $\bar{\vec{X}}^*$ is optimal for problem 4.3. Also note that if $\bar{J}(\bar{X}_v^*) \neq J(\bar{X}_v^*)$, then by slightly increasing or slightly decreasing the number of weapons one can obtain a problem for which the solution to the approximate problem (4.4) is also optimal for the true problem (4.3). We now state our result as a theorem:

Theorem 4.1 *Consider the Asset-Based WTA problem in which the kill and lethality probabilities depend solely on the target. Let* $\vec{\bar{X}}^*$ *be the optimal solution to the approximate problem defined in 4.4 obtained by the use of the MMR algorithm. Let* $\vec{X}^*$ *denote the optimal solution of the true problem 4.3 then*

$$J(\vec{\bar{X}}^*) + \bar{J}(\bar{X}_v^*) - J(\bar{X}_v^*) \geq J(\vec{X}^*) \geq J(\vec{\bar{X}}^*).$$

Furthermore if we let

$$\varepsilon = \max_k \max_{\{0 \leq X_k \leq \ell_k\}} \bar{J}_k(X_k) - J_k(X_k)$$

then

$$J(\vec{X}^*) - J(\vec{\bar{X}}^*) \leq \varepsilon.$$

Proof: The first part of the theorem has already been proved. The second part is obtained by upper bounding the difference $\bar{J}(\bar{X}_v^*) - J(\bar{X}_v^*)$ by its maximum possible value. ∎

Therefore we can obtain a suboptimal solution $\vec{\bar{X}}^*$ to the problem as well as an upper bound on the optimal value. Furthermore, if the number of weapons is slightly increased or slightly decreased then the algorithm produces the optimal solution for the corresponding problem.

5 Sensitivity Analysis

In this section we will present some sensitivity analysis results. These results will help us decide the importance of the role of each of the parameters in the optimization problem. This information will be useful in determining how accurately each of the parameters should be measured.

5.1 Optimal Value Sensitivity Analysis

We will present sensitivity analysis results in this subsection for the case of a single kill probability and a single lethality probability. The following baseline problem will be used:

Baseline Problem Definition

Number of weapons: $M = 200$,

Number of targets: $N = 100$,

Number of assets: $K = 10$,

Number of targets aimed at each asset: $n_k = 10, \quad k = 1, \ldots, 10$, Value of each asset: $W_k = 1, \qquad k = 1, \ldots, 10,$

Kill probability of each weapon-target pair: $p = 0.8,$

Lethality probability of each target: $\pi = 1.$

We will vary the parameters p, π, M and n_k individually and see how the optimal value of the problem changes. As we vary the kill probability p we will denote the optimal value by $J(p)$. Similar notation will be used for the other parameters. Since we do not have an algorithm that guarantees optimal solutions for the problem, we will compute upper and lower bounds on the optimal value. The algorithm presented in section 4.2 will be used to compute a solution to the problem as well as an upper bound on the optimal value. The expected value of the sub-optimal solution will be plotted with a solid line. The upper bound will be plotted with a dashed line. The plot for the optimal value will lie between these plots. Note that for some of the plots the algorithm produces the optimal solution. In these cases no dashed curve will be visible.

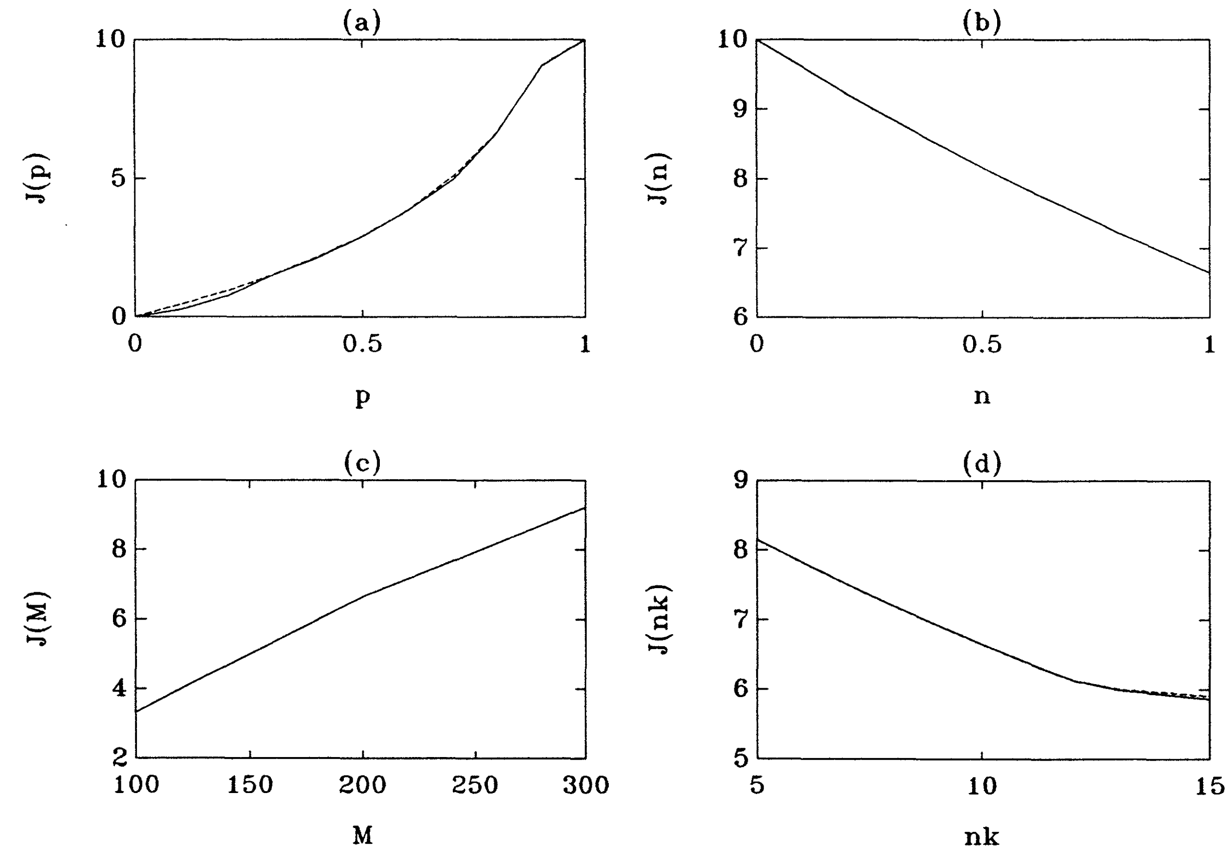

Figure 4: Upper and lower bounds on the optimal value for the baseline problem, as a function of (a) the kill probability (b) the lethality probability (c) the number of weapons and (d) the number of targets aimed for an asset (with $M = 2N$).

In figure 4 we have four plots. In plot (a) the upper and lower bounds on the optimal value is plotted versus the kill probability p which is the same for all weapon-target pairs. Note that the dashed curve is almost identical to the solid curve. This means that the solution produced by the algorithm is almost optimal for all values of the kill probability. Also note that for the values of interest to us ($0.5 \leq p \leq 0.9$) the optimal value is very sensitive to the kill probability. Small increases in the kill probability can result in large increases in the optimal value.

In plot (b) the optimal value is plotted versus the lethality probability π. Note that there is no dashed curve because the solution was optimal. Here we find that the optimal value decreases almost linearly with the lethality probability. It therefore appears that the lethality probability does not play an important role in the optimization problem.

Plot (c) contains the optimal value versus the number of weapons for $M = 100, 150, 200, 250, 300$. For these values of M the algorithm produced the optimal solution. We find that the optimal value increases almost linearly with the number of weapons.

In plot(d) the upper and lower bounds on the optimal value is plotted versus the number of targets aimed for each of the assets. We kept the weapon-target ratio fixed at 2:1. Again note that the algorithm is optimal for most of the plot. Here we find that the plot appears to be that of a piecewise-linear convex function. We also find that as the number of targets aimed for each asset increases, the optimal value decreases. This implies that, if the number of assets is kept fixed then as the size of the attack increases (i.e n_k increases for each k) the defense's arsenal must be increased at a *greater* rate to maintain the same level of performance. This gives the offense a tremendous advantage because, if we keep the kill and lethality probabilities fixed, then a small increase in the offense's arsenal has to be countered by a larger increase in the defense's arsenal if the defense wishes to maintain the same level of performance.

5.2 Optimal Solution Sensitivity Analysis

In the previous subsection we considered what happens to the optimal value as various problem parameters were varied. In this section we will see what happens to the optimal solution of the problem as each of the parameters is varied.

We will use the same baseline problem that was used in the previous subsection Because of the symmetry of the problem, the solution can be completely characterized by the number of defended assets. Note that in the solution to the problem, the same number of weapons is used to defend each of the defended assets except one. The number of weapons assigned to this special asset is less than the number assigned to each of the others. We will include this special asset, as a fraction, in the number of assets defended. This fraction is the ratio of the number of weapons assigned to defend the asset and the number of weapons assigned to each of the other defended assets. In figure 5 we have plotted the number of defended assets vs each of the parameters p, π, M and n.

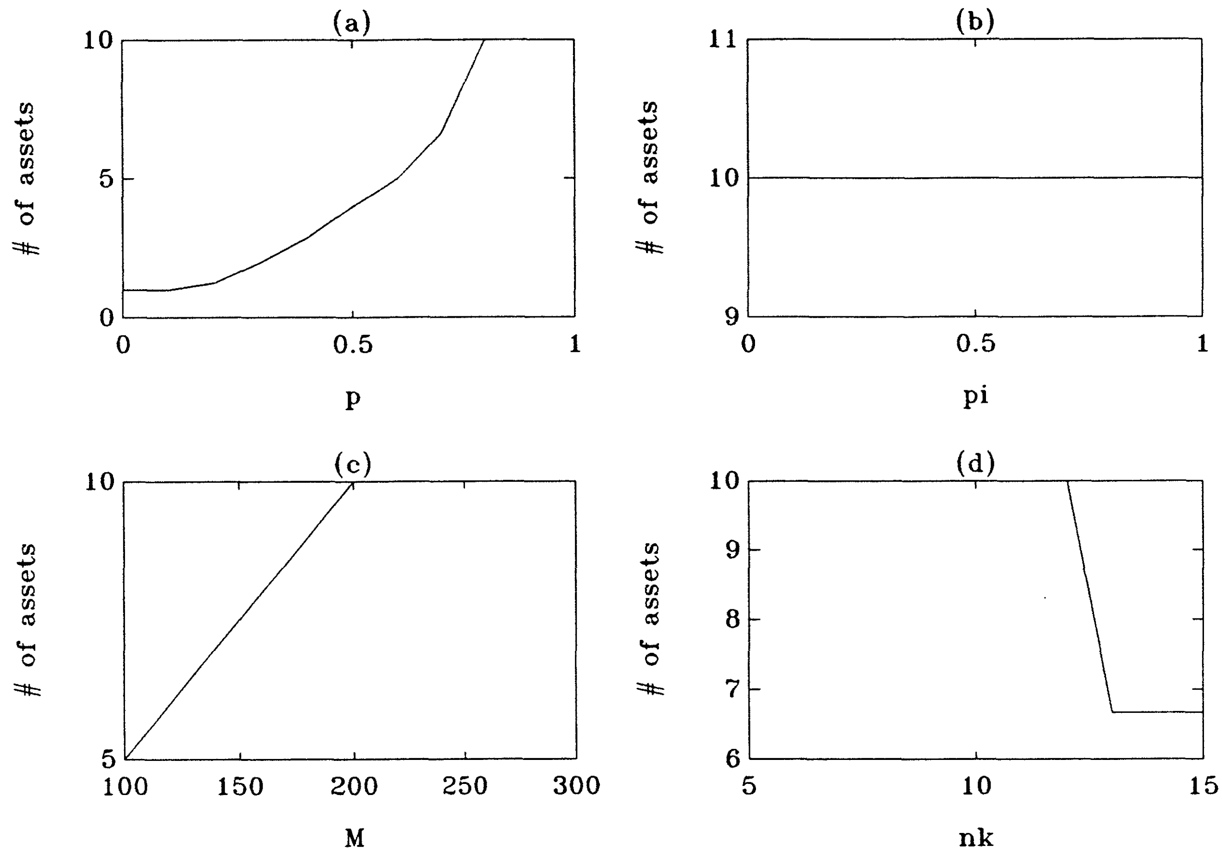

Figure 5: Number of defended assets as a function of (a) the kill probability (b) the lethality probability (c) the number of weapons and (d) the number of targets aimed for each asset (with $M = 2N$).

In plot (a) of figure 5 we have plotted the number of defended assets versus the kill probability. Note that small changes in the kill probability can result in significant changes in the strategy. Plot (b) contains the plot for the lethality probability. Here we find that changes in π do not affect the optimal strategy. This suggests that the lethality probability plays a small role in the optimization problem. Plot (c) contains the plot for the number of weapons. As the number of weapons increases more assets are defended until they are all defended. Plot (d) contains the plot for the number of targets per asset with a fixed 2:1 weapon to target ratio. Note the sudden change in the defense strategy as n_k changes from 12 to 13. For the case $n_k = 12$ the defense assigns two weapons per target to defend its assets. However, for the case $n_k = 13$, two weapons per target is not enough so it has to start using three weapons per target for 6 of the assets and two weapons per target for one of the assets (which will be included as a fraction of 2/3). Therefore the defense only defends $6\frac{2}{3}$ assets.

6 Conclusions

We have presented an algorithm for producing sub-optimal solutions to the Asset-Based WTA problem under the assumption of target dependent kill and lethality probabilities. Computational experimentation suggests that the solution produced by this algorithm is either optimal or near optimal for most problems. We conjecture that, if this approach is used as a heuristic for the case of multiple weapon classes then the resulting solution will also be near-optimal.

We also presented some sensitivity analysis results which will prove helpful in choosing parameter values for the problem. The optimal value and optimal solution of the problem is quite sensitive to changes in the kill probability, but appears to be insensitive to changes in the lethality probability. Furthermore, if the number of assets, and the kill and lethality probabilities are kept fixed then, as the number of offense weapons increases, the number of defensive weapons must increase at a greater rate if the defense wishes to maintain the same level of performance.

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