A KALMAN FILTER SOLUTION OF THE INVERSE SCATTERING

PROBLEM WITH A RATIONAL REFLECTION COEFFICIENT

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ABSTRACT

This paper presents a new inverse scattering method for reconstructing the reflectivity function of symmetric two-component wave equations, or the potential of a Schrodinger equation, when the reflection coefficient is rational. This method relies on the so-called Chandrasekhar equations which implement the Kalman filter associated to a stationary state-space model. These equations are derived by using first a general layer stripping principle to obtain some differential equations for reconstructing a general scattering medium, and by specializing these recursions to the case when the probing waves have a state-space model.

1. Introduction

Over the years, several methods have been developed to solve the inverse scattering problem for the one-dimensional Schrodinger equation and for two-component wave equations. These inverse scattering methods can be divided roughly in two categories, depending on whether they use integral equations [1] - [8] or a differential formulation [9] - [14] to reconstruct the potential of the Schrodinger equation, or the reflectivity function of the two-component wave system. However, even the most efficient of these techniques such as the fast Cholesky recursions described in [11] - [14] require a large volume of computations. It is therefore desirable to exploit any additional property that the scattering data may possess. Such a case occurs when the left reflection coefficient of the scattering medium is a rational function. In this case, several inverse scattering procedures have been proposed [15] - [19]. However, these methods were primarily concerned with the problem of obtaining closed-form solutions of the inverse scattering problem for reflection coefficients with one, two, three or more poles, rather than with that of obtaining recursive and computationally efficient reconstruction algorithms.

In this paper we present a new inverse scattering procedure for the case when the reflection coefficient is rational which relies on the so-called Chandrasekhar equations [20] - [21] of linear filtering theory. These equations are recursive and require only $0(n)$ operations per discretization step, where

n is the number of poles of the reflection coefficient. To obtain these equations, the inverse scattering problem for symmetric two-component wave equations is formulated in Section 2, and by using a layer stripping principle based on the method of propagation of singularities for hyperbolic partial differential equations, the fast Cholesky recursions are derived for reconstructing the reflectivity function of the two-component wave system layer by layer. By specializing these recursions to the case when the probing waves have a state-space model, the Chandrasekhar equations are obtained in Section 3. The fact that these recursions arise both in inverse scattering theory and in the Kalman filter implementation for a stationary state-space model is then interpreted by using earlier results of Dewilde and his coworkers [11], [22] - [23] which formulate the input-output estimation problem for a stationary stochastic process as an inverse scattering problem.

2. Fast Cholesky Recursions

The lossless scattering medium that we consider is described by symmetric two-component wave equations

$$p_x + p_t = - r(x)q(x,t) \tag{1a}$$

$$q_x - q_t = - r(x)p(x,t) \tag{1b}$$

which are of the type discussed by Zakharov and Shabat [7], and Ablowitz and Segur [8]. Here $r(x)$ is the reflectivity function and $p(x,t)$ and $q(x,t)$ are the rightward and leftward propagating waves in the medium at point x and time

t. Note that if $r(x) \equiv 0$ over some interval, then

$$p = p(x-t), \quad q = q(x+t) \tag{2}$$

over this interval, so that p and q correspond effectively to waves propagating

rightward and leftward with unit velocity. In the following, it will be

assumed that $r(x) = 0$ for $x < 0$, and that $r \in L_1[0,\infty)$, so that for $x < 0$

and as $x \to \infty$, the waves $p(x,t)$ and $q(x,t)$ have the form (2).

By taking the Fourier transform of (1), we obtain

$$\frac{d}{dx} \begin{bmatrix} \hat{p} \\ \hat{q} \end{bmatrix} = \begin{bmatrix} -j\omega & -r(x) \\ -r(x) & j\omega \end{bmatrix} \begin{bmatrix} \hat{p} \\ \hat{q} \end{bmatrix}, \tag{3}$$

and the waves $\hat{p}(x,\omega)$, $\hat{q}(x,\omega)$ are such that

$$\hat{p}(x,\omega) = \hat{p}_L(\omega)e^{-j\omega x}, \qquad \hat{q}(x,\omega) = \hat{q}_L(\omega)e^{j\omega x} \tag{4a}$$

for $x < 0$, and

$$\hat{p}(x,\omega) = \hat{p}_R(\omega)e^{-j\omega x}, \quad \hat{q}(x,\omega) = \hat{q}_R(\omega)e^{j\omega x} \tag{4b}$$

as $x \to \infty$. The outgoing waves $(\hat{p}_R(\omega), \hat{q}_L(\omega))$ can be expressed in function of

the incoming waves $(\hat{p}_L(\omega), \hat{q}_R(\omega))$ as

$$\begin{bmatrix} \hat{p}_R(\omega) \\ \hat{q}_L(\omega) \end{bmatrix} = S(\omega) \begin{bmatrix} \hat{p}_L(\omega) \\ \hat{q}_R(\omega) \end{bmatrix}, \tag{5}$$

where

$$S(\omega) = \begin{bmatrix} \hat{T}_L(\omega) & \hat{R}_R(\omega) \\ \hat{R}_L(\omega) & \hat{T}_R(\omega) \end{bmatrix} \tag{6}$$

denotes the scattering matrix associated to the medium (1).

Since the medium (1) is lossless, the matrix $S(\omega)$ has the property of being unitary, i.e.

$$S^H(\omega) S(\omega) = I \qquad (7)$$

for ω real, and also satisfies the reciprocity relation

$$\hat{T}_L(\omega) = \hat{T}_R(\omega) \quad . \qquad (8)$$

Physically, this relation means that the transmission loss through the medium is the same going in either direction. In addition, it can be shown [8] that the assumption that $r \in L_1[0,\infty)$ implies that the system (3) has no bound states, an observation which, when combined with (7) and (8) implies [5] that the entries of $S(\omega)$ can all be computed from either $\hat{R}_L(\omega)$ or $\hat{R}_R(\omega)$. The objective of the inverse scattering problem is to reconstruct $r(x)$ from $\hat{R}_L(\omega)$.

Over the years, a variety of methods have been devised to obtain $r(x)$ from $\hat{R}_L(\omega)$. One of these methods, on which we will focus our attention, is the fast Cholesky or downward continuation procedure described in Dewilde et al. [11], Bube and Burridge [12], Bruckstein, Levy and Kailath [13], and Yagle and Levy [14]. In this approach it is assumed that the medium is originally at rest, and is probed from the left by a known rightward propagating wave

$$p(0,t) = \delta(t) + \tilde{p}(0,t)u(t) \qquad (9)$$

which is incident on the medium at $t = 0$. Here $\delta(\cdot)$ denotes the Dirac delta

function and

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ \\ 0 & \text{for } t < 0 \end{cases} \tag{10}$$

is the unit step function. Note that the main feature of $p(0,t)$ is that it contains a leading impulse which is used as a tag indicating the wavefront of the probing wave. The measured data is the reflected wave

$$q(0,t) = \tilde{q}(0,t)u(t) \tag{11}$$

recorded at $x = 0$. In the special case when $\tilde{p}(0,t) \equiv 0$, $\tilde{q}(0,t) = R_L(t)$ is the impulse response of the scattering medium, where $R_L(t)$ denotes the inverse Fourier transform of the left reflection coefficient $\hat{R}_L(\omega)$.

Since the medium is causal and is originally at rest, the waves $p(x,t)$ and $q(x,t)$ inside the medium have the form

$$p(x,t) = \delta(t-x) + \tilde{p}(x,t)u(t-x) \tag{12a}$$

$$q(x,t) = \tilde{q}(x,t)u(t-x) \tag{12b}$$

where $\tilde{p}(x,t)$ and $\tilde{q}(x,t)$ are smooth functions. By substituting (12) inside (1), and identifying coefficients of the impulse $\delta(t-x)$ on both sides of (1b), we find that

$$\tilde{p}_x + \tilde{p}_t = - r(x)\tilde{q}(x,t) \tag{13a}$$

$$\tilde{q}_x - \tilde{q}_t = - r(x)\tilde{p}(x,t) \tag{13b}$$

and

$$r(x) = 2\tilde{q}(x,x), \quad r^2(x) = - 2 \frac{d}{dx} \tilde{p}(x,x) \quad . \tag{14}$$

The recursions (13) - (14) constitute the <u>fast Cholesky or downward continuation</u> <u>recursions</u>. The initial data for these recursions are the measured waves $\tilde{p}(0,t)$ and $\tilde{q}(0,t)$. The identities (13) - (14) can be viewed as using a layer stripping principle to identify the parameters of the scattering medium. Thus, assume that the waves $\tilde{p}(x,t)$ and $\tilde{q}(x,t)$ at depth x have been computed. The reflectivity function r(x) is obtained from (14) and is used in (13) to compute the waves $\tilde{p}(x+\Delta,t)$ and $\tilde{q}(x+\Delta,t)$ at depth $x+\Delta$, as shown in Fig. 1. The effect of the recursions (13) - (14) is therefore to identify and then strip away the layer $[x, x+\Delta)$.

The fast Cholesky recursions have a large number of applications, such as for the factorization of a Toeplitz operator in causal times anticausal Volterra operators [24]. These recursions have the feature of being computationally very efficient. If L is the maximum depth over which we want to reconstruct the scattering medium, and if we use a difference scheme with step size L/N to propagate the recursions (13) - (14), the total number of operations required to recover $r(\cdot)$ is $O(N^2)$ [13]. In addition, it was shown by Bultheel [25] that these recursions are numerically stable.

To apply the inverse scattering procedure described above to the case when the underlying physical system is modeled by a Schrodinger equation instead of the two-component wave system (1), we observe that if p(x,t) and q(x,t) satisfy (1), then

$$\phi(x,t) = p(x,t) + q(x,t) \tag{15}$$

satisfies the wave-like equation

$$\phi_{xx} - \phi_{tt} = V(x)\phi(x,t) \tag{16}$$

where

$$V(x) = -\frac{d}{dx} r(x) + r^2(x) \quad . \tag{17}$$

Taking the Fourier transform of (16) gives the Schrodinger equation

$$\hat{\phi}_{xx} + (\omega^2 - V(x)) \hat{\phi}(x,\omega) = 0 \tag{18}$$

and the special form of the transformation (15) implies that the scattering matrix $S(\omega)$ associated to (18) is identical to that of (3). Consequently, given the left reflection coefficient $\hat{R}_L(\omega)$, to reconstruct the potential $V(\cdot)$ we can first use the fast Cholesky recursions (13) - (14) to compute $r(\cdot)$, and then use the expression (17) to obtain $V(\cdot)$.

Alternately, we may observe that if the waves $p(x,t)$ and $q(x,t)$ have the form (12), then

$$\phi(x,t) = \delta(t-x) + \tilde{\phi}(x,t) u(t-x) \tag{19}$$

where $\tilde{\phi}(x,t) = \tilde{p}(x,t) + \tilde{q}(x,t)$, and substituting (14) inside (17), the potential can be expressed as

$$V(x) = -2 \frac{d}{dx} \tilde{\phi}(x,x) \quad . \tag{20}$$

3. Kalman Filter Solution

In spite of their relative efficiency, the fast Cholesky recursions still require a large volume of computations in order to recover $r(\cdot)$. It is therefore desirable to exploit as much as possible any additional structure that the left reflection coefficient $\hat{R}_L(\omega)$ may have. In the case when $\hat{R}_L(\omega)$ is rational, several methods have been proposed by Kay [15], Szu et al. [16], Jordan and Ahn [17], and Pechenick and Cohen [18], to reconstruct the potential

$V(\cdot)$. A review of these inverse scattering procedures, as well as a study

of the case when $V(x) \neq 0$ for $x < 0$, is given in Sabatier [19]. However, these

methods are not particularly attractive from a computational point of view,

since they require the computation of a matrix determinant, or the solution

of a system of linear equations of size equal to the number of poles of $\hat{R}_L(\omega)$,

for each value of x.

In this section, we derive a new reconstruction procedure for $r(\cdot)$ and $V(\cdot)$,

which is more efficient, and which relies on the introduction of a state-space

model for the waves $\tilde{p}(x,t)$ and $\tilde{q}(x,t)$. This reconstruction procedure is

recursive and takes a form identical to the Kalman filter of linear filtering

theory.

Our starting point is the assumption that the measured waves $\tilde{p}(0,t)$ and

$\tilde{q}(0,t)$ can be represented as

$$\tilde{p}(0,t) = C\, e^{At}\, K(0) \tag{21a}$$

$$\tilde{q}(0,t) = C\, e^{At}\, L(0) \tag{21b}$$

where A is an nxn matrix, C is a 1xn vector, and K(0) and L(0) are some nx1

vectors. In the Fourier domain, this corresponds to assuming that

$$\hat{p}(0,\omega) = 1 + C(j\omega I - A)^{-1}K(0) \tag{22a}$$

$$\hat{q}(0,\omega) = C(j\omega I - A)^{-1}L(0) \tag{22b}$$

where I denotes the nxn identity matrix. Two special cases will be of interest:

(i) $K(0) = 0$, $L(0) = B$; and (ii) $K(0) = L(0) = B$, where B is some nx1 vector.

In <u>case (i)</u>

$$\hat{R}_L(\omega) = \hat{q}(0,\omega) = C(j\omega I - A)^{-1} B \qquad (23)$$

so that the triple (A, B, C) is a <u>state-space model</u> of the left reflection coefficient $\hat{R}_L(\omega)$, which is therefore <u>rational</u>. Conversely, given a rational $\hat{R}_L(\omega)$, there exists a variety of ways [26] to realize it in state-space form as in (23). Furthermore, if this realization is minimal, i.e. if (A, B) is controllable and if (C, A) is observable, the size n of the state is equal to the number of poles of $\hat{R}_L(\omega)$. In <u>case (ii)</u>

$$\hat{R}_L(\omega) = \hat{q}(0,\omega)/\hat{p}(0,\omega) = \hat{k}(\omega)/(1 + \hat{k}(\omega)) \qquad (24)$$

with

$$\hat{k}(\omega) = C(j\omega I - A)^{-1} B \quad , \qquad (25)$$

so that the left reflection coefficient $\hat{R}_L(\omega)$ is also rational, but instead of computing a state-space realization for $\hat{R}_L(\omega)$, we construct one for $\hat{k}(\omega)$. In this case, the relation

$$\tilde{p}(0,t) = \tilde{q}(0,t) = k(t) \qquad (26)$$

where k(t) denotes the inverse Fourier transform of $\hat{k}(\omega)$, indicates that a perfect reflector is located to the left of the scattering medium at x = 0. This reflector corresponds for example to the surface of the earth for the case of a land seismogram in geophysics. Such a reflector appears also in the inverse scattering formulation of the linear filtering problem for a stationary stochastic process, as shown in Dewilde et al. [11] and [14] (see also Fig. 2).

The special form (21) of the waves $p(0,t)$ and $q(0,t)$ suggests that the waves at depth x should be written as

$$\tilde{p}(x,t) = C\,e^{A(t-x)}K(x) \tag{27a}$$

$$\tilde{q}(x,t) = C\,e^{A(t-x)}L(x) \quad . \tag{27b}$$

By substituting (27) inside the fast Cholesky recursions (13) - (14), and assuming that the pair (C, A) is observable, we find that the nx1 vector functions $K(x)$ and $L(x)$ must satisfy the differential equations

$$\frac{d}{dx}K(x) = -\,r(x)L(x) \tag{28a}$$

$$\frac{d}{dx}L(x) = -\,2AL(x) - r(x)K(x) \tag{28b}$$

with

$$r(x) = 2\tilde{q}(x,x) = 2CL(x) \quad . \tag{29}$$

Conversely, if $K(x)$ and $L(x)$ obey (28) - (29), then the waves $\tilde{p}(x,t)$ and $\tilde{q}(x,t)$ given by (27) satisfy the fast Cholesky recursions (13) - (14). The equations (28) - (29), along with the initial conditions $K(0) = 0$, $L(0) = B$ for case (i), and $K(0) = L(0) = B$ for case (ii) constitute our reconstruction procedure, which is therefore recursive. These equations can also be used to reconstruct the potential $V(\cdot)$ by observing from (20) that

$$V(x) = -\,2C\,\frac{d}{dx}\,(K(x) + L(x))$$

$$= 4(-\,CAL(x) + CL(x)(CL(x) + CK(x)) \quad . \tag{30}$$

Since $K(x)$ and $L(x)$ have size n, the recursions (28) - (30) require $0(n^2)$ operations per step if the matrix A has no special structure, and $0(n)$ if A

is in canonical form [26]. By comparison, the fast Cholesky recursions (13) - (14) require O(N) operations per step, where N is the total number of points used to discretize the interval [0,L] and where L is the maximum depth over which we want to reconstruct $r(\cdot)$ and $V(\cdot)$. In general n << N, so that the recursions (28) - (30) are computationally quite efficient.

By eliminating $r(x)$ from equations (28) - (29), we obtain

$$\frac{d}{dx} K(x) = - L(x)L^T(x)C^T \tag{31a}$$

$$\frac{d}{dx} L(x) = 2(A-K(x)C)L(x) \tag{31b}$$

which are the so-called <u>Chandrasekhar equations</u> of linear filtering theory. These recursions were originally developed in more general form by Chandrasekhar in the context of radiative transfer [20], and were then used by Kailath [21] to obtain the Kalman filter associated to a stationary state-space model. Thus, let $\xi(\cdot)$ be a stationary process described by

$$\frac{d}{dx} \xi(x) = A\xi(x) + u(x) \tag{32}$$

where A is a constant matrix and $u(\cdot)$ a white noise process with constant intensity. Given some scalar observations

$$y(x) = C\xi(x) + v(x), \quad x \geq 0 \tag{33}$$

where C is a constant row vector and $v(\cdot)$ a white noise process uncorrelated with $u(\cdot)$ and with unit intensity, the linear filtering estimate $\hat{\xi}(x)$ of $\xi(x)$ is given by the Kalman filter

$$\frac{d}{dx} \hat{\xi}(x) = A\hat{\xi}(x) + K(x)(y(x) - C\hat{\xi}(x)) \tag{34}$$

with $\hat{\xi}(0) = 0$. It is shown in [21] that the Kalman gain $K(2x)$ satisfies the recursions (31), where if $P(x)$ denotes the state error variance, $L(x)$ is the square-root of $-\frac{d}{dx} P(2x)$, i.e.

$$-\frac{d}{dx} P(2x) = L(x)L^T(x) \quad , \tag{35}$$

and where the initial conditions are

$$K(0) = L(0) = \Pi c^T \tag{36}$$

with $\Pi \underline{\underline{\Delta}}$ steady-state variance of $\xi(\cdot)$.

The fact that the same algorithm can be used to solve the inverse scattering problem with a rational reflection coefficient, and the linear filtering problem with a rational state-space model is not a coincidence. It was observed in [11], [14] that given some observations

$$y(x) = z(x) + v(x) \quad , \quad x \geq 0 \tag{37}$$

of a stationary process $z(\cdot)$ with covariance $k(x) = E[z(x)z(0)]$, the problem of finding the filtering estimate of $z(x)$ from the observations $y(\cdot)$ could be formulated as a modeling problem where the objective is to model $y(\cdot)$ as the output of a scattering medium described by symmetric two-component wave equations and driven by white noise, as shown in Fig. 2. The left reflection coefficient of this scattering medium was shown to be

$$\hat{R}_L(\omega) = \hat{k}(\omega)/(1+\hat{k}(\omega)) \tag{38}$$

where $\hat{k}(\omega)$ denotes the Fourier transform of the one-sided covariance $k(t)u(t)$.

Then, by applying the fast Cholesky recursions to the waves $p(0,t) = \delta(t) +$ $k(t)u(t)$, $q(0,t) = k(t)u(t)$ corresponding to (38), the modeling filter for $y(\cdot)$ was reconstructed, and the associated reflectivity function $r(\cdot)$ was used to obtain the optimum estimation filter for $z(x)$ via the so-called Krein-Levinson recursions [11].

In the special case considered here, we have $z(x) = C\xi(x)$ and $k(x) = C e^{Ax}\Pi C^T$, which explains why the recursions (31) with initial conditions (36) can be used to solve the linear filtering problem for $z(x)$ and $\xi(x)$.

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FIGURE CAPTIONS

Fig. 1: a) Update of $p(x,t)$; and b) update of $q(x,t)$ via the fast
         Cholesky recursions.

Fig. 2: Two-component scattering model of the process $y(\cdot)$.
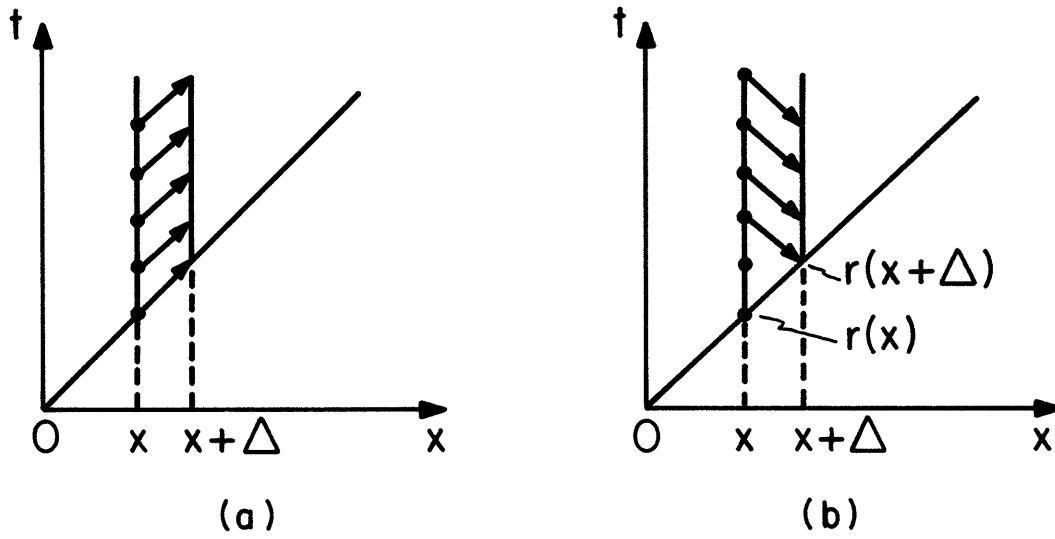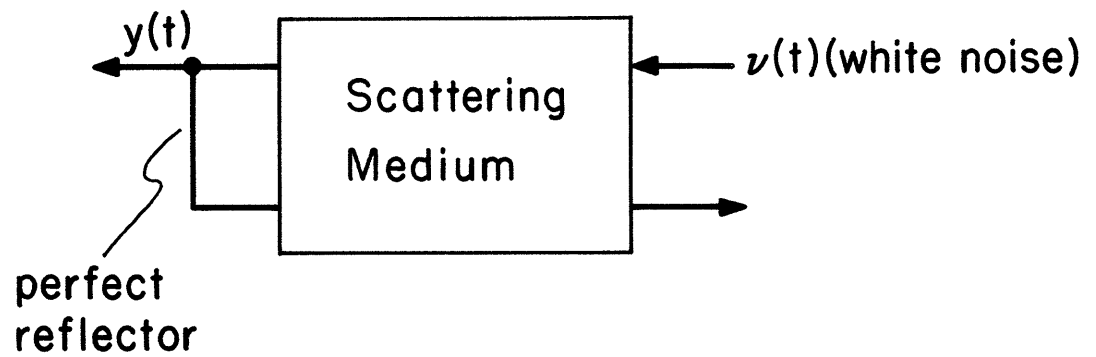
Fig. 1

a)   Update of p(x,t); and b) update of σ(x,t) via
    the fast Cholesky recursions.

Fig. 2

Two-component scattering model of the process y(·).