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LINEAR ESTIMATION OF BOUNDARY VALUE PROCESSES

by

Milton B. Adams*
C.S. Draper Laboratory
Cambridge, MA. 02139

Bernard C. Levy** Alan S. Willsky**
Department of Electrical Engineering and
Computer Science, M.I.T., Cambridge, MA. 02139

ABSTRACT

In this paper we discuss the problem of estimating boundary value processes in one or several dimensions. The estimator dynamics are described, and by using operator transformations for these dynamics, several implementations are obtained which either diagonalize or triangularize the linear least-squares estimator. These implementations enable us to compute the estimate of the process by using two-filter type of smoothing formulas, or more general smoothing formulas similar to those used for solving the smoothing problem for 1-D causal processes.

1. INTRODUCTION

In this paper we describe some results related to noncausal estimation problems. The starting point for our work in this area was the class of one-dimensional (1-D) two-point boundary value processes introduced and first analyzed by Krener [1]. In [2] we presented a brief description of a solution to this estimation problem using the method of complementary models which was first introduced by Weinert and Desai [3]. In this work we described the central role played by Green's theorem in determining a two-point boundary-value problem description of the smoother. While the details of the computations in [2] rely heavily on the specific 1-D model under consideration, the general approach used is not restricted to this model, and, as developed in detail in [4], is applicable to processes in several spatial dimensions. Specifically, in [4], the estimate is shown to be the solution to a boundary value problem, and consequently an issue is the construction of efficient methods to implement that solution. In [5] a detailed solution in the 1-D case is developed by diagonalizing the dynamics of the smoother boundary value differential equation (see also [2] for a brief description). The computations involved in these results relied heavily on the specific 1-D problem. In this paper we extend the ideas underlying that diagonalization approach by describing the diagonalization of estimator dynamics in an operator framework applicable to problems in several dimensions as well.

After reviewing the form of the estimator solution in Section 2, we describe some equivalent dynamical representations for the differential operator description of the estimator dynamics in Section 3. The diagonal form we seek, if it exists, is in the class of equivalent differential operator representations, and in Section 4 we present the conditions which define the class of transformation operators which lead to such forms. As an alternative to diagonalization, we outline a method for triangularizing the dynamics which leads to a representation of the estimator which is similar to smoothers obtained for 1-D and 2-D causal processes by the innovations approach [6] and [7]. This approach has some advantages over diagonalization. In particular, in constrast to diagonalization, no operator inversions are required in computing the estimate by triangularization. Having established the conditions for diagonalization we formulate matrix Riccati equations which lead to stable diagonal forms for the estimator for 1-D boundary value processes. Finally, questions of existence and uniqueness of diagonalizing transformations for 2-D estimators are discussed.

2. THE PROBLEM STATEMENT

2.1 Differential Operators and Green's Identity

The process to be estimated is defined in terms of a linear differential operator acting on a Hilbert space of square-integrable functions as follows. Let Ω_N be a bounded convex region in R^N with smooth boundary [16]. The space of nx1 vector functions which are square-integrable on Ω_N is represented by $L_2^n(\Omega_N)$. Let L be a formal differential operator mapping into $L_2^n(\Omega_N)$ and defined on D(L), the subspace of sufficiently differentiable elements of $L_2^n(\Omega_N)$.

Green's identity for L is

$$\langle Lx,\lambda\rangle_{L_2^n(\Omega_N)} = \langle x,L^\dagger\lambda\rangle_{L_2^n(\Omega_N)} + \langle x_b,E\lambda_b\rangle_{H_b} \quad (1)$$

where $L^\dagger$ is referred to as the formal adjoint differential operator [8], x_b and λ_b are elements of a Hilbert space H_b of processes defined on $\partial\Omega_N$, and E is a mapping from H_b into itself; $E:H_b\rightarrow H_b$. In particular, these processes are defined through the action of an operator Δ_b: $L_2^n(\Omega_N) \rightarrow H_b$, so that

$$x_b = \Delta_b x \quad \text{and } \lambda_b = \Delta_b\lambda \quad . \quad (2)$$

The nature of H_b, Δ_b, and E all depend upon L and Ω_N. For a discussion of Green's identity for ordinary differential operators see [9] and Chapter 3 of [10]; for elliptic, hyperbolic and parabolic second order partial differential operators see [8] and Chapter 7 of [10]. In this paper, we will restrict our discussions to operators L and regions Ω_N that admit a Green's identity.

The boundary condition associated with L is defined by a mapping V:

$$V:H_b \rightarrow R(V) \quad (3)$$

where the nature of the range space R(V) is determined by the following well-posedness condition. We will say that the pair (L,V) leads to a well-posed boundary value problem if the differential operator Λ formed by augmenting the formal differential operator L and boundary mapping V

$$\Lambda = \begin{bmatrix} L \\ V\Delta_b \end{bmatrix} \quad (4a)$$

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has a unique continuous left inverse $\Lambda^{\#}$. We denote the components of the left inverse by

$$\Lambda^{\#} = [\, G_u \vdots G_v \,] \tag{4b}$$

where G_u: $L_2^n(\Omega_N) \rightarrow D(L)$ and G_v: $L_2^{n_v}(\partial\Omega_N) \rightarrow D(L)$. For a well-posed problem the value of the vector dimension n_v depends on the type and order of the operator L and the dimensions N and n. In this case, the equation

$$\Lambda x = \begin{bmatrix} u \\ v \end{bmatrix} \tag{5a}$$

with u and v in the domains of G_u and G_v, respectively, has a unique solution which can be written as

$$x = G_u\, u + G_v\, v \quad . \tag{5b}$$

It will be assumed that all problems considered here are well-posed.

A description nearly identical to that given above holds for a class of discrete processes defined by linear boundary-value partial difference equations. In this case L is a partial difference operator and Ω_N is a multi-dimensional discrete-valued index set. It is shown in [10] that the estimation problem statement and solution presented in this paper apply as well for this class of discrete processes.

2.2 The Problem Statement

Let u be an mxl vector white noise on Ω_N with an invertible correlation operator Q (i.e. the correlation matrix of u is thought of as the kernel of an operator). Let v be an $n_v$xl vector second order process over $\partial\Omega_N$, uncorrelated with u and with invertible correlation operator Π_v. Then the process to be estimated is formally defined by

$$Lx = Bu \tag{6a}$$

with boundary condition

$$Vx_b = v \quad . \tag{6b}$$

The observations are defined as follows. Let C(t) be a pxn matrix continuous in $t \in \Omega_N$. Let W be an operator mapping elements of H_b into R(W), a space of $n_w$xl vector functions defined over the index set $\partial\Omega_N$. Let r be a pxl vector white noise over Ω_N with invertible correlation operator R, and let r_b be a $n_w$xl vector process with invertible correlation operator Π_b. It will be assumed that u, v, r and r_b are multiply uncorrelated. The set of observations of x is given by:

$$y = Cx + r \qquad \text{on } \Omega_N \tag{7a}$$

and

$$y_b = Wx_b + r_b \qquad \text{on } \partial\Omega_N \ . \tag{7b}$$

We will need to make some assumptions with respect to the relationship between the operators V and W. In particular, we will assume that if the operator obtained by augmenting V and W as

$$\begin{bmatrix} V \\ W \end{bmatrix} \tag{8a}$$

is not invertible, then there exists an operator W_c such that

$$\begin{bmatrix} V \\ W \\ W_c \end{bmatrix} \tag{8c}$$

is invertible.

Our estimation problem is to find the linear minimum variance estimate of x given the oberservation set

$Y = \{y,\ y_b\}$.

3. THE ESTIMATE AND ESTIMATION ERROR

The solution to this estimation problem has been obtained in [4] by an application of the method of complementary models. The complementary process has the property that it is orthogonal to the observations and that, when combined with the observations, contains information equivalent to the boundary conditions, driving noise and measurement noise, i.e. all of the underlying variables which determine the system state and observations. By establishing an internal differential realization for the complementary process associated with the observation set Y and augmenting that realization with the differential realization for Y given by (6) and (7), the estimator is shown in [4] to be given in differential operator form as the solution of the inverse of the augmented system projected onto the span of Y.

In particular the estimate $\hat{x}$ is the solution of

$$\begin{bmatrix} L & \vdots & -BQB^* \\ - & - & - \\ C^*R^{-1}C & \vdots & L^{\dagger} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ - \\ C^*R^{-1}y \end{bmatrix} \tag{9}$$

$$W^*\Pi_b^{-1}y_b = [\, W^*\Pi_b^{-1}W + V^*\Pi_v^{-1}V \ :\ E\,] \begin{bmatrix} \hat{x}_b \\ \hat{\lambda}_b \end{bmatrix} \tag{10}$$

where a superscript asterisk denotes the Hilbert adjoint [11] of an operator.

Note that since L and $L^{\dagger}$ are of the same order, the order of the estimator is twice that of L. Also note the remarkable fact that in addition to the original problem statement, we only need to know E and $L^{\dagger}$ from Green's identity in (1) to completely define the differential realization for the estimator. That is, it is not necessary to actually determine the internal differential realization for the complementary process.

As established in [4], the estimation error $\tilde{x} = x - \hat{x}$ is obtained as the solution of the inverted augmented system projected onto the span of the complementary process rather than that of the observations. A differential realization of the estimation error which is driven by the underlying processes $\{u, v, r, r_b\}$ whose probability law is known is given by

$$\begin{bmatrix} L & : & -BQB^* \\ - & - & - \\ C^*R^{-1}C & : & L \end{bmatrix} \begin{bmatrix} \tilde{x} \\ -\hat{\lambda} \end{bmatrix} = \begin{bmatrix} Bu \\ - \\ -C^*R^{-1}r \end{bmatrix} \tag{11}$$

with boundary condition

$$[V^*\Pi_v^{-1}v - W^*\Pi_b^{-1}r_b] = [W^*\Pi_b^{-1}W + V^*\Pi_v^{-1}V \ :\ E] \begin{bmatrix} \tilde{x}_b \\ -\hat{\lambda}_b \end{bmatrix} . \tag{12}$$

We have chosen to write (11) and (12) in terms of $-\hat{\lambda}_b$ instead of $\hat{\lambda}_b$ to highlight the similarity between the structure of the dynamics and boundary condition for the estimation error and that of the estimator in (9) and (10). One should be able to take advantage of these similarities when computing the estimate and its error covariance. For example, see the discussion of the implementation of the estimator and the computation of the error covariance for 1-D noncausal processes in [5].

4. DIAGONAL REPRESENTATIONS

4.1 Equivalent Differential Operator Representations

Within the class of equivalent differential operator representations for the estimator dynamics is, if it exists, the stable decoupled form we seek. Here we describe the class of equivalent representations for the differential operator form of the estimator in (9). Our goal is, if possible, to diagonalize these dynamics into two decoupled systems, each of which is stable. The stability property is desirable for purposes of numerical implementation of the estimator. The operator diagonalization is analogous to what has been done previously in [12] and [13] for differential realizations of the estimator dynamics for 1-D processes. We start by investigating equivalent operator representations of dynamics described by equations such as (9).

Consider the general differential operator form

$$\Lambda z = Bu \ ; \quad \Lambda : X \rightarrow Y \tag{13}$$

where Λ is a differential operator whose domain and range X and Y are two inner product spaces. Consider the invertible operator $T:X \rightarrow X$ which gives rise to the equivalence transformation

$$\xi = Tz \quad . \tag{14}$$

Defining

$$\tilde{\Lambda} = F\Lambda T^{-1} \quad \text{and} \quad \tilde{B} = FB \quad , \tag{15a}$$

where $F:Y \rightarrow Y$ is invertible, the dynamics of ξ in (14) can be written in the form of (13) as

$$\tilde{\Lambda}\xi = \tilde{B}u \quad . \tag{15b}$$

4.2 Operator Diagonalization

In decoupling the estimator dynamics, our objective is to find invertible operators T and F where the action of T defines the equivalent process

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = T \begin{bmatrix} \hat{x} \\ \hat{\lambda} \end{bmatrix} \tag{16a}$$

whose dynamics are diagonalized as (see (15a))

$$F\left[\begin{array}{c:c} L & -BQB^* \\ \hdashline C^*R^{-1}C & L^{\dagger} \end{array}\right] T^{-1} = \left[\begin{array}{c:c} L_1 & 0 \\ \hdashline 0 & L_2 \end{array}\right] \tag{16b}$$

with L_1 and L_2 stable.

To obtain the most general expression for (16b) we could define partitions of F and T^{-1} and carry out the indicated product. For example,

$$F = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \tag{17}$$

The first set of conditions for decoupling would be given by setting the upper right and lower left partitions of the product in (16b) to zero. Next the constraint that L_1 and L_2 must have some appropriate stability properties would be added.

Unfortunately, determining the complete class of F and T which are compatible with this most general statement of the problem is decidely nontrivial. With the benefit of our previous work on diagonalization of 1-D estimator dynamics, we will constrain the problem statement by assuming the following form for the operator T

$$T = \begin{bmatrix} \theta_1 & -I \\ \theta_2 & I \end{bmatrix} \tag{18a}$$

with its inverse written as

$$T^{-1} = \begin{bmatrix} I & I \\ -\theta_2 & \theta_1 \end{bmatrix} \begin{bmatrix} P_s & 0 \\ 0 & P_s \end{bmatrix} \tag{18b}$$

where

$$P_s = (\theta_1 + \theta_2)^{-1} \quad . \tag{19}$$

Although we have not yet determined the form of either θ_1 or θ_2, we will assume for the time being that their sum is invertible.

Substituting (18b) into (16b) and carrying out the product for an arbitrary operator F (see (17)), one finds that in order to achieve the diagonal form of (16b), we require:

$$F_1L - F_1BQB^*\theta_1 + F_2C^*R^{-1}C + F_2L^{\dagger}\theta_1 = 0 \tag{20a}$$

and

$$F_3L + F_3BQB^*\theta_2 + F_4C^*R^{-1}C - F_4L^{\dagger}\theta_2 = 0 \quad . \tag{20b}$$

Expressoins for L_1 and L_2 are obtained by substituting from (20a) and (20b), yielding

$$L_1 = - F_2L^{\dagger} + F_1BQB^* \tag{21a}$$

$$L_2 = F_4L^{\dagger} - F_3BQB^* \quad . \tag{21b}$$

Thus we must find (if possible) θ_1, θ_2, and the four partitions of F such that:

(i) (20a) and (20b) are satisfied,
(ii) the sum of θ_1 and θ_2 is invertible (i.e. T is invertible),
(iii) the operator F is invertible

and

(iv) the diagonal elements L_1 and L_2 are stable differential operators.

Assuming that F is given by (see section 4.4)

$$F = \begin{bmatrix} \theta_1 & I \\ \theta_2 & -I \end{bmatrix} \quad , \tag{22}$$

the equations (20a) and (20b) for θ_1 and θ_2 are the following Riccati equations (in the 2-D case, "operator Riccati" equations [14]):

$$(\theta_1L + L^{\dagger}\theta_1 - \theta_1BQB^*\theta_1 + C^*R^{-1}C)\xi = 0 \tag{23a}$$

and

$$(\theta_2L + L^{\dagger}\theta_2 + \theta_2BQB^*\theta_2 - C^*R^{-1}C)\xi = 0 \tag{23b}$$

for arbitrary ξ. Here we have used the fact that an operator $M = 0$ if and only if $M\xi = 0$ for arbitrary ξ in the domain of M.

Later we consider the diagonalization of the estimator dynamics for 1-D continuous problems. The motivation for looking at the 1-D problem in this light is to gain insight into how one might go about performing the diagonalization (i.e., determining the operators F and T) for 2-D cases.

4.3 Operator Triangularization

Here we show how the estimator could be implemented by triangularizing the estimator dynamics rather than by diagonalizing. The principal advantage of the triangularized representation as compared to the diagonalized representation is that the former can be developed in such a way that no operator inverses are required in recovering the estimate of the process x from the values of the transformed processes (in diagonalization the inverse of $\theta_1 + \theta_2$ is required). In addition, only a single operator equation must be solved as opposed to the two in (20a) and (20b).

To (lower) triangularize the estimator dynamics, we seek transformations T and F which lead to

$$F\left[\begin{array}{c:c} L & -BQB^* \\ \hdashline C^*R^{-1}C & L^\dagger \end{array}\right] T^{-1} = \left[\begin{array}{c:c} L_1 & 0 \\ \hdashline L_{21} & L_2 \end{array}\right] \qquad (24)$$

with a similar form for an upper triangularization. The following structures for F and T will lead to a triangularization with no operator inverses:

$$T = \begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} \quad \text{with inverse} \quad T^{-1} = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix}, \qquad (25a)$$

$$F = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} \quad . \qquad (25b)$$

Then it can be shown by direct substitution into (24) that the estimator dynamics become

$$\left[\begin{array}{c:c} L_1 & 0 \\ \hdashline L_{21} & L_2 \end{array}\right] = \left[\begin{array}{c:c} L + PC^*R^{-1}C & 0 \\ \hdashline C^*R^{-1}C & L^\dagger + C^*R^{-1}CP \end{array}\right] . \qquad (26)$$

The condition for this triangular form is the existence of a solution to the single operator equation

$$(LP + PL^\dagger + PC^*R^{-1}CP - BQB^*) \quad \xi = 0 \quad . \qquad (27)$$

4.4 The 1-D Continuous Case

In this section we solve the diagonalization problem for 1-D continuous estimators given the assumed form for T in (18a). Temporarily no assumptions will be made as to the form of F. In this case the differential operator L and its adjoint $L^\dagger$ are given by

$$(L\eta)(t) = \dot{\eta}(t) = A(t)\eta(t) \qquad (28a)$$

and

$$(L^\dagger\eta)(t) = -\dot{\eta}(t) - A'(t)\eta(t) \ . \qquad (28b)$$

Each of these has the form of a diffusion in t. The action of the operators C, B, R and Q is simply multiplication by the matrices C(t), B(t), R(t) and Q(t). The adjoints C* and B* are given by the matrix transposes C'(t) and B'(t). To simplify the notation, hereafter we will omit reference to the independent variable t.

Here we assume that F and T are time-varying invertible matrices. Our problem is to find dynamics governing their elements so that the conditions in (20a) and (20b) are met. The condition in (20a) is equivalent (for arbitrary ζ) to

$$(F_1 - F_2\theta_1)\dot{\zeta} + (-F_1A - F_2A'\theta_1 - F_2\dot{\theta}_1$$
$$- F_1BQB'\theta_1 + F_2C'R^{-1}C)\zeta = 0 \quad . \qquad (29)$$

Since this equation must be true for arbitrary ζ in the space of continuously differentiable functions, the coefficients of both ζ and its derivation must be zero. Considering the coefficient of the derivative of ζ first, we have

$$F_1 = F_2\theta_1 \quad . \qquad (30)$$

Substituting this into the coefficient for ζ and setting that coefficient to zero gives

$$F_2(\dot{\theta}_1 + \theta_1A + A'\theta_1 + \theta_1BQB'\theta_1 - C'R^{-1}C) = 0 \quad . \qquad (31)$$

A similar application of (20b) results in

$$F_3 = -F_4\theta_2 \qquad (32)$$

and

$$F_4(\dot{\theta}_2 + \theta_2A + A'\theta_2 - \theta_2BQB'\theta_2 + C'R^{-1}C) = 0 \quad . \qquad (33)$$

If we choose F as

$$F = \begin{bmatrix} \theta_1 & I \\ \theta_2 & -I \end{bmatrix} , \qquad (34)$$

then (31) and (33) are the usual Riccati equations for the dynamics of θ_1 and θ_2 and uniform complete controllability and reconstructability of the triple {A,B,C} guarantees the invertibility of both F and T. Furthermore, it can be shown by substituting from (34) into (21a) and (21b) that L_1 and L_2 are given by

$$L_1 = -L^\dagger + \theta_1BQB^* \qquad (35a)$$

and

$$L_2 = -L^\dagger - \theta_2BQB^* \quad . \qquad (35b)$$

These result in the same dynamics as those obtained by the Hamiltonian diagonalization in [5]. Note that we have not specified the boundary conditions for either θ_1 or θ_2. As discussed later these boundary conditions should be chosen to simplify the transformed estimator boundary conditions.

4.5 The 2-D case

First express the differential operator L in the form of a diffusion

$$L = L_t + A \qquad (36)$$

where $L_t = \frac{\partial}{\partial t}$ and A contains no partials with respect to t. This representation for L is typical of that employed in studies of distributed parameter systems (see [7], for example). However, for a truly noncausal process, the variable t would denote a spatial variable rather than time. See [10] for a further discussion of representing L in this form.

Given the representation for L in (36) the operator

Riccati equations (22a) and (22b) become

$$(\theta_1 L_t - L_t\theta_1 - A^{\dagger}\theta_1 - \theta_1 A - \theta_1 BQB^*\theta_1 + C^*R^{-1}C)\xi = 0 . \tag{37a}$$

and

$$(\theta_2 L_t - L_t\theta_2 - A^{\dagger}\theta_2 - \theta_2 A + \theta_2 BQB^*\theta_2 - C^*R^{-1}C)\xi = 0 \tag{37b}$$

Assuming the existence of a solution to these equations (to be discussed below), the diagonal operators L_1 and L_2 are

$$L_1 = L_t + (A + \theta_1 BQB^*) \tag{38a}$$

and

$$L_2 = L_t + (A^{\dagger} - \theta_2 BQB^*) \tag{38b}$$

and the transformed decoupled estimator dynamics become

$$L_1 q_1 = C^*R^{-1}y \quad \text{and} \quad L_2 q_2 = - C^*R^{-1}y \quad . \tag{39}$$

Of course the solutions q_1 and q_2 could be coupled through their boundary conditions as discussed in the next section.

Existence of solutions to the operator Riccati equations (37a) and (37b) remains an open issue in general. If the differential operator A were the infinitesimal generator of a strongly continuous semigroup (something which in general is difficult to establish, e.g. see the Hille-Yosida theorem in [15]), then it has been shown that there exists a solution to these equations. This is a sufficient but not a necessary condition. Given existence, there still remains the question of realizations for θ_1 and θ_2 which for the parabolic case have been shown to be integral operators whose kernels are the solutions to Riccati integro-differential equations [15].

4.6 Boundary Conditions for q_1 and q_2

If we assume that the existence and representation questions regarding the operator Riccati equations have been resolved, then we would choose boundary conditions for these operators in such a way that the boundary conditions for q_1 and q_2 are simplified. Substituting for q_1 and q_2 into the boundary condition (10) gives

$$W^*\Pi_b^{-1}y_b = [W^*\Pi_b^{-1}W + V^*\Pi_v^{-1}V \; \vdots \; E]\; T_b^{-1}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{40}$$

where T_b is the transformation T on the boundary $\partial\Omega_N$. The objective is to choose T_b (this amounts to selecting boundary conditions for the operators θ_1 and θ_2) in such a way that the transformed boundary condition (40) together with the decouple dynamics in (39) have a stable, efficient numerical implementation. Although an approach to this selection procedure is suggested in [10], this subject remains an open research topic.

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