

A NEW BEHAVIORAL PRINCIPLE FOR  
URBAN TRANSPORTATION NETWORKS

by

Stanley B. Gershwin

David M. Orlicki

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Laboratory for Information  
and Decision Systems  
Massachusetts Institute of Technology  
Cambridge, Massachusetts 02139

## ABSTRACT

A new hypothesis on traveler behavior in a network is described which is based on empirical findings in the theory of travel budgets. It is translated into an assignment principle for characterizing the distribution of travelers, as well as the demand and mode split. A numerical technique is proposed, and it is applied to several examples to illustrate qualitative features.

The new hypothesis is significant because it considers all travel decisions--whether or not to travel, where to go, and what mode to use--in a single, unified way. The feedback from travel time and money costs on links and modes to traveler behavior is explicitly considered.

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## 1. INTRODUCTION

### 1.1. Purpose

The purpose of this paper is to introduce a new model of traveler behavior in a transportation network. This model incorporates a mechanism which treats route, mode and destination choices, and the decision of whether or not to travel, in a single, unified formulation.

Models and computation techniques have been studied (Dafermos, 1974; Dafermos, 1980; Florian and Nguyen, 1974; Leblanc and Abdulaal, 1980) which treat these issues in an integrated way. However the models of individual decisions are based on different assumptions. (For example, route choice is often based on the logit model.) Some of the choice models (such as the logit models) appear to be convenient methods of summarizing local survey data rather than representations of universal human behavioral mechanisms.

We use the term "behavioral principle" to be a set of statements that characterize the choices made by individuals in a transportation system. We use "assignment principle" to mean a set of statements that characterize flows of vehicles or travelers in a system. One of the purposes of this paper is to emphasize the relationship between the two. An assignment principle is limited in its validity and predictive power unless it is based on a single, unified behavioral principle.

The models presented in here are based on the following behavioral principle. Travelers have maximum amounts of time and money that they are willing and able to spend traveling during a day. They may spend less but not more. Within these constraints, they maximize the benefit they obtain from traveling. This benefit is determined by the links and nodes of the network a traveler passes and on the mode he employs.

An important concept is that of a journey. Travelers start and end their journeys at the same point: a residence. They travel in a continuous path, possibly with several loops. This differs from existing models, in which the fundamental unit of travel is a trip.

## 1.2 Outline

Section 2 of this paper derives two versions of an assignment principle from the behavioral principle mentioned above. A numerical technique is presented in Section 3. Sections 4 and 5 contain a set of examples. The small examples in Section 4 are intended to help explain the behavioral and assignment principles; the large example of Section 5 demonstrates their realism and applicability. Further research topics are suggested in Section 6.

## 2. BEHAVIORAL AND ASSIGNMENT PRINCIPLES

Traffic Assignment is the computation of vehicle and traveler flows in a transportation network. It is based on data on the travelers such as their origins, possible destinations, and car ownership; and on the physical attributes of the network such as its structure and its link flow capacities. Because travelers make important choices that affect the distribution of flows (such as whether or not to travel, and where to go), all assignment schemes should be based, explicitly or otherwise, on a model as a behavioral principle; when it is translated into a statement about flows it is an assignment principle.

The major theoretical advance described in this report is a new assignment principle. This principle combines the travel budget theory of Zahavi (1979) and others (Kirby, 1981) with a detailed representation of a network. It extends the current state of the art by combining demand, mode split, and route assignment in an integrated formulation. Because this formulation is based on the travel budget theory, it has solid empirical backing.

In Section 2.1, we review the user-optimization assignment principle of Wardrop (1952). We show the relationship between an assignment about the behavior of individuals in a transportation network and the widely used statement about flows in the networks on which most modern assignment computation techniques are based.

We state a new behavioral principle which is not new, it is the basis of Zahavi's (1979) UMOT (Unified Mechanism of Travel) model. Its novelty comes from its new context. The new principle is based on the concept of travel budgets: travelers and potential travelers are assumed to have certain maximum daily expenditures of time and money for transportation. These budgets depend on such attributes as household sites and household income. Following the same development as for user optimization, a set of statements about flows in a network is derived from the individual behavioral principle.

Two versions of the new assignment principle are presented. The first, described in Section 2.2, assumes that budgets are equal for all members of each class of travelers. The second, in Section 2.3, allows budgets to differ for various members of the same class. The second formulation, which is more nearly in agreement with reality, allows the same traveler to choose different travel patterns on different days. Important features of the new principle are discussed in Section 2.4.

The new principle differs from formulations based on Wardrop's principle in many ways. First, daily travel behavior is considered, not single trips. That daily travel is important is indicated by the work of Clarke et al. (1981) whose empirical evidence shows that trip taking behavior must be treated in the context of household size and age, and of the full day's transportation needs. Consequently, instead of assuming fixed origins and destinations, we assume a fixed origin, the traveler's residence location. Travelers are assumed to have daily journeys that start and end at that point. Second, following Zahavi, travel is viewed not as a disutility, but rather as a utility, providing travelers with access to opportunities, and paid for out of limited budgets.

## 2.1 Review of Wardrop's Assignment Principle

### 2.1.1 Assumptions on Individual Behavior

Wardrop (1952) asserted two principles which were intended to characterize the behavior of travelers in a transportation network. One, which has since been called user optimization, is based on the premises that

- a. Travelers have fixed origins and destinations.
- b. Travelers seek paths which minimize their travel time.

This principle has been of great importance in the development of transportation models. Its evident limitations have provoked many researchers to suggest extensions to include elastic demand (Florian and Nguyen, 1974) multiple modes (Leblanc and Abdulaal, 1980) and other modifications.

### 2.1.2 Resultant Statements about Flows

To calculate flows, the above statements about individuals must be translated into a set of statements about flows. For the purposes here, it is convenient to discuss path flows; these statements can also be expressed in terms of link flows (Dafermos, 1980).

Let  $i$  be an index representing an origin node and a destination node in the network. Let  $D^i$  be the demand, in vehicles per time unit (most often hours) for origin-destination pair  $i$ . That is,  $D^i$  travelers per hour wish to go from  $a_i$  (the origin) to  $b_i$  (the destination). Let  $j$  be a path in the network that connects  $a_i$  and  $b_i$ , and let  $J_i$  be the set of all such paths. Let  $x_j$  be the flow rate of vehicles in path  $j$ .

The flow rates satisfy

$$x_j \geq 0, \quad j \in J_i, \quad \text{all } j, \quad (2.1)$$

$$\sum_{j \in J_i} x_j = D^i. \quad (2.2)$$

Let  $t_j$  be the travel time of path  $j$ . This travel time is the sum of the travel times of the links that constitute path  $j$ . The choice of the shortest path can be represented by considering the flows and travel times of any pair of paths.

Let  $j, k \in J_i$ . Then,

$$x_j > 0, \quad x_k > 0 \Rightarrow t_j = t_k. \quad (2.3a)$$

$$t_j > t_k \Rightarrow x_j = 0. \quad (2.3b)$$

That is, since travelers choose the shortest path, if both paths have flow, they must have equal travel time. If one path has greater travel time than the other, no travelers will take it.

Path travel time is the sum of link travel times:

$$t_j = \sum_{\ell} A_{j\ell} \tau_{\ell}, \quad (2.4)$$



where  $\tau_\ell$  is the travel time of link  $\ell$ , and  $A_{j\ell}$  is the path-link incidence matrix  $x$ . That is,

$$A_{j\ell} = \begin{cases} 1 & \text{if link } \ell \text{ is in path } j, \\ 0 & \text{otherwise.} \end{cases} \quad (2.5)$$

Link travel time is a function of link flow. Let  $f_\ell$  be the flow rate of vehicles on link  $\ell$ . Then,

$$f_\ell = \sum_j A_{j\ell} x_j. \quad (2.6)$$

This is, the flow on a link is the sum of the flows on all paths that pass through that link.

Finally,  $\tau_\ell$  is a function of  $f$ , the vector of link flows in the network.

### 2.1.3 Limitations

This principle neglects certain important features of traveler behavior. In reality, origins and destinations are not fixed; the decision if and where to travel (i.e., the demand) depends on congestion which in turn depends on the demand on the network. Demand also depends on the money cost of travel which is not considered at all.

Ways of treating some of these features have been proposed by many researchers. However the resulting formulations often require parameters that are difficult or impossible to obtain. They also require a new calibration for each application, so they have limited predictive value (Zahavi, 1981).

## 2.2 New Principle - Deterministic Version

### 2.2.1 Assumptions on Individual Behavior

Many authors have observed certain regularities about travelers' behavior. (Kirby, 1981). In developed countries throughout the world, the average traveler spends between 1.0 and 1.5 hours every day in the transportation system. Furthermore, travelers who own cars spend about 11 percent of their income on travel, and those who do not own cars spend 3 to 5 percent (Zahavi, 1979). This must

influence the amount and kind of travel that each individual uses.

A behavioral principle that takes these empirical observations into account must treat daily travel, not single trips. This is because the travel time budget is an amount of time spent each day. We define a journey to be the path in the network that an individual takes during a day. It is made up of several trips.

In the course of day, most travelers start and end their journeys at the same point: their residences. (A small number of people do not, including travelers starting or returning from business or vacation trips.) These journeys need not be simple loops. For example, some workers go home for lunch; some people travel to work and home, and then out again for shopping or entertainment.

We assume that such journeys are feasible for an individual only if the total travel time is less than his daily budget  $T$ , and the total money cost is less than his budget  $M$ . The money costs include fixed costs for a car as well as running costs if he drives; or fares if he takes transit or, of course, both if he takes a car for part of his journey (evening shopping or entertainment) and transit for the rest (work trips).

Here, we assume that each traveler has fixed budgets which do not vary from day to day, and which are the same as those of all others in this class. In Section 2.3, we relax these assumptions.

Classes are defined as sets of travelers with the same origin node (residence location) and with the same budgets for time and money to be spent each day for travel. These quantities are determined by such socioeconomic indicators as household income and number of travelers per household. (Zahavi (1979) asserts that one's daily travel time and money budgets are influenced by such factors as travel speed; we treat both budgets as fixed at this stage of the analysis.)

Of all the journeys that satisfy these budget constraints, which will be chosen by a traveler? Zahavi suggests that travelers attempt to maximize their access to spatial and economic opportunities. He indicates that the daily travel distance may not be their precise objective, but is quite adequate as a first approximation for a given urban structure and transport network.

Because we are treating a detailed representation of a network, we can represent other objectives. For example, we can assign a value to each link and each node in the network, and add the values encountered on a journey to yield the value of the journey. These values can differ from class to class: that is, the value of a given node (e.g., the location of an extremely expensive shopping mall) can depend on whether the traveler is rich or poor. Other formulas for calculating the value of a trip are also amenable to this analysis.

To summarize: a traveler in class  $a$  chooses a journey  $p$ , among all journeys  $P^a$  available to him, to

$$\text{maximize } w_p, \text{ the value or utility of journey } p \quad (2.7)$$

$$p \in P^a,$$

subject to the time and money budget constraints:

$$t_p \leq T^a, \quad (2.8)$$

$$m_p \leq M^a. \quad (2.9)$$

### 2.2.2 Resulting Statement about Flows

Equations (2.7) to (2.9) by themselves are not adequate to characterize network flows. For this purpose, they must be expressed in a form which is analogous to that of Section 2.1.1: a set of equations, inequalities, and logical relations involving flows.

Let  $x_p$  be the flow of class  $a$  travelers on journey  $p$ . Then,  $x_p$  must certainly satisfy

$$x_p \geq 0, \quad (2.10)$$

$$\sum_{p \in P^a} x_p = D^a, \quad (2.11)$$

where  $D^a$  is the total number of travelers available in class  $a$ . As we indicate below,  $D^a$  can include people who will choose not to travel. We measure  $x_p$  and  $D^a$

in units of travelers per day.

To characterize flows, we must specify a set of relations that are consistent with (2.7) to (2.9). That is, if the demand  $D^a$  is changed by a small amount, representing one more traveler, the new distribution must be consistent with the behavior of the new traveler.

Assume that for each class  $a$ , the journeys  $p \in P^a$  are indexed in order of increasing utility. That is, if  $p_1 > p_2$ ,  $w_{p_1} > w_{p_2}$ . We assume that no two journeys have the same utility value and that  $D^a > 0$ . (Pairs of journeys are allowed to have the same utilities by Gershwin, Orlicki, and Platzman, 1981.)

The journeys  $p \in P^a$  are divided into four sets:

1). Infeasible journeys: are those for which

$$t_p > T^a,$$

and/or

$$m_p > M^a.$$

If  $p$  is an infeasible journey,  $x_p = 0$ .

2). Constrained journeys are those for which

$$t_p = T^a \text{ and } m_p \leq M^a,$$

or

$$t_p \leq T^a \text{ and } m_p = M^a.$$

3). There is at most one special journey  $p^*$  such that

$$x_{p^*} > 0,$$

$$t_{p^*} < T^a,$$

$$m_{p^*} < M^a.$$

If no special journey exists, then we define  $p^*$  as the smallest index of the constrained journeys.

4). Unutilized journeys are those for which

$$p < p^* \text{ (i.e., } w_p < w_{p^*}\text{)}$$

and

$$x_p = 0.$$

To demonstrate that these conditions are consistent with (2.7) to (2.9), add a small increment  $\delta D^a$  to the demand  $D^a$ . Then, the time and money costs associated with whatever journeys  $p$  are chosen by the new users comprising  $\delta D^a$  are  $t_p + \delta t_p$ , and  $m_p + \delta m_p$  respectively. Assume that  $\delta t_p > 0$ , and  $\delta m_p > 0$ .

Consider the effect of  $\delta t_p$  and  $\delta m_p$  on the four possible sets of journeys:

1). Infeasible journeys remain infeasible since  $m_p + \delta m_p > m_p > M_p^a$  and/or  $t_p + \delta t_p > t_p > T_p^a$ . Therefore,  $\delta x_p = 0$ .

2). Constrained journeys may not accept more flow and still remain feasible since  $t_p + \delta t_p > T_p^a$ . Therefore,  $\delta x_p = 0$ . e

3). If a special journey exists, it accepts more flow since

$$t_p^* + \delta t_p \leq T_p^a,$$

$$m_p^* + \delta m_p \leq M_p^a$$

for sufficiently small  $\delta D^a$ . Note that  $\delta D^a$  may cause the special path to become a constrained path. ie

If there does not exist a special journey for the original equilibrated system, new flow is then assigned to an unutilized journey which then becomes a special journey.

4). Unutilized journeys remain unaffected since the newly reduced flow will be assigned to the higher utility, still available, special path.

### 2.3 New Principle - Stochastic Version

While the principle in Section 2.2 captures a greater variety of phenomena than the user optimization principle of Section 2.1, there are several important

features with which it cannot deal.

The first difficulty is the fact the different members of the same socio-economic class may have different budgets on the same day, and that the same person may have different budgets on different days. Zahavi (1979) shows that there is a consistent coefficient of variation in both budgets among a wide variety of populations.

Therefore, we redefine a class to be a set of people with the same residence location and with a common probability density function for time and money budgets. The probability that an individual of class a will have a time budget between  $u$  and  $u + \delta u$  and a money budget between  $n$  and  $n + \delta n$  is

$$f^a(u,n) \delta u \delta n . \quad (2.12)$$

The number of travelers of class a who have time budgets between  $u$  and  $u + \delta u$  and money budgets between  $n$  and  $n + \delta n$  is

$$D^a f^a(u,n) \delta u \delta n . \quad (2.13)$$

Let  $R$  be a region in  $(u,n)$  space. The number of travelers whose budgets fall in that region is

$$D^a \int_R f^a(u,n) \delta u \delta n . \quad (2.14)$$

To state an assignment principle, we relate flows to integrals of the form (2.14). Let

$$P^a = \{p_1^a, p_2^a, \dots, p_k^a\}$$

be the set of journeys available to travelers in class a and, following the convention stated in Section 2.2, let  $p_1^a$  be the least desirable journey and  $p_k^a$  be the most desirable. The utilities of all the journeys are assumed to be distinct. Let  $x_1^a, \dots, x_k^a$  be the flows on those journeys and define

$$X_i^a = \sum_{j=i}^k x_j^a . \quad (2.15)$$

That is,  $X_i^a$  is the total demand on journeys  $p_i^a, p_{i+1}^a, \dots, p_k^a$ . Let  $t_i^a$  and  $m_i^a$  be the time and money costs of journey  $p_i^a$ .

Since  $p_k^a$  is the most desirable journey, the total flow on this journey is the number of travelers whose time budgets are greater than, or equal to,  $t_k^a$ , and whose money budgets are greater than, or equal to,  $m_k^a$ . That is,

$$X_k^a = D^a \int_{\substack{u > t_k^a \\ n > m_k^a}} f(u, n) du dn . \quad (2.16)$$

Define

$$R_k^a = \{(u, n) \mid u \geq t_k^a, \quad n \geq m_k^a\}, \quad (2.17)$$

$$U_k^a = R_k^a . \quad (2.18)$$

This region is illustrated in Figure 2.1. Using the new notation, (2.16) can be written

$$X_k^a = D^a \int_{U_k^a} f(u, n) du dn . \quad (2.19)$$

To characterize the rest of the flows, we define

$$R_i^a = \{(u, n) \mid u \geq t_i^a, \quad n \geq m_i^a\}, \quad (2.20)$$

the set of expenditure levels that equal or exceed the required expenditures on path  $i$ . The people who can afford journey  $p_i^a$  or better are those whose budgets fall in any region  $R_j^a, i \leq j \leq k$ . Define

$$U_i^a = \bigcup_{j \geq i} R_j^a . \quad (2.21)$$

Then the number of travelers  $X_i^a$  that take journey  $p_i^a$  or better is the number whose budgets fall in region  $U_i^a$ . That is,

$$X_i^a = D^a \int_{U_i^a} f(u, n) du dn . \quad (2.22)$$

Equations (2.20) to (2.22) are a complete statement of the assignment principle in the stochastic case. The statement is remarkably concise

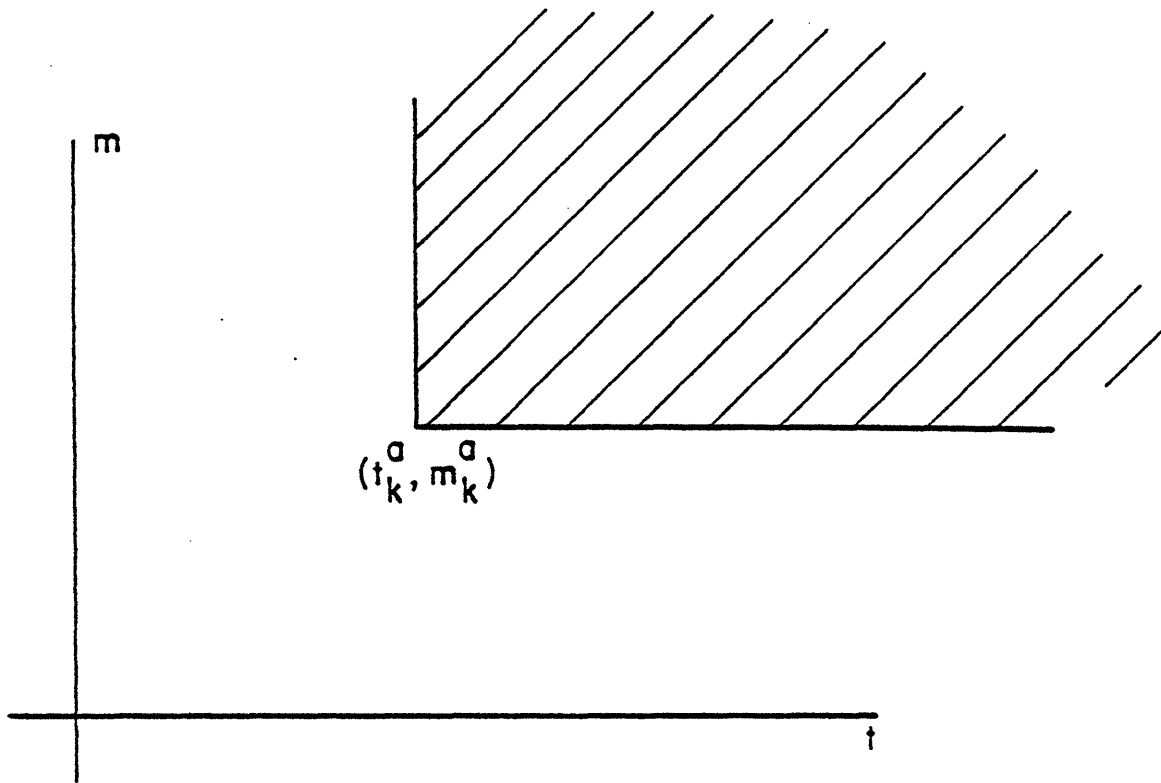


Figure 2.1: Region of Integration for Most Desirable Journey



compared with that required in the deterministic case, especially in light of its greater information content. In addition, a simple numerical technique for solving (2.20) to (2.22) suffices (Section 3).

Equation (2.21) can also be written

$$U_i^a = U_{i+1}^a \cup R_i^a, \quad 0 < i \leq k-1. \quad (2.23)$$

Figure 2.2 illustrates  $U_{i+1}^a$ ,  $R_i^a$ , and  $U_i^a$ . Note that the boundary of  $U_i^a$  always has a staircase-like structure, with a vertical half-line at the left and a horizontal half-line at the right.

The flow on path  $i$  satisfies

$$x_i^a = X_i^a - X_{i+1}^a,$$

so that

$$x_i^a = D^a \int_{U_i^a - U_{i+1}^a} f^a(u, n) du \, dn. \quad (2.24)$$

The region  $U_i^a - U_{i+1}^a$  is the oddly shaped rectangular polygon in Figure 2.2 whose lower left-hand corner is  $(t_i^a, m_i^a)$ .

Note that if  $t_i^a$  and  $m_i^a$  are sufficiently large,  $R_i^a$  falls entirely inside of  $U_{i+1}^a$ . In that case,  $U_i^a - U_{i+1}^a = 0$  and  $x_i^a = 0$ . This is intuitively reasonable. Journey  $i$  is the least desirable of those currently under consideration ( $i, i+1, \dots, k$ ) because of the ranking contention. If both its travel time and its money cost are greater than those of all better paths, it attracts no flow.

Equations (2.20) to (2.22) have been constructed to be consistent with (2.7) to (2.9). The region  $U_i^a$  contains all budgets that are greater than, or equal to, the expenditures on journeys  $p_i^a, p_{i+1}^a, \dots, p_k^a$ . Equation (2.22) implies that if an individual has budgets that are in  $U_i^a$ , he will choose one of these journeys and not journeys  $p_1^a, p_2^a, \dots, p_{i-1}^a$ .

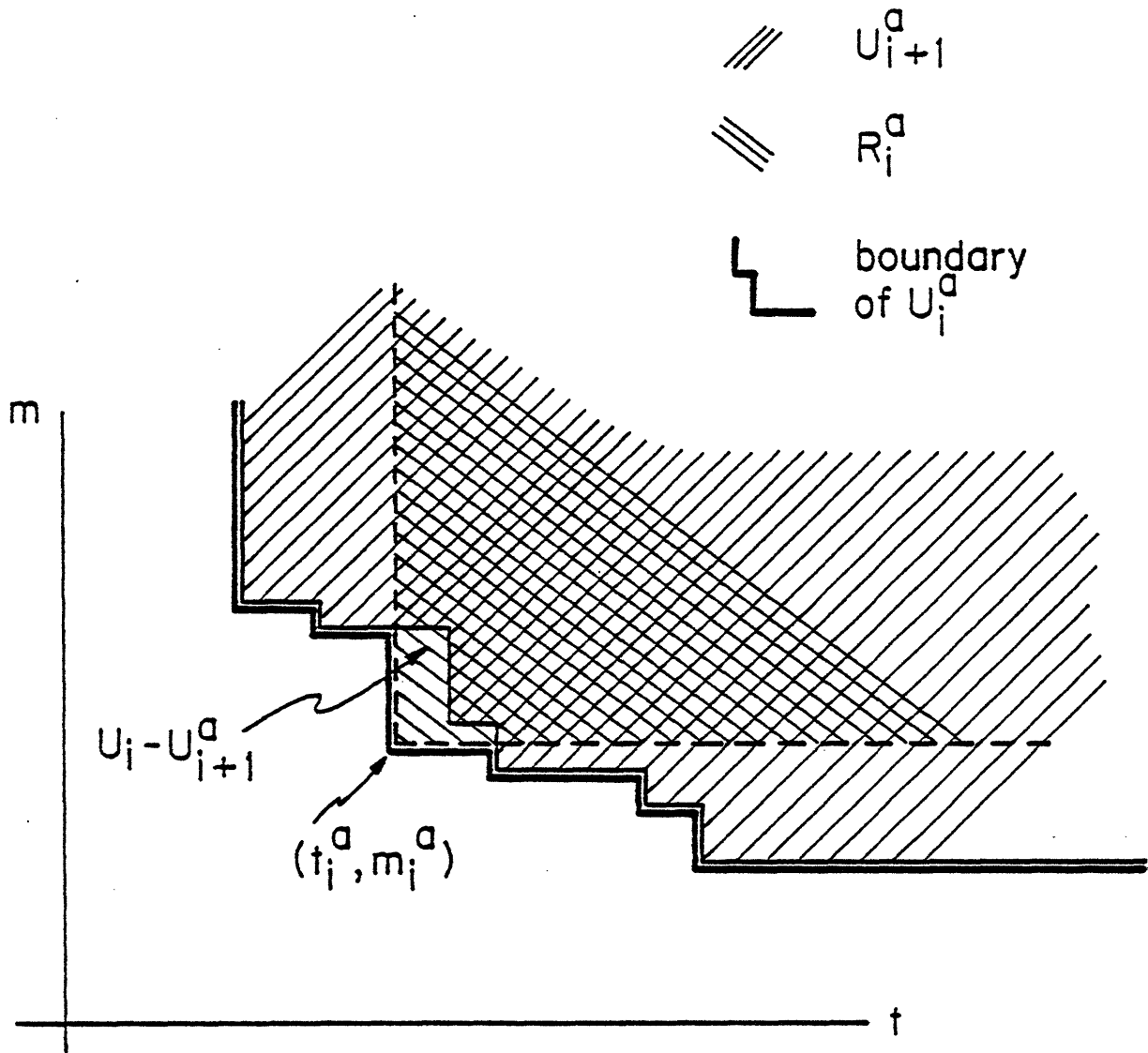


Figure 2.2: Region of Integration for Ordinary Journey

In particular (2.24) implies that if his budgets fall in  $U_i^a - U_{i+1}^a$ , he will take journey  $p_i^a$ . If he can afford to take journey  $p_i^a$ , but he cannot afford  $p_{i+1}^a, \dots, p_k^a$ , he will choose  $p_i^a$ . This is exactly what the formulation (2.7) to (2.9) says:  $p_i^a$  is the best journey (the ordering convention requires that the utility values of  $p_1^a, \dots, p_{i-1}^a$  are less than that of  $p_i^a$ ) whose costs are less than, or equal to, his budgets.

## 2.4 Discussion of New Principles

There are several features of the new principles that are not determined by the reasoning presented above. Certain choices have been made in constructing all the examples to follow. These choices are described here. Alternative choices could easily have been made with no difficulty.

### 2.4.1 Null Journey

It is convenient to introduce a null journey for each class. This is a journey that requires no travel time or money, that has less utility than any other journey available to its class, and whose flow does not affect the travel or money costs on any other journey in the network. The people who take the null journey are the people who stay home; their budgets are insufficient to allow them to do any traveling at all.

### 2.4.2 Costs

In the numerical examples to follow, the travel time for a journey is the sum of the travel times on its constituent links. The money cost of travel is computed in the same way with an additional term due to either the fixed cost for owning a car or a fare for transit. Neither the statement of the equilibrium conditions nor the algorithm in Section 3 depends on the method of calculating the costs.

### 2.4.3 Utility of Journeys

We have assumed that utility is constant, depending on the geography of the network, and not depending on flows, delays, or money costs. We assume that the utilities of journeys available to the same class are distinct, and that journeys may be ranked in order of utility, with the null journey the worst.

In this model, what is important about a journey is not its utility value, but rather its utility ranking. It is important that one journey be better than another; not how much better.

This feature obviates a precise determination of utility. Such a determination may be expensive or impossible. It is a reasonable assumption at least on an individual basis: each individual chooses the best journey he can afford, without being concerned with how much better it is than other journeys. On the other hand, different individuals may rank journeys differently, particularly if they are similar.

In the examples discussed below, it is assumed that utility, like travel time, is accumulated as one traverses a journey. A utility value is assigned to each node, and the value of the journey is the sum of the utilities of the nodes passed through.

This choice is made in an effort to build on, and further refine, Zahavi's ideas. Zahavi (1979) suggests that a reasonable approximation to travelers' behavior is to assume that they maximize the distance they cover within their time and money budgets. He points out that distance is only a surrogate for travelers' real objectives, which are to work, shop, be entertained and generally take advantage of as many of the facilities in the region as possible, within bounds of time and money constraints. The utility value of a node is simply a measure of the number of these facilities that can be found at each location.

We say that utilities are additive if the utility of a journey is the sum of the utility values of the nodes and links through which the journey passes.

Again, neither the statement of the equilibrium conditions nor the algorithms presented in the next section depend on the formula for calculating the utility ranking.

## 2.5 Related Work

There are similarities between the model suggested here and a model discussed by Schneider (1968) and Dial (1979). In their model, both time and money costs are important, but individuals do not have budgets. Rather, each traveler seeks to choose a trip (not a day's journey)  $i$  that minimizes a disutility which is a linear combination  $k_t t_i + k_m m_i$ .

Travelers differ by having different coefficients  $k_t$  and  $k_m$ . This principle leads them to a graph like Figure 2.2, but where  $U_i^a$  is replaced by a convex polyhedron. Trips that have both greater time and greater money costs than other trips get no flow. This is also true in the present model unless such journeys have superior utility values.

3. SOLUTION TECHNIQUE FOR STOCHASTIC VERSION  
OF NEW ASSIGNMENT PRINCIPLE

3.1 Statement of Equilibration Procedure

Let  $x$  be the vector of all path flows in the network. Define  $g(x)$  to be the vector of the same dimensionality as  $x$  whose component corresponding to journey  $i$  of class  $a$  is

$$g_i^a(x) = D^a \int_{U_i^a - U_{i+1}^a} f^a(u,n) du dn \quad . \quad (3.1)$$

The dependence of  $g$  on  $x$  is due to the dependence of  $U_i^a$  and  $U_{i+1}^a$  on  $x$ . These sets depend on  $x$  because they are determined by the time and money costs of journey  $p_i^a$  and of all the journeys better than  $p_i^a$  for class  $a$ . These costs, in turn, depend on the costs of the links that make up those journeys, and the link costs depend on the link flows, which are the sums of all the journey flows that pass through those links. It is clear that, in general,  $g_i^a$  depends on all the journey flows in the network.

Equations (2.20), (2.21), and (2.24) (which together are equivalent to (2.20) to (2.22)) can be written

$$x = g(x) \quad . \quad (3.2)$$

Thus, we seek a solution to a fixed point problem. It is observed that  $g(x)$  is a continuous function on the compact set

$$x_i^a \geq 0 \quad , \quad (3.3)$$

$$\sum_i x_i^a = D^a \quad . \quad (3.4)$$

to which  $x$  is restricted. Consequently, at least one solution to (3.2) exists (Dunford and Schwartz, 1957),

Equation (3.2) suggests a procedure;

$x(0)$  specified,

$$x(q+1) = (1-\lambda)x(q) + \lambda g(x(q)) \quad , \quad (3.6)$$

where

$$0 \leq \lambda \leq 1 . \quad (3.7)$$

In our experience, (3.5), (3.6) converges whenever  $\lambda$  is sufficiently small. Convergence is considered to have occurred when

$$\max_{a,i} |x_i^a(q) - g_i^a(x(q))| < \epsilon . \quad (3.8)$$

The main effort in executing (3.6) is evaluating the regions  $U_i^a$ , and then performing the required integrals. This is described in the following section.

### 3.2 Calculation of Integrals

It is convenient to write the iteration process as

$$x_i^a(q+1) = (1-\lambda)x_i^a(q) + \lambda \int_{U_i^a(q)} f^a(u,n) du dn, \quad (3.9)$$

$$x_k^a(q+1) = x_k^a(q+1), \quad (3.10)$$

$$x_i^a(q+1) = x_i^a(q+1) - x_{i+1}^a(q+1), \quad (3.11)$$

where  $x_i^a(q)$  is the class a flow on journey  $p_i^a$  or better. The index k refers to the best (highest utility) journey for class a and, as usual, the journey index numbers increase with increased utility.

In Section 3.2.1, we characterize the regions  $U_i^a(q)$ . In 3.2.2, we demonstrate how to evaluate the integral in equation (3.9).

#### 3.2.1 Regions of Integration

Region  $U_i^a$  is defined by equations (2.18), (2.20), and (2.23) (where the argument  $q$  is suppressed). It always has the characteristic staircase shape of Figure 2.2, of which Figure 2.1 is a special case. To perform the integration in (3.9), a list of the corners of Figure 2.2 is required. The corners of  $U_{i+1}^a$  are displayed in Figure 3.1.

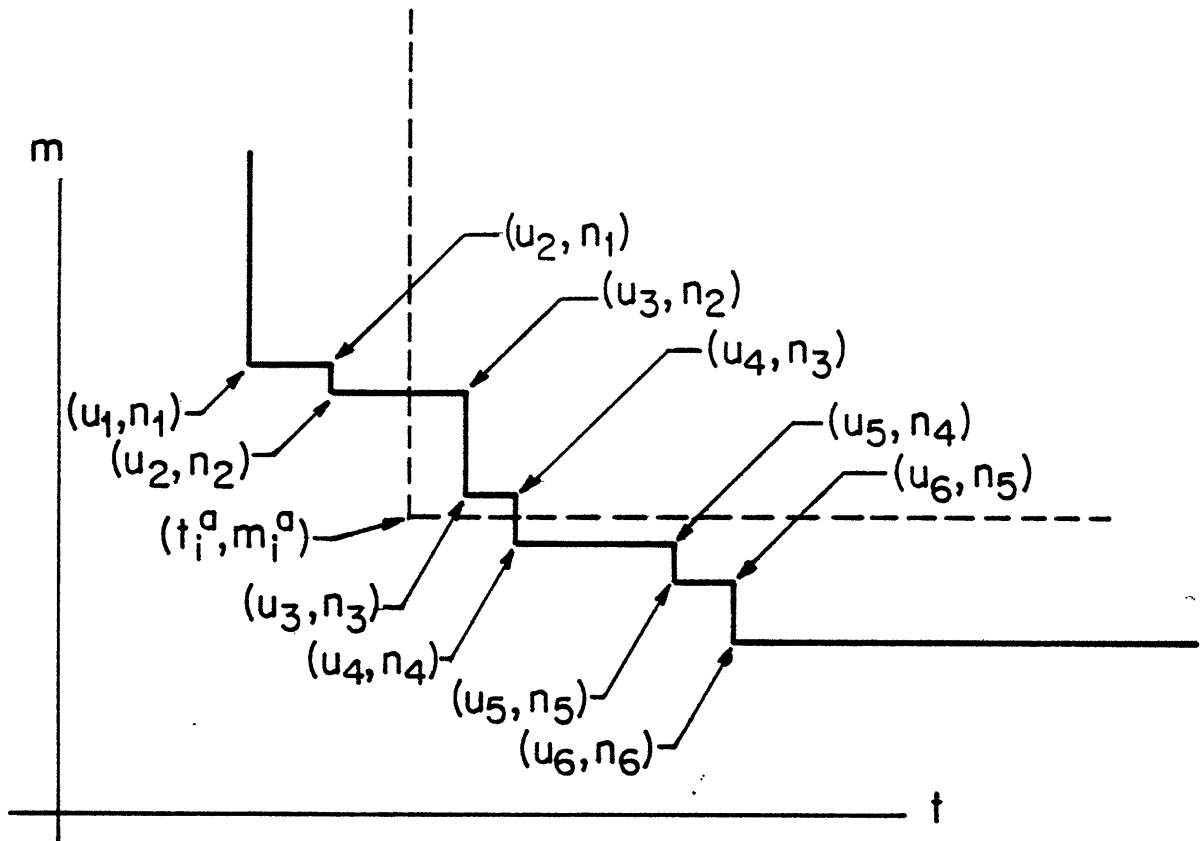


Figure 3.1: Corners of  $U_{i+1}^a$  and  $U_i^a$



The components of a corner of the form  $(u_j, n_j)$  are the travel time and travel money costs of some journey of class a better than journey i. Heretofore, they have been listed in order of increasing utility. To perform the integration, they must be listed as they are in Figure 3.1, in order of increasing travel time and decreasing money (or vice versa). Furthermore, the costs of journeys i such that  $R_i^a$  is a subset of  $U_{i+1}^a$  must not appear in this list. (See the remarks following (2.24).)

Let  $L_i^a$  be the list required to perform the integration over  $U_i^a$ . It is defined inductively from  $L_{i+1}^a$  based on (2.23). From (2.17) and (2.18), we define

$$L_k^a = [(t_k^a, m_k^a)]. \quad (3.12)$$

where  $p_k^a$  is the best path. Let

$$L_{i+1}^a = [(u_1, n_1), \dots, (u_r, n_r)], \quad (3.13)$$

where

$$u_1 < u_2 < \dots < u_r, \quad (3.14)$$

and

$$n_1 > n_2 > \dots > n_r. \quad (3.15)$$

If, for some  $j$ ,  $1 \leq j \leq r$

$$\begin{aligned} t_i^a &> u_j \\ m_i^a &> n_j \end{aligned} \quad (3.16)$$

then,

$$L_i^a = L_{i+1}^a \quad (3.17)$$

This is because journey  $i$  has lower utility than the journey whose costs are  $(u_j, n_j)$ . Since its costs are also greater, there is no reason to take this journey. Geometrically,  $R_i^a$  is a subset of  $U_{i+1}^a$ , and  $(t_i^a, m_i^a)$  falls inside of  $U_{i+1}^a$ .

Otherwise, let  $\alpha$  be the largest integer such that

$$u_\alpha < t_i^a \quad (3.18)$$

and let  $\beta$  be the smallest index such that

$$n_\beta < m_i^a \quad (3.19)$$

(In Figure 3.1,  $\alpha = 2$ ,  $\beta = 4$ .) Then, the list  $L_i^a$  is constructed from list  $L_{i+1}^a$  by replacing all the points  $(u_{\alpha+1}, n_{\alpha+1}), \dots, (u_{\beta-1}, n_{\beta-1})$  with  $(t_i^a, m_i^a)$ . That is, if

$$\begin{aligned} L_{i+1}^a &= [c_1^{i+1}, \dots, c_r^{i+1}] \\ L_i^a &= [c_1^i, \dots, c_s^i] \end{aligned} \quad (3.20)$$

then,

$$\begin{aligned} c_j^i &= c_j^{i+1} \quad j = 1, \dots, \alpha \\ c_{\alpha+1}^i &= (t_i^a, m_i^a) \\ c_j^i &= c_{\beta-\alpha-2+j}^{i+1} \quad j = \alpha+2, \dots, s \end{aligned} \quad (3.21)$$

$$s = r - \beta + \alpha + 2$$

In Figure 3.1,

$$\begin{aligned} L_{i+1}^a &= [(u_1, n_1), \dots, (u_6, n_6)], \\ L_i^a &= [(u_1, n_1), (u_2, n_2), (t_i^a, m_i^a), (u_4, m_4), \\ &\quad (u_5, m_5), (u_6, m_6)] . \end{aligned}$$

Note that the numbers of corners in  $U_i^a$  (i.e.,  $s$ ) may be greater than the number in  $U_{i+1}^a$  (i.e.,  $r$ ) by 1, or it may be equal, or it may be fewer if  $\beta > \alpha + 2$ .

### 3.2.2 Calculation of Integrals

Once list  $L_i^a$  is determined, the integral in (3.9) can be written

$$\int_{U_i^a} f^a(u,n) du dn = \sum_{j=1}^{s-1} \int_{S_j} f^a(u,n) du dn + \int_Q f^a(u,n) du dn, \quad (3.22)$$

where  $s$  is the number of corners of  $U_i^a$ ,  $S_j$  is the strip,

$$S_j = \{(u,n) \mid u_j \leq u \leq u_{j+1}, n \geq n_j\}, \quad (3.23)$$

and  $Q$  is the quadrant

$$Q = \{(u,n) \mid u \geq u_s, n \geq n_s\}. \quad (3.24)$$

In all the examples described below, we have assumed that the budgets are independent. As a result,

$$f^a(u,n) = \phi^a(u)\psi^a(n), \quad (3.25)$$

and

$$\int_{S_j} f^a(u,n) du dn = \left[ \int_{u_j}^{u_{j+1}} \phi^a(u) du \right] \left[ \int_{n_j}^{\infty} \psi^a(n) dn \right], \quad (3.26)$$

$$\int_Q f^a(u,n) du dn = \left[ \int_{u_s}^{\infty} \phi^a(u) du \right] \left[ \int_{n_s}^{\infty} \psi^a(n) dn \right]. \quad (3.27)$$

The integrals in (3.26), (3.27) are particularly easy to calculate when the  $\phi$  and  $\psi$  density functions are piecewise linear. We have observed that numerical

results are not sensitive to the detailed shapes of these distributions although they are sensitive to their means and variances.

### 3.3 Algorithm Behavior

In the following sections, we discuss a number of numerical examples. (Other examples are presented in Gershwin, Orlicki, and Zahavi, 1981.) We restrict our attention to the behavior of the solution of the system (2.20) to (2.22). Here, we present an informal summary of our observations on the behavior of the iteration procedure (3.6).

The convergence properties of the algorithm are sensitive to two sets of quantities:  $\lambda$ , and the variances of  $\phi^a(\cdot)$  and  $\psi^a(\cdot)$ .

The larger the variances, the greater the reliability of convergence. When the variances are small, it is very easy for the algorithm to overshoot the equilibrium distribution and oscillate wildly.

We conjecture that this behavior is due to the fact that, when the variance is small, the travelers in a given class are similar to one another. If the cost of a given journey is too high, all the members of that class will react in the same way, and most of the flow will be removed from that journey by the integral in (3.9). If the costs are too low for the present flow on a journey, the integral in (3.9) will tend to redistribute most of the flow of that class onto that journey. If the variances are large, only a small amount of flow will be affected by an error in cost.

The step size  $\lambda$  also affects algorithm behavior. When  $\lambda$  is small, the change from iteration to iteration is small, and although convergence is likely it is time-consuming. When  $\lambda$  is large, overshoot is a danger, and oscillations can be observed.

The amount of computer time that the algorithm requires is also greatly affected by the computation required to evaluate the integrals on the right-hand side of (3.26) and (3.27). Distributions that require a great deal of arithmetic (such as a normal, Erlang, or gamma) require much more total computer time than others (such as the uniform or a piecewise linear density).

It is clear that these observations can be combined to enhance the efficiency of the algorithm. First,  $\lambda$  can be increased as the algorithm shows signs of slowing down; i.e., reaching equilibrium. Second, the variances can be initialized at much larger values than the required variances, and decreased gradually when the algorithm seems to be converging. Finally, if it is necessary that results be computed with difficult-to-calculate distributions, the algorithm can be restarted after first converging with an easy distribution.

#### 4. NUMERICAL EXAMPLES

In this section, we describe several examples of small-size networks that we analyze using the numerical technique of Section 3. These examples illustrate various qualitative properties of the assignment principles of Section 2 and of the iteration process. A network of more realistic size and complexity is presented in Section 5.

##### 4.1 Four-Link Network

There are five journeys in the network of Figure 4.1: the null journey and the four that each traverse one link once. The flow that takes journey  $i$ , which traverses link  $i$ , is  $x_i$ . The flow on the null journey is  $x_0$ . The four links have identical delay functions given by

$$t_i = .1 + \left( \frac{x_i}{250} \right)^4. \quad (4.1)$$

The journeys have different utility values: journey 4 is better than journey 3, and so forth.

Figure 4.2 displays the flow levels as a function of total demand when the time budget has a triangle distribution with mean 2.00 and variance 1.95. The triangle density function is displayed in Figure 4.3. The money budget is set high so as to be a non-binding constraint.

When the demand is near zero, nearly all flow is attracted to journey 4. Journey 4 is the best journey and the delay is small, so nearly all travelers can afford to choose it.

As the demand increases, the flow on journey 4 increases. However, some travelers are excluded from it since its delay has increased, so they take journey 3. Eventually, journey 3 becomes expensive and travelers move to journey 2, journey 1, and the null journey. In the limit as  $D \rightarrow \infty$ ,

$$x_i \rightarrow c^*, \quad (4.2)$$

$$x_0 \rightarrow D - 4c^*, \quad (4.3)$$

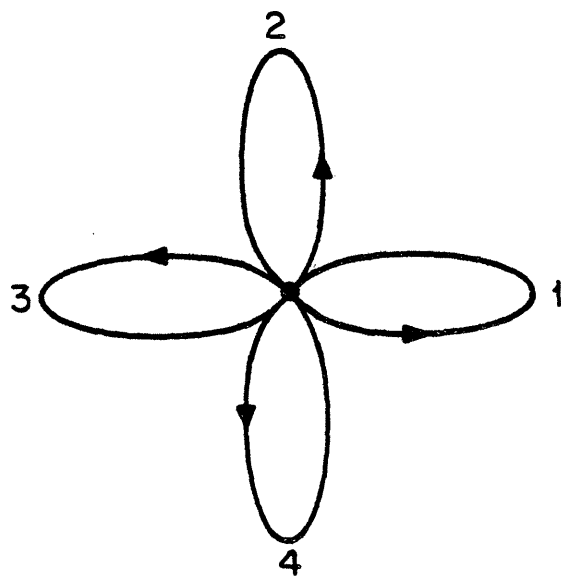


Figure 4.1: Four-Link Network

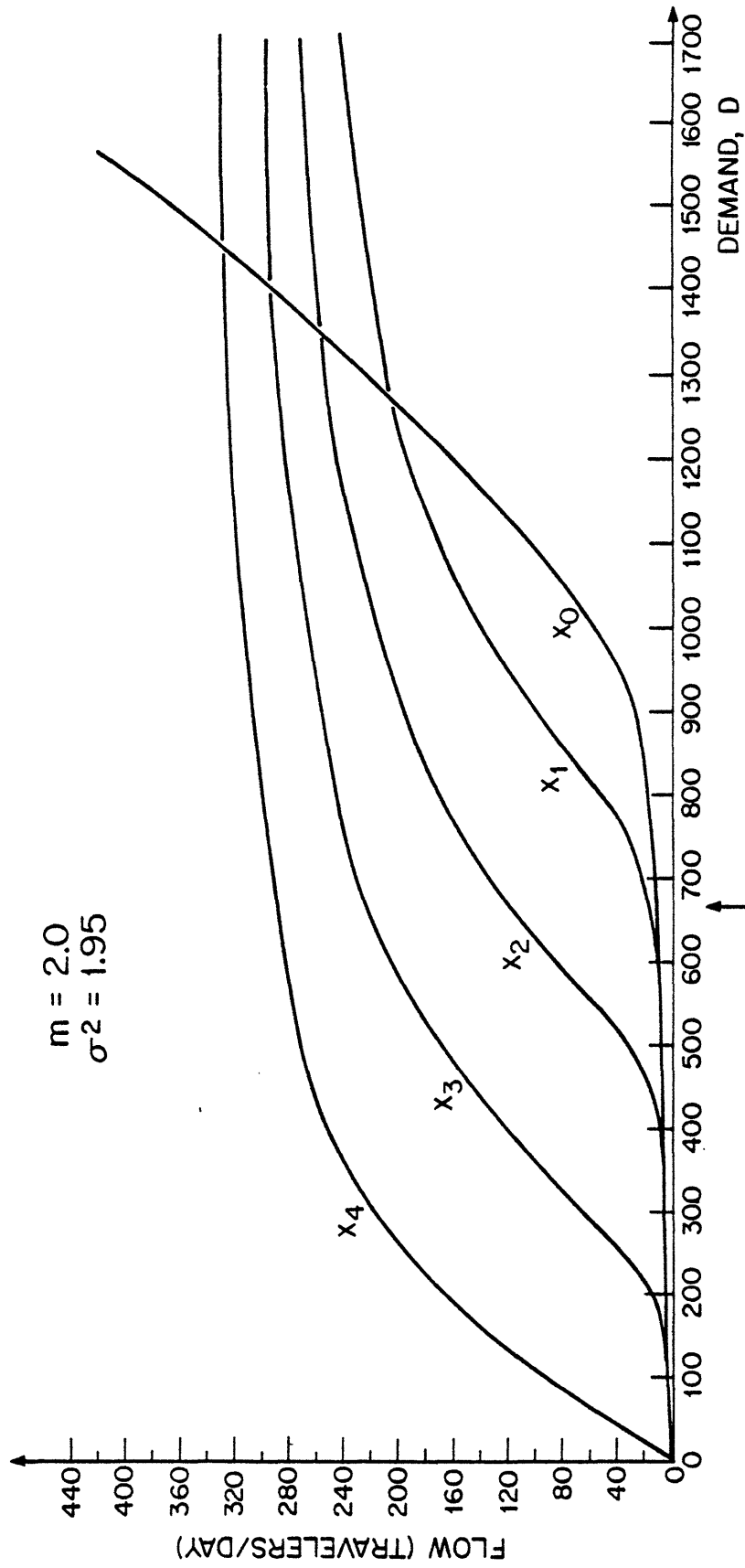


Figure 4.2: Flow vs. Demand -- High Variance



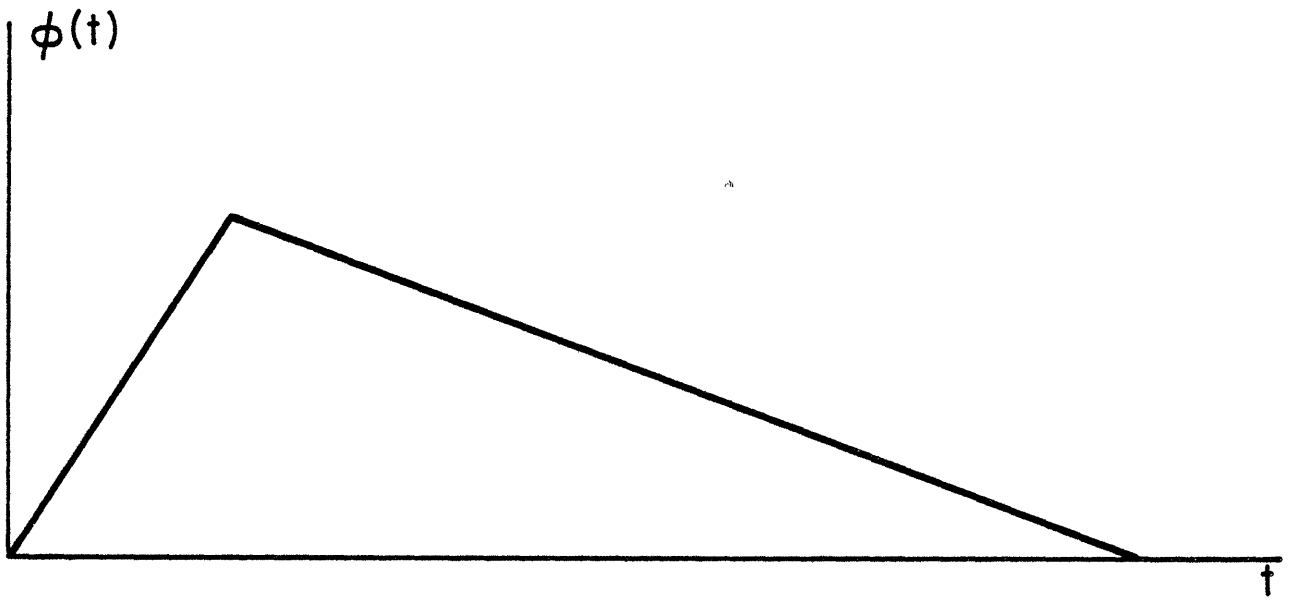


Figure 4.3: Triangle Density Function

where  $c^*$  is a limiting flow, and  $x_0$  contains all the travel demand that is unmet.

Figure 4.2 illustrates the relationship between the deterministic and stochastic versions of the principle. When the demand  $D$  is as indicated by the arrow, two journeys are approximately at budget levels (constrained journeys), one has less travel time than the budget level (special journey), and two have nearly no flow (unutilized journeys).

In the deterministic case, Figure 4.4 describes the behavior of the flows. Again, limits (4.2), (4.3) are valid. Here  $c^*$  is the solution to

$$T = .1 + \left( \frac{x_i}{250} \right)^4, \quad (4.4)$$

where  $T = 2.00$  is the value of the time budget so that  $c^* = 293.5$ . Clearly, at the arrow, journeys 4 and 3 are constrained, journey 2 is special, and journeys 1 and 0 are unutilized.

#### 4.2 Small Network with Interacting Journeys

Figure 4.5 is a network with three nodes and five links. Node and link numbers are indicated. The link time cost functions are given by

$$\tau_{\ell}(f) = \tau_{\ell 1} + \left( \frac{f}{\tau_{\ell 2}} \right)^4, \quad (4.5)$$

and the link money cost functions by

$$\mu_{\ell}(f) = \mu_{\ell 1} + f^{1.5}, \quad (4.6)$$

where the parameters are listed in Table 4.1.

Several examples are treated that are based on this network. In each, the budget density functions  $\phi$  and  $\psi$  are uniform.

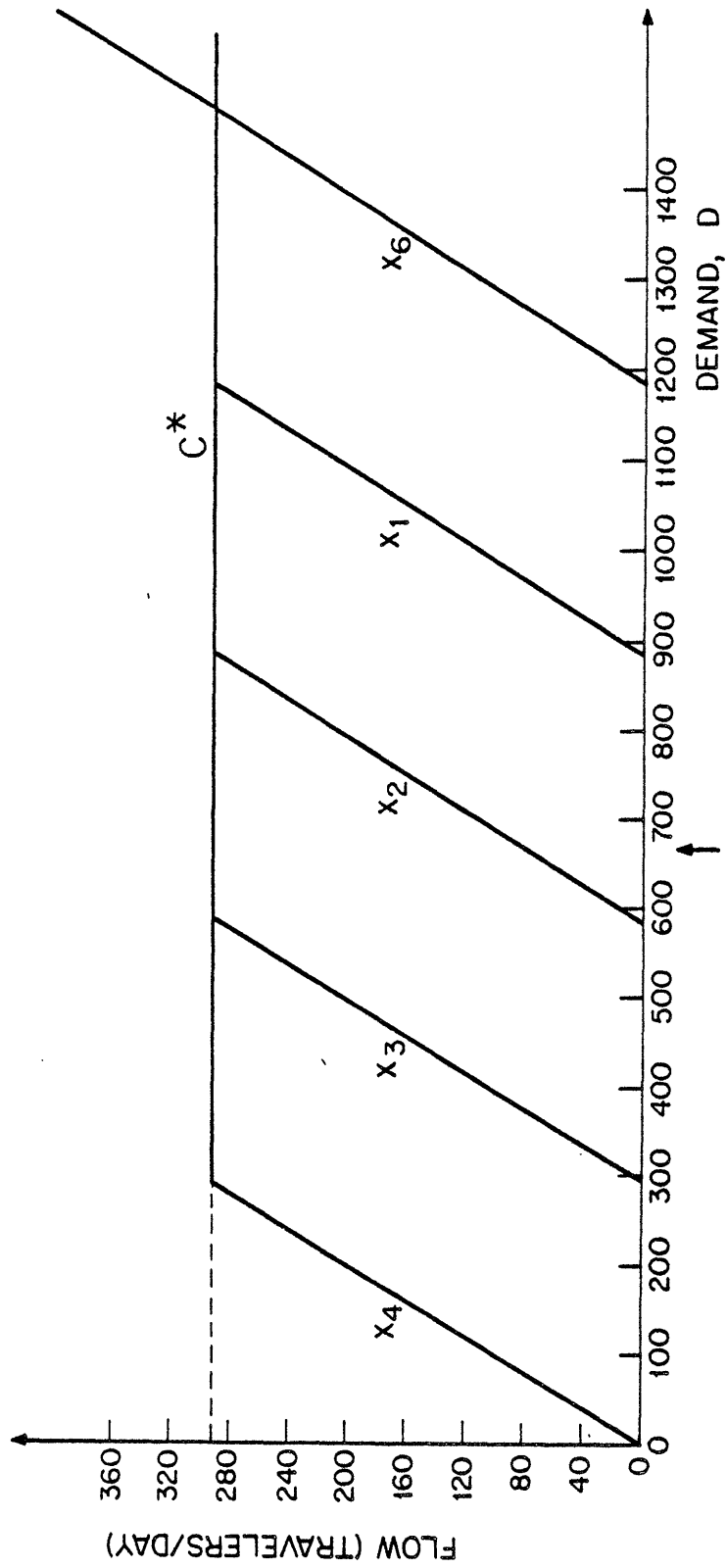


Figure 4.4: Flow vs. Demand -- Deterministic Principle

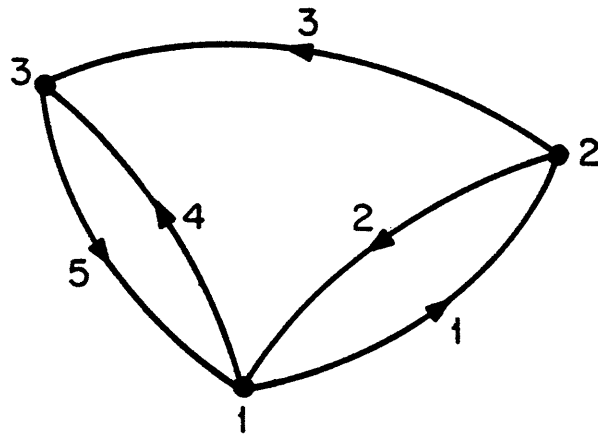


Figure 4.5: Three-Node Network

Table 4.1 - Network Parameters

link $l$	$\tau_{l1}$	$\tau_{l2}$	$\mu_{l1}$
1	0.25	200	0
2	0.25	200	0
3	0.50	400	0.50
4	0.50	200	0
5	0.50	200	0

Case 1

The time budget is uniform between 2.0 and 2.5 and the money budget is uniform between 3.0 and 3.5. The total demand is 200. Table 4.2 lists the journeys in increasing order of utility, as well as the equilibrium flow, travel time, and money cost.

Table 4.2 - Results of Case 1

Journey	Flow	Time	Cost
null (1)	0	0	0
1 - 3 - 1 (2)	0	1.03	.74
1 - 2 - 1 (3)	114.31	1.61	1.61
1 - 2 - 3 - 1 (4)	85.69	2.29	2.64

Journey 4, which is the most desirable, is like a constrained journey in that there are some travelers on it who are spending their entire travel-time budgets. Journey 3 is analogous to the special path of the deterministic formulation since no traveler is spending either of his full budgets on travel. Journeys 1 and 2 are unutilized. It is noted that this interpretation of the relationship between the stochastic and deterministic principles does not always hold. It works here because the uniform density is zero outside of a certain range.

Case 2

Case 2 is the same as Case 1 except that the demand is raised to 300. Results are presented in Table 4.3.

Table 4.3 Results of Case 2

Journey	Flow	Time	Cost
null (1)	0	0	0
1 - 3 - 1 (2)	97.12	1.20	.93
1 - 2 - 1 (3)	176.24	2.16	2.29
1 - 2 - 3 - 1 (4)	26.65	2.46	2.87

In the previous network, when the demand is increased, it fills up the current "special" path, and then overflows onto the next. Here, all the flow is redistributed. In particular, there is less flow on the best path in Case 2 than in Case 1.

The reason for this is that the time required to traverse journey 4 has increased from 2.29 to 2.46, which is almost at the upper limit of the time-budget distribution. This increase in time is due to the increased flow on journey 3, which shares link 1 with journey 4; and on journey 2, which shares link 5. Again, the stochastic version of the assignment principle mimics the deterministic in that there are constrained, special, and unutilized journeys in the proper order. Note that the flow on journey 4 is now smaller than it was in case 1.

### Case 3

It is observed that journey 2 has time and money costs which are less than one-half of the upper limits on the budget distributions. Some travelers might therefore be expected to go around the loop twice. Consequently we have added an additional journey: 1 - 3 - 1 - 3 - 1. The utility of this journey is greater than that of journey 2 and less than the utility of journey 3.

To our great surprise, no redistribution takes place after the new journey is added. That is, the new equilibrium is such that the flow on the new journey is 0, and the flows on the others are the same as those of Table 4.3. The time cost of the journey, 2.40, is greater than that of journey 3 (2.16) although its utility is less. (The money costs are irrelevant since they are less than the lower limit of the money budget density function). Consequently, no travelers who are capable

of switching from 1 - 2 - 1 to 1 - 3 - 1 - 3 - 1 (i.e., those on 1 - 2 - 1 whose time budgets are greater than 2.40) have any incentive for doing so.

This indicates the marketing aspect of providing travel service. When a new travel alternative is offered (in this case, a new journey), it is not enough to ascertain that it is better than some existing alternatives that are used, and that it is affordable by (i.e., within the budget of) some travelers. The new alternative will be used only when there are some travelers that can afford it, and for whom it is better than their current transportation choice.

This result may be seen in terms of the integration regions  $U_i$ . Figure 4.6 shows the uniform joint-density distribution of budgets for travelers in the network and the  $U_i$  regions for journeys 2, 3, 4, and the new journey. The integrals of regions of the budget joint density are performed in order of utility, with the region for the highest utility journey evaluated first. The new journey we propose has utility greater than journey 2, and less than those of journeys 3 and 4 of the original network. Figure 4.11 shows that by the time the integral of  $f(\cdot)$  over the region  $U_{\text{new}}$  for the proposed flow is evaluated, all potential users of the new journey have already been assigned to journeys 3 and 4. The travelers having budgets in the joint density contained in the region  $U_3 - U_2$  are the only candidates still unassigned, and they cannot afford the new journey. Thus, even though a journey with higher utility than a presently employed path is available which has costs in the feasible budget region, it may go unused depending on its value to travelers relative to other journey choices.

#### 4.3 Circular Network

Figure 4.7 contains a network consisting of 13 nodes and 40 links in the shape of two concentric circles. The details of this network are presented in Gershwin, Orlicki, and Zahavi (1981).

Here, the effect of the shape of the distribution is investigated. Two runs are compared that are identical -- 5 journeys for each of two classes; time budgets means are 1.25 hours and variances  $0.5 \text{ (hour)}^2$ ; money budgets means are 3.00 dollars and variances  $1.5 \text{ (dollars)}^2$  -- in all respects except one. One run has gamma distributions and the other has uniform distributions. See Table 4.7.

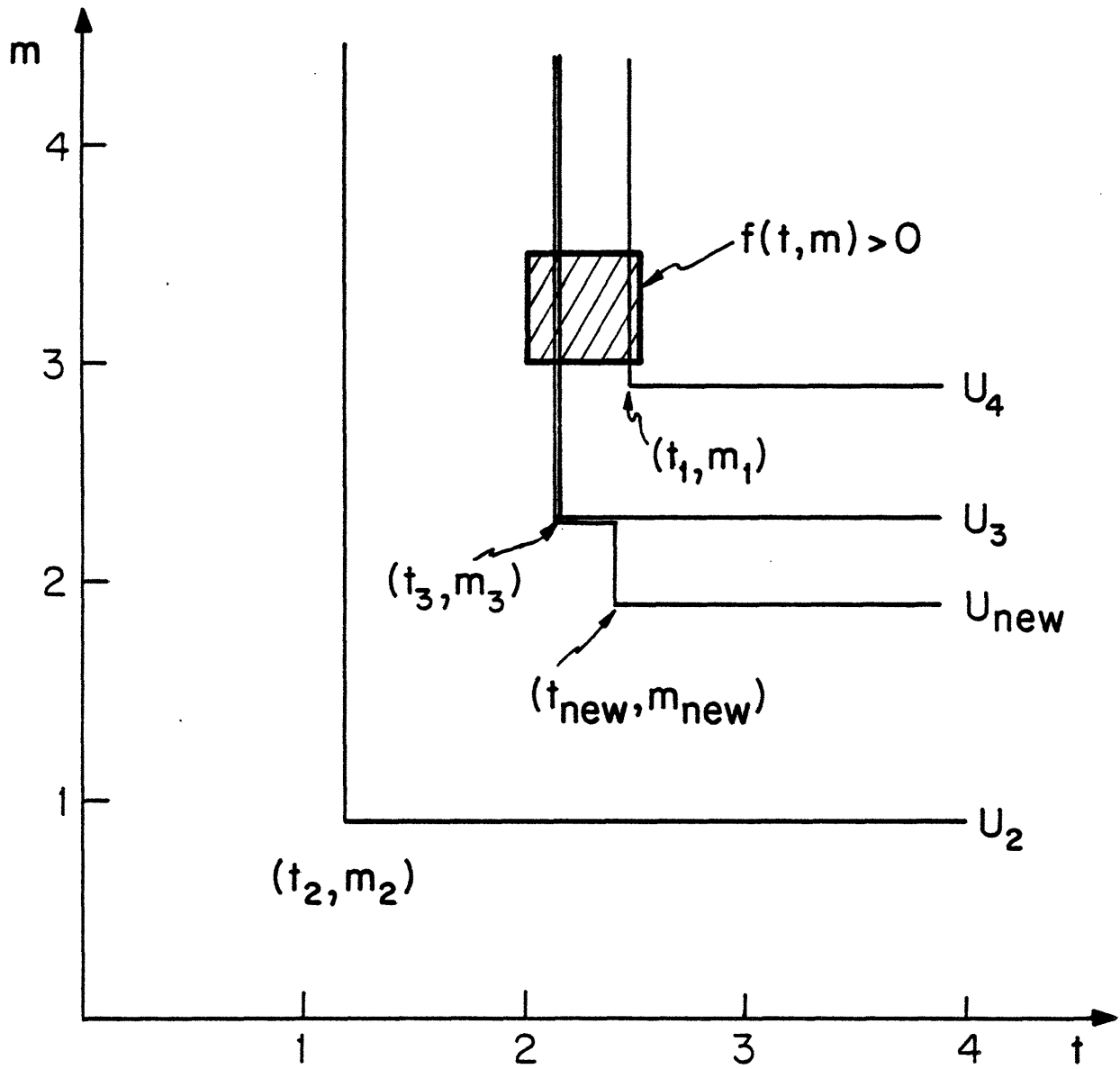


Figure 4.6: Regions of Integration for Cases 2 and 3



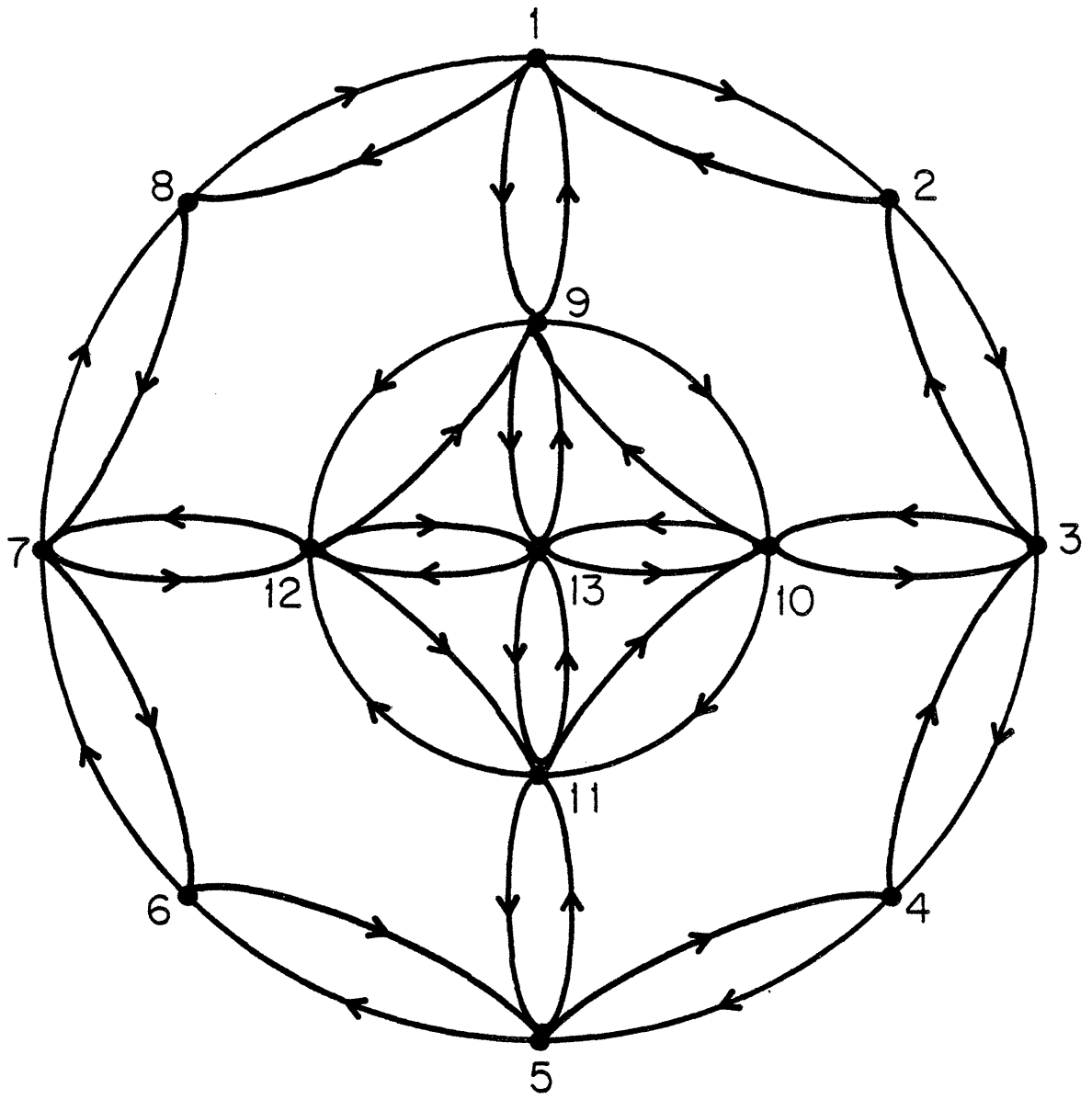


Figure 4.7: Circular Network

Table 4.7

Journey	Flow (gamma)		Time (gamma)		Time (uniform)		Money Cost	
	(gamma)	(uniform)	(gamma)	(uniform)	(gamma)	(uniform)	(gamma)	(uniform)
null	13405.39	14773.79	0	0	0	0	0	0
2-1-9-13-9-1-2	4802.91	3651.29	.84	.81	.81	1.55	1.52	1.52
2-3-10-13-10-3-2	678.12	0	1.01	1.01	1.01	1.79	1.80	1.80
2-1-9-13-10-3-2	6441.36	5326.32	1.04	1.00	1.00	1.83	1.78	1.78
2-1-8-7-12-13-10-3-2	9672.22	11248.59	1.29	1.29	1.29	2.31	2.31	2.31

The resulting corresponding journey costs are nearly identical. The corresponding flows are also close to one another. If larger variances were used, we assume that the flows would be even closer. (Larger, more realistic, variances could not be used because one of the distributions was uniform. There is a limit on the variance of a uniform distribution whose mean is specified and whose lower limit is required to be zero or positive.) This assumption follows from the observation that the journey costs are similar in the two cases and from our experience that large variances tend to reduce the sensitivity of the distribution of flows to journey costs.

## 5. LARGE NETWORK

In this section, we show that a system of realistic size and complexity can be treated using the technique described here and that it yields reasonable results.

Relatively crude models for mode delays and costs are used. For example, parking fees are not included. Only a single kind of car is represented, and a single global average of 1.5 occupants per car is used for all economic classes and all residential locations. Walking, which is an important transportation mode in the dense downtown area, is not considered.

For details, and more variations on the basic network, see Gershwin, Orlicki, and Zahavi (1981).

### 5.1 Discussion of System

#### 5.1.1 Network

The network is presented in Figure 5.1. It has 144 nodes and 264 links. It is symmetric about its vertical and horizontal axes. Its total length is 343.2 kilometers. Lengths of links are indicated in the figure. Roads are most dense near the city center and become less dense toward the periphery. All streets are one-way.

We calculate the utility of a journey by counting the number of nodes that the journey passes through. This count is multiplied by 1.1 for car journeys to represent the attractiveness of a car over that of a bus. All nodes are equally desirable. Because of the greater density of nodes near the city center, journeys going inward are more valuable than those going outward.

Journeys are paths in the network that start and end at the same residential node. Between 76 and 81 journeys are made available to each class. No mixed-mode journeys have been considered.

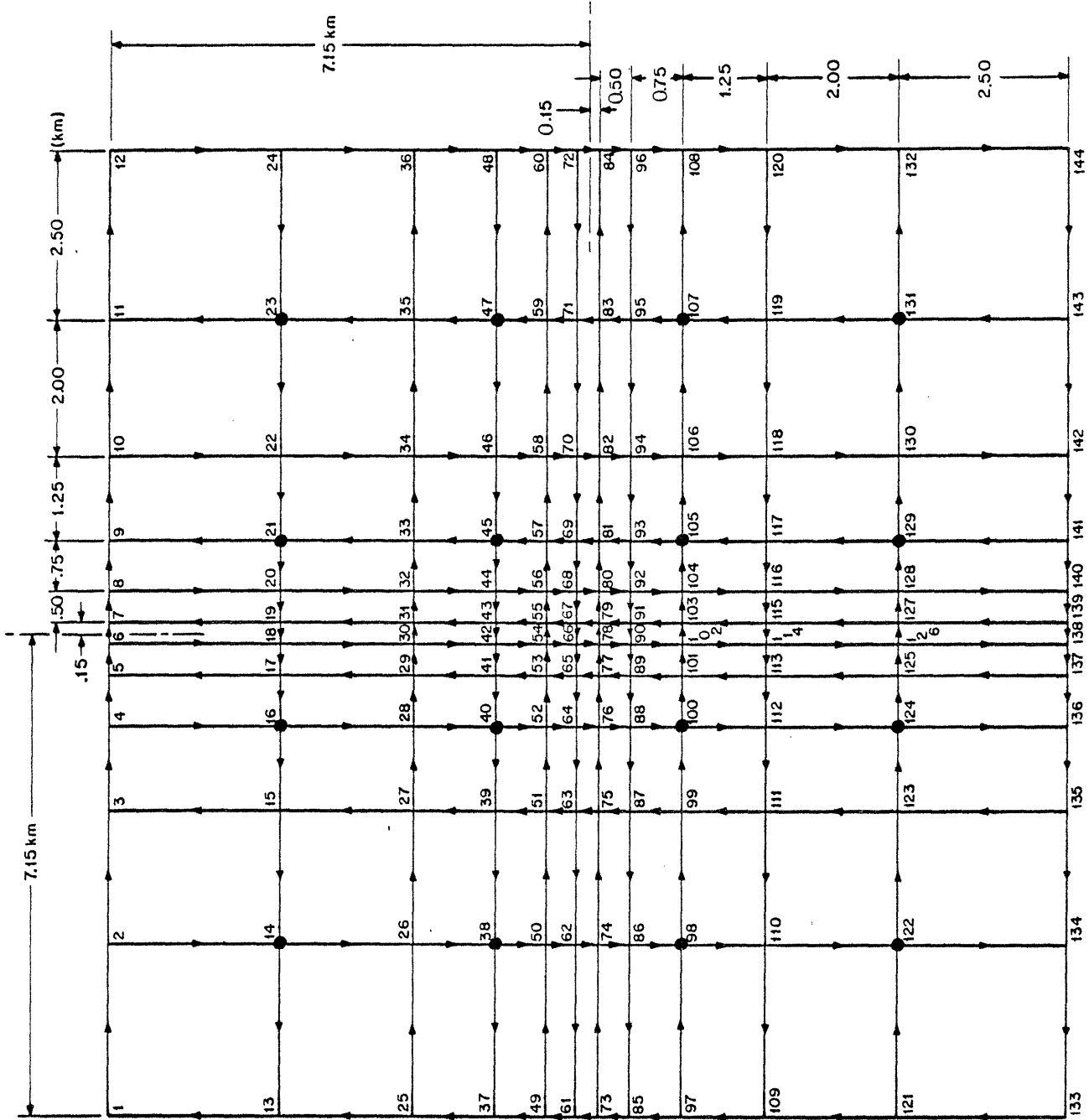


Figure 5.1: Large Network

Car journeys are allowed to go anywhere in the network as long as path-continuity and link directions are respected. Bus journeys are more restricted. They have the same freedom as cars in the inner city; that is, among nodes 40, 45, 100, and 105. However, outside of that region they may travel only on certain links. These are the links that connect the following nodes: 13-24; 37-48; 97-108; 121-132; 2, 14, 26, ..., 134; 4, 16, 18, ..., 136; 9, 21, 33, ..., 141; 11, 23, 35, ..., 143.

### 5.1.2 Population

The population consists of 300,000 people divided into 2 income groups and 16 residential locations. The wealthier group has a household income of 35,000 dollars per year and the poorer has a household income of 15,000 dollars per year. Households are assumed to contain 3 people, on the average; and years have, on the average, 320 days of regular travel, so these incomes are 36.46 dollars per day per person and 15.63 dollars per person respectively.

The locations at which people live and their class designations are listed below. Classes 1 to 16 are the wealthier people; classes 17 to 32 are the poorer. The number of people in each class is 9375.

Node	Class
14	1, 17
16	2, 18
21	3, 19
23	4, 20
38	5, 21
40	6, 22
45	7, 23
47	8, 24
98	9, 25
100	10, 26
105	11, 27
107	12, 28
122	13, 29
124	14, 30
129	15, 31
131	16, 32

Because of the great size of this system, techniques have been sought to reduce computation time. A very effective technique is to use a triangle density function (Figure 4.3) rather than a gamma or other distribution. Calculations involving the triangle are simple and, as we argued in Section 4.3, probably accurate when variances are large.

The time budget distribution for all classes is triangular with mean 1.1 hours and variance  $0.43 \text{ (hours)}^2$  for each traveler, which yields a coefficient of variation of 0.6. This is consistent with Zahavi's (1979) empirical findings. The money budget is also triangular, with means of 11 percent, or 6.00 and 2.55 dollars per traveler. The variances are  $17.50 \text{ dollars}^2$  and  $3.10 \text{ dollars}^2$  respectively.

### 5.1.3 Time Costs

The time for a journey is the sum of the times for its constituent links. In the case of bus journeys, an additional delay of 0.2 hours per transfer is added to represent the time spent waiting for a bus.

Link travel time for cars is given by

$$\tau_{\ell} = \tau_{\ell 1} \left( 1 + 0.15 \left( \frac{f_{\ell}}{c_{\ell}} \right)^4 \right), \quad (5.1)$$

where  $\tau_{\ell 1}$ , and  $c_{\ell}$  are constants. The capacity,  $c_{\ell}$ , is 24000 vehicles per day. The free-flow time,  $\tau_{\ell 1}$ , is the length of each link (as indicated in (Figure 5.1) divided by the free-flow speed, which is assumed to be 50 km/h. Link travel time for buses is assumed to be twice that for cars.

The flow,  $f_{\ell}$ , is given as

$$f_{\ell} = \alpha_c f_{\ell}^c + \alpha_b f_{\ell}^b, \quad (5.2)$$

where  $f_{\ell}^c$  and  $f_{\ell}^b$  are the total car-traveler flow and bus-traveler flow through link  $\ell$ . That is,

$$f_{\ell}^c = \sum_a \sum_{j \in \text{car}} A_{j\ell}^c x_j^a, \quad (5.3)$$

$$f_{\ell}^b = \sum_a \sum_{j \in \text{bus}} A_{j\ell}^b x_j^a, \quad (5.4)$$

where the second sum in (5.3) is over all car journeys, and the second sum in (5.4) is over all bus journeys. The quantities  $\alpha_c$  and  $\alpha_b$  represent the impact of a single traveler on the total car-equivalent flow.

We choose

$$\alpha_c = 1/1.5$$

since we assume a car occupancy of 1.5 travelers, and

$$\alpha_b = 0.125.$$

This is obtained by dividing the bus equivalency factor (3) by an average bus occupancy of 24.

#### 5.1.4 Money Costs

The cost of most bus journeys is a fare of 1.00 dollars. There are a small number of journeys that involve multiple loops. That is, some nodes, other than the residence node, are visited more than once. We assume that travelers taking such journeys are making stops that do not allow free transfers, so they are charged one dollar per loop.

The cost of a car journey is given by

$$m_j = \frac{1}{1.5} (m^F + m^D + m^T t_j), \quad (5.5)$$

in which  $m^F$  is the fixed cost of owning a car and is assumed to be 5.00 dollars/day. This value has been chosen to account for the cost of purchasing a car as well as other fixed costs such as insurance. The divisor 1.5 is the average car occupancy and reflects the occupants sharing the car's costs. The other terms are due to the empirical findings of Evans and Herman (1978) who found that the fuel cost of a car is linear in the distance and the time it travels. We have increased their coefficients to account for other running costs such as repairs and maintenance. We assume  $m^D$  is 0.0672 dollars per kilometer times the length of a journey, and that  $m^T$  is 2.322 dollars per hour.



## 5.2 Basic Network Results

Table 5.1 lists summary results for this network. Note that more wealthy people take cars than buses; more poor people take buses. More wealthy people travel. Note that car behavior is similar for all people who take cars, and bus behavior is the same for all bus riders; the major difference in behavior is due to the different fractions of each class who take each mode.

Note that more people travel in the inner city (i.e., more of those who originate at nodes 40, 45, 100, and 105) and more stay at home in the suburbs.

Wealthy people gain more benefits, i.e., utility, from travel, than poor people. They spend more money, and more of them take cars. Poor people spend more time traveling. Note that all times and speeds are door-to-door times and speeds.

## 5.3 Fare Change

Table 5.2 summarizes the behavior of the network when bus fares are raised by 50 percent. Bus riders are profoundly affected, particularly those of lower income. Car commuters are almost unaffected. It is important to see that, in this network, when fares are raised, people do not switch from buses to cars (except for a very small number of the poor). Instead, they switch from traveling in buses to not traveling at all.

It should be noted that a crucial modeling assumption may have had a profound effect on this result. Travelers are assumed to be independent, with independent time and money budgets. In reality, members of a household may share the money available to the entire household. If this is the case, increasing bus fare may have little effect on bus travel in households where some members travel by car. Instead it may increase the expenditures of the bus traveling members which reduces the funds available for the car travelers. Thus, it may decrease car travel.

Table 5.1: Summary of Basic Network Results

Class	Number of Travelers	Distance Per Traveler, km	Time Per Traveler, hours	Money Per Traveler, dollars	Utility Per Traveler,	Velocity Per Traveler, km/h	Mode
Wealthy	76,700	25.39	0.80	6.19	45.56	31.89	Auto
Poor	20,609	15.63	0.44	4.91	27.94	35.22	
Wealthy	35,453	16.75	1.47	1.47	34.62	11.37	Bus
Poor	57,535	16.24	1.41	1.39	32.93	11.49	
Wealthy	112,153	22.65	1.01	4.70	42.10	22.44	Both
Poor	78,143	16.08	1.16	2.32	31.61	13.89	

Class	Traveling	Fraction by Car	Fraction by Bus	Area
Wealthy	0.87	0.62	0.38	inner city
Poor	0.71	0.21	0.79	
Wealthy	0.71	0.71	0.29	outer city
Poor	0.46	0.29	0.71	

Table 5.2: Summary of Effects of Fare Change

Class	Number of Travelers	Distance Per Traveler, km	Time Per Traveler, hours	Money Per Traveler, dollars	Utility Per Traveler,	Velocity Per Traveler, km/h	Mode
Wealthy	76,606	25.43	0.79	6.19	45.68	31.99	Auto
Poor	20,539	15.65	0.44	4.91	28.02	35.35	
Wealthy	31,161	16.66	1.47	2.10	34.40	11.30	Bus
Poor	48,165	16.08	1.41	1.95	32.43	11.43	
Wealthy	107,766	22.89	0.99	5.01	42.42	23.10	Both
Poor	68,704	15.95	1.12	2.83	31.11	14.26	

#### 5.4 Tolls

An experiment has been run to assess the effect of adding tolls on certain links entering the inner city. The intention is to discourage the use of cars in the inner city, so only cars are subject to the fee. The links are 45-44, 100-101, 40-52, 105-93, and the amount charged is 1.00 dollar.

As Table 5.3 indicates; the desired effect is achieved. Fewer people travel by car, and those who do drive tend to drive less. The effect of tolls is greater on the wealthy than on the poor since it is the wealthy that drive more. The average daily money expenditure of wealthy car drivers goes up, while that of all others is nearly constant. The automotive travel of the poorer people who do use cars, however, is affected much more than that of the wealthier. Bus ridership increases enough so that there is only a small decrease in total travel.

#### 5.5 Prohibitions

Table 5.4 displays the effects of forbidding cars from traveling on certain links. The links are the same as those studied in Section 5.4. (In fact, the method used is to impose a 50 dollar toll on those links, which very effectively limits their use.)

The effect of prohibiting travel is only a little greater than that of imposing tolls on the same links. Most of the numbers in Table 5.4 are quite close to those of Table 5.3. One significant difference is the average daily expenditure of wealthy car riders. When a 1.00 dollar toll was imposed, enough of them were willing to pay it that the expenditure went up. However, a 50.00 dollar toll was unreasonable and they refused. Because they lost opportunities to spend their money, their expenditure decreased.

It seems reasonable to conclude that as the toll increases from zero (the base case in Section 5.2) the expenditure of wealthy car riders will increase, reach a maximum, and decrease. The maximum expenditure is limited by the budget distributions. Additional study is required to verify this conclusion and to make it precise.

Table 5.3 Effects of Tolls

Class	Number of Travelers	Distance Per Traveler, km	Time Per Traveler, hours	Money Per Traveler, dollars	Utility Per Traveler,	Velocity Per Traveler, km/h	Mode
Wealthy	73,609	25.27	0.77	6.66	44.29	32.78	Auto
Poor	18,615	16.61	0.42	4.93	26.08	39.24	
Wealthy	39,043	16.78	1.47	1.50	35.55	11.43	Bus
Poor	60,337	16.16	1.39	1.39	33.27	11.61	
Wealthy	112,652	22.33	1.01	4.87	41.26	22.05	Both
Poor	78,952	16.26	1.16	2.23	31.57	13.98	

Table 5.4 Link Prohibitions

Class	Number of Travelers	Distance Per Traveler, km	Time Per Traveler, hours	Money Per Traveler, dollars	Utility Per Traveler,	Velocity Per Traveler, km/h	Mode
Wealthy	72,707	25.27	0.76	6.08	40.67	33.32	Auto
Poor	18,323	16.43	0.43	4.87	25.33	38.13	
Wealthy	39,990	16.84	1.47	1.52	36.02	11.47	Bus
Poor	60,923	16.17	1.38	1.39	33.45	11.68	
Wealthy	112,697	22.28	1.01	4.46	39.02	22.05	Both
Poor	79,246	16.23	1.16	2.20	31.62	13.94	

## 5.6 Computer Costs

The results described here were calculated using the MIT Multics timesharing system on a Honeywell 68/DPS computer. Typical runs required 5-8 iterations when executed from scratch, i.e. when the initial flows were set to 0. The number of iterations depended on the value of  $\lambda$ . Values of  $\lambda$  in the range of .5-.8 were used. Variational runs, such as those in Sections 5.3-5.5 typically required only 1-2 iterations. Such runs were executed with  $\lambda$  near .9.

A typical run took 40 cpu seconds for initialization and post-processing and 22 seconds per iteration.

Memory requirements are dominated by the number and size of journeys. For this example, approximately 110 kilobytes were required.

Considering the substantial size of the network, these costs are remarkably small. Four-fold symmetry was exploited in exploring this example but we see no obstacle to the study of larger unsymmetric networks or the implementation of these results on a smaller computer.

## 6. CONCLUSIONS AND FURTHER RESEARCH

The demand/distribution/assignment principles described in Section 2 and the algorithm presented in Section 3 promise to be valuable tools for the understanding and prediction of travel behavior in a city. Additional research can enhance the usefulness of these tools. This research is of three types: modeling, analytical, and empirical.

### 6.1 Modeling

#### 6.1.1 Time-stratified Travel

As Stephenson and Teply (1981) show, changes in network conditions can lead to changes in the time of day that people travel. Clarke et al (1981) demonstrate that travel decisions vary during the course of the day and depend on the scheduling of events such as work, school, and shopping hours. Consequently, the variation of the distribution of traffic in a network as a function of hour of the day can be important.

In principle, this can be treated using the approach introduced here. Journeys now become paths in space-time and not merely in space. That is, to describe a journey, one must not only list the links and nodes that the journey passes through, but the time of day that each link and node is reached. Two journeys that pass through the same points in the network are now considered different if they reach those points at different times. Utilities may be assigned according to when a visit to a node or link takes place. (This is important: food shopping after work is worth much more than shopping before work if there is no intervening stop at home). Link costs vary with the time of day according to the time-varying congestion.

The difficulty is that the number of possible journeys increases enormously. A formulation is required that summarizes the effects of the availability of an infinite number of travel alternatives without enumerating.

#### 6.1.2 Subjective Utility Ranking

In the present model, the number of people in a class that choose a journey depends on the journey's costs and on the desirability ranking of all the journeys. The utility values of the journeys are not considered, except to determine the ranking.



This is reasonable when journeys are not similar. However, when they are nearly the same, we can expect that different people will have different opinions on relative rankings. Therefore, the model should be changed to represent the effects of having journeys with nearly equal utilities. One approach is to incorporate into the density function  $f^a(\dots)$  an indication of ranking. That is, if there are  $k$  journeys available to class  $a$  let  $\pi$  be a permutation of  $\{1, \dots, k\}$ , which represents a possible ranking of the journeys. Then let

$$D^a f^a(u, n, \pi) du dn,$$

be the number of people in class  $a$  with time and money costs between  $u$  and  $u + du$  and  $n$  and  $n + dn$ , respectively, and who have preference order  $\pi$ . The analysis that has led to equations (2.20) to (2.22) must be repeated for each possible preference order. If, as we suspect, there are not many likely permutations, the increase in complexity that this causes will be limited and manageable.

### 6.1.3 Probabilistic Preference Modeling

It may be argued that preferences are almost never as clear cut as we have modeled them here. That is, there is rarely a universally agreed preference order among travel alternatives, even among those of the same socio-economic class.

In that case, a better behavior principle may be the following. For class  $a$  and path  $i$ , let  $\Pi_i^a$  be a probability that depends on  $W_i$ , the utility of path  $i$ , in a known way. For example,  $\Pi_i^a$  may be proportional to  $W_i$ . Let the probability that a given traveler of class  $a$  who can afford to take path  $i$  be equal to  $\Pi_i^a$ .

### 6.1.4 Correlations Within Households

We have modeled all individuals as being completely independent. In fact, members of a household share resources and responsibilities. For example, they share income so a reduction in one member's expenditure may lead to an increase in another's budget. In addition, if one member goes to a supermarket on a given day, others are less likely to go to a supermarket. On the other hand, if one

member goes to a movie, others are more likely to.

Incorporating this correlation may have an important effect on network results. For example, as pointed out above, reducing bus fares may increase car travel, since the amount of bus travel is likely to be unchanged and more money is available for car travelers.

#### 6.1.5 Residential Choice

We have considered only some of the choices available to travelers that affect their travel behavior. In particular, we have not allowed them the freedom to choose their residences; we assume that their residential locations are fixed. A more realistic and comprehensive model would include a representation of residential choice.

The choice of residential location is greatly influenced by the transportation system: when a household considers a new residence, it carefully considers how much access it has to locations of work, shopping, and where other needs are fulfilled. Other factors must be considered as well, such as housing cost. Housing cost itself is determined in part by these same transportation considerations.

It is desirable to include residential choice in the model presented here to assess long-term effects of changes in the transportation system. To do this, the following modifications of the method must be considered:

- a. Classes no longer have fixed origins but rather can choose journeys anywhere in the network. Some of the nodes on each resulting journey become residential locations.
- b. A precise model of individual behavior incorporating both residential and travel decisions is required. This may be an extension of (2.7) to (2.9). It is hoped that the utility can be extended to include the intrinsic utility (i.e., independent of transportation considerations) of a house or apartment, and that the money budget can be enlarged to include housing as well as travel costs. It is also

hoped that such a model can be transformed into a principle that describes flows and demands for housing which is analogous to (2.20) to (2.22).

- c. A model of housing cost is required. The cost of owning or renting a house or apartment depends on its physical characteristics (such as its size, the size of the plot of land it is on, its state of repair), and on market factors such as interest rates. It also depends on the demand for residences in its vicinity which in turn depends on transportation considerations. These modifications will extend the model described here. Travelers will be represented as making both travel decisions and residence decisions. Both phenomena must be considered simultaneously if long-term changes in urban structure are to be understood.

## 5.2 Analytical Research

A major limitation to the new assignment principle is in its computational complexity and large data base requirements when large networks are treated. Most important, as the network grows, the number of journeys (possible journeys as well as those actually used) increases. The lengths of these journeys, measured in the number nodes passed through, may also grow. In the example reported here, the journeys were all generated manually. In the large network, simple loops were generated by hand and the computer examined all concatenations to form journeys. An efficient exact or heuristic method is required to generate journeys in large networks.

The computational difficulties are exacerbated by some of the extensions described above. Further research to accelerate the equilibration process, or to replace it altogether, will help mitigate these difficulties. Of greater value will be a procedure for aggregating the network so that a reduced network can be used which will produce approximately equivalent results.

There are important qualitative issues that must be understood, such as the number of equilibria. We have briefly touched on the effect of the shape of the budget distributions. Other sensitivity issues, such as the effect of the shape of the link delay functions should be considered.

If the full benefits of symmetry are used, very large networks can be treated, particularly concentric circle networks, such as in Figure 4.7. Although such studies will have no direct application, they will be useful for the study of the behavior of travelers in large abstract city networks. That is, general principles for understanding transportation systems can be developed by considering such special cases.

Optimization techniques based on this model are required. Such techniques will seek the best of a class of network modifications or the best of a class of control policies. They will minimize such costs as total funds expended, aggregate delay, energy consumption, or maximize such objectives as mobility while taking into account travelers' responses to the change. These optimization techniques will be of great value to system planners and managers.

### 6.3 Empirical Research

Various assumptions have been made in formulating the examples. Most important may be the assumption that utility is additive along journeys. Such assumptions must be tested and verified or modified.

A study can be undertaken to apply the methods described here to a real city. The city, of course, should be small and well-documented in order to assess the predictive power of the model.

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