

DIFFUSION APPROXIMATIONS OF TRANSFER LINES WITH
UNRELIABLE MACHINES AND FINITE STORAGE ELEMENTS

by

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ABSTRACT

In this paper, we develop an approximate model for the flow of parts in a transfer line with unreliable machines. Using a stochastic model, we establish that the normalized flow converges weakly to a diffusion process in a bounded domain as the storage capacities increase. This diffusion process is reflected at oblique directions on the boundary. We develop a strong sample-path characterization of the boundary process of this reflected diffusion, and use this characterization to establish weak convergence of the local time processes. The approximation results are shown to be consistent with exact analytical results for two machine transfer lines. We use the approximate model to develop equations which describe the ergodic distribution and the average lost production for a three machine transfer line.

1. INTRODUCTION

An important class of systems which arises in manufacturing, chemical processes, computer networks and power systems, is where material moves through a network of unreliable links between storage stations. Transfer lines are networks where all of the storage stations are arranged sequentially; figure 1 describes a typical line network. The presence of storage stations serves to compensate for link failures by maintaining the flow upstream and downstream of a failure, thereby decreasing the effect of a failure on the rest of the network. When the operation of a link is modeled as a random process, exact analysis of the flow of material is a difficult task. In this research we develop an aggregate model of the flow through the network based on the physical assumption that the storage capacities are large but finite. This aggregate model is developed as the limit of a sequence of probabilistic models for the flow of material through the line network. Based on this aggregate model, we can approximate properties of the long-term behavior of the line network. Although storage capacities are assumed large, saturation of individual storage stations occurs and is considered in the method here.

Analytical studies of line networks using a probabilistic approach were first studied by Vladzievskii (1952). A number of authors have studied the flow rates of lines with storages of infinite capacity; some of these are Hunt [1956], Suzuki [1964], Barlow and Proschan [1975]. Unreliable line networks with one storage station have been studied by a number of authors (Buzacott and Hanifin [1978], Gershwin and Schick [1980a], Gershwin and Berman [1981]). These papers have bibliographies of work in this area.

Systems with more storage stations are difficult to analyze because of the

complexity of interfaces when storage are either full or empty. For some special systems, Soyster, Schmidt and Rohrer [1979] have obtained exact probabilistic analysis of networks with more than one storage. Gershwin and Schick's results [1980b] are more general, but still limited. Nevertheless, exact analysis of networks with more than one storage is a difficult computational task.

The aggregate model described in this paper is established as a consistent long-term approximation by verifying that an exact model based on the formulation of Gershwin and Schick [1980b] converges weakly to the aggregate model in a probabilistic sense. For a discussion of weak convergence of probabilistic measures, the reader should consult Billingsley [1968]. The arguments of convergence depend heavily on the averaging results of Khasminskii [1966a,b].

The aggregate model obtained in this paper is a diffusion process. Diffusion approximations in queueing networks have been studied by a number of authors, notably Borovkov [1965], Iglehart and Whitt [1970], Kobayashi [1974], Reiman [1977], Burman [1979] and Harrison [1978]. Although queueing networks feature storages of infinite capacity, many of the techniques used in the analysis of these networks are used here. In particular, the construction of reflected Brownian motion in Harrison and Reiman [1979] provides a valuable introduction to these results.

2. MATHEMATICAL MODEL OF MATERIAL FLOW

In this paper, we will assume that individual objects are of infinitesimal size, so that the flow of objects through a network is a continuous variable. Using the diagram of figure 1 as reference, objects flow from an

infinite source to an infinite sink across storage stations and unreliable links. The failure and repair processes of the links are assumed to be independent jump processes with constant failure and repair rates. It is also assumed that there is no creation or destruction of objects in the line.

Let x_i , $i=1, \dots, k-1$ denote the amount of material in storage element i . Let α_j , $j=1, \dots, k$ denote the state of the link preceding storage element j . The variable α_j can take two values, 1 or 0, indicating respectively that link j is operating or not. By assumption, α_j is a random process, with transition probabilities

$$\text{Prob}\{\alpha_j(t+\Delta) = 1 \mid \alpha_j(t) = 0\} = r_j\Delta + o(\Delta) \quad (2.1)$$

$$\text{Prob}\{\alpha_j(t+\Delta) = 0 \mid \alpha_j(t) = 1\} = p_j\Delta + o(\Delta)$$

From the theory of representation of jump processes (Davis [1976]), we can describe α_j by a stochastic differential equation driven by Poisson processes. Thus, one obtains

$$d\alpha_j(t) = -\alpha_j(t)dF_j(t) + (1 - \alpha_j(t))dR_j(t) \quad (2.2)$$

where F_j , R_j , F_i are independent Poisson processes with transition rates p_j , r_j , p_i for any j, i .

Let N_j denote the capacity of storage j . Denote by N_j the flow capacity on link j . The flow rate is assumed to be of maximum capacity whenever possible. Since no objects are created or destroyed, we can describe the storage process by the differential equation

$$\frac{dx_i}{dt} = \mu_i \alpha_i - \mu_{i+1} \alpha_{i+1} ; \quad 0 < x_i < N_i \quad (2.3)$$

$$i=1, \dots, k-1$$

Define the vectors $\underline{x} = (x_1, \dots, x_k)^T$, $\underline{\alpha} = (\alpha_0, \dots, \alpha_k)^T$ as the state of the system. Let $s = (\underline{x}, \underline{\alpha})$. Equations (2.2) and (2.3) provide a system of stochastic differential equations which describes the evolution of the probabilistic state $s(t)$ whenever all of the storage elements are away from their limits. However, when a storage element is either empty or full, equation (2.3) must be modified so that conservation of flow through the line network applies.

Consider the situation when storage i becomes full. Then, equation (2.3) must become

$$\frac{dx_i}{dt} \leq 0 \quad (2.4)$$

Since the storage element filled up, the incoming flow must be reduced to match the outgoing flow. That is, the rate μ_i is modified so that

$$\mu_i' \alpha_i \leq \mu_{i+1} \alpha_{i+1} \quad (2.5)$$

This implies

$$\mu_i' \leq \mu_{i+1} \alpha_{i+1} \quad \text{if } \alpha_{i+1} = 1$$

Consequently

$$\mu_i' = \min(\mu_i, \mu_{i+1} \alpha_{i+1}) \quad (2.6)$$

if $\alpha_i=1$ and $x_i=N_i$.

Notice that $\alpha_i(t)$ cannot equal 0 if storage i just fills up.

When storage i empties, the outgoing flow b_{i+1} must be reduced to match the incoming flow. That is,

$$\mu'_{i+1} = \min(\mu_{i+1}, \mu_i \alpha_i) \quad (2.7)$$

when $\alpha_{i+1} = 1$ and $x_i = 0$. Note that α_{i+1} is not zero when storage i empties.

When more complex combinations of full and empty storages occur, new production rates are defined to enforce conservation of flow. The full stochastic differential equations for the x process is given by

$$\frac{dx_i}{dt} = \mu_i(s)\alpha_i - \mu_{i+1}(s)\alpha_{i+1} \quad (2.8)$$

where $\mu_i(s)$ satisfies the boundary conditions described by equations (2.6),

(2.7) and their extensions to higher order cases. These extensions are discussed in greater detail in section 5.

3. SCALING

In order to develop an aggregate model of the system, we will assume that all of the storage capacities are large. Mathematically, we assume

$$N_i = \frac{B_i}{\epsilon}, \quad i=1, \dots, k-1 \quad (3.1)$$

for some small ϵ , and constants B_i . Without loss of generality, we will assume that all B_i are equal to 1. Otherwise we can introduce constants to keep track of the relative scaling. Define a scaled variable $y_i(t)$ as the fraction of storage used:

$$y_i(t) = \frac{x_i(t)}{N_i} \quad (3.2)$$

thus, equation (2.8) becomes

$$N_i \frac{d}{dt} y_i = \alpha_i \mu_i(s) - \alpha_{i+1} \mu_{i+1}(s) \quad (3.3)$$

Equation (3.3) represents a random evolution for the $\underline{y}(t)$ process, with a discontinuity in drift when the process exits the open domain $D = (0,1)^{k-1}$. Aggregation of random evolutions has been studied by a number of authors; Hersh [1975] has compiled a comprehensive survey of the work in that area. However, none of that work can incorporate the local discontinuity of the drift as the process reaches the boundary.

The process $\underline{y}(t)$ has coordinates with values between 0 and 1, representing the fraction of capacity used in storage. The boundary effects described in section 2 will occur whenever one of the coordinates of $\underline{y}(t)$ is either 0 or 1. Let γ denote the time of first exit of the $\underline{y}(t)$ process from its interior. That is,

$$\gamma(\omega) = \inf\{t > 0 \mid y(t,\omega) \notin D\}$$

We will develop an approximation to the $\underline{y}(t)$ process until its time of first exit from the domain D .

Denote by $\underline{z}(t)$ the process in \mathbb{R}^{k-1} whose evolution described by

$$N_i \frac{dz_i}{dt} = \alpha_i \mu_i - \alpha_{i+1} \mu_{i+1} \quad (3.4)$$

$$z_i(0) = y_i(0)$$

where μ_i are the constant flow rates when y is in D .

Note that the sample paths of the $\underline{z}(t)$ process agree with the sample

paths of the $\underline{y}(t)$ process until time $\gamma(w)$. The process $\underline{z}(t)$ represents the evolution of the normalized storage process if no boundary adjustments were made.

Define τ as ϵt . In this time scale equation (3.4) becomes

$$\frac{dz_i}{d\tau}(\tau) = \alpha_i \mu_i - \alpha_{i+1} \mu_{i+1} \quad (3.5)$$

Notice that the Markov process $(\underline{z}(\tau), \underline{\alpha}(\tau))$ has components varying in two different time scales. The $\underline{z}(\tau)$ process has variations on the slow scale τ , and the $\underline{\alpha}(\tau)$ process has transitions in the t scale. This separation of scales is a consequence of the assumption that N_i is large, and will be exploited to obtain aggregate models. In the next section, we will establish that the process $\underline{z}(\tau)$ can be approximated by a Markov process which does not depend on the jump process $\underline{\alpha}(\tau)$; this approximation can be used in computing expectations of the process $\underline{z}(\tau)$.

4. AGGREGATION

The $\underline{\alpha}(t)$ process described in equation (2.2) is a jump process with a finite number of states. Each of the components has independent transitions, and is strongly ergodic. The ergodic measure of the j th component is

$$\bar{P}_j(\alpha_j) = \frac{(1-\alpha_j)p_j + \alpha_j r_j}{p_j + r_j} \quad (4.1)$$

The overall ergodic measure is given by

$$\bar{P}(\underline{\alpha}) = \prod_{j=1}^k \bar{P}_j(\alpha_j) \quad (4.2)$$

As the parameter ϵ approaches zero, the separation between the time

scales τ and t increases. Hence, more transitions of the $\underline{\alpha}$ process occur between significant changes in the $\underline{y}(\tau)$ process. One would expect that a good approximation for the evolution of the $\underline{z}(\tau)$ process would be provided by the expected drift, in terms of the ergodic measure of the $\underline{\alpha}(t)$ process. This result is established in this section.

Define the average drift \bar{F}_i as

$$\bar{F}_i = \sum_{\underline{\alpha}} (\alpha_i \mu_i - \alpha_{i+1} \mu_{i+1}) \bar{P}(\underline{\alpha}) \quad (4.3)$$

Combining equations (4.1) and (4.3) yields

$$\bar{F}_i = \frac{r_i \mu_i}{r_i + p_i} - \frac{r_{i+1} \mu_{i+1}}{r_{i+1} + p_{i+1}} \quad (4.4)$$

Define $\underline{z}^0(\tau)$ as

$$z_i^0(\tau) = z_i(0) + \bar{F}_i \tau \quad (4.5)$$

The processes $\underline{z}^0(\tau)$ represents the average evolution of the $\underline{z}(\tau)$ process. The next results specify the accuracy of this approximation.

Theorem 4.1. Let T be an arbitrary finite positive number. Consider the processes $\underline{z}(\tau)$ and $\underline{z}^0(\tau)$, $0 \leq \tau \leq T$. As $\epsilon \rightarrow 0$, the process $\underline{z}(\cdot)$ converges uniformly in the mean to \underline{z}^0 . That is,

$$\lim_{\epsilon \rightarrow 0} \sup_{0 \leq \tau \leq T} E\{|\underline{z}(\tau) - \underline{z}^0(\tau)|\} = 0 \quad (4.6)$$

Proof. The proof is a straightforward application of Theorem 1.1 of Khasminskii (1966).

The fact that the rates μ_i are constant enables us to establish a stronger result than uniform convergence in the mean. We can establish that $\underline{z}(\cdot)$ converges to $\underline{z}^0(\cdot)$ almost surely, and examine the distribution of its deviations.

Theorem 4.2. Under the conditions of Theorem 4.1, the process $\underline{z}(\cdot)$ converges to the process $\underline{z}^0(\cdot)$ almost surely as $\varepsilon \rightarrow 0$. Furthermore, let

$$v_i(\tau) = \frac{1}{\sqrt{\varepsilon}} (z_i(\tau) - z_i^0(\tau)) \quad (4.7)$$

The process $\underline{v}(\tau)$ converges weakly to a zero-mean Wiener process \underline{w} with covariance

$$\begin{aligned} E \underline{w}(\tau) \underline{w}^T(s) &= \underline{\Sigma} \min(\tau, s) \\ \Sigma_{ii} &= 2 \left\{ \frac{\mu_i^2 p_i r_i}{(p_i + r_i)^3} + \frac{\mu_{i+1}^2 p_{i+1} r_{i+1}}{(p_{i+1} + r_{i+1})^3} \right\} \\ \Sigma_{i+1,i} &= \Sigma_{i,i+1} = \frac{-2\mu_{i+1}^2 p_{i+1} r_{i+1}}{(p_{i+1} + r_{i+1})^3} \\ \Sigma_{ij} &= 0, \quad |i-j| \geq 2 \end{aligned} \quad (4.8)$$

Proof. The proof is included in the appendix. The weak convergence of the $\underline{v}(\tau)$ process is a direct result of Khasminskii [1966], Theorem 3.1.

Theorems 4.1 and 4.2 define aggregate models for the evolution of the $\underline{z}(\tau)$ process, independent of the $\underline{q}(\tau)$ process. These aggregate models are established as consistent by the convergence of the true process as $\varepsilon \rightarrow 0$.

The models are developed in the slow time scale $\tau = \epsilon t$; they are most useful when the line network is unbalanced in the mean. That is, when the average drift in the system, \bar{F} , is of order 1.

When all of the drifts in the system, \bar{F}_i , are of order ϵ , the approximation given by equation (4.5) is not of much use, because no significant trends occur in times of order $1/\epsilon$. Such cases are referred to as balanced line networks. However, in a still slower time scale, an aggregate model can be obtained.

Let $\tau_1 = \epsilon^2 t$ be a slow time scale. In the τ_1 scale, equation (3.4) becomes

$$\frac{d}{d\tau_1} z_i(\tau_1) = \frac{-\alpha_{i+1}(\tau_1)\mu_{i+1} + \alpha_i(\tau_1)\mu_i}{\epsilon} \quad (4.10)$$

Assume additionally that

$$\bar{F}_i = \epsilon f_i, \quad i = 1, \dots, k-1 \quad (4.11)$$

Then, we can write (4.10) as

$$\frac{d}{d\tau_1} z_i = \frac{\alpha_i \mu_i - \alpha_{i+1} \mu_{i+1} - \epsilon f_i}{\epsilon} + f_i \quad (4.12)$$

Let Q denote the infinitesimal generator of the Markov process $\underline{\alpha}(t)$. The operator Q can be viewed as a singular matrix mapping $R^{2^k} \rightarrow R^{2^k}$. Denote vectors in R^{2^k} by the functions $g(\underline{\alpha})$. Suppose that

$$g_i(\underline{\alpha}) = -\alpha_{i+1}\mu_{i+1} + \alpha_i\mu_i - \epsilon f_i$$

$$h_i(\underline{\alpha}) = \frac{\mu_i \alpha_i}{r_i + p_i} - \frac{\mu_{i+1} \alpha_{i+1}}{r_{i+1} + p_{i+1}}$$

By its definition, the matrix Q can be expressed as

$$\begin{aligned}
 Qh(\underline{\alpha}) &= \sum_{\substack{i \\ \alpha_i=0}} r_i (h(\underline{\alpha}_i^c) - h(\underline{\alpha})) \\
 &+ \sum_{\substack{i \\ \alpha_i=1}} p_i (h(\underline{\alpha}_i^c) - h(\underline{\alpha})) \\
 &= \sum_j \{r_j(1-\alpha_j) + p_j\alpha_j\} \{h(\underline{\alpha}_j^c) - h(\underline{\alpha})\}
 \end{aligned}$$

where

$$\underline{\alpha}_j^c = (\alpha_1, \dots, \alpha_{i-1}, 1-\alpha_i, \alpha_{i+1}, \dots, \alpha_k)$$

Then,

$$\begin{aligned}
 Qh_i(\underline{\alpha}) &= (r_i(1-\alpha_i) + p_i\alpha_i) \left(\frac{\mu_i(1-\alpha_i)}{r_i + p_i} - \frac{\mu_i\alpha_i}{r_i + p_i} \right) \\
 &- \frac{(r_{i+1}(1-\alpha_{i+1}) + p_{i+1}\alpha_{i+1})}{\left\{ \frac{\mu_{i+1}(1-\alpha_{i+1})}{r_{i+1} + p_{i+1}} - \frac{\mu_{i+1}\alpha_{i+1}}{r_{i+1} + p_{i+1}} \right\}} \\
 &= \frac{\mu_i r_i}{r_i + p_i} - \mu_i \alpha_i + \mu_{i+1} \alpha_{i+1} - \frac{\mu_{i+1} r_{i+1}}{r_{i+1} + p_{i+1}} \\
 &= -g_i(\underline{\alpha})
 \end{aligned}$$

Consider now an arbitrary bounded function $h(\underline{z})$ in $C^2(\mathbb{R}^{k-1})$, the space of real valued, twice continuously differentiable functions of \mathbb{R}^{k-1} . Denote by L the infinitesimal generator of the Markov process $(\underline{z}, \underline{\alpha})$ in the τ_1 time scale. Then

$$L = \frac{Q}{\varepsilon^2} + \frac{1}{\varepsilon} \sum_{i=1}^{k-1} (\alpha_i \mu_i - \alpha_{i+1} \mu_{i+1} - \varepsilon f_i) \frac{\partial}{\partial z_i} + \sum_{i=1}^{k-1} f_i \frac{\partial}{\partial z_i}$$

Let \bar{L} denote the diffusion operator

$$\bar{L} = \sum_{i=1}^{k-1} f_i \frac{\partial}{\partial z_i} + \frac{1}{2} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \Sigma_{ij} \frac{\partial^2}{\partial z_i \partial z_j} \quad (4.13)$$

where Σ is defined in (4.8) and (4.9).

Notice that \bar{L} is the generator of a pathwise unique strong Markov process in \mathbb{R}^{k-1} (Stroock-Varadhan [1979]).

Theorem 4.3 When the line network is nearly balanced, the process $\underline{z}(\tau_1)$, $0 \leq \tau_1 \leq T$, for arbitrary finite T , converges weakly as $\varepsilon \rightarrow 0$ to the unique diffusion Markov process \underline{v} whose infinitesimal generator is \bar{L} . Moreover, all the moments of \underline{z} converge to the moments of \underline{v} as $\varepsilon \rightarrow 0$.

The proof of these results is a direct application of Theorem 1 in Papanicolaou-Kohler, [1974] because the $\underline{\alpha}$ process is ergodic, hence it is strongly mixing.

5. DIFFUSION APPROXIMATIONS WITH BOUNDARY CONDITIONS

The results of section 4 provide an approximation to the normalized storage process $\underline{y}(t)$ until its time of first exit from the interior of the region D . In this section, those approximations will be extended to cover arbitrary intervals of time. In this case, the boundary conditions described in section 2 have to be explicitly considered.

Consider the process $\underline{z}(\tau)$ defined in equation (3.5). Define the compensating processes $C_0(t, z)$, $C_1(t, z)$ for any continuous real valued function z as

$$C_0(t, z) = \min_{0 \leq s < t} \{0, z(s)\} \quad (5.1)$$

$$C_1(t, z) = \max_{0 \leq s < t} \{0, z(s) - 1\} \quad (5.2)$$

The functions $C_0(t, z)$ and $C_1(t, z)$ represent the excesses of the function $z(t)$ outside the interval $[0, 1]$. Hence, for any function $z(t)$, we can define the compensated function $z^1(t)$ as

$$z^1(t) = z(t) - C_0(t, z) - C_1(t, z) \quad (5.3)$$

The function $z^1(t)$ does not take its values in the unit interval, because the effect of two compensating processes drive the new function outside. However, one can define a sequence of functions $z^j(t)$ inductively as

$$z^{j+1}(t) = z^j(t) - C_0(t, z^j(t)) - C_1(t, z^j(t)) \quad (5.4)$$

For any bounded interval $[0, T]$, and any continuous function $z(t)$ on $[0, T]$, $z_j(t)$ is a continuous function.

Consider the process $\underline{z}(t)$ defined in section 2. The failure-repair process $\underline{\alpha}(t)$ is a Markov jump process which describes the evolution of $\underline{z}(t)$. Since the rates of evolution of $\underline{z}(t)$ are constant except for the effects of $\underline{\alpha}$, the probabilistic distribution of increments of $\underline{z}(t)$ is independent of the value of $\underline{z}(t)$; that is,

$$\begin{aligned} \Pr\{\underline{z}(t+\Delta) - \underline{z}(t) \in B \mid \underline{z}(t), \underline{\alpha}(t)\} &= \\ \Pr\{\underline{z}(t+\Delta) - \underline{z}(t) \in B \mid \underline{\alpha}(t)\} & \end{aligned}$$

The process $\underline{y}(t)$ has a similar property, except for the effects of the

boundary conditions. We would like to incorporate the effects of these boundary conditions as compensating processes, in the manner of equations (5.3) and (5.4). This is the purpose of the next result.

Consider an arbitrary sample path $\underline{z}(t)$, $t \in [0, T]$. Define the sequence of times t_i as

$$\begin{aligned} t_0 &= \inf\{t \mid C_0(t, z_j) \neq 0 \text{ or } C_1(t, z_j) \neq 0 \text{ for some } j\} \\ t_i &= \inf\{t \mid C_0(t, z_j^i) \neq 0 \text{ or } C_1(t, z_j^i) \neq 0 \text{ for some } j\} \end{aligned} \quad (5.5)$$

Assume that, at time t_0 , the trajectory $\underline{z}(\cdot)$ must be compensated or else it will leave \bar{D} . Let us consider the effect of compensation. If $C_1(t, z_j) > 0$ for $t < t_0$, we need to define

$$z_j^1(t) = z_j(t) - C_1(t, z_j)$$

in order to maintain $z_j^1(t)$ in \bar{D} . The compensator $C_1(t, z_j)$ represents the excess flow which is blocked due to the capacity of storage j . This excess flow must accumulate in the previous storage. That is,

$$z_{j-1}^1(t) = z_{j-1}(t) + C_1(t, z_j)$$

Similarly, if $C_0(t, z_j) < 0$, then

$$z_j^1(t) = z_j(t) - C_0(t, z_j)$$

$$z_{j+1}^1(t) = z_{j+1}(t) + C_0(t, z_j)$$

For more complicated boundary conditions, when more than one storage level is on the boundary, we proceed to the general construction.

The times t_i represent times when the compensated processes \underline{z}^i would require additional compensation to stay in \bar{D} . Now, define an integer valued function on the time sequence t_i as

$$n(t_i) = \max_{1 \leq j \leq k-1} \{j \mid z_j^i(t_i) = 1 \text{ and } C_1(t, z_j^i) > 0, t > t_i\} \quad (5.6a)$$

If the set of such indices j is empty, let $n(t_i)$ be

$$n(t_i) = \min_{1 \leq j \leq k-1} \{k-1+j \mid z_j^i(t_i) = 0 \text{ and } C_0(t, z_j^i) < 0, t > t_i\} \quad (5.6b)$$

Notice that, if only one storage level reaches the boundary at time t_i , then $n(t_i)$ identifies that storage, and indicates whether it is empty or full. Whenever two or more storage levels reach the boundary simultaneously at time t_i , the function $n(t_i)$ selects a storage by the following rule:

Select the storage which saturated farthest downstream. If there is no storage which is saturated, then select the storage which emptied farthest upstream.

This selection rule serves to ensure that the compensation process at any one time requires no more than $2k$ iterations. This is because the effects of saturation propagate upstream, whereas the effects of starvation propagate downstream.

We can now define a sequence of compensated functions \underline{z}^i recursively, as

$$\begin{aligned} \underline{z}^0(t) &= \underline{z}(t) \\ z_j^{i+1}(t) &= z_j^i(t) - C_0(t, z_j^i) I\{n(t_i) = k-1+j\} \\ &\quad - C_i(t, z_j^i) I\{n(t_i) = j\} + C_0(t, z_{j-1}^i) I\{n(t_i) = k+j-z, j \neq 1\} \\ &\quad + C_1(t, z_{j+1}^i) I\{n(t_i) = j+1, j \neq k-1\} \end{aligned} \quad (5.7)$$

Equation (5.7) expresses the conservation of flow equations throughout most of the domain \bar{D} . Notice that, because of our convention for selecting $n(t_i)$, the process $\underline{z}^i(t)$ is compensated in a finite number of steps near most corners. This is due to the one-directional propagation of saturation and starvation effects. In fact, the exceptional corners of \bar{D} can be characterized as corners where this construction breaks down. These corners correspond to situations where adjacent storages x_i and x_{i+1} are respectively empty and full. Define the neighborhood set N_δ as:

$$N_\delta = \{ \underline{x} \in \bar{D} : \text{For some } i=1, \dots, k-1, x_i < \delta, x_{i+1} > 1 - \delta \}$$

Define the stopping time $T_{1\delta}(\underline{z})$ as

$$T_{1\delta}(\underline{z}) = \min \{ t > 0 : \underline{z}^{i+1}(t) \in \bar{N}_\delta \} \quad (5.8)$$

Equation (5.7) describes the evolution of the flow up to time $T_{1\delta}(\underline{z})$, for each sample path \underline{z} . Assume that, for $t > T_{1\delta}$, we have

$$C_i(t, z_j^\ell) = C_i(T_{1\delta}(\underline{z}), z_j^\ell) \quad \text{for all } j, \ell \quad (5.9)$$

This corresponds to stopping the compensator processes when the compensated trajectories enter N_δ . Denote the compensated trajectory, for $0 \leq t \leq T$, as $\underline{z}^{(1)}(t)$. Note that, due to the construction of the compensating processes, $\underline{z}^{(1)}(t) \in \bar{D}$ for $t \leq T_{1\delta}$.

Lemma 5.1 The map $G^{(1)}: C\{[0, T]; R^{k-1}\} \rightarrow C\{[0, T]; R^{k-1}\}$ which maps $\underline{z}(\cdot) \rightarrow \underline{z}^{(1)}(\cdot)$ is continuous in the supremum topology.

Proof: Due to the definitions of the compensating processes, the neighborhood N_δ , and conditions (5.8) and (5.9), the map $G^{(1)}$ is a finite composition of continuous maps (C_0 and C_1 and I), hence it is itself continuous.

We now proceed to describe the evolution of the compensated flow process while it resides in N_δ . Divide N_δ into regions of the form N_δ^i , where

$$N_\delta^i = \{ \underline{x} \in \bar{D} \mid x_i > \delta, x_{i+1} < 1 - \delta \}$$

Within each N_δ^i , we construct the compensated process in a manner similar to equation (5.7), except for the storages i and $i+1$, which must be treated separately. Figure 2 illustrates the difficulties associated with compensating for excesses in storage $i+1$ in N_δ^i .

Assume for simplicity that there is only one i such that

$$z_i^{(1)}(T_{1\delta}) < \delta, z_{i+1}^{(1)}(T_{1\delta}) > 1 - \delta.$$

The more complicated cases require greater enumeration, but offer no conceptual problems. Define the compensator process

$$U(t, x, y) = \max_{s \leq t} \{ -x(s), y(s) - 1, 0 \} \quad (5.10)$$

Define the sequence of times

$$t_\ell^{(1)} = \inf \{ t, T \mid C_n(t, z_j^{(1)\ell}) \neq 0 \text{ or } U(t, z_i^{(1)\ell}, z_{i+1}^{(1)\ell}) \neq 0, \text{ or } \underline{z}^{(1)\ell}(t) \in \bar{D} - N_{2\delta}^i, j=1, \dots, k-1, j \neq i, i+1; n=0,1 \} \quad (5.11)$$

where $\underline{z}^{(1)0} = \underline{z}^{(1)}$, and $\underline{z}^{(1)\ell}$ will be defined recursively. Define the integer valued function

$$n^{(1)}(t_m^{(1)}) = \max_{\substack{1 \leq j \leq k-1 \\ j \neq i, i+1}} \{ j : z_j^{(1)m}(t_m^{(1)}) = 1 \text{ and } C_1(t, z_j^{(1)m}) > 0 \text{ for } t > t_m^{(1)} \} \quad (5.12)$$

As in equation (5.6b), if this set is empty, $n^{(1)}(t_m^{(1)})$ would be redefined accordingly. Notice that equation (5.11) represents the proper indices for recursively computing the coupling of boundary effects between neighboring storages, as in equation (5.7), except for the boundary effects of storages i and $i+1$. If $n^{(1)}(t_m^{(1)})$ is still undefined, it means that one of the last two conditions of equation (5.11) is in force. Let

$$n^{(1)}(t_m^{(1)}) = 2k \quad \text{if } U(t, z_i^{(1)m}, z_{i+1}^{(1)m}) > 0 \quad \text{for } t > t_m^{(1)}. \quad (5.13)$$

$$n^{(1)}(t_m^{(1)}) = 2k+1 \quad \text{otherwise.}$$

Define the compensated process $\underline{z}^{(1)m+1}(t)$ as follows:

Starting with $\underline{z}^{(1)m}(t)$, apply the compensation algorithm described in equation (5.7) for $n^{(1)}(t_m^{(1)})$ smaller than $2k - 1$. For $n^{(1)}(t_m^{(1)}) = 2k$, define $\underline{z}^{(1)m+1}$ as

$$z_{i+1}^{(1)m+1}(t) = z_{i+1}^{(1)m}(t) - U(t, z_i^{(1)m}, z_{i+1}^{(1)m})$$

$$z_i^{(1)m+1}(t) = z_i^{(1)m}(t) + U(t, z_i^{(1)m}, z_{i+1}^{(1)m+1}) \quad (5.14)$$

For $n(t_m^{(1)}) = 2k + 1$, all of the compensating processes are stopped, including U , for times t larger than $t_m^{(1)}$. The resulting compensated trajectory from $t = 0$ to $t = T$ is denoted as $\underline{z}^{(2)}(t)$. Note that $\underline{z}^{(2)}(t_m^{(1)})$ is outside of the neighborhood N_δ^i in this case. Define $T_{2\delta}$ as equal to $t_m^{(1)}$ if $n^{(1)}(t_m^{(1)}) = 2k + 1$.

Lemma 5.2 The map $G^{(2)}: C\{[0,T];R^{k-1}\} \rightarrow C\{[0,T];R^{k-1}\}$ which maps $\underline{z}^{(1)}(\cdot) \rightarrow \underline{z}^{(2)}(\cdot)$ is continuous in the supremum topology. Furthermore, $\underline{z}^{(2)}(t) \in \bar{D}$ for $0 \leq t \leq T_{2\delta}$.

The proof is a consequence of the one-directional propagation of saturation and starvation effects. Essentially, a finite number of compensators must be added to take into account the boundary effects.

The construction of the process with boundary $\underline{y}^\delta(t)$ from an arbitrary sample path $\underline{z}(t)$ in $C\{[0,T];R^{k-1}\}$ can be completed inductively. Away from N_δ , the process is adjusted using the compensators (5.5)-(5.7). In N_δ , the process is adjusted using the algorithm of equations (5.8)-(5.11). Let $T_{i\delta}$ be the sequence of entrance times into N_δ , and exit times from $N_{2\delta}$, and let $\underline{z}^{(i)}$ be the resulting compensated process. For any finite $\delta > 0$, the continuity of any sample path $\underline{z}(\cdot)$, coupled with the results of Lemmas 5.1 and 5.2, will guarantee that the sequence of stopping times $T_{i\delta}$ is unbounded; hence, for any $\delta > 0$ the above construction describes \underline{y}^δ as a continuous mapping of \underline{z} in the supremum topology in $C\{[0,T];R^{k-1}\}$. The next result establishes that, for sufficiently small δ , $\underline{y}^\delta(\cdot)$ is independent of δ .

Lemma 5.3 There exists $w > 0$ such that, for $\delta \leq w$,

$$\begin{aligned} \underline{y}^\delta(t) &= \underline{y}^w(t) \quad \text{for } 0 \leq t \leq T, \text{ where } \underline{y}^\delta(t) \text{ is defined as} \\ \underline{y}^\delta(t) &= \lim_{i \rightarrow \infty} \underline{z}^{(i)}(t). \end{aligned}$$

The proof of this result is in the appendix. The main idea is to establish that the compensating processes used in N_δ correspond exactly to the compensating processes used in $\bar{D} - N_\delta$, modulo a nondifferentiable change of coordinates which is reflected in the definition of U . The only

points at which any difference is observed corresponds to corners where $y_i(t) = 0, y_{i+1}(t) = 1$, for some i . It is easily seen from (5.10) that the compensator U is independent of δ at such corners. In fact, our compensation procedure is entirely based on an additive decomposition which is essentially independent of δ , since the values of U and the respective C_i 's agree up to the time when the above corners are reached.

The reason for introducing the neighborhoods N_δ into the construction of the process is to isolate points where the map between $\underline{z}(\cdot)$ and the compensating processes C_i at each boundary is discontinuous. Although the map from \underline{z} into U is continuous, the decomposition of U into compensating processes at each boundary, $C_0(t, z_i)$ and $C_1(t, z_{i+1})$ is discontinuous due to the nondifferentiable coordinate transformation. This implies that we must treat the corners in a special way, requiring the previous construction.

The next lemma is a consequence of Lemmas 5.1-5.3:

Lemma 5.4 The map $G: \underline{z}(\cdot) \rightarrow \underline{y}(\cdot)$ specified as

$$\underline{y}(t) = \lim_{i \rightarrow \infty} z^{(i)}(t)$$

is a continuous map from $C\{[0, T]; R^{k-1}\}$ into $C\{[0, T]; \bar{D}\}$ for any finite T .

Notice that Lemmas 5.1 to 5.4 establish that the trajectories of the normalized storage process with boundary conditions are a continuous map of the trajectories of the process without boundary. Furthermore, Theorems 4.1 and 4.2 establish weak convergence, as $\epsilon \rightarrow 0$, of the process without boundary to a diffusion process with support in $C\{[0, T]; \bar{D}\}$.

Denote this diffusion process as $\underline{y}(t)$, $0 \leq t \leq T$. Then, theorem 5.1 of Billingsley [1968] establishes that, for an arbitrary interval, the process $\underline{y}(t)$ converges weakly as $\epsilon \rightarrow 0$ to the process with support in $C \{ [0, T]; \bar{D} \}$, whose distributions are given from the map G of Lemma 5.4. This discussion can be formalized as

Theorem 5.5 Assume that the process $\underline{z}(\cdot)$ converges weakly in $C \{ [0, T]; R^{k-1} \}$ as $\epsilon \rightarrow 0$ to $\underline{v}(\cdot)$, a diffusion process. Then, the process $\underline{y}(\cdot)$ converges weakly in $C \{ [0, T]; R^{k-1} \}$ to the process $G(\underline{v})$.

We can establish a stronger result. For any fixed trajectory of $\underline{z}(\cdot)$, we can write the process $\underline{y}(\cdot)$ as

$$\underline{y}(t) = \underline{z}^{(i)}(t), \quad t < T_{i\delta}.$$

Consider the time interval $t < T_{1\delta}$, and assume that $t < t_i$ as defined by (5.5). Then,

$$\begin{aligned} y_j(t) = & z_j(t) - \sum_{m=1}^i C_0(t, z_j^{m-1}) I \{ n(t_{m-1}) = k-1+j \} \\ & - \sum_{m=1}^i C_1(t, z_j^{m-1}) I \{ n(t_{m-1}) = j \} \\ & + \sum_{m=1}^i C_0(t, z_{j-1}^{m-1}) I \{ n(t_{m-1}) = k+j-2, j \neq 1 \} \\ & + \sum_{m=1}^i C_1(t, z_{j+1}^{m-1}) I \{ n(t_{m-1}) = j+1, j \neq k-1 \} \end{aligned} \quad (5.15)$$

It is easy to establish inductively that the first sum is constant except when $y_j(t) = 0$. Similarly, the second, third and fourth sums are constant except when $y_j(t) = 1$, $y_{j-1}(t) = 0$, and $y_{j+1}(t) = 1$ respectively.

Hence, we can represent $y_j(t)$ implicitly as

$$y_j(t) = z_j(t) + U_j^0(t) + U_{j+1}^1(t) - U_j^1(t) - U_{j-1}^0(t) \quad (5.16)$$

where $U_j^0(t), U_j^1(t)$ are increasing processes which increase only when $y_j(t) = 0$ or $y_j(t) = 1$. This representation holds up to time $T_{1\delta}$.

For times t in $[T_{1\delta}, T_{2\delta}]$, the process $\underline{y}(t)$ lies in $N_{2\delta}^i$ for some i . A similar expression to equation (5.15) can be obtained, with the exception of the effect of the compensator U , which can be rewritten as

$$\begin{aligned} U(t, z_i, z_{i+1}) &= \max \left\{ \max_{s \leq t} \{-z_i(s), 0\}, \max_{s \leq t} \{z_{i+1}(s) - 1, 0\} \right\} \\ &= \max \{-C_0(t, z_i), C_1(t, z_{i+1})\} \\ &= C_1(t, z_{i+1}) - C_0(t, z_i) - \min \{-C_0(t, z_i), C_1(t, z_{i+1})\} \end{aligned} \quad (5.17)$$

Note that the last term is an increasing term which increases only when $y_i(t) = 0$ and $y_{i+1}(t) = 1$. Hence, for $T_{1\delta} < t < T_{2\delta}$, we can represent \underline{y} as

$$y_j(t) = z_j^{(1)}(t) + U_j^0(t) - U_j^1(t) - U_{j-1}^0(t) + U_{j-1}^1(t) - V_{j,j+1}(t) \quad (5.18)$$

where $U_j^0, U_j^1, V_{j,j+1}$ are increasing processes which increase only when $y_j = 0, y_j=1, \text{ or } y_j=0 \text{ and } y_{j+1} = 1$ simultaneously, respectively. We can combine these processes with the processes obtained from equations (5.15) and (5.16) to obtain a global description of the compensating processes up to times $T_{2\delta}$. This construction can be extended inductively to define the compensating processes for all times t in $[0, T]$.

The unique feature of this construction is the presence of corner compensators $V_{i, i+1}(t)$, which are basically defined in N_δ . These

compensators motivated us to treat compensation in N_δ as a separate problem. Although it is important to recognize the existence of these corner compensators, the next result will enable us to ignore them in the approximation.

Theorem 5.6 The set of all trajectories $\underline{z}(\cdot)$ in $C\{[0, T]; \mathbb{R}^{k-1}\}$ such that $V_{i, i+1}(t, \underline{z}) = 0$ for all $i < k-1$, for all $t \leq T$, has Wiener measure 1.

The proof follows from the fact that, at any corner $y_i = 0$, $y_{i+1} = 1$, the only set of admissible directions which keep the process at that corner is $-dy_i = dy_{i+1} > 0$. Unless the Wiener process is degenerate, this implies that the local time at the corner will vanish, thereby establishing the theorem.

When the process $\underline{z}(t)$ is nearly balanced, the process $\underline{y}(t)$ will be a diffusion process, with instantaneous oblique reflection at the boundary. The equation describing $\underline{y}(t)$, which neglects $V_{i, i+1}(t)$ terms because of Theorem 5.6, is

$$y_j(t) = z_j(t) + U_j^0(t) - U_j^1(t) - U_{j-1}^0(t) + U_{j+1}^1(t) \quad (5.19)$$

The compensating processes U_j^0, U_j^1 , are related to the local time of this diffusion process on the boundary of D . For a detailed explanation of this relation, the reader should consult Watanabe [1971].

The directions of reflection can be obtained directly from equation (5.19)

For instance, on the face

$$y_j = 0$$

the equations for the evolution of $y_j(t)$ are

$$\left. \begin{aligned} \frac{dy_\ell}{dt}(t) &= \frac{dz_\ell}{dt}(t) & \ell \neq j, j+1 \\ \frac{dy_j}{dt}(t) &= \frac{dz_j}{dt}(t) + \frac{dU_j^0}{dt}(t) \\ \frac{dy_{j+1}}{dt}(t) &= \frac{dz_{j+1}}{dt}(t) - \frac{d}{dt} U_j^0(t) \end{aligned} \right\} \quad (5.20)$$

Hence, the direction of oblique reflection on the face $y_j = 0$ is given by the effect of the compensating processes U_j^0 , corresponding to reflection in the direction

$$\underline{d} = \underbrace{(0, \dots, 0)}_{j-1}, +1, -1, 0, \dots, 0$$

When the transfer line is nearly balanced, the limiting process spends no scaled time on the boundary, on the time scale $\tau = \epsilon^2 t$. However, the limiting process has a local time function at the boundary, which can be used to obtain an expression for the real time t spent on the boundary. This characterization will be useful in later sections, when we evaluate expressions for the throughput of the transfer line. From equation (2.8), the equation for throughput rate (in normalized units and scaled time) is given by

$$T(\tau) = \frac{1}{\tau} \left\{ \int_0^{\tau} \mu_k \alpha_k dt - U_{k-1}^0(\tau) \right\}$$

The quantity $\frac{U_{k-1}^0(\tau)}{\tau}$ represents the average lost production rate due to starvation of the last machine.

The result expressed in Theorem 5.5 defines a reflected diffusion process as the limit process. This process is defined uniquely in the weak sense, in terms of a continuous mapping on the sample paths of a standard diffusion process. This construction depends strongly on three assumptions: constant flow rates on links, constant failure and repair rates, and the geometry of line networks. When any of these three conditions are violated, the limit process must be constructed using a different argument. This is a nontrivial problem because of the lack of smoothness of the domain \bar{D} , a closed unit cube.

6. APPROXIMATION WITH LEVEL DEPENDENT FAILURE RATES

In this formulation of the previous sections, the failure and repair processes of the machines in the transfer line are independent of the levels of storage. However, a common practice in manufacturing networks is to turn off machines which are either starved or blocked, thereby eliminating the possibility of a machine failure during intervals of time when that machine is not processing any material. A mathematical model with these properties is described in Gershwin and Schick (1980b), and Forestier [1980].

The main difference in such a model is to introduce a feedback path from the continuous storage level \underline{x} to the discrete state process $\underline{\alpha}$, occurring when \underline{x} reaches its boundary. In terms of the normalized storage process \underline{y} , there are two situations where a machine is on, but not processing any material. The first situation, called blockage, occurs when machine $i+1$ is

off, and storage i is full. Then, the adjustment process described by (2.6) yields $\mu_i = 0$. Hence, machine i is assumed not to fail.

The second situation occurs when machine i is off, and storage i is empty. The adjustment process for machine $i+1$ yields

$$\mu_{i+1} = 0.$$

We call such a machine starved, and assume it cannot fail.

The equations for the α process can be modified to describe starvation and blocking as follows.

$$d\alpha_i = (1-\alpha_i)dR_i + \alpha_i(1-I\{\mu_i=0\})dF_i \quad (6.1)$$

where the last term has been modified to prevent failures during non-production intervals. The function $\mu_i(\underline{y}, \underline{\alpha})$ depends on the complete state of the system in a memoryless, fashion, given by the adjustment rules for conservation of flow.

Essentially, the description of the y process is decomposed into an internal description, describing the evolution of the process away from the boundary, and a boundary description which illustrates what happens to the process near a boundary. Our purpose in this section is to show that the modified $(\underline{y}, \underline{\alpha})$ process given by (5.8) and (6.1) converges weakly to the same diffusion process given in Theorem 5.4.

Throughout this section, we assume that the transfer line is nearly balanced, so that the appropriate time scale τ is $\varepsilon^2 t$. Let $\underline{y}^{1\varepsilon}(\tau; \varepsilon)$ denote the scaled process defined in section 5, and $P^{1\varepsilon}$ the induced probability measure on $C\{[0, T]; \mathbb{R}^{k-1}\}$. Similarly, denote by $\underline{y}^{2\varepsilon}(\tau; \varepsilon)$ the resulting

scaled process when starvation and blockage affect the probability rates, and $P^{2\varepsilon}$ its corresponding measure. The main result of this section is stated in the following theorem.

Theorem 6.1. In the topology of weak convergence on $C\{[0,T], R^{k-1}\}$

$$\lim_{\varepsilon \rightarrow 0} P^{1\varepsilon} = \lim_{\varepsilon \rightarrow 0} P^{2\varepsilon}$$

The proof is given in the Appendix. Basically, Theorem 6.1 is a consequence that, as $\varepsilon \rightarrow 0$, the process spends less percent of the time at the boundary. The evolution of $\underline{y}^{1\varepsilon}$ and $\underline{y}^{2\varepsilon}$ are identical outside the boundary, and they leave the boundary in the same direction. Hence, as the time spent on the boundary decays, the two processes approach each other. The differences in the behavior of the $\underline{\alpha}$ processes associated with $\underline{y}^{1\varepsilon}$ and $\underline{y}^{2\varepsilon}$ do not appear in the slow time scale $\tau = \varepsilon^2 t$. If the transfer line was not nearly balanced, the appropriate time scale would be $\tau = \varepsilon t$, and these differences would be noticeable in the approximate model.

Theorem 6.1 has served additionally to establish that the limiting process is instantaneously reflected at the boundary ∂D , by showing that the Lebesgue measure of the occupation time has expectation zero. This is consistent with the representation of the limiting process as instantaneously reflected Brownian motion.

7. ERGODIC DISTRIBUTION OF TWO MACHINE TRANSFER LINES USING DISSUSION APPROXIMATIONS

The simplest network one can construct consists of two unreliable links with a storage center in the middle, connecting an infinite source

to an infinite sink, as depicted in figure 3. In the context of manufacturing networks, many authors have studied the long term behavior of this simple network. Gershwin and Schick (1980b) provide the basic equations for the description of the Markov processes $(x(t), \alpha_1(t), \alpha_2(t))$.

Assume that the flow rates on each link is equal to 1; that is

$$\mu_1 = \mu_2 = 1 .$$

Then, the basic flow equation for the storage process is

$$\frac{dx}{dt} = (\alpha_1 - \alpha_2) \tag{7.1}$$

when the storage buffer is neither empty nor full. Assuming that the capacity of the storage process N is large, the normalized storage equation is

$$N \frac{dy}{dt} = (\alpha_1 - \alpha_2)$$
$$y = \frac{x}{N} \tag{7.2}$$

The processes α_i are jump processes with failure and repair rates p_i, r_i respectively, $i = 1, 2$.

In Gershwin and Schick (1980a), this model is studied in detail, obtaining an exact expression for the ergodic probability distribution of the (x, α_1, α_2) process. We will assume that starvation and blockage prevent machines from failing as in section 6.

Let $N = 1/\epsilon$, and $\tau = \epsilon^2 t$. Then,

$$\frac{dy}{d\tau} = \frac{\alpha_1 - \alpha_2}{\epsilon} \tag{7.3}$$

Assume that

$$\frac{r_1}{r_1 + p_1} - \frac{r_2}{r_2 + p_2} = \epsilon m \quad (7.4)$$

Equation (7.4) indicates that the network is nearly balanced, validating the use of the $\epsilon^2 t$ time scale.

From Gershwin and Schick (1980a), the marginal ergodic distribution of the $x(t)$ process is given by:

$$p(x < d) = \int_0^d g(x) dx + P\{x=0\} + P\{x=N\} \cdot I\{N < d\} \quad (7.5)$$

$$p(x = 0) = C \frac{(r_1+r_2)}{p_2} \left(\frac{1}{r_1} + \frac{1}{p_1+p_2} \right) \quad (7.6)$$

$$p(x = N) = C e^{\lambda N} \frac{(r_1+r_2)}{p_1} \left(\frac{1}{r_2} + \frac{1}{p_1+p_2} \right) \quad (7.7)$$

$$g(x) = C e^{\lambda x} \left(1 + \frac{r_1+r_2}{p_1+p_2} \right)^2 \quad (7.8)$$

$$\lambda = (p_2 r_1 - p_1 r_2) \left(\frac{1}{p_2+p_1} + \frac{1}{r_2+r_1} \right) \quad (7.9)$$

Define $\hat{\lambda} = \lambda/\epsilon$. Then, a simple integral establishes

$$\begin{aligned} C^{-1} &= \frac{r_1 + r_2}{p_2} \left(\frac{1}{r_1} + \frac{1}{p_1+p_2} \right) + \frac{(r_1+r_2)}{p_1} \left(\frac{1}{r_2} + \frac{1}{p_1+p_2} \right) e^{\lambda N} \\ &\quad + \frac{1}{\lambda} (e^{\lambda N} - 1) \left(1 + \frac{(r_1+r_2)}{(p_1+p_2)} \right)^2 \end{aligned} \quad (7.10)$$

The ergodic distribution of $y(t)$ is given in the following equations:

$$P\{y = 0\} = P\{x = 0\}$$

$$P\{y = 1\} = P\{x = N\}$$

$$P\{y \in [y, y+dy)\} = \frac{1}{\epsilon} g(y/\epsilon) dy$$

Let $N = 1/\epsilon$. As $\epsilon \rightarrow 0$, λ is of order ϵ , hence we have

$$\lim_{\epsilon \rightarrow 0} C^{-1} = \lim_{\epsilon \rightarrow 0} \frac{1}{\hat{\lambda}} (e^{\hat{\lambda}} - 1) \left(1 + \frac{r_1 + r_2}{p_1 + p_2} \right)^2 \quad (7.11)$$

Thus,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} g(y/\epsilon) = \frac{\hat{\lambda}}{e^{\hat{\lambda}} - 1} e^{\hat{\lambda}y}. \quad (7.12)$$

Furthermore,

$$\lim_{\epsilon \rightarrow 0} P\{y = 0\} = \lim_{\epsilon \rightarrow 0} P\{y = 1\} = 0 \quad (7.13)$$

because λ is of order ϵ , by assuming that the transfer line is nearly balanced.

The ergodic distribution indicated by equation (7.13) reflects the long-term behavior of the $\underline{z}(t)$ process. The diffusion approximation $v(\cdot)$ generated in section 5 for the balanced line case has as its infinitesimal generator

$$L = m \frac{\partial}{\partial v} + \frac{1}{2} \sigma \frac{\partial^2}{\partial v^2} \quad (7.14)$$

$$\sigma = \frac{2p_1 r_1}{(p_1 + r_1)^3} + \frac{2p_2 r_2}{(p_2 + r_2)^3} \quad (7.15)$$

with domain

$$D(L) = \{f \mid f \text{ is bounded, twice differentiable on } (0,1)$$

$$\text{and } \frac{\partial f}{\partial v}(0) = \frac{\partial f}{\partial v}(1) = 0\}.$$

Hence, the ergodic distribution $p(v)$ is given by

$$\begin{aligned} -m \frac{\partial p}{\partial v} + \frac{1}{2} \sigma \frac{\partial^2 p}{\partial v^2} &= 0 \\ -m p(0) + \frac{1}{2} \sigma \frac{\partial p}{\partial v}(0) &= 0 \\ -m p(1) + \frac{1}{2} \sigma \frac{\partial p}{\partial v}(1) &= 0 \end{aligned} \tag{7.16}$$

The solution of (7.16) is

$$p(v) = \frac{K}{-1 + e^{+K}} e^{+Ky} \tag{7.17}$$

where $K = \frac{2m}{\sigma}$. To show that (7.17) and (7.12) are alike, we have to establish that

$$\lim_{\varepsilon \rightarrow 0} |K - \hat{\lambda}| = 0$$

From equations (7.4) and (7.11)

$$\begin{aligned}
 |K - \hat{\lambda}| &= \left| \frac{1}{\varepsilon} (p_2 r_1 - p_1 r_2) \right| \cdot \left| \frac{(p_1 + r_1)^2 (p_2 + r_2)^2}{p_1 r_1 (p_2 + r_2)^3 + p_2 r_2 (p_1 + r_1)^3} \right| \cdot \\
 &\left| 1 - \frac{(p_1 + p_2 + r_1 + r_2)}{(p_2 + p_1)(r_2 + r_1)} \cdot \left(\frac{p_1 r_1 (p_2 + r_2)}{(p_1 + r_1)^2} + \frac{p_2 r_2}{(p_2 + r_2)^2} (p_1 + r_1) \right) \right| \quad (7.18)
 \end{aligned}$$

Since the transfer line is nearly balanced, we have

$$\left. \begin{aligned}
 \frac{r_1}{r_1 + p_1} &= \frac{r_2}{r_2 + p_2} + 0(\varepsilon) \\
 r_1 p_2 - r_2 p_1 &= 0(\varepsilon) \\
 \frac{r_1}{r_1 + p_1} &= \frac{p_2}{r_2 + p_2} + 0(\varepsilon)
 \end{aligned} \right\} \quad (7.19)$$

Hence, the first two terms in the right hand side of equation (7.18) are bounded as $\varepsilon \rightarrow 0$. The last term can be expanded using equation (7.19) as

$$\begin{aligned}
 &\left| 1 - \frac{(p_1 + p_2 + r_1 + r_2)}{(p_1 + p_2)(r_1 + r_2)} \cdot \left(\frac{p_1 r_1 (p_2 + r_2)}{(p_1 + r_1)^2} + \frac{p_2 r_2}{(p_2 + r_2)^2} (p_1 + r_1) \right) \right| \\
 &= \left| 1 - \frac{(p_1 + p_2 + r_1 + r_2)}{(p_1 + p_2)(r_1 + r_2)} \left\{ \frac{p_1 r_2}{p_1 + r_1} + \frac{p_2 r_1}{p_2 + r_2} + 0(\varepsilon) \right\} \right| \\
 &= \left| 1 - \frac{1}{(p_1 + p_2)(r_1 + r_2)} \left\{ p_1 r_2 + p_2 r_1 + \frac{p_1 r_2}{p_1 + r_1} (p_2 + r_2) + \frac{p_2 r_1 (p_1 + r_1)}{p_2 + r_2} + 0(\varepsilon) \right\} \right| \\
 &= \left| 1 - \frac{1}{(p_1 + p_2)(r_1 + r_2)} \left\{ p_1 r_2 + p_2 r_1 + p_2 r_2 + p_1 r_1 + 0(\varepsilon) \right\} \right| \\
 &= \frac{1}{(p_1 + p_2)(r_1 + r_2)} \cdot 0(\varepsilon)
 \end{aligned}$$

which establishes

$$\lim_{\varepsilon \rightarrow 0} |K - \hat{\lambda}| = 0.$$

Hence, the ergodic distribution of the diffusion approximation is consistent with the ergodic distribution of the original model. Theorem 5.5 indicates that continuous functionals of the process, such as expected exit times, will converge in the same fashion.

8. THREE MACHINE TRANSFER LINES

The three machine transfer line is the first nontrivial example of coupling between the storage buffers. Figure 4 describes a typical three machine transfer line with two storages present. We will assume that blockage and starvation affect machine failure rates, as indicated in section 6.

Assuming that $N_1 = N_2 = 1/\varepsilon$, and that the transfer line is nearly balanced, the normalized equations of flow in the time scale $\tau = \varepsilon^2 t$ are

$$\frac{dy_1}{d\tau} = \frac{1}{\varepsilon} (\mu_1 \alpha_1 - \mu_2 \alpha_2) \tag{8.1}$$

$$\frac{dy_2}{d\tau} = \frac{1}{\varepsilon} (\mu_2 \alpha_2 - \mu_3 \alpha_3) \tag{8.2}$$

when $(y_1, y_2) \in (0,1) \times (0,1)$.

On the boundary, the adjustment rules for conservation of flow must apply.

In terms of the compensating processes, this means

$$\frac{dy_1}{d\tau} = \frac{1}{\varepsilon} (\mu_1 \alpha_1 - \mu_2 \alpha_2) + \frac{d}{d\tau} U_1^0(\tau; \varepsilon) - \frac{d}{d\tau} U_1^1(\tau; \varepsilon) + \frac{d}{d\tau} U_2^1(\tau; \varepsilon) \tag{8.3}$$

$$\frac{dy_2}{d\tau} = \frac{1}{\varepsilon} (\mu_2 \alpha_2 - \mu_3 \alpha_3) + \frac{d}{d\tau} U_2^0(\tau; \varepsilon) - \frac{d}{d\tau} U_2^1(\tau; \varepsilon) - \frac{d}{d\tau} U_1^0(\tau; \varepsilon) \quad (8.4)$$

where we have explicitly depicted the dependence of the compensators on τ . The results of Theorem 5.5 and 6.1 let us represent the approximating diffusion process as

$$\begin{aligned} v_1(\tau) &= \omega_1(\tau) + U_1^0(\tau) - U_1^1(\tau) + U_2^1(\tau) \\ v_2(\tau) &= \omega_2(\tau) + U_2^0(\tau) - U_2^1(\tau) - U_1^0(\tau) \end{aligned} \quad (8.5)$$

where (ω_1, ω_2) is a diffusion process with parameters $(\underline{m}, \underline{\Sigma})$, given by

$$\underline{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} \frac{\mu_1 r_1}{r_1 + p_1} & -\frac{\mu_2 r_2}{r_2 + p_2} \\ \frac{\mu_2 r_2}{r_2 + p_2} & -\frac{\mu_3 r_3}{r_3 + p_3} \end{pmatrix} \quad (8.6)$$

$$\underline{\Sigma} = \begin{pmatrix} 2 \frac{\mu_1^2 p_1 r_1}{(p_1 + r_1)^3} + \frac{2 \mu_2^2 p_2 r_2}{(p_2 + r_2)^3} & -\frac{2 \mu_2^2 p_2 r_2}{(p_2 + r_2)^3} \\ -\frac{2 \mu_2^2 p_2 r_2}{(p_2 + r_2)^3} & \frac{2 \mu_2^2 p_2 r_2}{(p_2 + r_2)^3} + \frac{2 \mu_3^2 p_3 r_3}{(p_3 + r_3)^3} \end{pmatrix} \quad (8.7)$$

Equation (8.5) corresponds to a diffusion process on the unit square with oblique reflection at the boundaries; the directions of reflection are illustrated in figure 5.

The processes U_i^0, U_i^1 are continuous, increasing processes, which are bounded almost surely at each time τ . This implies that the processes $v_1(\tau), v_2(\tau)$ are semimartingales, and thus we have a generalization of Ito's formula [Kunita-Watanabe [1967], Harrison-Reiman [1979]]. Let f be a twice continuously differentiable function on \bar{D} . Denote by \bar{L} the infinitesimal generator of (ω_1, ω_2) , that is

$$\bar{L} f(\omega_1, \omega_2) = \sum_{i=1}^2 m_i \frac{\partial}{\partial \omega_i} f(\omega_1, \omega_2) + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial^2}{\partial \omega_i \partial \omega_j} f(\omega_1, \omega_2) \quad (8.8)$$

Then, we have

$$\begin{aligned} f(v_1(\tau), v_2(\tau)) - f(v_1(0), v_2(0)) &= \\ & \int_0^\tau \bar{L} f(v_1(s), v_2(s)) ds + \\ & \int_0^\tau \frac{\partial}{\partial v_1} f(v_1(s), v_2(s)) d\omega_1(s) + \int_0^\tau \frac{\partial}{\partial v_2} f(v_1(s), v_2(s)) d\omega_2(s) \\ & + \int_0^\tau \left(\frac{\partial f}{\partial v_1} - \frac{\partial f}{\partial v_2} \right) (v_1(s), v_2(s)) dU_1^0(s) \\ & + \int_0^\tau - \left(\frac{\partial f}{\partial v_1} - \frac{\partial f}{\partial v_2} \right) (v_1(s), v_2(s)) dU_2^1(s) \\ & + \int_0^\tau - \frac{\partial f}{\partial v_1} (v_1(s), v_2(s)) dU_1^1(s) \\ & + \int_0^\tau \frac{\partial f}{\partial v_2} (v_1(s), v_2(s)) dU_2^0(s) \end{aligned} \quad (8.9)$$

where the last four terms represent the contributions of the four parts of ∂D . Notice that, if f were such that $f \in \mathcal{D}^0$, the set of all twice con-

tinuously differentiable functions such that

$$1. \quad \frac{\partial f}{\partial v_1} - \frac{\partial f}{\partial v_2} = 0 \text{ on } v_1 = 0$$

$$2. \quad \frac{\partial f}{\partial v_1} - \frac{\partial f}{\partial v_2} = 0 \text{ on } v_2 = 1$$

$$3. \quad \frac{\partial f}{\partial v_1} = 0 \quad \text{on } v_1 = 1$$

$$4. \quad \frac{\partial f}{\partial v_2} = 0 \quad \text{on } v_2 = 0,$$

the equation (8.9) implies

$$\begin{aligned} E\{f(v_1(\tau), v_2(\tau)) \mid v_1(0), v_2(0)\} - f(v_1(0), v_2(0)) = \\ E\left\{ \int_0^\tau \bar{L} f(v_1(s), v_2(s)) ds \mid v_1(0), v_2(0) \right\} \end{aligned} \quad (8.10)$$

The infinitesimal generator of the (v_1, v_2) process is thus seen to be \bar{L} , with its domain \mathcal{D} including the class of functions \mathcal{D}° .

The process $(v_1(\tau), v_2(\tau))$ is a diffusion process in a compact domain, with a positive probability of visiting all states, even the corners of \bar{D} . Hence, there exists a unique ergodic probability density function $p^*(v_1, v_2)$ such that

$$E_{p^*} \{f(v_1(\tau), v_2(\tau))\} = E_{p^*} \{f(v_1(0), v_2(0))\} \quad (8.11)$$

for all $\tau > 0$.

Using (8.10) and (8.11) yields a characterization of $p^*(v_1, v_2)$ as

$$E_{p^*} \{ \bar{L} f(v_1, v_2) \} = 0 \text{ for all } f \in \mathcal{D}^\circ \quad (8.12)$$

Representing the expectation as an integral gives

$$\int_0^1 \int_0^1 \bar{L} f(x,y) p^*(x,y) dx dy = 0 \quad (8.13)$$

for all $f \in \mathcal{D}^\circ$. If $p^*(x,y)$ is smooth enough, equation (8.13) can be integrated by parts to obtain an equation for $p^*(x,y)$. The smoothness of $p^*(x,y)$ in D follows from Weyl's lemma, as stated in McKean [1969]. Writing (8.13) at length yields

$$\begin{aligned} \int_0^1 \int_0^1 \bar{L} f(x,y) p^*(x,y) dx dy &= \int_0^1 \int_0^1 m_1 \frac{\partial f}{\partial x} \cdot p^*(x,y) dx dy \\ &+ \int_0^1 \int_0^1 m_2 \left(\frac{\partial f}{\partial y} \right) \cdot p^*(x,y) dx dy \\ &+ \int_0^1 \int_0^1 \frac{1}{2} \Sigma_{11} \left(\frac{\partial^2 f}{\partial x^2} \right) p^*(x,y) dx dy \\ &+ \int_0^1 \int_0^1 \Sigma_{12} \left(\frac{\partial^2 f}{\partial x \partial y} \right) p^*(x,y) dx dy \\ &+ \int_0^1 \int_0^1 \frac{1}{2} \Sigma_{22} \left(\frac{\partial^2 f}{\partial y^2} \right) p^*(x,y) dx dy \end{aligned} \quad (8.14)$$

We will integrate each term by parts. Denote by S_1 the surface $x = 0$; S_2 is $y = 0$, S_3 is $x_1 = 1$, S_4 is $y = 1$. Then, we can integrate (8.14) by parts, to obtain:

$$\begin{aligned}
 \int_D \bar{L} f \cdot p^* \, dx dy &= \\
 \int_D f \cdot \left\{ \frac{1}{2} \Sigma_{11} \frac{\partial^2}{\partial x^2} p^* + \Sigma_{12} \frac{\partial^2}{\partial x \partial y} p^* + \frac{1}{2} \Sigma_{22} \frac{\partial^2}{\partial x \partial y} p^* \right. \\
 &\quad \left. - m_1 \frac{\partial p^*}{\partial x} - m_2 \frac{\partial p^*}{\partial y} \right\} dx dy \\
 + \int_{s_1} f \left\{ -m_1 p^* + \frac{1}{2} \Sigma_{11} \frac{\partial p^*}{\partial x} \right\} dy - \int_{s_1} p^* \left\{ \Sigma_{12} \frac{\partial f}{\partial y} + \frac{\Sigma_{11}}{2} \frac{\partial f}{\partial x} \right\} dy \\
 + \int_{s_2} f \left\{ \frac{1}{2} \Sigma_{22} \frac{\partial p^*}{\partial y} + \Sigma_{12} \frac{\partial p^*}{\partial x} - m_2 p^* \right\} dx - \int_{s_2} p^* \frac{1}{2} \Sigma_{22} \frac{\partial f}{\partial y} dx \\
 + \int_{s_3} f \left\{ -\frac{1}{2} \Sigma_{11} \frac{\partial p^*}{\partial x} + m_1 p^* \right\} dy + \int_{s_3} p^* \left\{ \Sigma_{12} \frac{\partial f}{\partial y} + \frac{1}{2} \Sigma_{11} \frac{\partial f}{\partial x} \right\} dy \\
 + \int_{s_4} f \left\{ -\frac{1}{2} \Sigma_{22} \frac{\partial p^*}{\partial y} - \Sigma_{12} \frac{\partial p^*}{\partial x} + m_2 p^* \right\} dx + \int_{s_4} p^* \frac{1}{2} \Sigma_{22} \frac{\partial f}{\partial y} dx,
 \end{aligned} \tag{8.15}$$

where the arguments of the integrals are implicit in their integration sets.

Now, assume that f is twice continuously differentiable, and vanishes in a neighborhood of ∂D , denoted by $N(\partial D)$. It is clear that f is in the domain of \bar{L} , because all derivatives vanish near the boundary ∂D . For such f , equation (8.15) reduces to

$$\int_D \bar{L} f \cdot p^* dx dy = \int_{D-N(\partial D)} f \cdot \{L^* p^*\} dx dy \tag{8.16}$$

where

$$L^* p^* = \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \Sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} p^* - \sum_{i=1}^2 m_i \frac{\partial p^*}{\partial x_i} \tag{8.17}$$

Since f is an arbitrary smooth function in $D-N(\partial D)$, and the neighborhood $N(\partial D)$ can be selected arbitrarily small, equation (8.13) implies

$$L^*p^*(x,y) = 0 \quad (8.18)$$

for all (x,y) in D . Now, consider only function f which are in D^0 , and which are bounded, and are zero outside of a neighborhood $N(\partial D)$. For these functions, the definition of D^0 implies:

$$\text{On } S_1, \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y},$$

$$\text{On } S_2 \quad \frac{\partial f}{\partial y} = 0,$$

$$\text{On } S_3 \quad \frac{\partial f}{\partial x} = 0,$$

$$\text{On } S_4 \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}.$$

Hence,

$$\begin{aligned} \int_{S_1} p^* \left(\Sigma_{12} \frac{\partial f}{\partial y} + \frac{\Sigma_{11}}{2} \frac{\partial f}{\partial x} \right) dy &= (fp^*(0,1) - fp^*(0,0)) \left(\Sigma_{12} + \frac{\Sigma_{11}}{2} \right) \\ &\quad - \left(\Sigma_{12} + \frac{\Sigma_{11}}{2} \right) \int_{S_1} f \frac{\partial p^*}{\partial y} dy \end{aligned} \quad (8.19)$$

$$\int_{S_2} \frac{1}{2} p^* \Sigma_{22} \frac{\partial f}{\partial y} dx = 0 \quad (8.20)$$

$$\begin{aligned} \int_{S_3} p^* \left\{ \Sigma_{12} \frac{\partial f}{\partial y} + \frac{\Sigma_{11}}{2} \frac{\partial f}{\partial x} \right\} dy &= \\ \Sigma_{12} fp^*(1,1) - \Sigma_{12} fp^*(1,0) - \int_{S_3} \Sigma_{12} f \frac{\partial p^*}{\partial y} dy \end{aligned} \quad (8.21)$$

$$\begin{aligned} \int_{S_4} p^* \frac{1}{2} \Sigma_{22} \frac{\partial f}{\partial y} dx &= \frac{1}{2} \Sigma_{22} fp^*(1,1) - \frac{1}{2} \Sigma_{22} fp^*(0,1) \\ &\quad - \int_{S_4} \frac{1}{2} \Sigma_{22} f \frac{\partial p^*}{\partial x} dx \end{aligned} \quad (8.22)$$

Substituting (8.19)-(8.22) into (8.15) yields, for these functions f ,

$$\begin{aligned}
\int_D \bar{L} f p^* dx dy &= \int_{N(\partial D)} f \cdot L^* p^* dx dy \\
+ \int_{S_1} f \cdot \left\{ \frac{1}{2} \Sigma_{11} \frac{\partial p^*}{\partial x} + \frac{1}{2} \Sigma_{11} \frac{\partial p^*}{\partial y} + \Sigma_{12} \frac{\partial p^*}{\partial y} - m_1 p^* \right\} dy \\
+ \int_{S_2} f \cdot \left\{ \frac{1}{2} \Sigma_{22} \frac{\partial p^*}{\partial y} + \Sigma_{12} \frac{\partial p^*}{\partial x} - m_2 p^* \right\} dx \\
+ \int_{S_3} f \cdot \left\{ -\frac{1}{2} \Sigma_{11} \frac{\partial p^*}{\partial x} - \Sigma_{12} \frac{\partial p^*}{\partial y} + m_1 p^* \right\} dy \\
+ \int_{S_4} f \cdot \left\{ -\frac{1}{2} \Sigma_{22} \frac{\partial p^*}{\partial y} - \frac{1}{2} \Sigma_{22} \frac{\partial p^*}{\partial x} - \Sigma_{12} \frac{\partial p^*}{\partial x} + m_2 p^* \right\} dx \\
+ fp^*(0,0) \left(\Sigma_{12} + \frac{\Sigma_{11}}{2} \right) + fp^*(0,1) \left(-\frac{\Sigma_{11}}{2} - \Sigma_{12} - \frac{\Sigma_{22}}{2} \right) \\
+ fp^*(1,1) \left(\Sigma_{12} + \frac{\Sigma_{22}}{2} \right) + fp^*(1,0) (-\Sigma_{12})
\end{aligned} \tag{8.23}$$

Since f can be arbitrary in S_i , and the neighborhood $N(\partial D)$ can be reduced, equations (8.23) and (8.13) imply

$$\frac{1}{2} \Sigma_{11} \frac{\partial p^*}{\partial x} + \frac{1}{2} \Sigma_{11} \frac{\partial p^*}{\partial y} + \Sigma_{12} \frac{\partial p^*}{\partial y} - m_1 p^* = 0 \text{ on } S_1 \tag{8.24}$$

$$\frac{1}{2} \Sigma_{22} \frac{\partial p^*}{\partial y} + \Sigma_{12} \frac{\partial p^*}{\partial x} - m_2 p^* = 0 \text{ on } S_2 \tag{8.25}$$

$$\frac{1}{2} \Sigma_{11} \frac{\partial p^*}{\partial x} + \Sigma_{12} \frac{\partial p^*}{\partial y} - m_1 p^* = 0 \text{ on } S_3 \tag{8.26}$$

$$\frac{1}{2} \Sigma_{22} \frac{\partial p^*}{\partial y} + \frac{1}{2} \Sigma_{22} \frac{\partial p^*}{\partial x} + \Sigma_{12} \frac{\partial p^*}{\partial x} - m_2 p^* = 0 \text{ on } S_4 \tag{8.27}$$

$$\begin{aligned}
 p^*(0,0) &= 0 \text{ unless } \Sigma_{12} + \frac{\Sigma_{11}}{2} = 0 \\
 p^*(1,1) &= 0 \text{ unless } \frac{\Sigma_{22}}{2} + \Sigma_{12} = 0 \\
 p^*(0,1) &= 0 \text{ unless } \Sigma_{11} + 2\Sigma_{12} + \Sigma_{22} = 0 \\
 p^*(1,0) &= 0 \text{ unless } \Sigma_{12} = 0
 \end{aligned} \tag{8.28}$$

From equation (8.7), we can verify that $p^*(1,0) = p^*(0,1) = 0$. However, when machines 1 and 2, or machines 2 and 3 have identical failure and repair rates, the values of $p^*(0,0)$ and $p^*(1,1)$ can be nonzero. In these cases, the intensity of the coupling term Σ_{12} matches and cancels the oblique flow along the boundary, resulting in decoupled reflecting conditions. This can be seen from equations (8.24) and (8.27), which, when $\Sigma_{11} = \Sigma_{22} = -2\Sigma_{12}$, reduce to

$$\frac{1}{2} \Sigma_{11} \frac{\partial p^*}{\partial x} - m_1 p^* = 0 \text{ on } S_1 \tag{8.29}$$

$$\frac{1}{2} \Sigma_{22} \frac{\partial p^*}{\partial y} - m_2 p^* = 0 \text{ on } S_4 \tag{8.30}$$

Obtaining exact solutions for equation (8.18) satisfying (8.24)-(8.28) is a difficult problem, which can seldom be solved in closed form. However, the markov process $(v_1(\tau), v_2(\tau))$ can be approximated in the weak sense by a Markov Chain, as in Kushner [1976], and the ergodic distribution of this chain can be computed as an approximate solution to these equations.

Assume that the stationary probability distribution $p^*(x,y)$ has been determined. Let E^* denote the measure on the path space induced by p^* . Following the development of Harrison-Reiman [1980], let f be any bounded,

twice continuously differentiable function on \bar{D} . Then, equation (8.9) implies, from Fubini's theorem,

$$\begin{aligned} & \int_D \bar{L}f \cdot p^* \, dv_1 dv_2 + \int_0^\tau E^* \left\{ \left(\frac{\partial f}{\partial v_1} - \frac{\partial f}{\partial v_2} \right) dU_1^0(s) \right\} \\ & + \int_0^\tau E^* \left\{ \left(\frac{\partial f}{\partial v_1} - \frac{\partial f}{\partial v_2} \right) dU_2^1(s) \right\} + \int_0^\tau E^* \left\{ \left(-\frac{\partial f}{\partial v_1} \right) dU_1^1(s) \right\} \\ & + \int_0^\tau E^* \left\{ \frac{\partial f}{\partial v_2} dU_2^0(s) \right\} = 0 \end{aligned} \quad (8.31)$$

Define measures on S_1, S_2, S_3, S_4 as

$$\begin{aligned} \nu_1(B) &= \frac{1}{\tau} \int_0^\tau E^* \{ I(v_2(s) \in B) \mid dU_1^0(s) \} \text{ for } B \subset S_1 \\ \nu_2(B) &= \frac{1}{\tau} \int_0^\tau E^* \{ I(v_1(s) \in B) \mid dU_2^0(s) \} , \quad B \subset S_2 \\ \nu_3(B) &= \frac{1}{\tau} \int_0^\tau E^* \{ I(v_2(s) \in B) dU_1^1(s) \} , \quad B \subset S_3 \\ \nu_4(B) &= \frac{1}{\tau} \int_0^\tau E^* \{ I(v_1(s) \in B) dU_2^1(s) \} , \quad B \subset S_4 \end{aligned}$$

The measures ν_i are the occupation time, or local time, measures on the boundary, defined in Donsker-Vardhan [1975]. It is easy to show that, for any τ ,

$$0 < E^* \{ U_j^i(\tau) \} < \infty .$$

Hence, we can use Fubini's theorem to reduce equation (8.31) to

$$\begin{aligned} & \int_D \bar{L}f \cdot p^* \, dv_1 dv_2 + \int_{S_1} \left(\frac{\partial f}{\partial v_1} - \frac{\partial f}{\partial v_2} \right) \nu_1(dy) \\ & + \int_{S_2} \frac{\partial f}{\partial v_2} \nu_2(dx) + \int_{S_3} -\frac{\partial f}{\partial v_1} \nu_3(dy) \\ & + \int_{S_4} \left(\frac{\partial f}{\partial v_1} - \frac{\partial f}{\partial v_2} \right) \nu_4(dx) = 0 \end{aligned} \quad (8.32)$$

Equation (8.32) can be evaluated for selected functions f , to obtain the properties of the process when it reaches the stationary limit. For instance, recall that the throughput rate in section 5 was given by

$$T(\tau) = \left\{ \frac{1}{\tau} \int_0^{\tau} \mu_3 \alpha_3 ds - \frac{U_2^0(\tau)}{\tau} \right\} \quad (8.33)$$

The expected stationary throughput rate is just

$$\begin{aligned} T_{SS} &= E^*(T(\tau)) \\ &= E^* \left(\frac{\mu_3}{\tau} \int_0^{\tau} \alpha_3 ds \right) - E^* \{ U_2^0(\tau) \} \\ &= \frac{\mu_3 r_3}{r_3 + p_3} - E^* \left\{ \frac{U_2^0(\tau)}{\tau} \right\} \\ &= \frac{\mu_3 r_3}{r_3 + p_3} - v_2(S_2) \end{aligned} \quad (8.34)$$

Let $f = \alpha e^{-v_2/\alpha}$. f is bounded, and smooth, hence (8.32) implies

$$\begin{aligned} \int_D \left(\frac{1}{2} \frac{\Sigma_{22}}{\alpha} - m_2 \right) e^{-v_2/\alpha} p^*(v_1, v_2) dv_1 dv_2 \\ + \int_{S_1} e^{-x_2/\alpha} v_1 (dv_2) \\ + \int_{S_2} -v_2 (dv_1) \\ + \int_{S_4} e^{-1/\alpha} v_4 (dv_1) = 0 \end{aligned} \quad (8.35)$$

Letting α approach zero in (8.35) yields

$$v_2(S_2) = \int_{S_2} \frac{1}{2} \Sigma_{22} p^*(v_1, 0) dv_1 \quad (8.36)$$

Hence, knowledge of $p^*(v_1, v_2)$ would be sufficient for computation of $v_2(S_2)$, and thus the average throughput rate.

9. CONCLUSION

In this paper, we have presented a methodology for approximating the flow of material through a transfer line of unreliable machines with finite storage buffers. Under the assumption of large but finite storages, the flow of material is approximated by a diffusion process with reflecting boundary conditions, independent of the process which describes the failures and repairs of the machines. This approximation reduces the number of states which must be considered by a factor of 2^k , where k is the number of machines in the transfer line.

The structure of the approximation was exploited in the case of 2 and 3 machine transfer lines to obtain equations for the stationary distribution of the approximate diffusion process. In the two machine case, these equations were solved explicitly, and found to be consistent with the results of Gershwin and Schick [1980a]. The equations for the stationary distribution of the three machine transfer line were too complicated to solve in closed form, although numerical algorithms for their solution are currently under study.

The methodology derived in this paper can be applied to transfer lines of arbitrary length without ignoring the coupling effects of starvation and blockage. As such it represents a significant generalization of the previous works mentioned in the introduction. Work is currently in progress

to generalize these results to arbitrary network topologies with nonconstant flow rates and storage dependent failure rates. For these problems, the techniques used in this paper will not apply, because of the dependence of the failure-repair processes on the levels of storage. Key theoretical questions concerning the existence and uniqueness of the limit process must be answered. These problems are currently under investigation, and will be reported in later publications.

APPENDIX

Proof of Theorem 4.2

Weak convergence of the $\underline{y}(t)$ process is a consequence of Theorem 3.1 of Khasminskii [1966]. To establish the almost sure convergence of $\underline{z}(t)$, rewrite $\underline{z}(t)$ as

$$z_i(t) = z_i(0) + \epsilon \int_0^t \mu_i \alpha_i(s) ds - \epsilon \int_0^t \mu_{i+1} \alpha_{i+1}(s) ds$$

The process $\underline{z}(t)$ can be defined entirely in terms of the cumulative process $\underline{S}(t)$, where

$$S_i(t) = \int_0^t \alpha_i(s) ds$$

The strong law of large numbers for cumulative processes implies (Doob [1953])

$$\lim_{t \rightarrow \infty} \frac{S_i(t)}{t} = \frac{r_i}{r_i + p_i} \quad \text{a.s.}$$

Let $\tau = \epsilon t$. Then, for fixed τ ,

$$\begin{aligned} \lim_{t \rightarrow \infty} \left| \frac{S_i(t)}{t} - \frac{r_i}{r_i + p_i} \right| &= \lim_{\epsilon \rightarrow 0} \left| \frac{S_i(\tau/\epsilon)}{\tau/\epsilon} - \frac{r_i}{r_i + p_i} \right| \\ &= 0 \quad \text{a.s.} \end{aligned}$$

Hence, for any τ ,

$$\lim_{\epsilon \rightarrow 0} \left| z_i(\tau) - z_i(0) - \frac{\mu_i r_i \tau}{r_i + p_i} + \frac{\mu_{i+1} r_{i+1} \tau}{r_i + p_i} \right| = 0 \quad \text{a.s.}$$

Completing the proof.

Proof of Lemma 5.3

Consider any $\delta > 0$. Assume without loss of generality that $\underline{z}^{(1)}(\tau_{1\delta})$ is in a neighborhood N_δ^i , and that the process $\underline{z}^{(2)}(t)$ will leave $N_{2\delta}^i$ before entering any other N_δ^j . Under these assumptions, for coordinates $z_\ell, \ell \neq i, i+1$, the adjustment rules (5.6)-(5.7) represent a continuous map of $z_\ell^{(1)} \rightarrow z_\ell^{(2)}$, because the compensator U does not feed into these equations. Hence, we can focus our attention on (5.14).

Equations (5.10) and (5.14) imply that $z_{i+1}^{(1)j+1}(t) \leq 1$ and $z_i^{(1)j+1}(t) \geq 0$ for all $\tau_{1\delta} \leq t \leq \tau_{2\delta}$. Furthermore, if $z_i^{(1)j} > 0$, then $U(t, z_i^{(1)j}, z_{i+1}^{(2)j}) = C_1(t, z_{i+1}^{(1)j})$. A similar statement applies to $z_{i+1}^{(2)j} < 1$. Hence, we see that, independent of δ , $U(t, z_i^{(1)j}, z_{i+1}^{(2)j})$ is a continuous extension of C_0 and C_1 for $t \leq \tau_{1\delta}$ to the interval $\tau_{1\delta} \leq t \leq \tau_{2\delta}$. The only difference occurs when $z_i^{(1)j+1} = 0$, $z_{i+1}^{(2)j+1} = 1$, which occurs inside N_δ for all $\delta > 0$. Since the adjustment rule (5.14) is independent of δ , this establishes that $\underline{z}^{(2)}(\cdot)$ will not depend on δ for δ small enough.

Proof of Theorem 6.1

It is sufficient to establish that the finite dimensional distribution of $P^{1\epsilon}$ and $P^{2\epsilon}$ converge to the same limit, since the sequence $P^{1\epsilon}$ has been shown as tight. Let $A^{i\epsilon}$ denote the infinitesimal generators of the processes $(\underline{y}^{i\epsilon}, \underline{\alpha}^{i\epsilon})$ $i = 1, 2$ and $T^{i\epsilon}$ the associated semigroups. Consider a bounded continuous function $f(y)$ in the domain of $A^{1\epsilon}$. Note that $f(y)$ will be in the domain of $A^{2\epsilon}$ also, by its independence from α . Hence

$$T_t^{1\epsilon} f(y) - T_t^{2\epsilon} f(y) = \int_0^t T_{t-s}^{1\epsilon} (A_s^{2\epsilon} - A_s^{1\epsilon}) T_s^{2\epsilon} f(y) ds$$

Since $A^{1\epsilon}$ and $A^{2\epsilon}$ differ only on ∂D , we have

$$\|T_t^{1\epsilon} f - T_t^{2\epsilon} f\| \leq K \int_0^t T_{t-s}^{1\epsilon} I\{y \in \partial D\} ds$$

where

$$K = \|f\| \cdot \sup_{y \in \partial D} \|(A^{2\epsilon} - A^{1\epsilon}) f, (y, \alpha)\|$$

$$\|f\| = 1$$

$$f, (y, \alpha) \in D(A^{1\epsilon})$$

The constant K is finite because the difference between $A^{2\epsilon}$ and $A^{1\epsilon}$ can be expressed as a bounded matrix, because it consists of the $\underline{\alpha}$ transitions which are not allowed under $P^{2\epsilon}$. Hence, the proof is completed if

$$\lim_{\epsilon \rightarrow 0} \int_0^t T_{t-s}^{1\epsilon} I\{y \in \partial D\} ds = 0$$

Now,

$$\int_0^t T_{t-s}^{1\epsilon} I\{y \in \partial D\} ds = E\left\{ \int_0^t I\{y_s \in \partial D\} ds \right\}$$

Suppose we consider the boundary $y_i = 0$. Consider an arbitrary positive smooth bounded function $g(y_i)$ with $\frac{\partial g}{\partial y} > d$ on $y_i = 0$.

$$\begin{aligned} \frac{d T_t^{1\epsilon}}{dt} g(y_i) &= \frac{\alpha_i \mu_i - \alpha_{i+1} \mu_{i+1}}{\epsilon} \frac{\partial}{\partial y_i} g(y_i) I\{y_i > 0\} \\ &+ \frac{1}{\epsilon} \{ \max\{\alpha_i \mu_i - \alpha_{i+1} \mu_{i+1}, 0\} \frac{\partial}{\partial y_i} g(y_i) I\{y_i = 0\} \end{aligned}$$

neglecting the effects of corners. Hence,

$$T_t^{1\epsilon} g(y_i) - g(y_i) \int_0^t T_s^{1\epsilon} A^{1\epsilon} g(y_i) I\{y_i > 0\} ds = \int_0^t T_s^{1\epsilon} \bar{A}^{1\epsilon} g(y_i) I\{y_i = 0\} ds$$

As ϵ decreases, the terms on the right hand side are all bounded, since

$$\lim_{\epsilon \rightarrow 0} \int_0^t T_s^{1\epsilon} A^{1\epsilon} g(y) I\{y_i > 0\} ds = \int_0^t \bar{T}_s A g(y) I\{y_i > 0\} ds$$

That is, $T_s^{1\epsilon}$ converges to the corresponding diffusion operator \bar{T}_s with generator A in the interior of D . The terms on the right hand side, for ϵ small, converge to

$$\begin{aligned} & E_{Y,\alpha} \int_0^t \left\{ \frac{1}{\epsilon} \max\{\alpha_i \mu_i - \alpha_{i+1} \mu_{i+1}, 0\} \frac{\partial}{\partial y_i} g(y_i) I\{y_i = 0\} \right\} ds \\ &= \frac{1}{\epsilon} \int_0^t E_{Y,\alpha} \left\{ \max\{\alpha_i \mu_i - \alpha_{i+1} \mu_{i+1}, 0\} \frac{\partial}{\partial y_i} g(y_i) I\{y_i = 0\} \right\} ds \\ &\geq \frac{1}{\epsilon} d C_2 \int_0^t E_{Y,\alpha} \{ I(y_i = 0) \} ds \end{aligned}$$

where C_2 is a constant representing the net expected drift. That is,

$$C_2 = E_{\alpha} \{ \max\{\alpha_i \mu_i - \alpha_{i+1} \mu_{i+1}, 0\} \}$$

which is of order 1, if there exists a configuration $\underline{\alpha}$ such that

$\alpha_i \mu_i - \alpha_{i+1} \mu_{i+1} < 0$. This expectation can be computed using the ergodic measure for small ϵ . Hence,

$$\lim_{\epsilon \rightarrow 0} \int_0^t E_{Y,\alpha} \{ I(y_i(s) = 0) \} ds = 0 .$$

This argument can be extended to the entire domain ∂D , by recognizing that, there are no points in ∂D , where the allowed drifts $\frac{dy_i}{dt}$ will have the same ergodic average as the unconstrained drifts in D° , thereby establishing that the equivalent constant C_2 is of order 1.

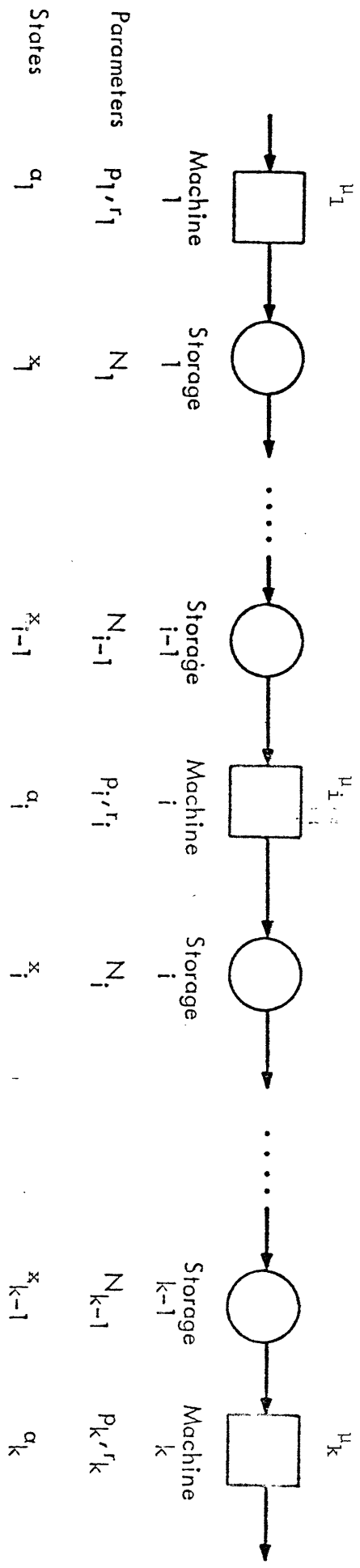


Fig. 1. A k-stage Transfer Line

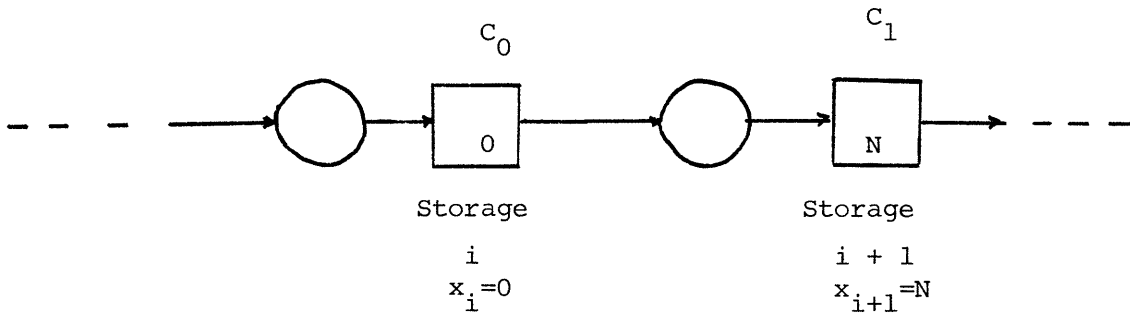


Figure 2: Interaction of overflow and underflow

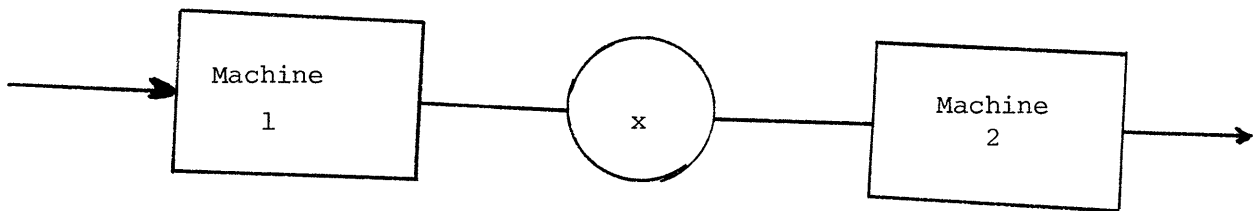


Figure 3: Two Machine Transfer Line

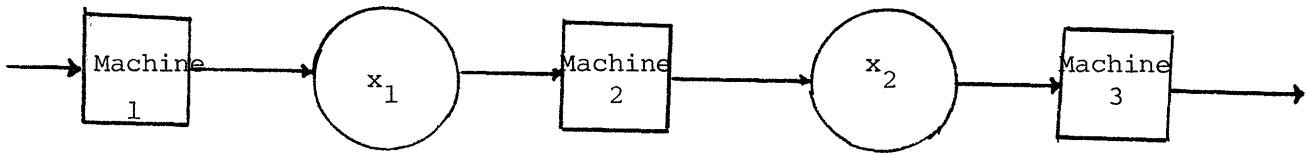


Figure 4. Three Machine Transfer Line

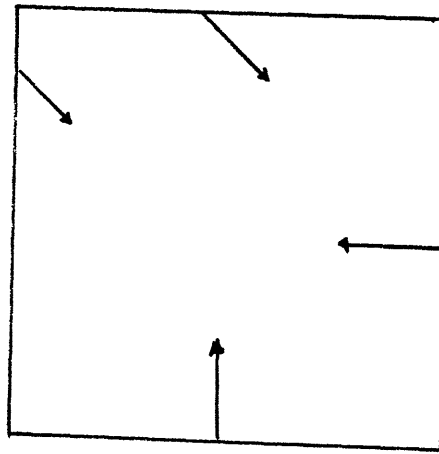


Figure 5. Directions of reflection of Diffusion Approximation for the Three Machine Transfer Line

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