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THE ALGEBRAIC REGULATOR PROBLEM FROM THE STATE-SPACE POINT OF VIEW

by

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ABSTRACT

We study the algebraic aspects of the regulator problem, using some new ideas in the state-space ('geometric') approach to feedback design problems for linear multivariable systems. Necessary and sufficient conditions are given for the solvability of a general version of this problem, requiring output stability, internal stability, and disturbance decoupling as well. An algorithm is given by which these conditions can be verified from the system parameters. A few remarks are added on the analytical and numerical aspects of the problem, and on the relative merits of the state-space and the transfer matrix approach.

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1. Introduction

The problem of making a given system follow a certain signal in the presence of disturbances is, of course, a basic one in controller design. Several versions have been under study since the very beginnings of control theory. In recent years, much attention has been paid to the underlying algebraic structure of the problem. The central issue here is to decide on solvability or non-solvability of the problem for a given set of parameters. Of course, in practice the parameters are not known precisely, and the yes-or-no answer which comes from the algebraic analysis is related in a nontrivial way to the hard/easy scale that is much more familiar to the engineer. Still, we may expect that a good understanding of the cases in which the problem is not solvable will be of help in identifying the crucial features of those control problems that should be classified as 'intrinsically difficult'. Moreover, if the answer to the algebraic problem is constructive in the sense that it provides an algorithm to find a solution if there exists one, then this algorithm may also be used as a starting point for the development of software that would be applicable under a minimum of assumptions on the system to be controlled.

Among other factors, these considerations have played a role in the development of several different approaches to, what we shall call, the algebraic regulator problem. State space methods were used in [1-7], resulting in a constructive solution for a fairly general version of the problem. It was felt, however, that a solution in terms of transfer functions would provide a better starting point for investigations involving (small) parameters changes, and this was one of the incentives for a number of papers using techniques like coprime factorization of transfer matrices ([8-16]). The solvability conditions obtained, however, are in part inattractive from the numerical point of view (cf. the conclusions of [15]).

The purpose of the present paper is to re-state the case for the state space approach. We shall use some new ideas to obtain a constructive solution for a general version of the regulator problem, involving output stability, internal stability, and disturbance decoupling. The main feature of the approach adopted here is that it incorporates (dynamic) observation feedback in a natural way. (The intricacy of working with observation feedback in earlier state-space treatments has sometimes been mentioned as a reason to prefer transfer matrix techniques: see [10]). We shall give several equivalent formulations of the main result, among which there will be an explicit matrix version that could be a starting point for calculations. This paper improves on the results in [18]. The organization of the paper is as follows. After having introduced some notation and preliminaries in section 2, we motivate our formulation of the regulator problem in Section 3. Section 4 contains necessary conditions for this problem to be solvable. These conditions are shown to be also sufficient in section 5, and hence we obtain our basic result. In section 6, we show that this result leads to a completely constructive solvability criterion. The 'internal model principle' is briefly discussed in Section 7, and conclusions follow in section 8. An appendix is added in which it is shown that the problem considered here is a strict generalization of the one considered in [1] (see also [2], Ch. 7).

2. Notation and Preliminaries

We shall consider only linear, finite-dimensional systems over $\mathbb{R}$. In general, vector spaces will be indicated by script capitals, linear mappings by Roman capitals and vectors by lower case letters. Further conventions in the use of letters are as follows. The generic description for a system is

$$x'(t) = Ax(t) + Bu(t) + Eq(t) \qquad x(t) \in \mathcal{X},\ u(t) \in \mathcal{U} \tag{2.1}$$

$$y(t) = Cx(t) \qquad y(t) \in \mathcal{Y} \tag{2.2}$$

$$z(t) = Dx(t) \qquad z(t) \in \mathcal{Z}\ . \tag{2.3}$$

Here, $x(t)$ is called the <u>state</u> of the system at time t, $u(t)$ is the <u>input</u>, $q(t)$ the <u>disturbance</u>, $y(t)$ the <u>observation</u>, $z(t)$ the <u>output</u>. Our controllers will be devices that produce a control function $u(t)$ from an observation function $y(t)$ in the following way:

$$w'(t) = A_c w(t) + G_c y(t) \qquad w(t) \in \mathcal{W} \tag{2.4}$$

$$u(t) = F_c w(t) + Ky(t)\ , \tag{2.5}$$

This is called a <u>compensator</u>; $w(t)$ is the <u>compensator state</u> and $\mathcal{W}$ is the <u>compensator state space</u>. We can combine the equations (2.1-3) and (2.4-5) to form the <u>extended system</u>:

$$\frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix}(t) = \begin{pmatrix} A+BKC & BF_c \\ G_c C & A_c \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}(t) + \begin{pmatrix} E \\ 0 \end{pmatrix} q(t) \tag{2.6}$$

$$z(t) = (D \quad 0)\begin{pmatrix} x \\ w \end{pmatrix}(t)\ . \tag{2.7}$$

We denote

$$A_e = \begin{pmatrix} A+BKC & BF_c \\ G_cC & A_c \end{pmatrix} \tag{2.8}$$

and call this the extended system matrix. This mapping acts on the extended state space $\mathcal{X}^e := \mathcal{X} \oplus \mathcal{W}$. There are two natural mappings between $\mathcal{X}^e$ and $\mathcal{X}$: the natural projection $P: \mathcal{X}^e \to \mathcal{X}$, defined by

$$P\binom{x}{w} = x \tag{2.9}$$

and the canonical imbedding $Q: \mathcal{X} \to \mathcal{X}^e$, defined by

$$Qx = \binom{x}{0} \quad . \tag{2.10}$$

A typical form of a control problem is now: given the system (2.1-3), find a compensator of the form (2.4-5) such that the closed-loop system (2.6-7) has certain properties. For the algebraic regulator problem, these properties can be specified in terms of invariant subspaces of the extended system matrix. We shall denote the 'bad subspace' of A_e by $\mathcal{X}_b^e(A_e)$, so

$$\mathcal{X}_b^e(A_e) = \sum_{\mathrm{Re}\ \lambda \geq 0} \ \sum_{n \in \mathbb{N}} \ker(\lambda I - A_e)^n \quad . \tag{2.11}$$

This subspace of $\mathcal{X}^e$ contains the 'unstable modes' of A_e, i.e., the eigendirections corresponding to non-decreasing solutions. We say that we have output stability in the closed-loop system if

$$\mathcal{X}_b^e(A_e) \subset \ker(D \quad 0) \ . \tag{2.12}$$

This means that the output $z(t)$ will converge to zero, if no external disturbance is present ($q(t) = 0$). Another property of interest is disturbance decoupling: we say that the closed-loop system had this property if there exists an A_e-invariant subspace M such that

$$\operatorname{im}\begin{pmatrix}E\\0\end{pmatrix} \subset \dot{M} \subset \ker(D \quad 0) \ . \tag{2.13}$$

This means that the behavior of z(t) is completely unaffected by that of q(t). If we have both output stability and disturbance decoupling, then the output z(t) converges to zero regardless of the behavior of q(t). Note that these properties can also be formulated in terms of subspaces of X: (2.12) is equivalent to

$$PX_b^e(A_e) \subset \ker D \tag{2.14}$$

and (2.13) is the same as

$$\operatorname{im} E \subset Q^{-1}M \subset PM \subset \ker D \ . \tag{2.15}$$

A third property will be discussed below.

For a while, let us concentrate on the pair (A,B) of <u>system mapping</u> and <u>input mapping</u> (see (2.1)). A subspace V of X is said to be <u>(A,B)-invariant</u> if there exists a 'state feedback mapping' F: $X \to U$ such that V is (A+BF)-invariant. If V is (A,B)-invariant, the set of all mappings $F: X \to U$ such that $(A+BF)V \subset V$ is denoted by $\underline{F}(V)$. An alternative characterization of (A,B)-invariance can be given as follows ([2], lemma 4.2):

<u>Lemma 2.1</u> A subspace V of X is (A,B)-invariant if and only if

$$AV \subset V + \operatorname{im} B. \tag{2.16}$$

From this, it is easily seen that the set of (A,B)-invariant subspaces is closed under subspace addition. Consequently, the set of (A,B)-invariant subspaces that are contained in a given subspace K

(which set is never empty, because the zero subspace is (A,B)-invariant) has a unique largest element which is denoted by $\mathcal{V}^*(\mathcal{K})$. An algorithm to construct $\mathcal{V}^*(\mathcal{K})$ for any given $\mathcal{K}$ can be found in [2], p.91.

Given an (A,B)-invariant subspace $\mathcal{V}$, it will be important for us to know how the eigenvalues of A+BF can be manipulated when F may be chosen from the class $\underline{F}(\mathcal{V})$. To describe the situation, it is convenient to introduce the following notation. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are invariant subspaces for some linear mapping T, and $\mathcal{L}_1 \subset \mathcal{L}_2$, then $T:\mathcal{L}_2/\mathcal{L}_1$ will denote the factor mapping induced on the quotient space $\mathcal{L}_2/\mathcal{L}_1$ by the restriction of T to $\mathcal{L}_2$. In matrix terms, this simply means that if the matrix of T can be written, with respect to a suitable basis, in the block form

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix} \tag{2.17}$$

then the matrix of $T:\mathcal{L}_2/\mathcal{L}_1$ is T_{22}. If $\mathcal{L}_1 = \{0\}$, we shall write $T:\mathcal{L}_2$ instead of $T:\mathcal{L}_2/\{0\}$. We can now formulate the following result ([24]; see also [2], Cor. 5.2 and Thm. 4.4):

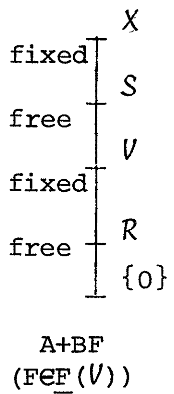
__Lemma 2.2__: *Let $\mathcal{V}$ be an (A,B)-invariant subspace. Then the smallest (A+BF)-invariant subspace containing im $B \cap \mathcal{V}$ is the same for all $F \in \underline{F}(\mathcal{V})$. Denote this subspace by $\mathcal{R}$, and let $\mathcal{S}$ be the smallest A-invariant subspace containing both im B and $\mathcal{V}$. Then $\mathcal{S}$ is (A+BF)-invariant for all F, and we have for all $F_1, F_2 \in \underline{F}(\mathcal{V})$:*

$$A + BF_1:\mathcal{X}/\mathcal{S} = A:\mathcal{X}/\mathcal{S} \tag{2.18}$$

$$A + BF_1:\mathcal{V}/\mathcal{R} = A + BF_2:\mathcal{V}/\mathcal{R} \,. \tag{2.19}$$

Moreover, for any real polynomials $p_1(\lambda)$ and $p_2(\lambda)$ with $\deg(p_1) = \dim S - \dim V$ and $\deg(p_2) = \dim R$, there exists an $F \in \underline{F}(V)$ such that the characteristic polynomials of $A+BF: S/V$ and $A+BF: R$ are equal to $p_1(\lambda)$ and $p_2(\lambda)$, respectively.

The content of this lemma can conveniently be expressed in the form of a diagram, in which the words 'free' and 'fixed' refer to the eigenvalues of A+BF when F may be chosen from $\underline{F}(V)$:



(2.20)

An (A,B)-invariant subspace V is called a <u>controllability subspace</u> if $\sigma(A+BF:V)$ is free ([2], p.102), and it is called <u>strongly invariant</u> if $\sigma(A+BF:X/V)$ is fixed. If there exists an $F \in \underline{F}(V)$ such that $\sigma(A+BF:V) \subset \{\lambda \in \mathbb{C} \mid \text{Re } \lambda < 0\}$, V is called a <u>stabilizability subspace</u>.

For brevity of notation, let us write

$$\mathbb{C}_g = \{\lambda \in \mathbb{C} \mid \text{Re } \lambda < 0\}, \quad \mathbb{C}_b = \mathbb{C} \setminus \mathbb{C}_g . \tag{2.21}$$

(Other partitionings of the complex plane may be used, for instance to express stronger stability requirements. The effects on the theory will be none, provided that the partitioning is symmetric with respect to the real axis, and $\mathbb{C}_g \cap \mathbb{R} \neq 0$.) We already introduced $X_b^e(A_e)$, and the notations

$X_g^e(A_e)$, $X_g(A)$, $X_b(A)$ etc. will refer in an obvious way to the modal subspaces corresponding to the part of $\mathbb{C}$ indicated by the subscript. For any subspace L, we use the following notation for the smallest A-invariant subspace containing L and for the largest A-invariant subspace contained in L:

$$\langle A | L \rangle := \sum_{k \in \mathbb{Z}_+} A^k L \tag{2.22}$$

$$\langle L | A \rangle := \bigcap_{k \in \mathbb{Z}_+} A^{-k} L \quad . \tag{2.23}$$

A strongly invariant subspace of particular interest is

$$X_{stab} := X_g(A) + \langle A | \text{im } B \rangle \tag{2.24}$$

which is easily seen to be the largest stabilizability subspace in X. More generally, one can prove ([19], p.26; [2], p.114) that the set of all stabilizability subspaces contained in a given subspace K has a unique largest element, which will be denoted by $V_g^*(K)$. Let V be an (A,B)-invariant subspace. It is seen from Lemma 2.2 that there exists $F \in \underline{F}(V)$ such that $\sigma(A+BF:X/V) \subset \mathbb{C}_g$ if and only if $S + X_g(A) = X$, where $S = \langle A | \text{im } B + V \rangle$. In this case, we shall say that V is <u>outer-stabilizable</u> . It is easily proved that $\langle A | \text{im } B + V \rangle = \langle A | \text{im } B \rangle + V$, and so we obtain the following characterization of outer-stabilizability.

<u>*Lemma 2.3.*</u> *An (A,B)-invariant subspace V is outer-stabilizable if and only if*

$$V + X_{stab} = X \tag{2.25}$$

Everything what has been said above about the pair (A,B) can be dualized to statements about the pair (C,A) of output mapping and state mapping. We shall quickly go through the most important notions. A subspace T of X is said to be (C,A)-invariant if there exists a mapping $G: Y \to X$ such that T is (A-GC)-invariant, or, equivalently, if

$$A(T \cap \ker C) \subset T \tag{2.26}$$

The set of all mappings $G: Y \to X$ such that $(A-GC)T \subset T$ is denoted by $\underline{G}(T)$. A (C,A)-invariant subspace T is said to be a detectability subspace if there exists $G \in \underline{G}(T)$ such that $\sigma(A-GC: X/T) \subset \mathbb{C}_g$. To every subspace E, there is a smallest detectability subspace containing it, which will be denoted by $T^*_g(E)$. We define

$$X_{det} := T^*_g(\{0\}) = X_b(A) \cap \langle \ker C | A \rangle . \tag{2.27}$$

We now return to the specification of properties for the closed-loop system (2.6-7). It is easily seen that the subspace QX_{det} is always A_e-invariant, and that $A: X_{det}$ is similar to $A_e: QX_{det}$. The subspace $P^{-1}(X_{det} + X_{stab})$ is also always A_e-invariant, and $A_e: X^e/P^{-1}(X_{det} + X_{stab})$ is similar to $A: X/X_{det} + X_{stab})$. This leads immediately to the following result.

Lemma 2.4 *For any compensator of the form (2.4.-5) applied to the system (2.1-3), the extended system matrix A_e given by (2.8) will satisfy*

$$\dim X^e_b(A_e) \geq \dim X_{det} + \operatorname{codim}(X_{det} + X_{stab}) . \tag{2.28}$$

We shall say that the closed-loop system (2.6.7) is internally stable if equality holds in (2.28). This nomenclature will be explained in the next section.

Finally, we shall need a concept that is related to the triple A,B and C. A <u>(C,A,B)- pair</u> ([17]) is an (ordered) pair of subspaces $(\mathcal{T},\mathcal{V})$ in which $\mathcal{T}$ is (C,A)- invariant, $\mathcal{V}$ is (A,B)-invariant, and $\mathcal{T} \subset \mathcal{V}$. The following result ([18], lemma 4.2) will be instrumental.

<u>*Lemma 2.5:*</u> *Let $(\mathcal{T},\mathcal{V})$ be a (C,A,B)-pair. Then there exists a mapping* $K:\mathcal{Y} \rightarrow \mathcal{U}$ *such that* $(A+BKC)\mathcal{T} \subset \mathcal{V}$.

This means that in situations where we are allowed to replace A by A+BKC (applying a preliminary static output feedback), it is no restriction of generality to assume that $A\mathcal{T} \subset \mathcal{V}$. Note that the properties we discussed above for the pair (A,B) are all feedback invariant: they would have been the same for any pair of the form (A+BF, B). Likewise, the properties relating to the pair (C,A) would have been the same for any pair of the form (C,A-GC). Consequently, the change from A to A+BKC changed neither the input-to-state nor the state-to-output structure, which makes it a transformation that is applicable under many circumstances. If we have to do with several (C,A,B)-pairs $(\mathcal{T}_i, \mathcal{V}_i)$ (i=1,...,k), there does not necessarily exist a K such that $(A+BKC)\mathcal{T}_i \subset \mathcal{V}_i$ for all i; we shall say that the pairs $(\mathcal{T}_i, \mathcal{V}_i)$ are <u>compatible</u> if such a K does exist.

3. Problem Statement

A common control set-up for a 'plant' to follow a reference signal in the face of disturbances is depicted in Fig. 3.1.

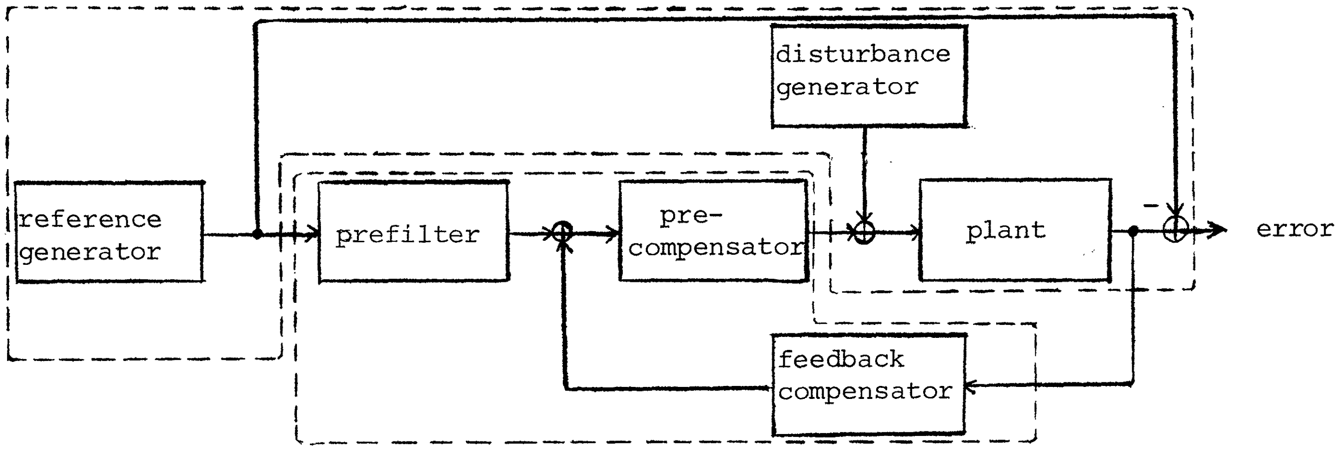


Fig. 3.1 Control Scheme

Here, the prefilter, the precompensator and the feedback compensator are elements that are to be constructed by the designer in such a way that the error will tend to zero for every choice of initial conditions in the reference generator, the disturbance generator and the plant. The diagram can be re-organized to display more clearly the interface between the given elements and the elements that are to be constructed, in the following way:

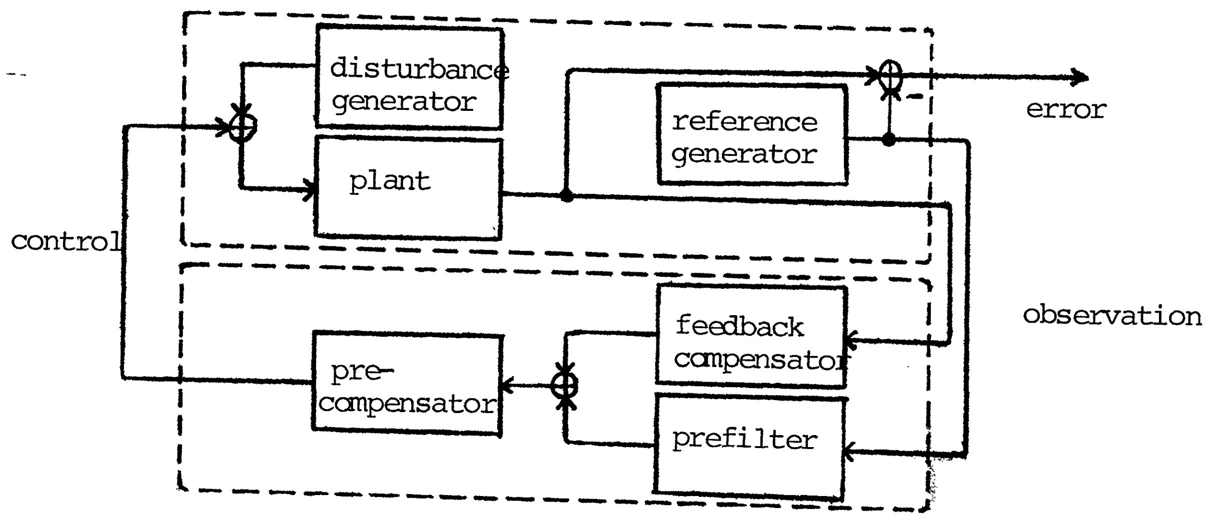


Fig. 3.2 Re-organized Control Scheme

The scheme can be simplified and generalized at the same time, as follows:

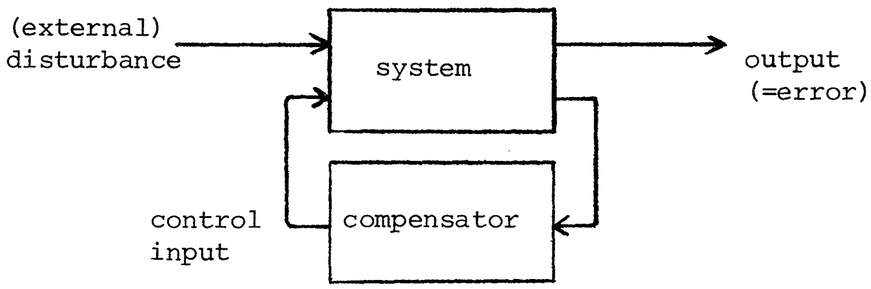


Fig. 3.3 Simplified and Generalized Control Scheme

All the given elements have been taken together under the name 'system', and the control elements are represented by one feedback processor called the 'compensator'. Also, an additional external disturbance has been added for which no knowledge of dynamics is assumed. (This may be quite natural, for instance, when this disturbance is used to model a lack of information about certain system parameters.) The error has been re-named as simply 'output'; the longer term 'variables-to-be-controlled' is also sometimes used.

We are now in the situation described in the previous section. The system is described by the equations (2.1-3), the compensator equations are given by (2.4-5), and the closed-loop system as a whole is described by (2.6-7). The question is, of course, whether we are still able to properly define our control objectives in the present context, in which the distinction between plant, disturbance and reference has seemingly disappeared.

To answer this question, we break down the system mapping A using the chain of invariant subspaces $\{0\} \subset X_{det} \subset X_{det} + X_{stab} \subset X$. Taking into account the facts that $\mathcal{X}_{det} \subset \ker C$ and that $\operatorname{im} B \subset \mathcal{X}_{det} + \mathcal{X}_{stab}$, this enables us to re-write the equations

$$x'(t) = Ax(t) + Bu(t) \tag{3.1}$$

$$y(t) = Cx(t) \tag{3.2}$$

in the following way:

$$x_1'(t) = A_{11}x_1(t) + A_{12}x_2(t) + A_{13}x_3(t) + B_1u(t) \tag{3.3}$$

$$x_2'(t) = A_{22}x_2(t) + A_{23}x_3(t) + B_2u(t) \tag{3.4}$$

$$x_3'(t) = A_{33}x_3(t) \tag{3.5}$$

$$y(t) = C_2x_2(t) + C_3x_3(t) \ . \tag{3.6}$$

Picturewise, we have:

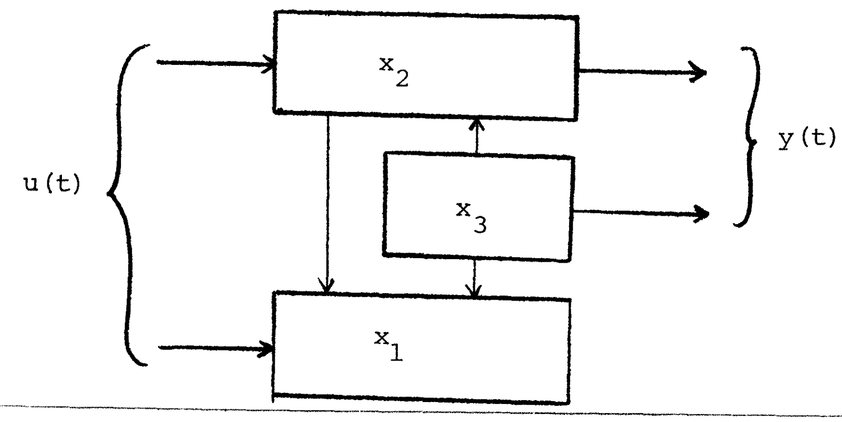


Fig. 3.4. Decomposition of a General Linear System

This makes it natural to interpret $x_2(t)$ (corresponding to $A\colon \mathcal{X}_{det}$) as representing irrelevant plant variables. That is, we assume that we are not in the fundamentally hopeless situation in which there are unobservable unstable relevant plant modes. The vector $x_3(t)$ is naturally interpreted as representing the state variables of the reference and (internal) disturbance generators. Again, supposing that $x_3(t)$ partly represents plant variables would bring us into a fundamentally wrong situation, this time because of the presence of unstable uncontrollable plant modes. It can be argued (see for instance [7]) that it is reasonable to asssume that $\mathcal{X}_{det} = \{0\}$, but we shall take the option of performing the mathematical analysis in full generality, to see if the outcome agrees with our interpretations.

With this background, it is now reasonable to formulate the following specifications for the closed-loop system. To ensure that the system output (which represents the difference between reference signal and actual plant behavior) will tend to zero in spite of the internal and external disturbances, we ask for *output stability* and *disturbance decoupling* ((2.12) and (2.13)). Moreover, we want the plant to be stabilized. Using the interpetation discussed above, this requirement is expressed by the condition of *internal stability*:

$$\dim \mathcal{X}_b^e(A_e) = \dim \mathcal{X}_{det} + \operatorname{codim} \mathcal{X}_{det} + \mathcal{X}_{stab}). \tag{3.7}$$

So the *algebraic regulator problem* that will be discussed in this paper is: Given a system of the form (2.1-3), find necessary and sufficient conditions for the existence of a compensator of the form (2.4-5) such that the closed-loop system (2.6-7) has the properties of output stability, disturbance decoupling, and internal stability; and give an algorithm to construct such a compensator, if there exists one. We use the qualifier 'algebraic' because this problem does not include issues like sensitivity to parameter changes, response of the system to signals other than which it has been designed for, efficient and numerically stable computational algorithms, and so on. It will be shown in the Appendix that the algebraic regulator problem as it is formulated here is a strict generalization of the problem considered in [1] (also in [2], Ch. 7).

4. Necessity

We start with the following simple but basic observation (cf. [17]).

Lemma 4.1 *Let A_e be an extended system matrix of the form (2.8), and suppose that $M_1, \ldots, M_k$ are A_e-invariant subspaces. Then the pairs $(Q^{-1}M_1, PM_1), \ldots, (Q^{-1}M_k, PM_k)$ are compatible (C,A,B)-pairs.*

Proof Take $i \in \{1, \ldots, k\}$, and let $x \in Q^{-1}M_i \cap \ker C$. Then $\binom{x}{0} \in M_i$, and consequently

$$\begin{pmatrix} A+BKC & BF_c \\ G_cC & A_c \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} Ax \\ 0 \end{pmatrix} \in M_i \quad . \tag{4.1}$$

We see that $Ax \in M_i$, showing that $Q^{-1}M_i$ is (C,A)-invariant. Next, let $x \in PM_i$ and take $w \in W$ such that $\begin{pmatrix} x \\ w \end{pmatrix} \in M_i$. Then

$$\begin{pmatrix} A+BKC & BF_c \\ G_cC & A_c \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} Ax+B(KCx + F_cw) \\ G_cCx + A_cw \end{pmatrix} \in M_i \quad . \tag{4.2}$$

Hence, $Ax + B(KCx + F_cw) \in PM_i$ which implies that $Ax \in PM_i + \operatorname{im} B$ and that PM_i is (A,B)-invariant. Finally, let $x \in Q^{-1}M_i$. We have

$$\begin{pmatrix} A+BKC & BF_c \\ G_cC & A_c \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} (A+BKC)x \\ G_cC_x \end{pmatrix} \in M_i \tag{4.3}$$

which shows that $(A+BKC)Q^{-1}M_i \subset PM_i$. Since K does not depend on i, this completes the proof.

We now want to bring in the aspect of eigenvalue assignment. First, recall the following standard result.

<u>*Lemma 4.2*</u> *Let $T: X \to X$ be a linear mapping, and let L_1 and L_2 be invariant subspaces for T, with $L_1 \subset L_2$.* Then *the following are equivalent:*

$$\sigma(T: L_2/L_1) \subset C_g \tag{4.4}$$

$$(\lambda I-A)^{-1}L_1 \cap L_2 = L_1 \quad \forall \lambda \in \mathbb{C}_b \tag{4.5}$$

$$(\lambda I-A)L_2 + L_1 = L_2 \quad \forall \lambda \in \mathbb{C}_b \ . \tag{4.6}$$

Suppose that V_1 and V_2 are (A,B)-invariant subspaces, and $V_1 \subset V_2$. We shall say that (A,B) is <u>stabilizable between V_1 and V_2</u> if there exists an $F \in \underline{F}(V_1) \cap \underline{F}(V_2)$ such that $\sigma(A+BF: V_2/V_1) \subset \mathbb{C}_g$. We have the following characterization of this property.

<u>*Lemma 4.3*</u> *Let V_1 and V_2 be (A,B)-invariant subspaces, with $V_1 \subset V_2$. Then (A,B) is stabilizable between V_1 and V_2 if and only if*

$$(\lambda I-A)V_2 + V_1 + \mathrm{im}\, B = V_2 + \mathrm{im}\, B \quad \forall\lambda \in \mathbb{C}_b \ . \tag{4.7}$$

<u>Proof</u>. First, suppose there exists $F \in \underline{F}(V_2) \cap \underline{F}(V_1)$ such that A+BF: V_2/V_1 is stable. According to Lemma 4.2, we then have

$$(\lambda I-(A+BF))V_2 + V_1 = V_2 \quad \forall\lambda \in \mathbb{C}_b \ . \tag{4.8}$$

Adding im B on both sides now leads immediately to (4.7), if one uses the obvious equality

$$(\lambda I - (A+BF))V_2 + \mathrm{im}\, B = (\lambda I-A)V_2 + \mathrm{im}\, B \ . \tag{4.9}$$

Next, suppose that (4.7) holds. Construct a mapping $F_0 \in \underline{F}(V_1) \cap \underline{F}(V_2)$

by first defining F_0 on V_1 such that $(A+BF_0)V_1 \subset V_1$, then extending F_0 on V_2 in such a way that $(A+BF_0)V_2 \subset V_2$, and finally extending F_0 in an arbitrary way to a mapping defined on all of X. Consider the controllability subspace

$$R_2 := \langle A+BF_0 | \text{im } B \cap V_2 \rangle \tag{4.10}$$

Define $A_0 := A+BF_0 : R_2$, and let $R: U \to U$ be such that $\text{im } BR = \text{im } B \cap V_2$. Write $B_0 := BR$. By definition, we have $\langle A_0 | \text{im } B_0 \rangle = R_2$, and so it follows from lemma 2.3 that every (A_0, B_0)-invariant subspace of R_2 is outer-stabilizable. In particular, there exists an $F_1: R_2 \to U$ such that $(A_0+B_0F_1)(R_2 \cap V_1) \subset R_2 \cap V_1$ and $\sigma(A_0+B_0F_1 : R_2/R_2 \cap V_1) \subset \mathbb{C}_g$. The mapping $F_0 + RF_1$, which is defined only on R_2, can be extended to a mapping $F: X \to U$ in such a way that $F \in \underline{F}(V_1) \cap \underline{F}(V_2)$. We claim that this mapping F satisfies $\sigma(A+BF: V_2/V_1) \subset C_g$.

To prove this, first note that $A+BF: (R_2+V_1)/V_1$ is similar to $A+BF: R_2/(R_2 \cap V_1) = A_0+B_0F_1 : R_2/(R_2 \cap V_1)$ which is stable by construction. Furthermore, we have given that (4.7) holds and this implies (using (4.9) again)

$$(\lambda I-(A+BF))V_2 + V_1 + \text{im } B = V_2 + \text{im } B \qquad \forall \lambda \in \mathbb{C}_b \tag{4.11}$$

Taking intersections with V_2 on both sides, we get

$$(\lambda I-(A+BF))V_2 + V_1 + (\text{im } B \cap V_2) = V_2 \qquad \forall \lambda \in \mathbb{C}_b . \tag{4.12}$$

Because $\text{im } B \cap V_2 \subset R_2 \subset V_2$, this implies

$$(\lambda I-(A+BF))V_2 + V_1 + R_2 = V_2 \qquad \forall \lambda \in \mathbb{C}_b \tag{4.13}$$

which by Lemma 4.2, means that $\sigma(A+BF: V_2/(V_1+R_2)) \subset \mathbb{C}_g$. The proof is done.

We are going to apply this lemma in the following way.

<u>*Lemma 4.4*</u> *Let A_e be an extended system matrix of the form (2.8). If M_1 and M_2 are both A_e-invariant subspaces satisfying $M_1 \subset M_2$ and $\sigma(A_e : M_2/M_1) \subset \mathbb{C}_g$, then the pair (A,B) is stabilizable between PM_1 and PM_2.*

<u>Proof</u> Take $x \in PM_2$, and let $w \in W$ be such that $\begin{pmatrix} x \\ w \end{pmatrix} \in M_2$. Also, take $\lambda \in \mathbb{C}_b$. By lemma 4.2, there exists vectors $\begin{pmatrix} x_1 \\ w_1 \end{pmatrix} \in M_1$ and $\begin{pmatrix} x_2 \\ w_2 \end{pmatrix} \in M_2$ such that

$$\begin{pmatrix} x \\ w \end{pmatrix} = (\lambda I - A_e) \begin{pmatrix} x_2 \\ w_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ w_1 \end{pmatrix} . \tag{4.14}$$

In particular, we get

$$x = (\lambda I - A)x_2 - B(KCx_2 + F_c w_2) + x_1 . \tag{4.15}$$

This shows that

$$PM_2 \subset (\lambda I - A)PM_2 + PM_1 + \text{im } B \qquad \forall \lambda \in \mathbb{C}_b . \tag{4.16}$$

By the (A,B)-invariance of PM_2, this is the same as

$$PM_2 + \text{im } B = (\lambda I - A)PM_2 + PM_1 + \text{im } B \quad \forall \lambda \in \mathbb{C}_b . \tag{4.17}$$

An application of Lemma 4.3 now gives the desired result.

Everything that has been said above about the pair (A,B) can be dualized into statements about the pair (C,A). If T_1 and T_2 are (C,A)-invariant subspaces such that $T_1 \subset T_2$, we shall say that the pair (C,A) is <u>detectable between T_1 and T_2</u> if there exists a $G \in G(T_1) \cap \underline{\underline{G}}(T_2)$ such that $\sigma(A-GC : T_2/T_1) \subset \mathbb{C}_g$. The following results correspond to Lemma 4.3 and Lemma 4.4, respectively:

Lemma 4.5 Let T_1 and T_2 be (C,A)-invariant subspaces, with $T_1 \subset T_2$. Then (C,A) is detectable between T_1 and T_2 if and only if

$$(\lambda I-A)^{-1}T_1 \cap T_2 \cap \ker C = T_1 \cap \ker C \qquad \forall \lambda \in \mathbb{C}_b . \qquad (4.18)$$

Lemma 4.6 Let A_e be an extended system matrix of the form (2.8). If M_1 and M_2 are both A_e-invariant subspaces, satisfying $M_1 \subset M_2$ and $\sigma(A_e:M_2/M_1) \subset \mathbb{C}_g$, then the pair (C,A) is detectable between $Q^{-1}M_1$ and $Q^{-1}M_2$.

It is useful to note the following result, which is a direct consequence of Lemma 4.3.

Corollary 4.7 Suppose that V_1, V_2 and V_3 are (A,B) - invariant subspaces, with $V_1 \subset V_2$. If the pair (A,B) is stabilizable between V_1 and V_2, then (A,B) is also stabilizable between $V_1 + V_3$ and $V_2 + V_3$.

After these preparations, it is easy to give an extensive list of necessary conditions for the algebraic regulator problem to be solvable.

Proposition 4.8 Suppose that the compensator (2.4-5) provides a solution to the algebraic regulator problem for the system (2.1-3); so there exists an A_e-invariant subspace M such that (2.13) holds and such that $\sigma(A_e:X^e/M) \subset \mathbb{C}_g$, and moreover the dimensional equality (3.7) holds. Write $V:=PM$, $T:=Q^{-1}M$, $V_0:=PX_b^e(A_e)$, and $T_0:=Q^{-1}X_b^e(A_e)$. Then the following is true:

(i) the pairs (T_0, V_0) and (T, V) are compatible (C,A,B)-pairs.

(ii) $T_0 \subset T$, and $V_0 \subset V$

(iii) $\operatorname{im} E \subset T \subset V \subset \ker D$

(iv) *(A,B) is stabilizable between V_0 and V and between V and X*

(v) *(C,A) is detectable between T_0 and T and between T and X*

(vi) $V_0 \cap (X_{det} + X_{stab}) = X_{det}$.

Proof The conditions (i) to (v) follow immediately from, respectively, Lemma 4.1, the fact that $X_b^e(A_e) \subset M$, the remark leading to (2.15), Lemma 4.4, and Lemma 4.6. To prove (vi), first note that T_0 is, by (v), a detectability subspace. Therefore, $X_{det} \subset T_0 \subset V_0$ and so we have

$$X_{det} \subset V_0 \cap (X_{det} + X_{stab}) \tag{4.19}$$

From (iv), it follows that V_0 is outer-stabilizable, so that $V_0 + X_{stab} = X$ (Lemma 2.3). Consequently, the following dimensional relations hold:

$$\begin{aligned} \dim(V_0 \cap (X_{det} + X_{stab})) &= \\ &= \dim V_0 + \dim(X_{det} + X_{stab}) - \dim X = \\ &= \dim V_0 - \operatorname{codim}(X_{det} + X_{stab}) \leq \\ &\leq \dim X_b^e(A_e) - \operatorname{codim}(X_{det} + X_{stab}) = \\ &= \dim X_{det} \quad . \end{aligned} \tag{4.20}$$

This shows that in fact equality holds in (4.19).

The list is not completely economical; for instance, it is easy to see that (ii) already implies that the (C,A,B)-pairs (T_0, V_0) and (T, V) are compatible. The extras have been obtained with little effort, however, and the form of the list is convenient for the next section where we are going to prove that the conditions given above are also sufficient.

5. Sufficiency, Main Result

There is a general method of compensator construction, in which it is also possible to keep track of the relation between invariant subspaces in the constructed closed-loop system and certain (C,A,B)-pairs in X. Here, we shall only need the following relatively simple result; more elaborate versions are given in [18], Thm. 4.1, and [19], pp. 63-64. The proof is basically easy, consisting mainly of using natural isomorphisms between subspaces of X and of X^e, and can be found in the cited references.

Lemma 5.1 *Let the system (2.1-3) be given. Suppose that we have a (C,A,B)-pair (T_c, V_c), an $F \in \underline{F}(V_c)$ such that $\ker F \supset T_c$, and a $G \in \underline{G}(T_c)$ such that $\operatorname{im} G \subset V_c$. Then a compensator of the form (2.4-5) can be defined as follows: Let W be a real vector space of dimension $\dim V_c - \dim T_c$. Let R be a mapping from V_c onto W such that $\ker R = T_c$, and let R^+ be any right inverse of R. Set $K = 0$, $F_c = FR^+$, $G_c = RG$, and $A_c = R(A+BF-GC)R^+$. The extended system matrix, that is obtained as*

$$A_e = \begin{pmatrix} A & BFR^+ \\ RGC & R(A+BF-GC)R^+ \end{pmatrix} \tag{5.1}$$

has the following eigenvalues:

$$\sigma(A_e) = \sigma(A+BF: V_c) \cup \sigma(A-GC: X/T_c) \quad . \tag{5.2}$$

Moreover, if (T,V) is a (C,A,B)-pair such that $AT \subset V$, $T_c \subset T \subset V \subset V_c$, $(A+BF)V \subset V$ and $(A-GC)T \subset T$, then the subspace M of X defined by

$$M := \left\{\begin{pmatrix} x \\ 0 \end{pmatrix} \,\middle|\, x \in T\right\} + \left\{\begin{pmatrix} x \\ Rx \end{pmatrix} \,\middle|\, x \in V\right\} \tag{5.3}$$

is A_e-invariant. The subspace

$$M_c := \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \middle| \, x \in V_c \right\} \tag{5.4}$$

is also A_e*-invariant, and the following similarity relations hold:*

$$A_e : X^e/(M+M_c) \cong A-GC: X/T \tag{5.5}$$

$$A_e : (M+M_c)/M_c \cong A_e : M/(M \cap M_c) \cong A-GC: T/T_c \tag{5.6}$$

$$A_e : (M+M_c)/M \cong A_e : M_c/(M \cap M_c) \cong A+BF: V_c/V \tag{5.7}$$

$$A_e : M \cap M_c \cong A+BF: V \ . \tag{5.8}$$

Picturewise, the relations (5.5-8) can be described as follows:

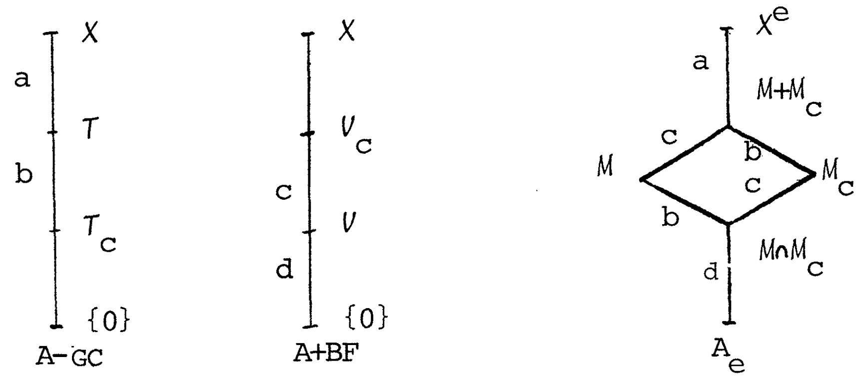


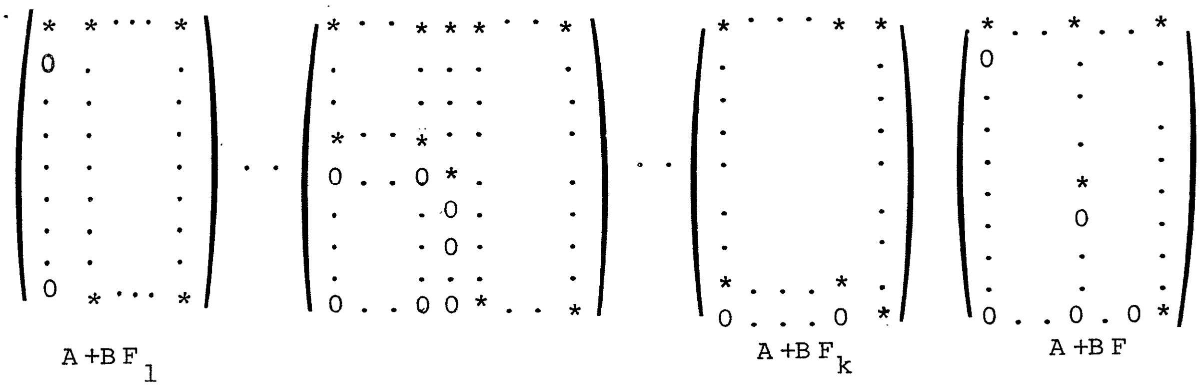
Fig. 5.1 Regulator Construction

In order to translate data on a chain of (A,B)-invariant subspaces into data on a feedback mapping, the following lemma is useful.

<u>*Lemma 5.2*</u> *Suppose we have a chain of (A,B)-invariant subspaces* $\{0\} = V_0 \subset V_1 \subset .. \subset V_{k-1} \subset V_k = X$. *Also, let mappings* $F_i \in \underline{F}(V_i) \cap \underline{F}(V_{i-1})$ *be given, for* $i = 1,\dots,k$. *Then there exists a mapping* $F \in \underline{F}(V_1) \cap .. \cap F(V_{k-1})$ *such that* $A+BF: V_i/V_{i-1} = A+BF_i : V_i/V_{i-1}$ *for all* $i \in \{1,\dots,k\}$.

<u>Proof</u> Select basis elements $\{x_1^1,\ldots,x_{n_1}^1,\ x_1^2,\ldots,x_{n_2}^2,\ldots,x_1^k,\ldots,x_{n_k}^k\}$ such that $\{x_1^1,\ldots,x_{n_1}^1,\ldots,x_1^i,\ldots,x_{n_i}^i\}$ forms a basis for V_i for all $i \in \{1,\ldots,k\}$. Define F by $Fx_j^i = F_i x_j^i$ $(i=1,\ldots,k;\ j=1,\ldots,n_i)$. Then F satisfies the requirements.

To illustrate the proof, consider the block matrix representations for the mappings $A+BF_i$ $(i=1,\ldots,k)$ and A+BF with respect to the selected basis:



It is now not difficult to show that the necessary conditions derived in the previous section are also sufficient.

<u>*Theorem 5.3*</u> *The algebraic regulator problem for the system (2.1-3) is solvable if and only if there exist two compatible (C,A,B)-pairs (T_0, V_0) and (T,V) such that*

(i) $T_0 \subset T,\quad V_0 \subset V$

(ii) $\mathrm{im}\, E \subset T \subset V \subset \ker D$

(iii) (A.B) is stabilizable between V_0 and V and between V and X

(iv) (C,A) is detectable between T_0 and T and between T and X

(v) $V_0 \cap (X_{det} + X_{stab}) = X_{det}$.

<u>Proof</u> In view of Prop. 4.8, it remains to show the sufficiency of the conditions. Employing a preliminary static output feedback if necessary, we may assume that $AT_0 \subset V_0$ and $AT \subset V$ (see Lemma 2.5 and the remarks following it). It follows that there exists an $F_1 \in \underline{F}(V_0)$ with ker $F_1 \supset T_0$. Using (iii) and lemma 5.2, we see that there exists an $F \in \underline{F}(V_0) \cap \underline{F}(V)$ such that ker $F \supset T_0$ and $\sigma(A+BF:X/V_0) \subset \mathbb{C}_g$. Using (iv) and the dual of lemma 5.2, we find that there exists a $G \in \underline{G}(T_0)$ $\underline{G}(T)$ such that $\sigma(A- C:X/T_0) \subset \mathbb{C}_g$. Now we apply the compensator construction of lemma 5.1, using T_0 for T_c and X for V_c. The invariant subspace M related to the pair (T,V) takes care of the disturbance decoupling property, by condition (ii). From the relations (5.5-8), we see that $X_b^e(A_e) \subset M \cap M_c = \{(\begin{smallmatrix} x \\ Rx \end{smallmatrix}) | x \in V\}$. Consequently, $PX_b^e(A_e) \subset V \subset \ker D$ and we have output stability. In fact, we see from (5.8) that $\dim X_b^e(A_e) \leq \dim V_0$. By condition (v), we have $\dim V_0 \leq \operatorname{codim}(X_{det} + X_{stab}) + \dim X_{det}$, and we conclude that the compensator constructed above leads to internal stability as well.

The proof is constructive once the pairs (T_0,V_0) and (T,V) are given. We shall now proceed to discuss how the existence of these pairs can be verified by an algorithm that will also construct such pairs, if they exist.

6. A Verification Algorithm

It may not seem easy to verify the conditions of Thm. 5.3 because they are stated in terms of two (C,A,B)-pairs, which gives us four variable subspaces. Without much effort, one can see that T_0 can always be replaced by X_{det} and T by $T_g^*(\text{im } E)$, but that still leaves us with two variable subspaces. It is possible to express the conditions in terms of V_0 (as in [18] and [19]), but concentrating on V will lead to a result that is more attractive from a numerical point of view. Before we come to this, some preliminary work is needed.

Lemma 6.1 *Let V be an (A,B)-invariant subspace, and let R be defined by* $R = \langle A+BF \mid \text{im } B \cap V \rangle$ $(F \in \underline{F}(V))$. (This defines R uniquely: *see Lemma 2.2) If* V_1 *is an (A,B)-invariant subspace such that* $R \subset V_1 \subset V$, *then* V_1 *is (A+BF)-invariant for all* $F \in F(V)$.

Proof Take $F \in \underline{F}(V)$, and $F_0 \in \underline{F}(V_1)$. We have $(A+BF)V_1 \subset (A+BF)V \subset V$, but also $(A+BF)V_1 \subset (A+BF_0)V_1 + B(F-F_0)V_1 \subset V_1 + \text{im } B$. Hence $(A+BF)V_1 \subset V \cap (V_1 + \text{im } B) \subset V_1 + R \subset V_1$.

Lemma 6.2. Let V_0 be an (A,B)-invariant subspace contained in a subspace K. The set of all (A,B)-invariant subspaces V contained in K and containing V_0, that are such that (A,B) is stabilizable between V_0 and V, contains a unique maximal element, which is given by $V_0 + V_g^*(K)$.

Proof. It follows immediately from Cor. 4.7 that (A,B) is stabilizable between V_0 and $V_0 + V_g^*(K)$. Conversely, let V be an (A,B)-invariant subspace with $V_0 \subset V \subset K$, such that (A,B) is stabilizable between V_0 and V.

Then there exists an $F \in \underline{F}(V_0) \cap \underline{F}(V) \cap \underline{F}(V^*(K))$ such that $\sigma(A+BF:V/V_0) \subset \mathbb{C}_g$. By lemma 6.1, we automatically have $F \in \underline{F}(V_g^*(K))$ as well. Now, on the one hand,

$$\sigma(A+BF:V/(V \cap (V_g^*(K) + V_0))) =$$

$$= \sigma(A+BF: (V_g^*(K) + V)/(V_g^*(K) + V_0)) \subset$$

$$\subset \sigma(A+BF: V^*(K)/V_g^*(K)) \subset \mathbb{C}_b,$$

but on the other hand,

$$\sigma(A+BF:V/\subset V \cap (V_g^*(K) + V_0))) \subset \tag{6.2}$$

$$\subset \sigma(A+BF:V/V_0) \subset \mathbb{C}_g .$$

It follows that

$$\sigma(A+BF:V/(V \cap (V_g^*(K) + V_0))) = \emptyset \tag{6.3}$$

or, $V \subset V_g^*(K) + V_0$.

We can now re-formulate Thm. 5.3 as follows.

<u>Thm. 6.3</u> The algebraic regulator problem for the system (2.1-3) is solvable if and only if there exists an (A,B)-invariant subspace V such that

$$V \subset \ker D \tag{6.4}$$

$$V + X_{stab} = X \tag{6.5}$$

$$V \cap (X_{det} + X_{stab}) = X_{det} + V_g^*(\ker D) \tag{6.6}$$

$$T_g^*(\text{im } E) \subset V . \tag{6.7}$$

Proof To prove the necessity, we assume that the conditions of Thm. 5.3 hold. So we have two compatible (C,A,B)-pairs $(\mathcal{T}_0, \mathcal{V}_0)$ and $(\mathcal{T}, \mathcal{V})$ satisfying (i-v). We shall show that $\mathcal{V}_0 + \mathcal{V}_g^*$ (ker D) satisfies the conditions (6.4-7). From (i) and (ii) we immediately have (6.4), (6.5) follows directly from (i) and (iii) with use of lemma 2.3, and (6.7) is obtained from (ii) and (iii) by an application of lemma 6.2. Finally, the obvious fact that $\mathcal{V}_g^*$ (ker D) $\subset \mathcal{X}_{det} + \mathcal{X}_{stab}$ entails, by condition (v),

$$(\mathcal{V}_0 + \mathcal{V}_g^*(\ker D)) \cap (\mathcal{X}_{det} + \mathcal{X}_{stab}) = \tag{6.8}$$

$$= \mathcal{V}_0 \cap (\mathcal{X}_{det} + \mathcal{X}_{stab}) + \mathcal{V}_g^*(\ker D) =$$

$$= \mathcal{X}_{det} + \mathcal{V}_g^*(\ker D) \quad .$$

For the sufficiency, we note that $\mathcal{X}_{det} \subset \mathcal{T}_g^*$ (im E) $\subset \mathcal{V} \subset$ ker D, and we consider the chain of (A,B) - invariant subspaces $\{0\} \subset \mathcal{X}_{det} \subset \mathcal{X}_{det} + \mathcal{V}_g^*(\ker D) \subset \mathcal{V} \subset \mathcal{X}$. Cor. 4.7 shows that (A,B) is stabilizable between $\mathcal{X}_{det}$ and $\mathcal{X}_{det} + \mathcal{V}_g^*(\ker D)$, and (6.5) shows that (A,B) is also stabilizable between $\mathcal{V}$ and $\mathcal{X}$. By lemma 5.2, there exists an $F \in \underline{F}(\mathcal{X}_{det}) \cap \underline{F}(\mathcal{X}_{det} + \mathcal{V}_g^*(\ker D)) \cap \underline{F}(\mathcal{V})$ such that $\ker F \supset \mathcal{X}_{det}$, $\sigma(A+BF: (\mathcal{X}_{det} + \mathcal{V}_g^*(\ker D))/\mathcal{X}_{det}) \subset \mathbb{C}_g$ and $\sigma(A+BF: \mathcal{X}/\mathcal{V}) \subset \mathbb{C}_g$. Now, define $\mathcal{V}_0$ by

$$\mathcal{V}_0 = \mathcal{X}_b(A+BF) \quad . \tag{6.9}$$

It is clear that $\mathcal{X}_{det} \subset \mathcal{V}_0 \subset \mathcal{V}$, and also that $\mathcal{V}_0 \cap (\mathcal{X}_{det} + \mathcal{V}_g^*(\ker D)) = \mathcal{X}_{det}$. Using (6.6) we see that condition (v) of Thm. 5.3 is satisfied. We also see that (iii) holds. The other conditions are easily verified, defining $\mathcal{T}_0 = \mathcal{X}_{det}$ and $\mathcal{T} = \mathcal{T}_g^*(\text{im } E)$.

The following slight variation of this result will be useful.

Corollary 6.4 The algebraic regulator problem for the system (2.1-3) is solvable if and only if

$$\mathcal{V}^*(\ker D) + \mathcal{X}_{stab} = \mathcal{X} \tag{6.10}$$

and there exists an (A,B)-invariant subspace $\mathcal{V}$ such that

$$\mathcal{V} \subset \mathcal{V}^*(\ker D) \tag{6.11}$$

$$\mathcal{V} + (\mathcal{X}_{\det} + (\mathcal{V}^*(\ker D) \cap \mathcal{X}_{stab})) = \mathcal{V}^*(\ker D) \tag{6.12}$$

$$\mathcal{V} \cap (\mathcal{X}_{\det} + (\mathcal{V}^*(\ker D) \cap \mathcal{X}_{stab})) = \mathcal{X}_{\det} + \mathcal{V}_g^*(\ker D) \tag{6.13}$$

$$\mathcal{T}_g^*(\mathrm{im}\, E) \subset \mathcal{V} \tag{6.14}$$

Proof. Necessity: (6.10) follows from (6.4) and (6.5), (6.12) is obtained by intersecting both sides of the equality in (6.5) with $\mathcal{V}^*(\ker D)$, and (6.13) is obtained in the same way from (6.6). Sufficiency: for (6.5), add $\mathcal{X}_{stab}$ on both sides of (6.12) and use (6.10). Note that $\mathcal{X}_{\det} \subset \mathcal{V}$ by (6.13) (or (6.14)), and consequently

$$\begin{aligned} \mathcal{V} \cap (\mathcal{X}_{\det} + (\mathcal{V}^*(\ker D) \cap \mathcal{X}_{stab})) &= \\ = \mathcal{X}_{\det} + (\mathcal{V} \cap \mathcal{X}_{stab}) &= \\ = \mathcal{V} \cap (\mathcal{X}_{\det} + \mathcal{X}_{stab}) &. \end{aligned} \tag{6.15}$$

Now use (6.13) again to obtain (6.6).

We see from Thm. 6.3 that the subspace $\mathcal{V}$ that we are looking for must be in between $\mathcal{X}_{\det} + \mathcal{V}_g^*(\ker D)$ and $\mathcal{V}^*(\ker D)$, and the advantage of the corollary is that the crucial conditions (6.12-13) are formulated in terms of these subspaces and of another subspace that is in between the two, $\mathcal{X}_{det} + (\mathcal{V}^*(\ker D) \cap \mathcal{X}_{stab})$. Picturewise, the situation we are trying to establish looks as follows:

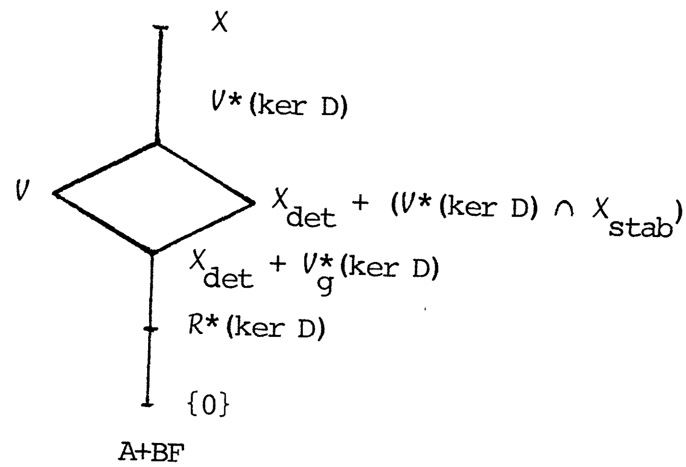


Fig. 6.1 The Split-Off Property

The important point to note here is that we are talking about (A,B)-invariant subspaces that all contain the subspace

$$\mathcal{R}^*(\ker D) := \langle A+BF \mid \operatorname{im} B \cap \mathcal{V}^*(\ker D) \rangle \quad (F \in \underline{F}(\mathcal{V}^*(\ker D))) \qquad (6.16)$$

and which are therefore, by Lemma 6.1, all invariant for each $F \in \underline{F}(\mathcal{V}^*(\ker D))$. This means that we can pick any $F \in \underline{F}(\mathcal{V}^*(\ker D))$ and see if a subspace $\mathcal{V}$ can be 'split off' as depicted in Fig. 6.1. This comes down to requiring that the subspace

$$(\mathcal{X}_{det} + (\mathcal{V}^*(\ker D) \cap \mathcal{X}_{stab}))/(\mathcal{X}_{det} + \mathcal{V}^*_g(\ker D)) \qquad (6.17)$$

must decompose the quotient space $\mathcal{V}^*(\ker D)/(\mathcal{X}_{det} + \mathcal{V}^*_g(\ker D))$ with respect to the mapping induced by A+BF on this space. This is well-known to be equivalent to a linear matrix equation (see for instance [2], p.21). The conclusion that we have now reached should be compared to Thms. 7.3 and 7.4 in [2]. In particular, the problem is trivial under the minimum phase condition $\mathcal{V}^*_g(\ker D) = \mathcal{V}^*(\ker D) \cap \mathcal{X}_{stab}$: in this case, the only solution of (6.11-13) is $\mathcal{V} = \mathcal{V}^*(\ker D)$. (This condition is often assumed in classical control theory, be it not quite in this formulation.)

To obtain a computational criterion, we may proceed as follows. Noting that it is necessary that $\mathcal{X}_{det} \subset \mathcal{V}^*(\ker D)$, we may set up a basis for $\mathcal{X}$ that is adapted to the chain of subspaces $\{0\} \subset \mathcal{X}_{det} + \mathcal{V}^*_g(\ker D) \subset \subset \mathcal{X}_{det} + (\mathcal{V}^*(\ker D) \cap \mathcal{X}_{stab}) \subset \mathcal{V}^*(\ker D) \subset \mathcal{X}$. Next, we form block matrix representations for the relevant mappings and subspaces. We get

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \\ 0 \\ B_4 \end{pmatrix}, \quad \mathcal{T}^*_g(\operatorname{im} E) = \operatorname{sp}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix}. \tag{6.18}$$

Here, the fact that $A_{21} = 0$ and $A_{31} = 0$ is explained by noting that

$$\begin{aligned} &A(\mathcal{X}_{det} + \mathcal{V}^*_g(\ker D)) \cap \mathcal{V}^*(\ker D) \subset \\ &\subset (\mathcal{X}_{det} + \mathcal{V}^*_g(\ker D) + \operatorname{im} B) \cap \mathcal{V}^*(\ker D) \subset \\ &\subset \mathcal{X}_{det} + \mathcal{V}^*_g(\ker D) + (\operatorname{im} B \cap \mathcal{V}^*(\ker D)) = \\ &= \mathcal{X}_{det} + \mathcal{V}^*_g(\ker D) \quad . \end{aligned} \tag{6.19}$$

The same explanation goes for $A_{32} = 0$. We used the fact that $\text{im } B \cap \mathcal{V}^*(\ker D) \subset \mathcal{X}_{det} + \mathcal{V}^*_g(\ker D)$, which also entails $B_2 = 0$ and $B_3 = 0$.

A subspace $\mathcal{V}$ satisfies (6.11-13) if and only if it can be represented, with respect to the selected basis, in the following way:

$$\mathcal{V} = \text{sp}\begin{pmatrix} I & 0 \\ 0 & X \\ 0 & I \\ 0 & 0 \end{pmatrix} \tag{6.20}$$

where X may be any matrix of suitable size. Such a subspace is (A,B)-invariant, by Lemma 2.1, if and only if there exist matrices Q and R such that

$$\begin{pmatrix} A_{11} & A_{12}X + A_{13} \\ 0 & A_{22}X + A_{23} \\ 0 & A_{33} \\ A_{41} & A_{42}X + A_{43} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & X \\ 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \\ 0 \\ B_4 \end{pmatrix} (R_1 R_2) . \tag{6.21}$$

By elementary calculations, and using the fact that there exist, by the (A,B)-invariance of $\mathcal{V}^*(\ker D)$, matrices F_i such that $A_{4i} + B_4 F_i = 0$ $(i=1,\ldots,3)$, we find that such matrices Q and R exist if and only if

$$A_{22}X + A_{23} = XA_{33} \tag{6.22}$$

(This is, of course, Sylvester's equation ([2], p.21).) Furthermore , the condition (6.14) holds if and only if there exists a matrix S such that

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & X \\ 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \qquad (6.23)$$

This is true if and only if $T_4 = 0$ and

$$T_2 = X\, T_3 \quad . \qquad (6.24)$$

Our conclusion is as follows.

Corollary 6.5. The algebraic regulator problem for the system (2.1-3) is solvable if and only if the following conditions hold.

$$\mathcal{V}^*(\ker D) + \mathcal{X}_{stab} = \mathcal{X} \qquad (6.25)$$

$$\mathcal{T}_g^*(\mathrm{im}\, E) \subset \mathcal{V}^*(\ker D) \qquad (6.26)$$

and there exists a matrix X satisfying

$$XA_{33} - A_{22}X = A_{23} \qquad (6.27)$$

$$XT_3 = T_2 \qquad (6.28)$$

where the A- and T-matrices are defined as in (6.18).

For any $F \in \underline{F}(\mathcal{V}^*(\ker D))$, A_{33} is the matrix of

$$A+BF:\ \mathcal{V}^*(\ker D)/(\mathcal{X}_{det} + (\mathcal{V}^*(\ker D) \cap \mathcal{X}_{stab})) \qquad (6.29)$$

and A_{22} is the matrix of

$$\text{A+BF}: (X_{det} + (\mathcal{V}^*(\ker D) \cap \mathcal{X}_{stab}))/(\mathcal{X}_{det} + \mathcal{V}^*_g(\ker D)). \qquad (6.30)$$

Under the conditions (6.25-26), the mapping in (6.29) is similar to $A:\mathcal{X}/(\mathcal{X}_{det} + \mathcal{X}_{stab})$, and so we can say that A_{33} represents the signal dynamics. In view of the interpretations of section 3 and of [2], Section 5.5, the eigenvalues of the mapping in (6.30) may be identified as the relevant unstable plant zeros. In particular, since we know that the equation (6.27) has a unique solution if and only if the matrices A_{22} and A_{33} have no eigenvalues in common ([26], p.225), we can say that a sufficient condition for (6.27) to be solvable is that the signal poles and the relevant unstable plant zeros are distinct.

It should be emphasized that quite a bit of numerical techniques are available to verify the conditions (6.25-28). The computation of $\mathcal{V}^*(\ker D)$ and related subspaces and mappings is discussed from the numerical point of view in [20-22]. The equation (6.27) can be solved efficiently, at least in the case where the eigenvalues of A_{22} and A_{33} are distinct, by the method of [23]. Note that the size of A_{22} is the number of unstable plant zeros whereas the size of A_{33} is the number of signal poles, and both numbers will be moderate in very many situations. Finally, if (6.27) has a unique solution, then (6.28) is just a matter of checking . All this gives hope that the solution provided by Cor. 6.5 will be a good foundation for developing numerical software for general regulator problems.

7. The Internal Model

Francis [7] proved that, in the special case where the output is the same as the observation (C=D), any compensator that solves the algebraic regulator problem must contain a copy of the signal dynamics, the so-called "internal model". A similar result was derived by Bengtsson [8] in a frequency-domain setting. Another form of the internal model principle, which involves a certain reduplication of signal dynamics, can be derived from strong robustness requirements: see [2], Ch. 8. Below, we shall show how the internal model can be obtained from the set-up presented here. Our result is slightly more general than that of Francis.

Proposition 7.1. *Suppose that the compensator (2.4-5) provides a solution to the algebraic regulator problem for the system (2.1-3), in which* $\ker D \subset \ker C$. *Then there exists an* A_c*-invariant subspace* $\mathcal{W}_0$ *of* $\mathcal{W}$ *such that* $A_c:\mathcal{W}_0$ *is similar to* $A:\mathcal{X}/(\mathcal{X}_{det} + \mathcal{X}_{stab})$.

Proof Write $\mathcal{T}_0 = Q^{-1}\mathcal{X}_b^e(A_e)$, $\mathcal{V}_0 = P\mathcal{X}_b^e(A_e)$. We first show that $\mathcal{T}_0 = \mathcal{X}_{det}$. Being a (C,A)-invariant subspace in $\mathcal{V}_0 \subset \ker D \subset \ker C$, $\mathcal{T}_0$ must in fact be A-invariant. So we have $\mathcal{T}_0 \subset \langle \ker C | A \rangle$. It is easily checked that $Q(\langle \ker C|A\rangle)$ is A_e-invariant and that Q intertwines $A: \langle \ker C|A\rangle$ and $A_e: Q(\langle \ker C|A\rangle)$. From this, it follows that $\mathcal{T}_0 = \mathcal{X}_{det}$.

From Prop. 4.8, Lemma 2.3, and the formulation of internal stability in (3.7), we see that

$$\dim \mathcal{V}_0 = \dim \mathcal{X}_b^e(A_e) \quad . \tag{7.1}$$

This implies that there exists a mapping $L:\mathcal{V}_0 \to \mathcal{W}$ such that

$$\mathcal{X}_b^e(A_e) = \{ \binom{x}{Lx} \mid x \in \mathcal{V}_0 \} \quad . \tag{7.2}$$

Because $\mathcal{V}_0 \subset \ker D \subset \ker C$, we have, for $x \in \mathcal{V}_0$,

$$A_e \begin{pmatrix} x \\ Lx \end{pmatrix} = \begin{pmatrix} (A+BF_cL)x \\ A_cLx \end{pmatrix} \in \mathcal{X}_b^e(A_e) \quad . \tag{7.3}$$

Writing $F_0 := F_cL$, we find that $(A+BF_0)\mathcal{V}_0 \subset \mathcal{V}_0$ and $A_cLx = L(A+BF_0)x$ for $x \in \mathcal{V}_0$. This means that $\mathcal{W}_0 :=$ im L is A_c-invariant, and that $A_c:\mathcal{W}$ similar to $A+BF_0:\mathcal{V}_0/(\ker L)$. Because $\ker L = Q^{-1}\mathcal{X}_b^e(A_e) = \mathcal{X}_{det}$, it remains to show that $A+BF_0:\mathcal{V}_0/\mathcal{X}_{det}$ is similar to $A:\mathcal{X}/(\mathcal{X}_{det}+\mathcal{X}_{stab})$. By Prop. 4.8 and Lemma 2.3, we have

$$\begin{aligned} & A+BF_0:\mathcal{V}_0/\mathcal{X}_{det} \cong \\ & \cong A+BF_0:\mathcal{V}_0/(\mathcal{V}_0 \cap (\mathcal{X}_{det}+\mathcal{X}_{stab})) \cong \\ & \cong A+BF_0: (\mathcal{V}_0 + (\mathcal{X}_{det} + \mathcal{X}_{stab}))/(\mathcal{X}_{det} + \mathcal{X}_{stab}) \cong \\ & \cong A:\mathcal{X}/(\mathcal{X}_{det} + \mathcal{X}_{stab}) \quad . \end{aligned} \tag{7.4}$$

This concludes the proof.

The internal model principle does not have to hold if there are observations available that are independent from the output ($\ker D \not\subset \ker C$). Indeed, the disturbance decoupling problem may be solvable in non-trivial cases, and then any relation between the compensator dynamics and the disturbance dynamics is quite effectively precluded, since we did not make any assumption on the dynamics of the disturbance entering through E. The point is, of course, that the availability of observations independent from the output allows for a certain freedom of design, which may be used to advantage.

The often-used assumption C=D is to be considered as a serious specialization. Let us conclude by showing, picturewise, how the internal model fits into the structure discussed in previous sections:

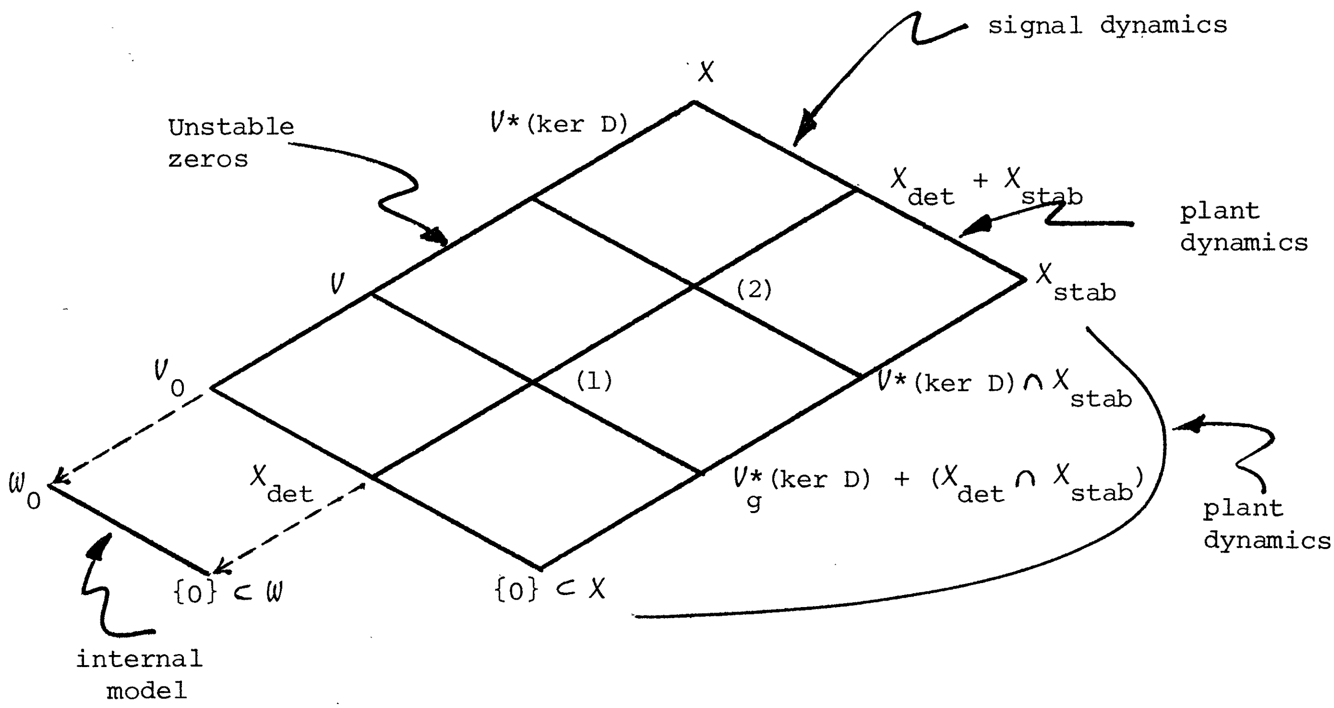


Fig. 7.1 Structure of the Algebraic Regulator Problem

(1) $\mathcal{X}_{det} + \mathcal{V}^*_g(\ker D)$

(2) $\mathcal{X}_{det} + (\mathcal{V}^*(\ker D) \cap \mathcal{X}_{stab})$

In this picture, only the presence of the internal model depends on the assumption $\ker D \subset \ker C$; all the rest holds in general.

8. Conclusions

We have been able to solve a general version of the algebraic regulator problem, requiring output stability, internal stability, and disturbance decoupling as well. The basic Thm. 5.3 has been derived in a quite straightforward way, using material that is essentially elementary, as it is also likely to be useful for the analysis of other feedback design problems. Among this material, especially useful are lemma 4.1, which gives the connection between closed-loop invariant subspaces and (C,A,B)-pairs, and lemma 4.4, which adds the stability aspects to this connection. On the constructive side, the versatile compensator construction of Lemma 5.1 is important, and the 'paste-together' lemma 5.2 comes in handy. The main drawback of the results that we get from this type of analysis, like Thm. 5.3, is that the solvability condition involves the existence of a number of (C,A,B)-pairs having certain properties, so that it remains to be seen how this condition is going to be verified. For some problems, it is possible to have canonical choices for the (C,A,B)-pairs in terms of computable subspaces like X_{det}, T_g^* (im E), etc. (Examples of this are the disturbance decoupling problem with stability ([27], [28], [19]), which is obtained as a special case of the problem treated here by taking $X_{stab} = X$, or the regulator problem under the minimum phase condition.) For the general problem under hand, this turned out to be not completely possible. It was possible, however, to select one pivot subspace in which the solvability condition could be expressed (Thm. 6.3), and to derive a computational criterion for this subspace (Cor. 6.5) which also had a geometric interpretation as a decomposability condition. In this way, we obtained a fully effective solution.

Life was also made somewhat easier by the use of the subspaces X_{stab} and X_{det} rather than the subspaces $\langle A|\mathrm{im}\, B\rangle$ and $\langle \ker C|A\rangle$ which were employed in [1] and [2]. There is a clear parallel here with the recent trend in transfer matrix analysis to use factorizations in stable rationals instead of polynomials (see for instance [12] and [14]).

The ultimate goal in solving the regulator problem is to identify the essential properties that not only determine when the problem is solvable for a given system, but that also lead to a concept of 'closeness' of systems. This concept should enable one to set bounds for the deterioration of performance when the given system is replaced by another one that is 'close' to it, while the controller remains the same. Other concepts that will inevitably enter the discussion include a measure of performance, a notion of distance between controllers, system sensitivity, and controller robustness. Ideas like these are, of course, to be found in the classical control literature (gain margin, phase margin, steady-state error, minimum phase, etc.) and there is also important recent work on the subject (see [29], for just one instance), but there are still quite a few issues to be cleared up. The context here is analytic rather than algebraic; nevertheless, as emphasized in the introduction, studying the algebraic problem should give us a a lead.

This, in turn, may be used as an argument as to what algebraic approach should be preferred. We have used the state-space method here, and we have built our most concrete results on explicit matrix representations. One might object that a large amount of arbitrariness and irrelevance is introduced by the choice of basis, and, moreover, there seems to be no way to compare systems of different order. The transfer matrix approach seems to offer better perspectives. However, the objective is to cut out as much

Appendix

The purpose of this appendix is to prove that the algebraic regulator problem in our formulation is a strict generalization of the regulator problem with internal stability as studied in [1] (also [2], Ch. 7). In fact, RPIS is obtained from the problem solved here by setting E=0 in (2.1).

Proposition A.1 *Consider the system (2.1-3) with E set equal to zero. The algebraic regulator problem is solvable if and only if RPIS is solvable in the sense of [1], i.e., if there exists a feedback mapping* $F\colon \mathcal{X}\to\mathcal{U}$ *such that*

$$\ker F \supset \langle \ker C \mid A\rangle \tag{A.1}$$

$$\mathcal{X}_b(A+BF)\cap(\langle A \mid \operatorname{im} B\rangle + \langle \ker C \mid A\rangle) \subset \langle \ker C \mid A\rangle \tag{A.2}$$

$$\mathcal{X}_b(A+BF) \subset \ker D. \tag{A.3}$$

Proof. First assume that there exists an F such that (A.1-3) is true, and write $\mathcal{V}_0 = \mathcal{X}_b(A+BF)$. We shall show that $\mathcal{V}_0$ satisfies the conditions of Thm. 6.1. Note that condition (ii) of this theorem becomes subsumed under condition (iv) in the special case E=0. It is clear that conditions (i) and (iii) hold ($\mathcal{X}_b(A+BF)$ is outer-stabilizable, of course). Also, it follows from (A.1) that $\langle \ker C \mid A\rangle$ is (A+BF)-invariant, and that $\mathcal{X}_{det} \subset \mathcal{X}_b(A+BF)$. So we get $\mathcal{X}_{det} \subset \mathcal{V}_0 \cap (\mathcal{X}_{det} + \mathcal{X}_{stab})$. To prove the reverse inclusion, note that (A.2) is equivalent to

$$\sigma(A+BF : (\langle A \mid \operatorname{im} B\rangle + \langle \ker C \mid A\rangle)/\langle \ker C \mid A\rangle) \subset \mathbb{C}g \tag{A.4}$$

$$\leftrightarrow \sigma(A+BF : \langle A \mid \operatorname{im} B\rangle/(\langle A \mid \operatorname{im} B\rangle \cap \langle \ker C \mid A\rangle)) \subset \mathbb{C}g$$

$$\leftrightarrow \mathcal{X}_b(A+BF) \cap \langle A \mid \operatorname{im} B\rangle \subset \langle A \mid \operatorname{im} B\rangle \cap \langle \ker C \mid A\rangle$$

$$\leftrightarrow \mathcal{X}_b(A+BF) \cap \langle A \mid \operatorname{im} B\rangle \subset \langle \ker C \mid A\rangle .$$

irrelevant information as possible, in order to arrive at the essential properties mentioned above. Even if the transfer matrix is used, the information contained in it must be cut considerably to obtain useful criteria. In general, there is no 'natural' all-purpose topology either in the state-space formulation or in the frequency domain that is suitable for all problems, since it is a fact of every-day life that things that are quite well interchangeable as a solution to one problem may give very different results when used for another purpose. (A telephone and a paper-weight, for instance.) So in either one of both approaches, one has to work to get at the crucial properties. Use of the tranfer matrix is an easy way to get rid of a bit of information that is irrelevant for control problems, but the fact that this bit is part of the information that has to be cut out is known and can also be used by the researcher who employs the state-space description. The eigenvalues of the matrices A_{22} and A_{33} in (6.18), for instance, are called 'plant zeros' and 'signal poles' because, to the control engineer, they are better known from the transfer function point of view. However, they also come out of our state space analysis in a quite natural way, and this does suggest that this type of analysis can compete on an equal footing with the transfer matrix approach. Other factors, such as the numerical feasibility of algorithms, may then be decisive.

Because $\mathcal{X}_b(A+BF) \cap \langle \ker C|A \rangle = \mathcal{X}_{det}$, this is equivalent to

$$\mathcal{X}_b(A+BF) \cap \langle A|\text{im } B \rangle \subset \mathcal{X}_{det} \tag{A.5}$$

The subspace $\mathcal{X}_{stab}$ is the same for the pair (A+BF, B) as it is for the pair (A,B), so we have

$$\begin{aligned} \mathcal{X}_{stab} &= \mathcal{X}_g(A+BF) + \langle A|\text{im } B\rangle = \\ &= \mathcal{X}_g(A+BF) + (\mathcal{X}_b(A+BF) \cap \langle A|\text{im } B\rangle) \quad . \end{aligned} \tag{A.6}$$

Intersecting the extremes of (A.6) with $\mathcal{X}_b(A+BF)$, we obtain

$$\mathcal{X}_b(A+BF) \cap \mathcal{X}_{stab} = \mathcal{X}_b(A+BF) \cap \langle A|\text{im } B\rangle \quad . \tag{A.7}$$

The conclusion from (A.5) and (A.7) is

$$\mathcal{V}_0 \cap \mathcal{X}_{stab} \subset \mathcal{X}_{det}. \tag{A.8}$$

We already proved that $\mathcal{X}_{det} \subset \mathcal{V}_0$, and under this condition (A.8) is equivalent to

$$\mathcal{V}_0 \cap (\mathcal{X}_{det} + \mathcal{X}_{stab}) \subset \mathcal{X}_{det} \tag{A.9}$$

which is what we wanted to prove.

Conversely, let us suppose that there exists an (A,B)-invariant subspace $\mathcal{V}_0$ such that the conditions (i-iv) of Thm. 6.1 hold. Then we have to construct an F satisfying (A.1-3). It follows from condition (iv), by intersection of both sides of the equality with $\mathcal{X}_g(A)$, that

$$\mathcal{V}_0 \cap \mathcal{X}_g(A) = \{0\}. \tag{A.10}$$

It is clear from this that we can define F on $\mathcal{V}_0 \oplus \mathcal{X}_g(A)$ in such a way that ker F $\supset \mathcal{X}_g(A)$ and $(A+BF)\mathcal{V}_0 \supset \mathcal{V}_0$. We can also arrange that ker F $\subset \mathcal{X}_{det}$, because $\mathcal{X}_{det} \subset \mathcal{V}_0$ by condition (iv). It follows from condition (iii) that $\mathcal{V}_0 \cap \mathcal{X}_g(A)$ is outer-stabilizable, and so lemma 5.2 shows that F can be extended to a mapping defined on all of $\mathcal{X}$ in such a way that $\sigma(A+BF:\mathcal{X}/(\mathcal{V}_0 \oplus \mathcal{X}_g(A)) \subset \mathbb{C}_g$. We then have $\mathcal{X}_b(A+BF) \subset \mathcal{V}_0 \subset \ker D$, which satisfies (A.3). Also, $\langle \ker C|A\rangle = \mathcal{X}_{det} \oplus (\langle \ker C|A\rangle \cap \mathcal{X}_g(A)) \subset \ker F$. Finally, it is seen from (A.4) and (A.7) that (A.2) is equivalent to

$$\mathcal{X}_b(A+BF) \cap \mathcal{X}_{stab} \subset \langle \ker C|A\rangle \ . \tag{A.11}$$

But this is immediate from condition (iv) and the fact that $\mathcal{X}_b(A+BF) \subset \mathcal{V}_0$.

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