Singular Perturbation, State Aggregation and Nonlinear Filtering

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Summary

Consider a state process $t \to x^{\epsilon}(t)$ evolving in R^n whose motion is that of a pure jump process in R^n, in the $O(1)$ time scale, upon which is superimposed a continuous motion along the orbits of a gradient-like vector field g in R^n, in the $O(1/\epsilon)$ time scale i.e. the infinitesimal generator of the state process is of the form $L + (1/\epsilon)g$. If we consider observations of the form

$$dy = h(x^{\epsilon}(t))dt + db(t) \ , \ t \geq 0,$$

then for each $\epsilon > 0$ the corresponding nonlinear filter is infinite dimensional. We show however that the projected motion $t \to \bar{x}^{\epsilon}(t)$ onto the equilibrium points of g is, in the limit as $\epsilon \downarrow 0$, a finite-state process governed by some explicit $\bar{L}$ on the finite state space consisting of the equilibrium points of g. We then show that the corresponding filters converge to a finite-state Wonham filter.

1. Variable Structure Systems

In situations where the structure of a dynamical system varies in time, it is often the case that the structural changes occur on a time scale that is much slower than the dynamics in any given mode of operation. For example, in the study of power systems, the swing equations are sometimes thought of as occuring on a fast time scale when compared to the relatively slow time scale of random faults or breakdowns.

Suppose that $g_1,\ldots,g_N$ are vector fields on R^n, suppose that $A(x) = (a_{ij}(x))$, $1 \leq i,j \leq N$, is an intensity matrix for each x in R^n. If there are N possible modes of operation of the system and if $a_{jk}(x)$ represents the infinitesimal transition probability that a structural change from mode j to mode k happens when the system is in state x, then a natural formulation of the above situation is to consider the trajectories of

$$\dot{x} = (1/\epsilon)\ g_i(x) \qquad (1)$$

where $x = x(t)$ is the state at time t and $i = i(t)$, a jump process on $\{1,\ldots,N\}$ governed by $(a_{jk}(x(t)))$, represents the mode in operation at time t. More accurately and more concisely, a natural formulation of the above situation is to say that $t \to (x(t), i(t))$ is a Markov process on $X = R^n \times \{1,\ldots,N\}$ governed by $A + (1/\epsilon)g$.

One can generalize (1) in various directions. For the purposes of system identification one may replace $\{1,\ldots,N\}$ by an arbitrary parameter space Λ (for related work see [3]). Alternatively, the state space need not be R^n and may be replaced by any smooth manifold. In fact all of these situations are subsumed by the following set-up:

Let X be a smooth manifold and let g be a smooth vector field on X. Let A be an integral operator on X given by

$$A\varphi(x) = \int_X (\varphi(x') - \varphi(x))\ \mu(x,dx'), \qquad (2)$$

for some measures $B \to \mu(x,B)$ depending on x in X. For each $\epsilon > 0$ let $t \to x^{\epsilon}(t)$ be a Markov process on X governed by $A + (1/\epsilon)g$.

The purpose of this paper is to study the limiting behaviour of these processes as $\epsilon \downarrow 0$. Our main result is that while the original motion $t \to x^{\epsilon}(t)$ clearly blows up as $\epsilon \downarrow 0$, in certain cases there is a reduced-order state space $\bar{X}$ and a projection $\pi : X \to \bar{X}$ such that $t \to \bar{x}^{\epsilon}(t) \equiv \pi(x^{\epsilon}(t))$ converges to a well-defined limit as $\epsilon \downarrow 0$. Thus X may be regarded as the full-order state space while A and g are the generators of the slow and fast dynamics respectively.

In general $\bar{X}$ should be chosen to be the limit set of the vector field g. In this paper we deal with the simplest kind of limiting behaviour, when the limit set of g is given by a finite number of states $\bar{X} = \{\bar{x}_1,\ldots,\bar{x}_N\}$ in X. Even in this case there are a number of novel features. Viewed as a singular perturbation problem, here there is no "fast variable" and thus the state space X is not a product of a fast variable and the slow variable $\bar{x} = \pi(x)$. Viewed as a state aggregation problem, here we have aggregation from a continuum of states X to a finite state space $\bar{X}$, a fact that radically

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changes the level of computational difficulty of nonlinear filtering and (partially or fully observable) stochastic control problems associated to the processes $t \to x^{\epsilon}(t)$.

As an application of our main result we shall see that while the nonlinear filters corresponding to the problem of estimating $t \to x^{\epsilon}(t)$ in the presence of additive white noise do not converge as $\epsilon \downarrow 0$, it turns out that the projected filters do in fact converge to a well-defined object, the (finite-dimensionally computable) finite-state Wonham filter.

Our treatment here is based on the martingale formulation an analogous theorem due to Papanicolaou, Stroock and Varadhan [4]. We therefore begin, for the sake of completeness, with a review of the martingale problem for $A + (1/\epsilon)g$. For a general treatment of the martingale problem for Levy processes, see [5].

2. The Martingale Problem for $A + (1/\epsilon)g$

Let X denote a smooth manifold and let g be a smooth complete vector field on X. Let $B(X)$ be the space of all bounded Borel functions on X, and let α_t, $-\infty < t < \infty$, denote the flow of g. The domain of g is the set $\mathcal{D}$ of all functions φ in $B(X)$ such that there is a ψ in $B(X)$ satisfying

$$\varphi(\alpha_t(x)) - \varphi(\alpha_s(x)) = \int_s^t \psi(\alpha_r(x))dr$$

for all x in X and $0 \leq s \leq t \leq T$. Any such ψ is then denoted by $g(\varphi)$ and we emphasize that there may be more than one $g(\varphi)$ associated to a given φ. If φ is sufficiently smooth, however, then there is a natural choice of $g(\varphi)$ given by

$$g(\varphi)(x) = \frac{d}{dt}\Big|_{t=0} \varphi(\alpha_t(x)).$$

We note for future reference that $\mathcal{D}$ is a vector space.

Let Ω denote the space of all right-continuous paths $\omega : [0,T] \to X$ having only a finite number of discontinuities of the first kind in any compact time interval. For each $0 \leq t \leq T$ let $x(t) : \Omega \to X$ be the evaluation map at time t: $x(t,\omega) = \omega(t)$. The Borel σ-algebra of Ω is then given by $\mathcal{F}_T$, where $\mathcal{F}_t$ is the σ-algebra generated by the maps $x(s)$, $0 \leq s \leq t$.

If $B \to \mu(x,B)$ is a finite positive Borel measure on X for each x in X such that $x \to \mu(x,B)$ is in $B(X)$ for each Borel set $B \subset X$, let $A\varphi$ be given by (2), for any φ in $B(X)$. A is then a bounded linear operator on $B(X)$ whose norm is less than or equal to twice the sup norm of λ where $\lambda(x) \equiv \mu(x,X)$. Let $C_0^{\infty}(X)$ denote the space of all smooth functions of compact support on X. Let $L_t : C_0^{\infty}(X) \to B(X)$ be a linear operator depending on t. We use the standard martingale definition of a Markov process [5]:

Definition. A Markov process on X governed by L_t is a probability measure P on Ω satisfying

$$E(\ \varphi(x(t)) - \varphi(x(s)) - \int_s^t L_r(\varphi)(x(r))dr \mid \mathcal{F}_s)=0$$

for all φ in $C_0^{\infty}(X)$ and all $0 \leq s \leq t \leq T$. (3)

Recall that this equivalent to the statement that for any bounded $\mathcal{F}_s$- measurable $\Phi : \Omega \to R$,

$$E(\{\varphi(x(t)) - \varphi(x(s)) - \int_s^t L_r(\varphi)(x(r))dr\}\ \Phi\ )=0$$

for $0 \leq s \leq t \leq T$. By abuse of notation, the measure P is referred to as the distribution of the Markov process $t \to x(t)$.

Let $G_t\varphi(x) \equiv \varphi(\alpha_t(x))$ and set $A_t = G_t A\ G_{-t}$. Consider the map $\alpha_\epsilon : \Omega \to \Omega$ given by $\alpha_\epsilon(\omega)(t) = \alpha_{-t/\epsilon}(\omega(t))$ and let $\tilde{P}_\epsilon$ be the image of a given measure P_ϵ under the map α_ϵ.

Lemma. P_ϵ is governed by $A + (1/\epsilon)g$ if and only if $\tilde{P}_\epsilon$ is governed by $A_{t/\epsilon}$.

This lemma is proven using integration by parts in (3) exactly as in the proof of theorem (2.1) of [6]. Since $A_{t/\epsilon}$ is an integral operator the methods of chapter 3,[2] yield the fact that there is one and only one measure $\tilde{P}_\epsilon$ for any given initial distribution governed by the operator $A_{t/\epsilon}$. Thus

Proposition. There is one and only one Markov process on X governed by $A + (1/\epsilon)g$, for any given initial distribution. Moreover (3) above holds with $L_t = A + (1/\epsilon)g$ for any φ in the domain $\mathcal{D}$ of g, and

P_ϵ($t \to x(t)$ is a finite disjoint union of compact trajectories of g) = 1.

proof. The sample paths of P_ϵ are as stated because P_ϵ is the image of the measure $\tilde{P}_\epsilon$ under the map α_ϵ^{-1} and the sample paths of $\tilde{P}_\epsilon$ are piecewise constant. To see that (3) holds with $L_t = A + (1/\epsilon)g$ for all φ in $\mathcal{D}$ first note that $\tilde{P}_\epsilon$ can be constructed so that (3) holds for all φ in $B(X)$, when $L_t = A_{t/\epsilon}$, and then note that the integration by parts trick referred to above still holds when φ is in $\mathcal{D}$.

Thus the martingale problem for $A + (1/\epsilon)g$ is well-posed. In particular if $\bar{X} = \{\bar{x}_1,\ldots,\bar{x}_N\}$ is a finite set then $\bar{X}$ can be considered to be a zero-dimensional manifold. Thus suppose (μ_{ij}) $1 \leq i, j \leq N$ is given and set

$$\bar{A}\bar{\varphi}(x_i) = \Sigma\ (\ \bar{\varphi}(\bar{x}_j) - \bar{\varphi}(\bar{x}_i))\ \mu_{ij} \qquad (4)$$

where the sum is over j, for all $\bar{\varphi}$ in $B(\bar{X}) = C_0^{\infty}(\bar{X})$. If in the above proposition we make the replacements $X \leftarrow \bar{X}$, $A \leftarrow \bar{A}$, $g \leftarrow 0$ then we conclude that the martingale problem for $\bar{A}$ is also well-posed. In closing this section, we note that the only property of g that we have used is the existence and uniqueness of an associated flow satisfying $C_0^{\infty}(X) \subset \mathcal{D}$.

3. Gradient-like Vector Fields

Recall that g is a complete smooth vector field on X with flow α_t. We assume that there are a finite number of points $\bar{x}_1,\ldots,\bar{x}_N$ in X such that for all x in X, $\alpha_t(x)$ converges to one of $\bar{x}_1,\ldots,\bar{x}_N$ as $t\uparrow\infty$. The set $\bar{X} = \{\bar{x}_1,\ldots,\bar{x}_N\}$ represents the reduced order state space. Let $B_i \subset X$ be the ith basin of attraction: B_i is the Borel set of all x in X such that $\alpha_t(x)$ converges to $\bar{x}_i$ as $t\uparrow\infty$. For x in B_i set $\pi(x) = \bar{x}_i$. The map $\pi : X \to \bar{X}$ is then in $\mathcal{D}$ and one choice of $g(\pi)$ is given by the zero function. For φ in $B(X)$ let $\bar{\varphi}$ denote the restriction of φ to $\bar{X}$.

Definition. The Fredholm alternative holds for φ in $B(X)$ iff there is a ψ in $\mathcal{D}$ satisfying

$$g(\psi) = \bar{\varphi}\circ\pi - \varphi. \tag{5}$$

Consider the following assumption.

(A) There is an integrable function $R(t)$, $0 \le t < \infty$ such that

$$|\mu(\alpha_t(x),B_j) - \mu(\pi(x),B_j)| \le R(t)$$

for $1 \le j \le N$, x in X, and $t \ge 0$, and $\int_0^\infty R(t)dt$ is finite.

Proposition. Under assumption (A), the Fredholm alternative holds for all functions of the form $A(\bar{\varphi}\circ\pi)$, for any given $\bar{\varphi}$ in $B(\bar{X})$.

proof. Set $\varphi \equiv \bar{\varphi}\circ\pi$ and

$$\psi(x) \equiv \int_0^\infty A\varphi(\alpha_s(x)) - A\varphi(\pi(x))\, ds.$$

Since $A\varphi$ is a finite linear combination of the functions $x \to \mu(x,B_j)$, assumption (A) guarantees that ψ is in $B(X)$. The rest of the proof follows from the above formula for ψ.

If the conclusion of this last proposition is true, then we shall say simply that the Fredholm alternative for g holds.

In what follows $\bar{\Omega}$ denotes the right-continuous path space of $\bar{X}$. Since the trajectories of P_ϵ are a finite disjoint union of compact trajectories of g, we see that $\pi : X \to \bar{X}$ induces a well-defined map $\Omega \to \bar{\Omega}$ given by $\omega \to \omega'$ where $\omega'(t) = \pi(\omega(t))$.

The swing equations arising in the study of power systems can be thought of as a vector field g on $X = T^n \times R^n$. For a study of equilibrium points of this vector field g, see [7].

4. Singular Perturbation

Let A and g be as before, and define $\bar{A}$ by

$$\bar{A}\bar{\varphi} = A(\bar{\varphi}\circ\pi)\big|_{\bar{X}}.$$

The linear map $\bar{A} : B(\bar{X}) \to B(\bar{X})$ is then given by equation (4) where

$$\mu_{ij} = \mu(\bar{x}_i,B_j),\ 1 \le i,j \le N.$$

What follows is the main result of the paper.

Theorem. Assume that the Fredholm alternative for g holds. Let $t \to x^\epsilon(t)$ be Markov processes on X governed by $A + (1/\epsilon)g$, all having a common initial distribution on X. Then the Markov processes $t \to \bar{x}^\epsilon(t)$ coverge in distribution to the unique Markov process on $\bar{X}$ governed by $\bar{A}$ and having the projected initial distribution, as $\epsilon\downarrow 0$. This means that for any bounded continuous functional $\bar{\Phi} : \bar{\Omega} \to R$

$$\bar{E}_\epsilon(\bar{\Phi}) \to \bar{E}(\bar{\Phi})$$

as $\epsilon\downarrow 0$.

The proof of this theorem is analogous to that of a theorem due to Papanicolaou, Stroock and Varadhan[4], and breaks naturally into two steps. The first step consists of showing that the distributions $\{\bar{P}_\epsilon\}$ of $t \to \bar{x}^\epsilon(t)$ are a relatively weakly compact family of measures on $\bar{\Omega}$, while the second step is the identification of the limiting distribution $\bar{P}$ via the Fredholm alternative and the well-posedness of the martingale problem for $\bar{A}$.

The topology on $\bar{\Omega}$ is the Skorokhod topology. This turns $\bar{\Omega}$ into a complete metric space and thus the Prohorov theory applies: A family of measures $\{\bar{P}_\epsilon\}$ on $\bar{\Omega}$ is relatively weakly compact iff $\{\bar{P}_\epsilon\}$ is uniformly tight: For each $\alpha > 0$ there is a compact set $K \subset \bar{\Omega}$ such that $\bar{P}_\epsilon(K) > 1 - \alpha$ for all $\epsilon > 0$. Since $\bar{X}$ may be considered as embedded in a real line the standard theory applies and so we conclude that $\{\bar{P}_\epsilon\}$ is relatively weakly compact, using a special case of proposition (A.1) of [5].

Now suppose that $\epsilon_k\downarrow 0$ and $\bar{P}_{\epsilon_k} \to$ some P' on $\bar{\Omega}$. Let $\bar{\varphi}$ be in $B(\bar{X})$ and choose ψ in $\mathcal{D}$ such that

$$g(\psi) = \bar{A}(\bar{\varphi})\circ\pi - A(\bar{\varphi}\circ\pi). \tag{6}$$

Since P_{ϵ_k} is governed by $A + (1/\epsilon)g$ and $\varphi + \epsilon_k\psi$ is in $\mathcal{D}$ we have ($\varphi \equiv \bar{\varphi}\circ\pi$)

$$0 = E_{\epsilon_k}(\{(\varphi+\epsilon_k\psi)(x(t)) - (\varphi+\epsilon_k\psi)(x(s))$$

$$- \int_s^t (A+(1/\epsilon_k)g)(\varphi+\epsilon_k\psi)(x(r))dr\}\ \bar{\Phi}\circ\pi\ )$$

for all bounded $\bar{\mathcal{F}}_s$- measurable $\bar{\Phi} : \bar{\Omega} \to R$. Thus by expanding and using (6) we see that

$$E_{\epsilon_k}(\{\ \bar{\varphi}(\bar{x}(t)) - \bar{\varphi}(\bar{x}(s)) - \int_s^t \bar{A}\bar{\varphi}(\bar{x}(r))dr\}\ \bar{\Phi}\ )$$

is $O(\epsilon_k)$ as $\epsilon\downarrow 0$, since ψ, $A(\psi)$, and $\bar{\Phi}$ are all bounded. Thus letting $k\uparrow\infty$, we see that any limiting probability measure of the set $\{\bar{P}_\epsilon\}$ is a Markov process governed by $\bar{A}$, and since there is a unique such Markov process, this shows that $\bar{P}_\epsilon \to \bar{P}$, the Markov process governed by $\bar{A}$.

With a little extra work, the above result still holds for all bounded $\bar{\Phi}$ that are continuous off a set of $\bar{P}$ measure zero. See [1].

5. Filtering

Suppose we are given noisy observations

$$y(t) = h(x^{\epsilon}(t)) + \text{white noise} \qquad (7)$$

of the Markov process $t \to x^{\epsilon}(t)$. The nonlinear filter corresponding to (7) is a well-defined map given by the Kallianpur-Striebel formula for example. Rather than use this formula, we shall use the robust form of the filter and simply define it to be the expectation of a certain bounded functional on $\bar{\Omega}$. For each y in $C([0,T])$ and φ, h in $B(X)$ let

$$\Phi_y(\varphi) = \varphi(x(t))\exp(-\int_0^t V(s,x(s))ds)$$

where

$$V(t,x) = \tfrac{1}{2}(y(t) - h(x))^2 .$$

It can be shown that Φ_y is a continuous map $\Omega \to R$ off a set of P_ϵ-measure zero.

Definition. The filter corresponding to a Markov process with distribution P is given by the map

$$\begin{aligned} C([0,T]) &\to C([0,T]) \\ y &\to E(\Phi_y(\varphi))/E(\Phi_y(1)). \end{aligned}$$

Since the distributions P_ϵ of the Markov processes $t \to x^{\epsilon}(t)$ do not converge, we do not expect the corresponding filters to converge. However the projected filters, obtained by replacing φ by $\bar{\varphi}\circ\pi$, h by $\bar{h}\circ\pi$, do in fact converge as $\epsilon \downarrow 0$:

Theorem. The projected filters converge to the finite-state Wonham filter corresponding to the problem of estimating the finite state process governed by $\bar{A}$, in the presence of additive white noise.

This follows immediatly from our main result.

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