

Feedforward Control,  
PID Control Laws,  
and  
Almost Invariant Subspaces

by

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Abstract

We define feedforward control as a control policy in which the exogenous disturbances are known for all time at the moment when the control is applied. It is shown that disturbance decoupling by feedforward control is possible iff it is possible by PID control or iff approximate disturbance decoupling by state feedback is possible.

Key Words: Feedforward control; Almost invariant subspaces; Disturbance decoupling; PID control; Impulsive control.

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## 1. Introduction

Consider the ubiquitous linear time-invariant system defined, in continuous and discrete time respectively, by

$$\Sigma_R: \dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) + G\underline{d}(t); \quad \underline{z}(t) = H\underline{x}(t) \quad (1)_R$$

$$\Sigma_Z: \underline{x}(t+1) = A\underline{x}(t) + B\underline{u}(t) + G\underline{d}(t); \quad \underline{z}(t) = H\underline{x}(t) \quad (1)_Z$$

with  $\underline{x} \in \mathcal{X} := \mathbb{R}^n$ , the state,  $\underline{u} \in \mathcal{U} := \mathbb{R}^m$ , the control,  $\underline{d} \in \mathcal{D} := \mathbb{R}^q$ , the (exogenous) disturbance, and  $\underline{z} \in \mathcal{Z} := \mathbb{R}^l$ , the controlled output. In the *disturbance decoupling problem* we are asked for a control such that in the closed loop system the disturbance has no influence on the controlled output.

The basic theory of this problem and its many variations has been the subject of numerous papers in the control journals (see WONHAM [1, Ch. 4&5] and WILLEMS & COMMAULT [2] for pointers and references to this literature). Recently this theory has been extended to treat the case when disturbance decoupling is possible up to any desired degree of accuracy. This extension uses the notion of almost invariant subspaces and is described in full detail in WILLEMS [3,4,5].

In the present paper we will give conditions under which disturbance decoupling is possible when knowledge of the whole disturbance trajectory  $\underline{d}$  is available to the controller. We think of this as (a form of complete) *feedforward control*: there is a mechanism for measuring the disturbance ahead of time and communicating it to the controller. One of the purposes of this paper is to relate feedforward control to approximate disturbance decoupling and to control policies using differentiators (PID-control) and predictors.

We will use common notation for  $\mathbb{C}$ ,  $\mathbb{R}$ ,  $\mathbb{Z}$ ,  $\mathbb{R}^+ := [0, \infty)$ , etc. Furthermore,  $C^\infty$  denotes the infinitely differentiable functions - their (co)domain will always be obvious from the context. We say that a map  $\underline{f}$  with domain  $\mathbb{R}$  or  $\mathbb{Z}$  has *left (right) compact support* if  $\exists t_0$  such that  $\underline{f}(t) = 0$  for  $t < t_0$  ( $t > t_0$ ).  $C_+^\infty$  denotes the  $C^\infty$  functions with left compact support.  $\sigma$  denotes the spectrum.

## 2. Problem Statement

There are a number of equivalent ways of formalizing the idea of feedforward control. Let us denote by  $\Sigma_{\mathbb{R}}$  and  $\Sigma_{\mathbb{Z}}$  all the trajectories which are compatible with the system (1). Formally:

$$\Sigma_{\mathbb{R}} := \{(\underline{d}, \underline{u}, \underline{x}, \underline{z}) : \mathbb{R} \rightarrow \mathcal{D} \times \mathcal{U} \times \mathcal{X} \times \mathcal{Y} \mid (\underline{d}, \underline{u}, \underline{x}, \underline{y}) \in C^\infty \text{ and } (1)_{\mathbb{R}} \text{ is satisfied}\}$$

and  $\Sigma_{\mathbb{Z}}$  is similarly defined. We will say that  $\Sigma_{\mathbb{R}}$  admits a *disturbance insensitive* trajectory if  $\forall \underline{d} \in C^\infty \exists \underline{u}, \underline{x}$  such that  $(\underline{d}, \underline{u}, \underline{x}, 0) \in \Sigma_{\mathbb{R}}$ . An analogous definition holds for  $\Sigma_{\mathbb{Z}}$ . We will also consider some specific classes of control laws:

(i) Feedforward control: Let  $\underline{\mathcal{F}}$  be an  $\mathbb{R}^{m \times q}$ -valued distribution with support on some half line  $[t_0, \infty)$  and let  $*$  denote convolution. For  $\underline{d} \in C_+^\infty$  define now

$$\underline{u}(t) = (\underline{\mathcal{F}} * \underline{d})(t) \quad (2)_{\mathbb{R}}$$

We will call this a *feedforward control* law with kernel  $\underline{\mathcal{F}}$ . We will say that  $\underline{\mathcal{F}}$  is *rational* if its Laplace transform is rational (in which case support  $\underline{\mathcal{F}} \subset [0, \infty)$  and  $\underline{\mathcal{F}}(t) = \sum_{k=0}^N F_k \delta^{(k)}(t) + \underline{F}(t)$  with  $\underline{F}$  a Bohl function and  $\delta$  the Dirac

delta). Analogously in the discrete time case we will call

$$\underline{u}(t) = \sum_{k=N}^{\infty} \mathcal{F}(t-k) \underline{d}(k) \quad (2)_Z$$

a *feedforward control law*.

(ii) state feedback:

$$\underline{u}(t) = \underline{F}\underline{x}(t) \quad (3)$$

(iii) PID control:

$$\underline{u}(t) = \sum_{k=0}^N F_k \underline{d}^{(k)}(t) + \underline{F}\underline{x}(t) \quad (4)_R$$

or its discrete time analogue

$$\underline{u}(t) = \sum_{k=0}^N F_k \underline{d}(t+k) + \underline{F}\underline{x}(t) \quad (4)_Z$$

(iv) PD control:

$$\underline{u}(t) = \sum_{k=0}^N F_k \underline{d}^{(k)}(t) \quad (5)_R$$

or the discrete time analogue, *finite window predictive control*:

$$\underline{u}(t) = \sum_{k=0}^N F_k \underline{d}(t+k) \quad (5)_Z$$

Consider the control law (4)<sub>R</sub>. This yields the closed loop system

$$\dot{\underline{x}}(t) = (A+BF)\underline{x}(t) + B \sum_{k=0}^N F_k \underline{d}^{(k)}(t) + G\underline{d}(t) ; \underline{z}(t) = H\underline{x}(t)$$

which yields, for all  $\underline{d} \in C_+^{\infty}$ , a (unique) solution  $\underline{x} \in C_+^{\infty}$ ,  $\underline{y} \in C_+^{\infty}$ . If  $\underline{y} = \underline{0}$  for all such  $\underline{d}$  then we will say that (4)<sub>R</sub> *decouples the disturbance d from z*.

Analogously for the other control laws.

The control law (3) in (1)<sub>R</sub> yields the closed loop system

$$\dot{\underline{x}}(t) = (A+BF)\underline{x}(t) + G\underline{d}(t) ; \underline{z}(t) = H\underline{x}(t)$$

which has the closed loop impulse response  $\underline{w}_F: t \in \mathbb{R}^+ \mapsto He^{(A+BF)t}G$ . Following [4] we will say that (1)<sub>R</sub> may be *almost* (or *approximately*) *disturbance decoupled* using a state feedback control law if  $\forall \epsilon > 0 \exists F$  such that  $\int_0^\infty \|\underline{w}_F(t)\| dt \leq \epsilon$ .

Our purpose is to give conditions for <sup>the solvability of</sup> the various disturbance decoupling problems given above and to show their interrelation.

### 3. Almost Invariant Subspaces

Consider the system  $\Sigma: \dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t); \underline{z}(t) = H\underline{x}(t)$ . Let  $\mathcal{V}_{\ker H}^*$  and  $\mathcal{R}_{\ker H}^*$  denote the classical notions of the supremal controlled invariant ('(A,B)-invariant') and controllability subspaces contained in  $\ker H$ . In [3,4] these notions have been generalized to almost invariance. If we measure being close to  $\ker H$  in the  $\mathcal{L}_\infty$ - sense, then we obtain  $\mathcal{V}_{a, \ker H}^*$  and  $\mathcal{R}_{a, \ker H}^*$  as respectively the supremal  $\mathcal{L}_\infty$ -almost controlled invariant and  $\mathcal{L}_\infty$ -almost controllability subspace contained in  $\ker H$ . If instead we measure being close to  $\ker H$  in the  $\mathcal{L}_1$ - sense then we arrive at  $\mathcal{V}_{b, \ker H}^*$  and  $\mathcal{R}_{b, \ker H}^*$  as the supremal  $\mathcal{L}_1$ -almost controlled invariant and  $\mathcal{L}_1$ -almost controllability subspace 'contained' in  $\ker H$ .

The subspaces  $\mathcal{R}_{a, \ker H}^*$  and  $\mathcal{R}_{b, \ker H}^*$  are readily computed. Indeed, consider the recursive algorithms (ACSA) and (ACSA)':

$$\boxed{\mathcal{R}_{\ker H}^{k+1} = \ker H \cap (A\mathcal{R}_{\ker H}^k + \text{im } B) ; \mathcal{R}_{\ker H}^0 = \{0\}} \quad \text{(ACSA)}$$

$$\boxed{\mathcal{P}_{\ker H}^{k+1} = \text{im } B + A(\ker H \cap \mathcal{P}_{\ker H}^k) ; \mathcal{P}_{\ker H}^0 = \{0\}} \quad \text{(ACSA)'}$$

then  $\lim_{k \rightarrow \infty} \mathcal{R}_{\ker H}^k =: \mathcal{R}_{\ker H}^\infty = \mathcal{R}_{a, \ker H}^*$  and  $\lim_{k \rightarrow \infty} \mathcal{P}_{\ker H}^k =: \mathcal{P}_{\ker H}^\infty = \mathcal{R}_{b, \ker H}^*$ .

These algorithms show also that  $\mathcal{R}_{a, \ker H}^*$  and  $\mathcal{R}_{b, \ker H}^*$  have a natural interpretation in discrete time. Indeed, consider  $\Sigma: \underline{x}(t+1) = A\underline{x}(t) + B\underline{u}(t); \underline{z}(t) = H\underline{x}(t)$ . Then we see immediately from these algorithms that (see Molinari [6]).

$$\mathcal{R}_{a, \ker H}^* = \{x_0 \in \mathcal{X} \mid \exists T < 0, \underline{u}, \underline{x} \text{ such that } \underline{x}(T) = 0, \underline{x}(0) = x_0, \\ \underline{x}(t+1) = A\underline{x}(t) + B\underline{u}(t); H\underline{x}(t) = 0 \text{ for } T < t < 0\}$$

$$\text{and } \mathcal{R}_{b, \ker H}^* = \{x_0 \in \mathcal{X} \mid \exists T < 0, \underline{u}, \underline{x} \text{ such that } \underline{x}(T) = 0, \underline{x}(0) = x_0, \\ \underline{x}(t+1) = A\underline{x}(t) + B\underline{u}(t); H\underline{x}(t) = 0 \text{ for } T < t < 0\}$$

Finally, we mention the following relations among the various subspaces introduced:

$$\mathcal{R}_{b, \ker H}^* = A\mathcal{R}_{a, \ker H}^* + \text{im} B; \quad \mathcal{V}_{a, \ker H}^* = \mathcal{V}_{\ker H}^* + \mathcal{R}_{a, \ker H}^*; \text{ and}$$

$$\mathcal{V}_{b, \ker H}^* = \mathcal{V}_{\ker H}^* + \mathcal{R}_{b, \ker H}^*.$$

#### 4. The Main Results

The results of this paper are:

Theorem 1: Consider  $\Sigma_R$ . Then the following statements are equivalent:

- (i)  $\text{im } G \subset \mathcal{V}_{b, \ker H}^*$
- (ii)  $\Sigma_R$  admits a disturbance insensitive trajectory
- (iii)  $\Sigma_R$  may be disturbance decoupled using feedforward control
- (iv)  $\Sigma_R$  may be disturbance decoupled using a PID control law
- (v)  $\Sigma_R$  may be almost disturbance decoupled using a state feedback control law.

The discrete time version gives us the following expected analogy:

Theorem 2: Consider  $\Sigma_Z$ . Then the following statements are equivalent:

- (i), (ii), and (iii) of Theorem 1
- (iv)  $\Sigma_Z$  may be disturbance decoupled using a control law of the type  $(4)_Z$

Bringing in stability, or, more generally, pole placement, yields the following refinements. We will say that *pole placement* holds if for any symmetric subset of  $\mathbb{C}_g$  with at least one point on the real axis, there exists  $F$  in a given class such that  $\sigma(A+BF) \subset \mathbb{C}_g$ . We will say that  $e^{(A+BF)t}$  has an *arbitrary rate of decay* if this holds for any  $\mathbb{C}_g$  of the type  $\mathbb{C}_g = \{s \in \mathbb{C} \mid \text{Re } s < M\}$ .

Theorem 3: Consider the system  $\Sigma_R$ . Then the following conditions are equivalent:

- (i)  $\text{im } G \subset \mathcal{R}_{b, \ker H}^*$
- (ii)  $\Sigma_R$  may be disturbance decoupled using a PD control law
- (iii) (Assume  $(A,B)$  controllable)  $\Sigma_R$  may be disturbance decoupled using a PID control law with pole placement on  $A+BF$
- (iv) (Assume  $(A,B)$  controllable)  $\Sigma_R$  may be approximately disturbance decoupled using a state feedback control law and requiring an arbitrary rate of decay on  $e^{(A+BF)t}$

An analogous theorem holds (without (iv)) for  $\Sigma_Z$  and with in (ii) a finite window predictive control law.

Finally, it is of interest to note the following interpretation of

$\mathcal{R}_{\ker H}^*$  (we state only the continuous time case):

Theorem 4: Consider  $\Sigma_R$ . Then the following conditions are equivalent:

- (i)  $\text{im } G \subset \mathcal{R}_{\ker H}^*$
- (ii)  $\Sigma_R$  may be disturbance decoupled both by using a PD and a state feedback control law
- (iii)  $\Sigma_R$  may be disturbance decoupled using a state feedback control law with pole placement on  $(A+BF)$

## 5. Discussion

5.1 Theorem 1 shows an interesting connection between feedforward control, PID control, and high gain feedback as it results in approximate disturbance decoupling [4]. We also note that it follows from the theorem that there exists any disturbance decoupling feedforward control law (i.e., any (nonlinear time-varying) map  $\mathcal{F}: \underline{d} \in C_+^\infty \mapsto \underline{u} \in C_+^\infty$  such that the (unique) solution  $\underline{x} \in C_+^\infty$  to  $(1)_R$  yields  $\underline{z} = H\underline{x} = \underline{0}$ ) iff there exists a PID control law or, as is easily seen to be equivalent, a rational convolution operator feedforward control law.

5.2 The maximal order of the differentiation,  $N$ , in the required PID control law is given by the smallest  $N$  such that, in the notation of (ACSA)',  $\text{im } G \subset \mathcal{P}_{\ker H}^{N-1} + \mathcal{V}_{\ker H}^*$ . This yields the known results in the cases  $N = 0$  and  $N < 0$ . An analogous statement holds for  $\Sigma_Z$ . All this indicates once more that differentiation should be considered as a predictive element even though it is hard to justify this formally.

5.3 In the continuous time case the unbounded nature of the differentiators is an intrinsic feature of the problem and cannot be traded for example for a smooth non-causal control law of the type

$$\underline{u}(t) = \int_{-\infty}^{+\infty} \underline{F}(t-\tau) \underline{d}(\tau) d\tau,$$



5.4 By suitably interpreting (ACSA)' it is easy to come up with algorithms for computing  $F, F_0, \dots, F_N$ 's which yield a disturbance decoupling PID control law.

5.5 The results obtained are symmetric in time and the same disturbance decoupling conditions hold if we consider the systems  $\Sigma_R$  or  $\Sigma_Z$  with inputs with right compact support.

5.6 A number of straightforward variations of Theorems 1-4 referring to stability regions, existence of PID control laws with  $N$  given, etc., can be stated. It may be of interest to give the condition for the existence of a finite window disturbance decoupling control law of the type

$$\underline{u}(t) = \sum_{k=N^-}^{N^+} F_k \underline{d}(t-k) \text{ with } N^- \leq 0 \leq N^+ \text{ or, in the continuous time case, a "classical" PID controller } \underline{u}(t) = \sum_{k=0}^{N^+} F_k^+ \underline{d}^{(k)}(t) + \sum_{k=1}^{N^-} F_k^- \int_0^t \int_0^{t-k-1} \dots \int_0^{t_1} \underline{d}(\tau) d\tau dt_1 \dots dt_{k-1}$$

The condition for disturbance decoupling is  $\text{im } G \subset \mathcal{R}_{b, \ker H}^* + \mathcal{V}_{d, \ker H}^*$  where  $\mathcal{V}_{d, \ker H}^*$  is the supremal 'deadbeat' controlled invariant subspace in  $\ker H$ , i.e.  $\sup \{ \mathcal{L} \mid \exists F: (A+BF)\mathcal{L} \subset \mathcal{L} \text{ and } \sigma((A+BF)|_{\mathcal{L}}) = \{0\} \}$ .  $\mathcal{V}_{d, \ker H}^*$  is just a bit larger than  $\mathcal{R}_{\ker H}^*$ .

## 6. An Outline of the Proofs

Theorems 1-4 follow without much difficulty from the results in Appendix A of [5]. In order to show the idea we will give a 'time domain' proof of Theorem 2.

Proof of Theorem 2: We will indicate the reasonings in the logical sequence (i)  $\Rightarrow$  (ii), (i)  $\Rightarrow$  (iii), (i)  $\Rightarrow$  (iv), (iii)  $\Rightarrow$  (i), (ii)  $\Rightarrow$  (i), (iv)  $\Rightarrow$  (iii).

(i)  $\Rightarrow$  (ii): Assume  $\text{im } G \subset \mathcal{V}_{b, \ker H}^* = \mathcal{R}_{b, \ker H}^* + \mathcal{V}_{\ker H}^*$ . Let  $\underline{d} \in C^\infty$  be given. We need to find  $\underline{u}, \underline{x}$  satisfying (1)<sub>Z</sub> such that  $H\underline{x} = \underline{y} = \underline{0}$ . Assume

first that  $\underline{d}$  is a pulse at 0 :  $\underline{d}(t) = 0$  for  $t \neq 0$ . Let  $d_0 := G\underline{d}(0)$ . Now,  $d_0$  may be written as  $d_0 = -d_1 + d_2$  with  $d_1 \in \mathcal{R}_{b, \ker H}^*$  and  $d_2 \in \mathcal{V}_{\ker H}^*$ .

Hence, by the characterization of  $\mathcal{R}_{b, \ker H}^*$  given at the end of Section 3,

$\exists \underline{u}$  of compact support such that the corresponding compact support  $\underline{x}$  satisfies  $H\underline{x}(t)=0$  for  $t \leq 0$  and  $A\underline{x}(0)+B\underline{u}(0)=d_1$ . Using this control yields  $\underline{x}(1) = A\underline{x}(0) + B\underline{u}(0) + G\underline{d}(0) = d_1 + d_0 = d_2 \in \mathcal{V}_{\ker H}^*$ . It is hence possible to choose  $\underline{u}$  such that also  $H\underline{x}(t) = 0$  for  $t > 0$ . By superposition this proves the result for any  $\underline{d}$  such that  $\underline{d}(t)=0$  for  $t < 0$ . By considering the reverse time trajectories (see [5, Sect. 8]) this conclusion also follows for  $\underline{d}$ 's such that  $\underline{d}(t)=0$  for  $t > 0$ . Since any  $\underline{d}$  may be written as  $\underline{d}=\underline{d}_+ + \underline{d}_-$  with  $\underline{d}_+(t) = 0$  for  $t < 0$  and  $\underline{d}_-(t) = 0$  for  $t > 0$ , this yields trajectories  $(\underline{d}_+, \underline{u}_+, \underline{x}_+, 0)$ ,  $(\underline{d}_-, \underline{u}_-, \underline{x}_-, 0) \in \Sigma_Z$  which, since  $\Sigma_Z$  is obviously linear, yields a  $(\underline{d}, \underline{u}, \underline{x}, 0) \in \Sigma_Z$  as required.

(ii)  $\Rightarrow$  (iii): is basically the first part of the proof of (i)  $\Rightarrow$  (ii)

(i)  $\Rightarrow$  (iv): use  $F$  such that  $(A+BF) \mathcal{V}_{\ker H}^* \subset \mathcal{V}_{\ker H}^*$  and the predictive law suggested by the first part of the proof of (i)  $\Rightarrow$  (ii).

(iii)  $\Rightarrow$  (i): Let  $\underline{u}(t) = \sum_{k=0}^N F_k \underline{d}(t-k) + F_N \underline{x}(t)$  be a disturbance decoupling control law. Take  $\underline{d}$  to be a pulse at 0. Then there exists a trajectory where  $\Sigma_Z^+$  denotes the elements of  $\Sigma_Z$  with left compact support.  $\{(\underline{d}, \underline{u}, \underline{x}, 0) \in \Sigma_Z^+\}^V$  Since  $\underline{d}$  is a pulse at 0, since  $\underline{x}$  has left compact support, and since  $\underline{x}(t) \in \ker H$  for  $t \leq 0$ , we have  $\underline{x}(0) \in \mathcal{R}_{a, \ker H}^*$ . Similarly, since  $\underline{x}(t) \in \ker H$  for  $t > 0$ ,  $\underline{x}(1) \in \mathcal{V}_{\ker H}^*$ . Hence,  $G\underline{d}(0) = \underline{x}(1) - A\underline{x}(0) - B\underline{u}(0) \in \mathcal{V}_{\ker H}^* + A\mathcal{R}_{a, \ker H}^* + \text{im} B = \mathcal{V}_{b, \ker H}^*$ , which yields  $\text{im} G \subset \mathcal{V}_{b, \ker H}^*$ , as desired.

(ii)  $\Rightarrow$  (i): is more similarly to (iii)  $\Rightarrow$  (i) except that now we arrive at  $\underline{x}(0) \in \mathcal{V}_{\ker H}^* + \mathcal{R}_{a, \ker H}^*$ . However, since  $\mathcal{V}_{\ker H}^* + A\mathcal{V}_{\ker H}^* + A\mathcal{R}_{a, \ker H}^* + \text{im} B = \mathcal{V}_{b, \ker H}^*$ , we arrive at the same conclusion.

(iv)  $\Rightarrow$  (i): is obvious, again by looking at a pulse for  $\underline{d}$ .

## 7. Extensions

7.1 Disturbance decoupled estimation. All what has been said up to now may be dualized and applied to the disturbance decoupled estimation problem. There, one considers  $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{G}\underline{d}$ ,  $\underline{y} = \underline{C}\underline{x}$ ,  $\underline{z} = \underline{H}\underline{x}$ , with  $\underline{y}$  the observation and  $\underline{z}$  the to-be-estimated output, and we are looking for conditions for the existence of an observer defined by the convolution  $\underline{z} = \underline{\mathcal{F}} * \underline{y}$  such that, with all variables in  $C_+^\infty$ , the map  $\underline{d} \mapsto \underline{e} := \underline{z} - \hat{\underline{z}}$  is zero. Theorems completely analogous to Theorems 1-4 may be stated for this case. The relevant solvability condition becomes  $\ker H \supset \mathcal{P}_{b, \text{im } G}^*$ , where  $\mathcal{P}_{b, \text{im } G}^*$  denotes the infimal  $\mathcal{L}_1$ -almost ('(A,C)') conditionally invariant subspace containing  $\text{im } G$  [5]. This condition is satisfied iff there exists a PID type of observer. Analogous results hold for the discrete time case.

7.2 Disturbance decoupling with output observations: If we consider again the system  $\Sigma_R$  but if we assume that instead of measuring  $\underline{d}$  or  $\underline{x}$ , we can only measure  $\underline{y}$ , given by  $\dot{\underline{x}}_1 = \underline{A}\underline{x}_1 + \underline{G}\underline{d}$ ;  $\underline{y} = \underline{C}\underline{x}_1$ , then we are able to find a disturbance decoupling feedforward control from  $\underline{y}$  to  $\underline{u}$  iff

$\text{im } G \subset \mathcal{V}_{b, \ker H}^*$  and  $\mathcal{P}_{b, \text{im } G}^* \subset \ker H$ . This is equivalent to the solvability, over  $\mathbb{R}(s)$ , of the equation  $H(Is - A)^{-1}B X(s) C(Is - A)^{-1}G = H(Is - A)^{-1}G$  in the unknown matrix  $X(s)$ . If we now make the (generic) assumption that there exists  $F(s)$  such that  $F(s) (I + C(Is - A)^{-1}B X(s)) = X(s)$  then we may conclude that there also exists a PID feedback control law which disturbance decouples for  $\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u} + \underline{G}\underline{d}$ ,  $\underline{y} = \underline{C}\underline{x}$ ,  $\underline{z} = \underline{H}\underline{x}$ . A similar result holds for the discrete time case but considering predictive elements in a feedback configuration poses some conceptual difficulties, however.

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