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Homogeneous Row-Continuous Bivariate Markov Chains with Boundaries

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In a previous paper [8] a theoretical treatment of finite row-continuous bivariate Markov chains $\underline{B}(t) = [J(t), N(t)]$ was developed, providing an algorithmic basis for finding their ergodic distributions and associated passage time moments. The continuous time chain $\underline{B}(t)$ with state space $\mathcal{B} = \{(j,n) : 0 \le j \le J, 0 \le n \le N\}$ was described as row-continuous in the sense that the marginal process N(t), indexed by row coordinate n, changed at transition epochs by at most one. In the present paper we restrict our discussion to those row-continuous chains for which the transition rate matrices, $\underline{\underline{\nu}}_n^0$, $\underline{\underline{\nu}}_n^+$, $\underline{\underline{\nu}}_n^-$ describing rates local to row n, are independent of n for each $1 \le n \le N - 1$. For $n = 0$, one has $\underline{\underline{\nu}}_0^- = \underline{\underline{0}}$, and for $n = N$, $\underline{\underline{\nu}}_N^+ = \underline{\underline{0}}$. Such processes may be described as row-homogenous, row-continuous processes modified by two retaining boundaries, as for earlier similar univariate contexts [2].

For all such processes the behavior of the bounded process is intimately related to that of the associated row-homogeneous process $\underline{B}^H(t) = [J(t), N^H(t)]$ on state space $\mathcal{B}^H = \{(j,n) : 0 \le j \le J, -\infty < n < \infty\}$ with $\underline{\underline{\nu}}_n^0$, $\underline{\underline{\nu}}_n^+$, $\underline{\underline{\nu}}_n^-$ independent of n for all integer n. The distributions governing the homogeneous process are as in [3] called Green's functions. The treatment of the bounded process via such functions is brought about through use of the same kind of compensation arguments and compensation functions employed in the earlier univariate studies. A quick introduction to the ideas is given in [3].

A treatment of the ergodic distribution for the single boundary process $\underline{B}(t)$ where $N = \infty$ has been given by M. Neuts [11], with some discussion of simple aspects of the transient behavior. For many applications in congestion theory, where finite buffering is present [9], the second boundary is of essential interest. Here we focus on two boundaries and study theoretically and algorithmically the transient behavior of such processes.

In section 1 the Green's function is defined and developed. An explicit generating function is derived and the functional relationship of the Green's function to the first passage time distribution is established. The compensation method is employed in section 2 to describe the dynamical behavior of the bounded process. In section 3, the invertibility of the Laplace transforms of the Green's function matrices and the first passage time matrix forms is demonstrated. The primary matrix geometric results are then exhibited in section 4, for both bounded and unbounded chains. Section 5 then develops algebraic and algorithmic means for finding the first passage time distributions. In the concluding section 6, the mean first passage times and exit times are obtained.

§1 The Homogeneous Row-Continuous Markov Chain and Related Green's Functions

To obtain the distribution of the process $\underline{B}(t)$ defined on the state space $\mathcal{B}$ we first obtain that for the associated row-homogeneous process $\underline{B}^H(t)$ defined on the infinite lattice $\mathcal{B}^H$. The latter distribution is the time-dependent Green's function $\underline{\underline{g}}_n(t)$ defined formally below. Two procedures for finding $\underline{\underline{g}}_n(t)$ will be developed. The ergodic Green's function $\underline{\underline{g}}_{\infty n} \overset{\underline{\underline{def}}}{} \int_0^\infty \underline{\underline{g}}_n(t)dt$ is shown to be finite for all n under simple, intuitive conditions on the row-homogeneous process.

The row-homogeneous row-continuous process $\underline{B}^H(t) = [J(t), N^H(t)]$ is defined on an infinite lattice $\mathcal{B}^H = \{(j,n) : 0 \le j \le J, -\infty < n < \infty\}$. The behavior of $\underline{B}^H(t)$ is governed by three transition rate matrices $\underline{\underline{\nu}}_H^+$, $\underline{\underline{\nu}}_H^-$, and $\underline{\underline{\nu}}_H^0$ which define the transitions between the states of a row and the two contiguous rows. Specifically, $\nu_{H;ij}^+$ is the hazard rate for transitions from (i,n) to (j, n+1), $\nu_{H;ij}^-$ is that from (i,n) to (j,n-1), and $\nu_{H;ij}^0$ is that from (i,n) to (j,n). As in [8], we uniformize by picking any $\nu > \max_i\{\sum_j(\nu_{H;ij}^+ + \nu_{H;ij}^- + \nu_{H;ij}^0)\}$ as a uniform rate. We then have, as in [ ],

<u>Def 1.1</u> $$\underline{\underline{a}}_H^+ = \frac{1}{\nu}\underline{\underline{\nu}}_H^+,\ \underline{\underline{a}}_H^- = \frac{1}{\nu}\underline{\underline{\nu}}_H^-,\ \underline{\underline{a}}_H^0 = \frac{1}{\nu}\underline{\underline{\nu}}_H^0 + \underline{\underline{I}} - \frac{1}{\nu}\underline{\underline{\nu}}_H^D$$

where

$$\underline{\underline{\nu}}_H^D = \text{Diag}((\underline{\underline{\nu}}_H^+ + \underline{\underline{\nu}}_H^- + \underline{\underline{\nu}}_H^0)\underline{1}) .$$

The transition probabilities for the row-homogeneous chain, from (i,0) to (j,n) are the components of the time-dependent matrix Green's function $\underline{\underline{g}}_n(t) = [g_{ij;n}(t)]$. In particular, we have

Def 1.2 $g_{ij;n}(t) = P[N^H(t) = n, J(t) = j \mid N^H(0) = 0,\ J(0) = i]$.

We will call the set $\{\underline{\underline{g}}_n(t): -\infty < n < \infty\}$ the time-dependent Green's function for $\underline{B}(t)$. Because of the row-homogeneity, $P[N^H(t) = n,\ J(t) = j \mid N^H(0) = m,\ J(0) = i] = g_{ij;n-m}(t)$. The generating function $\hat{\underline{\underline{g}}}(u,t) = \sum_{-\infty}^{\infty} \underline{\underline{g}}_n(t)u^n$ for the two-sided sequence $(\underline{\underline{g}}_n(t))_{n=-\infty}^{\infty}$ will be of value. Questions of summability and analytic structure have been discussed at length in [6,7].

Prop 1.3 The generating function $\hat{\underline{\underline{g}}}(u,t) = \sum_{n=-0}^{\infty} \underline{\underline{g}}_n(t)u^n$ satisfies

$$\hat{\underline{\underline{g}}}(u,t) = e^{-\nu t} e^{\nu t[\underline{\underline{a}}_H^0 + u\underline{\underline{a}}_H^+ + \frac{1}{u}\underline{\underline{a}}_H^-]} \ , \quad 0 < |u| < \infty$$

proof: The forward Kolmogorov equations for $\underline{B}^H(t)$ are

$$\frac{d}{dt}\underline{\underline{g}}_n(t) = -\nu\underline{\underline{g}}_n(t) + \nu\underline{\underline{g}}_{n-1}(t)\underline{\underline{a}}_H^+ + \nu\underline{\underline{g}}_n(t)\underline{\underline{a}}_H^0 + \nu\underline{\underline{g}}_{n+1}(t)\underline{\underline{a}}_H^- \ ,\forall_n \tag{1.4}$$

as the reader may verify from the component equations. Multiplying (1.4) by u^n and summing we get

$$\frac{d}{dt}\hat{\underline{\underline{g}}}(u,t) = -\nu\underline{\underline{g}}(u,t)[\underline{\underline{I}} - \{u\underline{\underline{a}}_H^+ + \underline{\underline{a}}_H^0 + \frac{1}{u}\underline{\underline{a}}_H^-\}] \tag{1.5}$$

and the result follows. □

Corollary 1.6 For the marginal process J(t) with transition probability matrix $\underline{\underline{p}}(t)$ we have

$\underline{\underline{p}}(t) = \hat{\underline{\underline{g}}}(1,t) = e^{-\nu t[I-\underline{\underline{a}}_H]}$ and the principal left eigenvector $\underline{e}^T$ of $\underline{\underline{a}}_H = \underline{\underline{a}}_H^0 + \underline{\underline{a}}_H^+ + \underline{\underline{a}}_H^-$ is the ergodic probability vector for J(t).

Proof: The proof follows directly from (1.3).

Although Proposition (1.3) is useful for its structural insights, it may also be used for direct calculation of the $g_n(t)$ when, for example, $a_H^+ >> a_H^-$ so that the cross-products converge quickly.

From (1.3) we have

$$(1.7) \qquad g_n(t) = \sum_{k=0}^{\infty} e^{-\nu t} \frac{(\nu t)^k}{k!} b_{k,n}$$

where $b_{k,n}$ is the matrix coefficient of u^n in $[a_H^0 + u a_H^+ + \frac{1}{u} a_H^-]^k$.

The following approach may be used to find $\gamma_n(s) = \mathcal{L}[g_n(t)] = \int_0^\infty e^{-st} g_n(t)dt$ when the upwards and downwards first passage time probability matrices are known. Formally, as in [8] we define the first passage times

Def 1.8 (a) $s_H^+(t) = [s_{ij}^+(t)]$, $s_H^-(t) = [s_{ij}^-(t)]$

(b) $\sigma^+(s) = \mathcal{L}[s_H^+(t)]$, $\sigma^-(s) = \mathcal{L}[s_H^-(t)]$

(c) where

$s_{ij}^+(t)dt$ = P[a first arrival at row n+1 in (t,t+dt) and that arrival is at state (j,n+1)| at state (i,n) at time 0]

$s_{ij}^-(t)dt$ = similary except to row n-1.

We first need a prelimary result

Lemma 1.9

$I - (\frac{\nu}{s+\nu})\{a_H^0 + a_H^+ \sigma^-(s) + a_H^- \sigma^+(s)\}$ is invertible for $s \geq 0$, if $\underline{e}^T[\underline{\underline{a}}_H^+ - \underline{\underline{a}}_H^-]\ \underline{1} \overset{def}{=} M_H \neq 0$, where $\underline{e}^T$ is the principal left eigenvector of $\underline{\underline{a}}_H = \underline{\underline{a}}_H^0 + \underline{\underline{a}}_H^+ + \underline{\underline{a}}_H^-$.

proof:

It is known [6,7] that when the asymptotic drift rate $M_H > 0$ we have $\underline{\underline{\sigma}}^+(0)$ stochastic and $\underline{\underline{\sigma}}^-(0)$ strictly substochastic. Similarly, when $M_H < 0$, $\underline{\underline{\sigma}}^+(0)$ is strictly substochastic and $\underline{\underline{\sigma}}^-(0)$ is stochastic.

In either case, $\underline{\underline{a}}_H$ is stochastic and thus $\underline{\underline{a}}_H^0 + \underline{\underline{\sigma}}^-(0)\underline{\underline{a}}_H^+ + \underline{\underline{\sigma}}^+(0)\underline{\underline{a}}_H^-$ is strictly substochastic. This quantity, therefore, has a spectral radius < 1 when $s=0$ and the lemma follows for that case. For s real, $s \geq 0$, the matrices $\underline{\underline{\sigma}}^+(s)$, $\underline{\underline{\sigma}}^-(s)$ are monotone decreasing (element-wise), as is $\nu/\nu+s$ so that the spectral radius of

$(\frac{\nu}{\nu+s})\{\underline{\underline{a}}_H^0 + \underline{\underline{\sigma}}^-(s)\underline{\underline{a}}_H^+ + \underline{\underline{\sigma}}^+(s)\underline{\underline{a}}_H^-\}$ is decreasing [1]

and the lemma is proven. □

Notice that $\underline{e}^T$ may be interpreted as the ergodic distribution of the underlying row process $J(t)$ governed by $\underline{\underline{\nu}} = \underline{\underline{\nu}}^+ + \underline{\underline{\nu}}^- + \underline{\underline{\nu}}^0$.

The next two propositions describe the second procedure for finding $\underline{\underline{g}}_n(t)$, in principle.

Prop 1.10 $\quad \underline{\underline{\gamma}}_0(s) = \frac{1}{s+\nu}\left[\underline{\underline{I}} - (\frac{\nu}{s+\nu})\{\underline{\underline{a}}_H^0 + \underline{\underline{a}}_H^+\underline{\underline{\sigma}}^-(s) + \underline{\underline{a}}_H^-\underline{\underline{\sigma}}^+(s)\}\right]^{-1}, s \geq 0.$

proof: The argument given in [8] is applicable immediately for the case $s > 0$ in our row-homogeneous setting. One need only make these identifications

(1.11) $\quad \underline{\underline{g}}_0(t) = \underline{\underline{p}}_{nn}(t)\ \forall_n$, since $\underline{\underline{p}}_{nn}(t) = \underline{\underline{p}}_{00}(t)$ from row-homogeneity

(1.12) $\quad (\underline{\underline{a}}_H^+, \underline{\underline{a}}_H^-, \underline{\underline{a}}_H^0) \longleftrightarrow (\underline{\underline{a}}_n^+, \underline{\underline{a}}_n^-, \underline{\underline{a}}_n^0)$

(1.13) $\underline{\underline{\sigma}}^{-}(s) \leftrightarrow \underline{\underline{\sigma}}^{-}_{n+1}(s)$; $\underline{\underline{\sigma}}^{+}(s) \leftrightarrow \underline{\underline{\sigma}}^{+}_{n-1}(s)$.

The invertibility for $s \geq 0$ is a consequence of (1.9)

Prop 1.14

(a) $\underline{\underline{\gamma}}_n(s) = [\underline{\underline{\sigma}}^{+}(s)]^n \, \underline{\underline{\gamma}}_0(s)$ for $n \geq 0$,

(b) $\underline{\underline{\gamma}}_n(s) = [\underline{\underline{\sigma}}^{-}(s)]^{-n} \, \underline{\underline{\gamma}}_0(s)$ for $n < 0$.

Proof: In the scalar setting, where J=0, the equation $\gamma_n(s) = [\sigma^{+}(s)]^n \gamma_0(s)$ is the familiar counterpart of the general equation $p_{mn}(t) = s_{mn}(t)*p_{nn}(t)$ for any chain. For a scalar, skip-free spatially homogeneous process, one has $s_{0n}(t) = s_{01}^{(n)}(t)$. In our matrix setting, $\underline{\underline{g}}_n(t) = \underline{\underline{s}}_{0n}(t)*\underline{\underline{g}}_0(t)$ where $\underline{\underline{s}}_{0n}(t)$ is, as in [8], the matrix p.d.f. for the first passage time from row 0 to row n, since to be in row n at time t, there had to be a first visit to row n at t', 0<t'<t, and one had to be in row n at time t. From the row-homogeneity and row-continuity $\underline{\underline{s}}_{0n}(t) = (\underline{\underline{s}}^{+}(t))^{(n)}$, so that $\underline{\underline{\sigma}}_{0n}(s) = \sigma^{+^n}(s)$ where n>0, and similarly, for n<0.

Thus, knowledge of $\underline{\underline{\sigma}}^{+}(s)$ and $\underline{\underline{\sigma}}^{-}(s)$ enables us to calculate $\underline{\underline{\gamma}}_n(s)$ for all n.

The ergodic Green's density is another quantity of interest. We next define the ergodic Green's density matrix $\underline{\underline{g}}_{\infty n}$.

Def 1.15

$$\underline{\underline{g}}_{\infty n} = [g_{\infty n;ij}] \; ; \; g_{\infty n;ij} = \int_0^\infty g_{ij;n}(t)dt$$

§2 The Bounded Process and Compensation

We are now in a position to relate the probability distribution for the bounded process $\underline{B}(t)$ to the Green's function $\underline{\underline{g}}(n,t)$ of section 1. By using the compensation method developed earlier [2] we are able to express the distribution of $\underline{B}(t)$ in terms of the probability vectors for the boundary rows at 0 and N.

The bounded process $\underline{B}(t)$ is the row-continuous process on $\mathcal{B} = \{(j,n) : 0 \le j \le J,\ 0 \le n \le N\}$ whose transition matrices coincide with those of $\underline{B}^H(t)$ for row 1,..,N-1, but differ at rows 0 and N. At those rows the transition rate matrices $\underline{\underline{\nu}}_0^-$ and $\underline{\underline{\nu}}_N^+$ are replaced by $\underline{\underline{0}}$.

(For some applications, the transition rate matrices $\underline{\underline{\nu}}_0^+$, $\underline{\underline{\nu}}_0^0$, $\underline{\underline{\nu}}_N^-$, and $\underline{\underline{\nu}}_N^0$ may have yet other values without affecting the applicability of the method. In this paper we consider only the process $\underline{B}(t)$ specified.)

Let ν, $\underline{\underline{a}}_H^+$, $\underline{\underline{a}}_H^-$, and $\underline{\underline{a}}_H^0$ be as for the underlying homogeneous matrix. Let $\underline{p}_n^T(t)$ be the probability vector for row n, i.e., let $\underline{p}_n^T(t) = (p_{j,n}(t))_0^J$ where $p_{j,n}(t) = P[J(t) = j,\ N(t) = n]$. The forward Kolmogorov equations for the process have the form

$$(2.1) \qquad \frac{d\underline{p}_0^T(t)}{dt} = -\nu\underline{p}_0^T(t) + \nu\underline{p}_0^T\underline{\underline{a}}_0^0 + \nu\underline{p}_1^T(t)\underline{\underline{a}}_H^-$$

$$\frac{d\underline{p}_n^T(t)}{dt} = -\nu\underline{p}_n^T(t) + \nu\underline{p}_n^T(t)\underline{\underline{a}}_H^0 + \nu\underline{p}_{n-1}^T(t)\underline{\underline{a}}_H^+ + \nu\underline{p}_{n+1}^T(t)\underline{\underline{a}}_H^- \qquad 1 \le n \le N-1$$

$$\frac{d\underline{p}_N^T(t)}{dt} = -\nu\underline{p}_N^T(t) + \nu\underline{p}_N^T(t)\underline{\underline{a}}_N^0 + \nu\underline{p}_{N-1}^T(t)\underline{\underline{a}}_H^+$$

where $\underline{\underline{a}}_0^0$ and $\underline{\underline{a}}_N^0$ differ from $\underline{\underline{a}}_H^0$ due to the modified boundary rates.

This finite set of equations may be replaced by the infinite set of vector differential equations,

$$(2.2) \qquad \frac{d\underline{p}_n^T(t)}{dt} = -\nu\underline{p}_n^T(t) + \nu\underline{p}_n^T(t)\underline{\underline{a}}_H^0 + \nu\underline{p}_{n-1}^T(t)\underline{\underline{a}}_H^+ + \nu\underline{p}_{n+1}^T(t)\underline{\underline{a}}_H^-$$

$$+ \delta_{n,0}\underline{C}_0^T(t) + \delta_{n,-1}\underline{C}_{-1}^T(t) + \delta_{n,N}\underline{C}_N^T(t) + \delta_{n,N+1}\underline{C}_{N+1}^T(t)$$

$$-\infty < n < \infty ,$$

where $\underline{C}_0^T(t)$, $\underline{C}_{-1}^T(t)$, $\underline{C}_N^T(t)$, and $\underline{C}_{N+1}^T(t)$ will be chosen in such a way as to have the resulting solution $\underline{p}_n^T(t)$ coincide with that for (2.1) and $\underline{p}_n^T(t) \equiv \underline{0}$ for $n \leq -1$ and $n \geq N+1$. The functions $\underline{C}_j^T(t)$ which do this are called compensation functions. Their intuitive meaning is associated with the point of view motivating the compensation method. The virtual transitions for the homogeneous dynamics are neutralized by the injection at rows $n = -1$ and $n = N+1$ of negative source density into the probability space and positive source density at rows $n = 0$ and $n = N$, reflecting the dynamics of the bounded process. The source injection rate at state (j,n) is $C_{j,n}(t)$ and the rate vectors are $\underline{C}_n^T(t)$. A more complete discussion of the procedure may be found in [2]. A quick introduction is given in [4]. A more recent alternate development that is concise and clear, is given in [5].

The probability vectors $\underline{p}_n^T(t)$ are the response, for the homogeneous process, to the initial distribution and to the injected sources. One then has, for any initial distribution $(\underline{f}_n^T)_0^N$,

$$(2.3)\qquad \underline{p}_n^T(t) = \sum_{m=0}^{N} \underline{f}_{-m}^T \underline{\underline{g}}_{n-m}(t) + \underline{C}_0^T(t) * \underline{\underline{g}}_n(t) + \underline{C}_{-1}^T(t) * \underline{\underline{g}}_{n-1}(t)$$

$$+ \underline{C}_N^T(t) * \underline{\underline{g}}_{n-N}(t) + \underline{C}_{N+1}(t) * \underline{\underline{g}}_{n-(1+N)}(t) \quad .$$

Equation (2.3) may also be obtained by taking the two-sided generating function of both sides of (2.2), taking the Laplace transform, solving for the resultant generating function transform and making the identification (2.3).

We now have the tools for the major result of this section.

Theorem 2.4 For any initial distribution $\{\underline{f}_n^T : 0 \le n \le N\}$ we have

$$\underline{p}_n^T(t) = \nu \underline{p}_0^T(t) \underline{\underline{a}}_D^- * \underline{\underline{g}}_n(t) - \nu \underline{p}_0^T(t) \underline{\underline{a}}_H^- * \underline{\underline{g}}_{n+1}(t) + \nu \underline{p}_N^T(t) \underline{\underline{a}}_D^+ * \underline{\underline{g}}_{n-N}(t)$$

$$- \nu \underline{p}_N^T(t) \underline{\underline{a}}_H^+ * \underline{\underline{g}}_{n-(N+1)}(t) + \sum_{m=0}^{N} \underline{f}_{-m}^T \, \underline{\underline{g}}_{n-m}(t) \ , \text{ for all } n$$

where $\underline{\underline{a}}_D^- = \text{diag}\,(\underline{\underline{a}}_H^- \underline{1})$, $\underline{\underline{a}}_D^+ = \text{diag}\,(\underline{\underline{a}}_H^+ \underline{1})$ and the * denotes convolution in time.

proof:

Examination of (2.1) and (2.2) shows that setting

$$(2.5)\qquad \underline{C}_N^T(t) = \nu \underline{p}_N^T(t)\ \text{diag}(\underline{\underline{a}}_H^+ \underline{1}),\ \underline{C}_0^T(t) = \nu \underline{p}_0^T(t)\ \text{diag}(\underline{\underline{a}}_H^- \underline{1})$$

converts the subset of equations $\{(2.2n) : 0 \le n \le N\}$ to those for the set of equations in (2.1). The formalism, as developed in [2], demonstrates that the compensation functions $\underline{C}_0^T(t)$, $\underline{C}_N^T(t)$ have precisely that form and that one needs

$$(2.6)\qquad \underline{C}^T_{N+1}(t) = -\nu\underline{p}^T_N(t)\underline{\underline{a}}^+_H\ ,\ \underline{C}^T_{-1}(t) = -\nu\underline{p}^T_0(t)\underline{\underline{a}}^-_H$$

with all other $\underline{C}^T_n(t) \equiv \underline{0}$. Equations (2.6) describe the neutralization of the virtual transitions to rows -1 and N+1, and (2.5) the return of the probability to rows 0 and N.

Theorem 2.4 provides the basis for relating $\underline{p}^T_n(t)$ to $\underline{p}^T_0(t)$ and $\underline{p}^T_N(T)$ and subsequently $\underline{e}^T_n$ to $\underline{e}^T_0$ and $\underline{e}^T_N$. By setting n=0 and n=N in (2.4) one obtains a coupled pair of integral equations for $\underline{p}^T_0(t)$ and $\underline{p}^T_N(t)$ permitting one in principle to evaluate these. An alternative procedure for finding $\underline{p}^T_0(t)$ and/or $\underline{p}^T_N(t)$ is to make use of the formalism developed in [8] and, in particular, in sections 2 and 3 there. A judicious combination of both procedures will be seen to be useful.

§3 Structure of First Passage Times and Invertibility of the Green's Function Matrices $\underline{\underline{\gamma}}_n(s)$

The Green's function matrices are vital to the solution of Theorem 2.4. They are available as functions of $\underline{\underline{g}}^+(s)$ and $\underline{\underline{g}}^-(s)$, which we now find equations for. We first demonstrate that $\underline{\underline{g}}^+(s)$ and $\underline{\underline{g}}^-(s)$ satisfy simple matrix quadratic equations.

Prop 3.1

(a) $\underline{\underline{g}}^+(s) = \underline{\underline{\alpha}}(s) + \underline{\underline{\beta}}(s)\ [\underline{\underline{g}}^+(s)]^2$

(b) $\underline{\underline{g}}^-(s) = \underline{\underline{\beta}}(s) + \underline{\underline{\alpha}}(s)\ [\underline{\underline{g}}^-(s)]^2$

where

(3.2) $\underline{\underline{\alpha}}(s) = \nu\underline{\underline{\gamma}}^*(s)\underline{\underline{a}}_H^+\ ,\ \underline{\underline{\beta}}(s) = \nu\underline{\underline{\gamma}}^*(s)\underline{\underline{a}}_H^-$

and

(3.3) $\underline{\underline{\gamma}}^*(s) = [(s+\nu)\underline{\underline{I}} - \nu\underline{\underline{a}}_H^0]^{-1}$

proof:

The proof is based on the results in section 2 of [8]. The matrix $\underline{\underline{\gamma}}^*(s)$ is the Laplace transform of the transition probability matrix for the loss process on $\{(j,0) : 0 \le j \le J\}$ governed by $\underline{\underline{\nu}}_H^0$, $\underline{\underline{\nu}}_H^+$, and $\underline{\underline{\nu}}_H^-$ when the adjacent rows n=1 and n=-1 are absorbing. Equation (3.3) is the analog of (2.4) of [8] in the notation of this paper.

Associated with the loss process on row 0 are the pair of exit time matrix densities $\underline{\underline{a}}(\tau) = [a_{jk}(\tau)]$ and $\underline{\underline{b}}(\tau) = [b_{jk}(\tau)]$ with the transforms $\underline{\underline{\alpha}}(s)$ and $\underline{\underline{\beta}}(s)$ respectively. Here $a_{jk}(\tau)$ is the probability density that exit is to state (k,1) at time τ given start at state (j,0) at time 0 and $b_{jk}(\tau)$ is the probability density that exit is to state (k,-1) at time τ given start at state (j,0) at time 0. The quadratic equation (a) states that to reach row 1 at state (k,1) one either goes to row 1 (at state k) before row -1, or one goes to row -1 and thence to row 0 and thence to row 1 at state (k,1). Equation (b) has the same meaning for downward transitions. □

We will now discuss the non-singularity of the matrices $\underline{\underline{\sigma}}^+(s)$, $\underline{\underline{\sigma}}^-(s)$, and $\underline{\underline{\gamma}}_0(s)$.

Prop 3.4

Let $\underline{B}^H(t)$ be irreducible. Let the asymptotic drift rate M_H be non-zero. Then, for $s \geq 0$

(a) $\underline{\underline{\sigma}}^+(s)$ is non-singular $\Longleftrightarrow$ $\underline{\underline{\nu}}_H^+$ is non-singular (or $\underline{\underline{a}}_H^+$)

(b) $\underline{\underline{\sigma}}^-(s)$ is non-singular $\Longleftrightarrow$ $\underline{\underline{\nu}}_H^-$ is non-singular (or $\underline{\underline{a}}_H^-$)

proof:

From the proof of (3.1) we note that $\underline{\underline{\alpha}}(0) + \underline{\underline{\beta}}(0)$ is stochastic and from irreducibility $\underline{\underline{\alpha}}(0) \neq 0$ and $\underline{\underline{\beta}}(0) \neq 0$. We have seen that $\underline{\underline{\sigma}}^+(s) = \underline{\underline{\alpha}}(s) + \underline{\underline{\beta}}(s)[\sigma^+(s)]^2$. Consequently $[\underline{\underline{I}} - \underline{\underline{\beta}}(s)\underline{\underline{\sigma}}^+(s)]\underline{\underline{\sigma}}^+(s) = \underline{\underline{\alpha}}(s)$ $= \underline{\underline{\gamma}}^*(s)\underline{\underline{\nu}}^+$.

Since $\underline{\underline{\sigma}}^+(s)$ is substochastic and $\underline{\underline{\beta}}(s)$ is strictly substochastic $[\underline{\underline{I}} - \underline{\underline{\beta}}(s)\underline{\underline{\sigma}}^+(s)]$ is non-singular. Moreover, since $\underline{\underline{a}}_H^0$ is substochastic, $\underline{\underline{\gamma}}^*(s)$ is non-singular by (3.3) hence (a). Part (b) follows the same way. □

Corollary 3.5

Under the conditions of Prop 3.2, for $s \geq 0$,

(a) $\underline{\underline{\gamma}}_n(s)$ is non-singular, $\forall_{n>0} \Longleftrightarrow \underline{\underline{a}}_H^+$ is non-singular

(b) $\underline{\underline{\gamma}}_n(s)$ is non-singular, $\forall_{n<0} \Longleftrightarrow \underline{\underline{a}}_H^-$ is non-singular.

Proof:

From Lemma 1.9 and Prop 1.10 we have that $\underline{\underline{\gamma}}_0(s)$ is non-singular for $s \geq 0$ when $M_H \neq 0$. From Prop 1.14 the result follows immediately.

§4 The Ergodic Distribution

The results obtained in section 2 can now be employed to provide ergodic probabilities. In particular, Theorem 2.4 leads to the following result:

Theorem 4.1

Let $\underline{B}(t)$ be irreducible and ergodic, and let $\underline{e}_n^T = \lim_{t\to\infty} \underline{p}_n^T(t)$. Then

$$\underline{e}_n^T = \nu\underline{e}_0^T\{\underline{\underline{a}}_D^- \underline{\underline{g}}_{\infty n} - \underline{\underline{a}}_H^- \underline{\underline{g}}_{\infty n+1}\} + \nu\underline{e}_N^T\{\underline{\underline{a}}_D^+ \underline{\underline{g}}_{\infty n-N} - \underline{\underline{a}}_H^+ \underline{\underline{g}}_{\infty n-(N+1)}\}$$

for $0 \le n \le N$

proof:

This follows from Laplace transformation of (2.4), multiplication by s, and passage to the limit $s\to 0^+$. In particular, $s\underline{\Pi}_n^T(s)\to\underline{e}_n^T$, $\underline{\underline{\gamma}}_n(s)\to\underline{\underline{g}}_{\infty n}$, and $s\underline{\underline{\gamma}}_n(s)\to 0$, by the usual Tauberian argument.

From 4.1 we can obtain at once the matrix geometric form of M. Neuts [10,11] for the ergodic distribution when the homogeneous process is modified by a single boundary at n=0. We have

Corollary 4.2

For $\underline{B}(t)$ ergodic with one boundary at n=0 (i.e., $N=\infty$) the ergodic row probabilities are matrix geometric, that is,

$$\underline{e}_n^T = \underline{e}_0^T \underline{\underline{\rho}}^n \quad \text{for all } n \ge 0$$

where

(4.3) $\underline{\rho} = \underline{\underline{g}}_{\infty 0}^{-1}\, \underline{\underline{\sigma}}^{+}(0)\, \underline{\underline{g}}_{\infty 0}$

and

$\underline{e}_0^T$ is an eigenvector satisfying

(4.4) $\underline{e}_0^T = \underline{e}_0^T\{\underline{\underline{a}}_H^0 + \underline{\underline{a}}_H^+ \underline{\underline{\sigma}}^-(0) + \underline{\underline{a}}_D^-\} \overset{\text{def}}{=\!=} \underline{e}_0^T\, \underline{\underline{b}}$

proof: By the same method as that for Theorem 4.1 we can write

(4.5) $\underline{e}_n^T = \nu \underline{e}_0^T\{\underline{\underline{a}}_D^- \underline{\underline{g}}_{\infty n} - \underline{\underline{a}}_H^- \underline{\underline{g}}_{\infty n+1}\}$. $n \geq 0$

In particular, letting n=0, and using (1.9) we have

(4.6) $\underline{e}_0^T = \nu \underline{e}_0^T\{\underline{\underline{a}}_D^- - \underline{\underline{a}}_H^- \underline{\underline{\sigma}}^+(0)\}\, \underline{\underline{g}}_{\infty 0}$.

We use prop 1.8 evaluated at s=0, then multiply (4.6) by $(\nu \underline{\underline{g}}_{\infty 0})^{-1}$ to get (4.4).

We recall, from (1.8) that $\underline{\underline{g}}_{\infty n} = (\underline{\underline{\sigma}}^+(0))^n \underline{\underline{g}}_{\infty 0}$ for $n \geq 0$, hence

(4.7) $\underline{e}_n^T = \nu \underline{e}_0^T\, \{\underline{\underline{a}}_D^- - \underline{\underline{a}}_H^- \underline{\underline{\sigma}}^+(0)\}\, \underline{\underline{g}}_{\infty 0}(\underline{\underline{g}}_{\infty 0}^{-1}\, \underline{\underline{\sigma}}^+(0) \underline{\underline{g}}_{\infty 0})^n$.

Setting n=0 gives

(4.8) $\underline{e}_0^T = \underline{e}_0^T[\nu\{\underline{\underline{a}}_D^- - \underline{\underline{a}}_H^- \underline{\underline{\sigma}}^+(0)\}\underline{\underline{g}}_{\infty 0}]$.

Substituting into (4.7) gives the desired equations. □

Remark:

The matrix $\underline{\underline{b}}$ of (4.4) is irreducible when $\underline{B}(t)$ is irreducible. Thus, $\underline{e}_0^T$ is the principal eigenvector of $\underline{\underline{b}}$. To see that $\underline{\underline{b}}$ is irreducible we present the following argument:

We wish to show, for all i,j there exists a $k \geq 0$ such that $(\underline{\underline{b}}^k)_{ij} \neq 0$. Now $\underline{B}(t)$ is irreducible, so there exists a path (in, say ℓ steps) from (i,0) to (j,0). There are two cases:

(1) the path stays on the row n=0.

If so, we have $(\underline{\underline{a}}_H^0)_{ij}^{\ell} \neq 0$. But, $\underline{\underline{b}} \geq \underline{\underline{a}}_H^0$, hence $(\underline{\underline{b}}^{\ell})_{ij} \neq 0$.

(2) the path leaves the row n=0 at the k^{th} step. We then see that $\{(\underline{\underline{a}}_H^0)^k \underline{\underline{a}}_H^+ \underline{\underline{g}}^-(0)\}_{ij} \neq 0$. But, $\underline{\underline{b}}^{k+1} \geq (\underline{\underline{a}}_H^0)^k \underline{\underline{a}}_H^+ \underline{\underline{g}}^-(0)$ and the argument concludes. □

There is a matrix geometric result for the two boundary $\underline{B}(t)$. We see that $\underline{e}_n^T$ decreases in relation to the maximal eigenvalue of $\underline{\underline{g}}^+(0)$.

Corollary 4.9

We have,

$$\underline{e}_n^T = \underline{e}_0^T(\nu\{\underline{\underline{a}}_D^- - \underline{\underline{a}}_H^- \underline{\underline{g}}^+(0)\}\underline{\underline{g}}_{\infty 0})\underline{\underline{p}}_1^n + \underline{e}_N^T(\nu\{\underline{\underline{a}}_D^+ - \underline{\underline{a}}_H^+ \underline{\underline{g}}^-(0)\}\underline{\underline{g}}_{\infty 0})\underline{\underline{p}}_2^{N-n}$$

where

$$(4.10) \qquad \underline{\underline{p}}_1 = \underline{\underline{g}}_{\infty 0}^{-1} \underline{\underline{g}}^+(0)\underline{\underline{g}}_{\infty 0}\ , \quad \underline{\underline{p}}_2 = \underline{\underline{g}}_{\infty 0}^{-1} \underline{\underline{g}}^-(0)\underline{\underline{g}}_{\infty 0} \quad .$$

Proof:

The proof is immediate from (4.1) and (1.9). □

We may let n=0,N in (4.9) to get 2 sets of matrix equations with 2 vector unknowns. The solvability of this set of equations has not been proven.

We now use Theorem 4.1 to find the ergodic distribution for the two boundary $\underline{B}(t)$ explicitly.

Prop 4.11

When $\underline{\underline{g}}^{+}(0)$ is strictly substochastic

$$\underline{e}_N^T = \underline{e}_0^T\{\underline{\underline{a}}_D^- - \underline{\underline{a}}_H^-\underline{\underline{g}}^+(0)\}\underline{\underline{g}}^+(0)^N\ [\underline{\underline{I}} - (\underline{\underline{a}}_H^0 + \underline{\underline{a}}_D^+ + \underline{\underline{a}}_H^-\underline{\underline{g}}^+(0))]^{-1}$$

Proof:

Let n=N in (4.1) to get

$$(4.12)\qquad \underline{e}_N^T = \nu\underline{e}_0^T\{\underline{\underline{a}}_D^-\underline{\underline{g}}_{\infty N} - \underline{\underline{a}}_H^-\underline{\underline{g}}_{\infty N+1}\} + \nu\underline{e}_N^T\{\underline{\underline{a}}_D^+\underline{\underline{g}}_{\infty 0} - \underline{\underline{a}}_H^+\underline{\underline{g}}_{\infty -1}\} .$$

Multiply by $(\nu\underline{\underline{g}}_{\infty 0})^{-1}$ and use (1.9) to get

$$(4.13)\qquad \frac{1}{\nu}\,\underline{e}_N^T\underline{\underline{g}}_{\infty 0}^{-1} = \underline{e}_0^T\{\underline{\underline{a}}_D^-\underline{\underline{g}}^+(0)^N - \underline{\underline{a}}_H^-\underline{\underline{g}}^+(0)^{N+1}\} + \underline{e}_N^T\{\underline{\underline{a}}_D^+ - \underline{\underline{a}}_H^+\underline{\underline{g}}^-(0)\} .$$

Now, we let s=0 in (1.8) and note that $\underline{\underline{a}}_H^0 + \underline{\underline{a}}_D^+ + \underline{\underline{a}}_H^-\underline{\underline{g}}^+(0)$ is strictly substochastic, because $\underline{\underline{g}}^+(0)$ is strictly substochastic, while $\underline{\underline{a}}_H^0 + \underline{\underline{a}}_D^+ + \underline{\underline{a}}_H^-$ is stochastic ($\underline{\underline{a}}_H^- \neq \underline{\underline{0}}$). Therefore, $\underline{\underline{I}} - (\underline{\underline{a}}_H^0 + \underline{\underline{a}}_D^+ + \underline{\underline{a}}_H^-\underline{\underline{g}}^+(0))$ is invertible and (4.11) follows. □

The analogous result is presented without proof.

Prop 4.14

When $\underline{\underline{g}}^-(0)$ is strictly substochastic

$$\underline{e}_0^T = \underline{e}_N^T\{\underline{\underline{a}}_D^+ - \underline{\underline{a}}_H^+\underline{\underline{g}}^-(0)\}\underline{\underline{g}}^-(0)^N\ [\underline{\underline{I}} - (\underline{\underline{a}}_H^0 + \underline{\underline{a}}_D^- + \underline{\underline{a}}_H^+\underline{\underline{g}}^-(0)]^{-1} .$$

§5 Algorithms for $\underline{\underline{g}}^+(0)$, $\underline{\underline{g}}^-(0)$ and $\underline{\underline{g}}_{\infty 0}$

The algorithms and results of section 4 assumed that $\underline{\underline{g}}^+(0)$, $\underline{\underline{g}}^-(0)$, and $\underline{\underline{g}}_{\infty 0}$ were known. We here present algorithms for their computation. A procedure is available permitting the full evaluation of $\underline{\underline{s}}^+(\tau)$ and $\underline{\underline{s}}^-(\tau)$ with substantial machine effort. This will be discussed elsewhere.

We first restate proposition 3.1 for the s=0 case.

Prop 5.1

(a) $\underline{\underline{g}}^+(0) = \underline{\underline{\alpha}}_0 + \underline{\underline{\beta}}_0(\underline{\underline{g}}^+(0))^2$

(b) $\underline{\underline{g}}^-(0) = \underline{\underline{\beta}}_0 + \underline{\underline{\alpha}}_0(\underline{\underline{g}}^-(0))^2$

where

(c) $\underline{\underline{\alpha}}_0 = [\underline{\underline{I}} - \underline{\underline{a}}_H^0]^{-1}\underline{\underline{a}}_H^+$, $\underline{\underline{\beta}}_0 = [\underline{\underline{I}} - \underline{\underline{a}}_H^0]^{-1}\underline{\underline{a}}_H^-$

Proof: by (3.1) □

We note that $(\underline{\underline{\alpha}}_0)_{ij}$ is the probability that, given start at (i,0), the first transition out of row 0 is to state (j,1). Similarly, $(\underline{\underline{\beta}}_0)_{ij}$ is the probability that, given start at (i,0), the first transition out of row 0 is to state (j,-1).

When $\underline{B}^H(t)$ is irreducible we have, of course, that $\underline{\underline{\alpha}}_0 + \underline{\underline{\beta}}_0$ is stochastic, and $\underline{\underline{\alpha}}_0$, $\underline{\underline{\beta}}_0$ are strictly substochastic. This may be verified by noting that

$$(\underline{\underline{\alpha}}_0 + \underline{\underline{\beta}}_0)\underline{1} = [I - a_H^0]^{-1} (\underline{\underline{a}}_H^+ + \underline{\underline{a}}_H^-)\underline{1}$$

$$= [I - a_H^0]^{-1} (\underline{1} - \underline{\underline{a}}_H^0\underline{1})$$

$$= [I - a_H^0]^{-1} [\underline{\underline{I}} - \underline{\underline{a}}_H^0]\underline{1} = \underline{1}.$$

<u>Theorem 5.2</u>

Assume, without loss of generality, that $\underline{\underline{\sigma}}^+(0)$ is strictly substochastic, $\underline{\underline{\sigma}}^-(0)$ is stochastic, then:

(5.3) the recursion: $\underline{\underline{\sigma}}_0 = \underline{\underline{0}}$, $\underline{\underline{\sigma}}_{K+1} = \underline{\underline{\alpha}}_0 + \underline{\underline{\beta}}_0 \underline{\underline{\sigma}}_K^2$

converges monotonically to $\underline{\underline{\sigma}}^+(0)$, and

(5.4) the recursion: $\underline{\underline{\sigma}}_0$ arbitrary substochastic

$$\underline{\underline{\sigma}}_{K+1} = \underline{\underline{\beta}}_0 + \underline{\underline{\alpha}}_0 \underline{\underline{}}_K^2$$

converges to $\underline{\underline{\sigma}}^-(0)$. The convergence is monotonic if $\underline{\underline{\sigma}}_0 = \underline{\underline{0}}$.

<u>Proof:</u>

This proof may be found in a slightly different form in [10,11]

We can find series expansions $\underline{\underline{\sigma}}^+(s)$ and $\underline{\underline{\sigma}}^-(s)$ for all real $s \geq 0$ by use of (3.1).

Theorem 5.5

Let $Z = \frac{\nu}{s+\nu}$, $\underline{\phi}(Z) \underline{\underline{def}} \frac{\underline{\sigma}^+(s)}{Z}$

then $\underline{\phi}(Z)$ is finite for real $0 < Z \leq 1$

and

(5.6) $\underline{\phi}(Z) = \underline{\underline{a}}_H^+ + Z\underline{\underline{a}}_H^0\phi(Z) + Z^2\underline{\underline{a}}_H^- \underline{\phi}^2(Z)$.

Proof:

By combining (3.1a) and (3.1e) we get:

(5.7) $\underline{\sigma}^+(s) = \nu[(s+\nu)\underline{\underline{I}} - \nu\underline{\underline{a}}_H^0]^{-1}\underline{\underline{a}}_H^+ + \nu[(s+\nu)\underline{\underline{I}} - \nu\underline{\underline{a}}_H^0]\underline{\underline{a}}_H^-(\underline{\sigma}^+(s))^2$.

Premultiplying by $\underline{\underline{I}} - (\frac{\nu}{s+\nu})\underline{\underline{a}}_H^0$ and substituting Z we have

(5.8) $[\underline{\underline{I}} - Z\underline{\underline{a}}_H^0]\underline{\sigma}^+(s) = Z\underline{\underline{a}}_H^+ + Z\underline{\underline{a}}_H^-[\underline{\sigma}^+(s)]^2$.

Hence, dividing by Z,

(5.9) $\frac{\underline{\sigma}^+(s)}{Z} = Z\underline{\underline{a}}_H^0 \frac{\underline{\sigma}^+(s)}{Z} + \underline{\underline{a}}_H^+ + Z^2\underline{\underline{a}}_H^-[\frac{\underline{\sigma}^+(s)}{Z}]^2$.

This is (5.7) when we substitute $\phi(Z)$ for $\frac{\underline{\sigma}(s)}{Z}$.

The convergence of $\phi(Z)$ is assured by the convergence of $\underline{\sigma}^+(s)$ for $s \geq 0$.

Note that (5.7) is easily solved for $\phi(Z)$ by the usual techniques. An expansion about Z=0 is available via the recursion:

Corollary 5.8

Let $\underline{\underline{g}}^{+}(s) = \sum_{j=0}^{\infty} \underline{\underline{b}}_j^{+} \left(\frac{\nu}{s+\nu}\right)^{j+1}$, then $\underline{\underline{b}}_0^{+} = \underline{\underline{a}}_H^{+}$, $\underline{\underline{b}}_1^{+} = \underline{\underline{a}}_H^{0}\underline{\underline{a}}_H^{+}$, and

$$(5.9)\quad \underline{\underline{b}}_{k+1}^{+} = \underline{\underline{a}}_H^{0}\underline{\underline{b}}_k^{+} + \underline{\underline{a}}_H^{-} \sum_{\substack{i+j=k-1\\ i,j\geq 0}} \underline{\underline{b}}_i^{+}\underline{\underline{b}}_j^{+} \quad \text{for } k\geq 1 .$$

Proof: Trivial by Theorem 5.5. □

Corollary 5.10

Let $\underline{\underline{g}}^{-}(s) = \sum_{j=0}^{\infty} \underline{\underline{b}}_j^{-} \left(\frac{\nu}{s+\nu}\right)^{j+1}$. Then, $\underline{\underline{b}}_0^{-} = \underline{\underline{a}}_H^{-}$, $\underline{\underline{b}}_1^{-} = \underline{\underline{a}}_H^{0}\underline{\underline{a}}_H^{-}$, and

$$(5.11)\quad \underline{\underline{b}}_{k+1}^{-} = \underline{\underline{a}}_H^{0}\underline{\underline{b}}_k^{-} + \underline{\underline{a}}_H^{+} \sum_{\substack{i+j=k-1\\ i,j\geq 0}} \underline{\underline{b}}_i^{-}\underline{\underline{b}}_j^{-} .$$

Proof:

This is the dual result to (5.8). □

When sufficient terms in the expansion of $\underline{\underline{g}}^{+}(s)$ and $\underline{\underline{g}}^{-}(s)$ have been calculated, we use (1.8) to find $\underline{\underline{\gamma}}_0(s)$ and then, via (1.9) we can get $\underline{\underline{\gamma}}_n(s)$ for any n. The convergence of Theorem 5.5 implies that $\underline{\underline{g}}_n(t)$ may be written as a matrix sum of gamma distributions. Similarly, $\underline{\underline{p}}_n^{T}(t)$ may be seen to be a vector sum of gamma distributions via (2.4).

§6. The Mean First Passage Times

In the earlier sections we described a procedure for finding the ergodic distribution of a homogeneous row-continuous chain. To discuss the dynamical behavior of a Markov process the first passage time densities are needed. These densities describe the reaction of the chain to perturbations as well as busy periods and saturation times.

Although the densities $\underline{\underline{g}}^{+}(s)$ and $\underline{\underline{g}}^{-}(s)$ are available for the homogeneous process, there is no generalizable method for deriving first passage times for the bounded process. We will therefore follow [8] in our approach towards a recursive method of obtaining mean first passage times.

Formally, as in (1.8) we define the first passage time densities

Def 6.1 Let $\underline{\underline{s}}_n^{+}(t) = [s_{n;ij}^{+}(t)]$ and $\underline{\underline{s}}_n^{-}(t) = [s_{n;ij}^{-}(t)]$ where $s_{n;ij}(t) = P$ [a first arrival at row n+1 at (t, t+dt) and that arrival is at state (j, n+1) $|$ at state (i,n) at time 0]. And let $s_{n;ij}^{-}(t)$ be similarly defined for arrival at row n-1.

Two related entities are the stochastic matrices

$$(6.2)\quad \underline{\underline{s}}_n^{+} = \int_0^\infty \underline{\underline{s}}_n^{+}(t)\,dt\ , \quad \underline{\underline{s}}_n^{-} = \int_0^\infty \underline{\underline{s}}_n^{-}(t)\,dt$$

and the matrix pdf moments

$$(6.3)\quad \underline{\underline{\mu}}_n^{+} = \int_0^\infty t\,\underline{\underline{s}}_n^{+}(t)\,dt\ , \quad \underline{\underline{\mu}}_n^{-} = \int_0^\infty t\,\underline{\underline{s}}_n^{-}(t)\,dt.$$

In this section we are primarily concerned with finding the means $\boldsymbol{\mu}_n^+$ and $\boldsymbol{\mu}_n^-$.

Because the approach so closely parallels [8] we present the results, without proof. Let $\mathbf{g}_n^+(s) \overset{\text{def}}{=} \mathcal{L}[\mathbf{s}_n^+(t)]$, $\mathbf{g}_n^- \overset{\text{def}}{=} \mathcal{L}[\mathbf{s}_n^-(t)]$, then

Theorem 6.4

(a) $\mathbf{g}_0^+(s) = [\mathbf{I} - (\frac{v}{s+v})(\mathbf{a}_H^0 + \mathbf{a}_D^-]^{-1}(\frac{v}{s+v})\mathbf{a}_H^+$

(b) $\mathbf{g}_n^+(s) = [\mathbf{I} - (\frac{v}{s+v})\{\mathbf{a}_H^0 + \mathbf{a}_H^- \mathbf{g}_{n-1}^+(s)\}]^{-1}(\frac{v}{s+v})\mathbf{a}_H^+ \qquad n = 1,\dots,N.$

Mote that this is a recursive means for finding $\mathbf{g}_n^+(s)$ and therefore $\mathbf{s}_n^+(t)$. The dual result for $\mathbf{g}_n^-(s)$ is

Theorem 6.5

(a) $\mathbf{g}_N^-(s) = [\mathbf{I} - (\frac{v}{s+v})(\mathbf{a}_H^0 + \mathbf{a}_D^+)]^{-1}(\frac{v}{s+v})\mathbf{a}_H^-.$

(b) $\mathbf{g}_n^-(s) = [\mathbf{I} - (\frac{v}{s+v})(\mathbf{a}_H^0 + \mathbf{a}_H^+\mathbf{g}_{n+1}^-(s)]^{-1}(\frac{v}{s+v})\mathbf{a}_H^- \qquad n=0,\dots,N-1.$

Of course, $\mathbf{g}_n^+(0) = \mathbf{s}_n^+$ and $\mathbf{g}_n^-(0) = \mathbf{s}_n^-$, so that we may immediately write

Corollary 6.6

(a) $\mathbf{s}_0^+ = [\mathbf{I} - (\mathbf{a}_H^0 + \mathbf{a}_D^-)]^{-1}\mathbf{a}_H^+ \; ; \; \mathbf{s}_N^- = [\mathbf{I} - (\mathbf{a}_H^0 + \mathbf{a}_D^+)]^{-1}\mathbf{a}_H^-$

(b) $\mathbf{s}_n^+ = [\mathbf{I} - (\mathbf{a}_H^0 + \mathbf{a}_H^-\mathbf{s}_{n-1}^+)]^{-1}\mathbf{a}_H^+ \; ; \; \mathbf{s}_n^- = [\mathbf{I} - (\mathbf{a}_H^0 + \mathbf{a}_H^+\mathbf{s}_{n+1}^-)]^{-1}\mathbf{a}_H^-$

$n=1,\dots,N. \qquad\qquad n=0,\dots,N-1.$

Using the previous results, along with the fact that

$$(6.7)\quad \underline{\underline{\mu}}_n^+ = -\underline{\underline{g}}_n^{+'}(0), \quad \underline{\underline{\mu}}_n^- = -\underline{\underline{g}}_n^{-'}(0),$$

one obtains a recursive algorithm for the mean first passage times.

Corollary 6.8

$$(a)\quad \underline{\underline{\mu}}_0^+ = \frac{1}{v}[\underline{\underline{I}} - (\underline{\underline{a}}_H^0 + \underline{\underline{a}}_D^-)]^{-1}\underline{\underline{s}}_0^+$$

$$\underline{\underline{\mu}}_n^+ = \frac{1}{v}[\underline{\underline{I}} - \{\underline{\underline{a}}_H^0 + \underline{\underline{a}}_H^- \underline{\underline{s}}_{n-1}^+\}]^{-1}[\underline{\underline{I}} - v\underline{\underline{a}}_H^- \underline{\underline{\mu}}_{n-1}^+]\underline{\underline{s}}_n^+ \qquad n=1,\ldots,N$$

$$(b)\quad \underline{\underline{\mu}}_n^- = \frac{1}{v}[\underline{\underline{I}} - (\underline{\underline{a}}_H^0 + \underline{\underline{a}}_D^+)]^{-1}\underline{\underline{s}}_n^-$$

$$\underline{\underline{\mu}}_n^- = \frac{1}{v}[\underline{\underline{I}} - \{\underline{\underline{a}}_H^0 + \underline{\underline{a}}_H^+\underline{\underline{s}}_{n+1}^-\}]^{-1}[\underline{\underline{I}} - v\underline{\underline{a}}_H^+ \underline{\underline{\mu}}_{n+1}^-]\underline{\underline{s}}_n^-, \qquad n=0,\ldots,N-1.$$

The last two corollaries provide an efficient recursive method for finding the mean single-step passage times. The storage requirements are only $O(J^2)$ and the computation is $O(NJ^3)$.

The sojourn time [3] T_{vm} on the row-set $\{0,\ldots,m\}$ has density

$$(6.9)\quad \underline{\underline{s}}_{vm}(t) = \frac{\underline{e}_{m+1}^T\ \underline{\underline{a}}_H^-\ \underline{\underline{s}}_m^+(t)\underline{1}}{\underline{e}_{m+1}^T\ \underline{\underline{a}}_H^-\ \underline{1}}$$

The corresponding mean sojourn time is, by [3],

$$(6.10)\quad E[T_{vm}] = \frac{P_\infty(\{0,\dots,m\})}{i_{m+1,m}} = \frac{1}{v}\cdot\frac{\sum_{j=0}^{m} \underline{e}_j^T \underline{1}}{\underline{e}_{m+1}^T \underline{\underline{a}}_H^- \underline{1}}$$

The ergodic exit time [3] T_{Em} on the row-set $\{0,\dots,m\}$ has density

$$(6.11)\quad \underline{\underline{s}}_{Em}(t) = \frac{\sum_{n\le m} \underline{e}_n^T \underline{\underline{s}}_{n,m+1}(t)\ \underline{1}}{\sum_{n\le m} \underline{e}_n^T \underline{1}}$$

with mean

$$(6.12)\quad E[T_{Em}] = \frac{\sum_{j=0}^{m} \underline{e}_j^T \underline{\underline{\mu}}_{jm+1} \underline{1}}{\sum_{j=0}^{m} \underline{e}_j^T \underline{1}} .$$

In a queueing context where row 0 is associated with no queue the sojourn time on $\{1,\dots,N\}$ corresponds to the busy period. As in (6.10), this mean busy period is

$$(6.13)\quad E(T_B] = \frac{1-\underline{e}_0^T \underline{1}}{\underline{e}_0^T \underline{\underline{a}}_H^+ \underline{1}}$$

The passage time density from row n to row m, $\underline{\underline{s}}_{nm}(t)$, may be found by recalling that, for $0 \le m < n \le N$.

(6.14) $\underline{\underline{s}}_{mn}(t) = \underline{\underline{s}}_m^+(t)*\underline{\underline{s}}_{m+1}^+(t)*\dots*\underline{\underline{s}}_{n-1}^+(t)$. We have, therefore, that, for $m < n$, the mean passage time $\underline{\underline{\mu}}_{mn}$ from row m to row n is

$$(6.15)\quad \underline{\underline{\mu}}_{mn} = \sum_{j=m}^{n-1} \Big(\prod_{m\le k<j} \underline{\underline{s}}_k^+\Big)\, \mu_j^+ \Big(\prod_{j<k\le n-1} \underline{\underline{s}}_k^+\Big)$$

and a similar result holds when n<m. Thus, arbitrary first passage

times are readily available.

A more efficient approach is available when many arbitrary first passage times are required, particularly when $\underline{\underline{a}}_H^+$ (or $\underline{\underline{a}}_H^-$) is invertible. We refer the reader to [8], Section 6.

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