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QUANTUM ESTIMATION THEORY[1]

by

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1. INTRODUCTION.

In classical communication theory, the message to be transmitted is modulated and the resulting signal propagates through a given channel to produce a received waveform. The function of the receiver is to recover the signal from the received waveform, perhaps in an optimum manner, optimum being defined by some performance criterion. The input to the receiver may have some additive noises added to the received waveform. It is assumed that the receiver can be constructed independently of the model of the received waveform and the additive noise. Moreover, it is assumed that the optimum receiver can be physically realized.

In communication at optical frequencies neither of these two assumptions are valid. No matter what measurement we make of the received field, the outcome is random whose statistics depend on the measurement being made. This is a reflection of the laws of quantum physics. Furthermore, there is no guarantee that the measurement characterizing the receiver can be actually implemented.

In this paper, we present a theory of quantum estimation problems. Full details will be published elsewhere [1]. For related work, see [2] and [3] and [4].

It will be assumed that the reader is familiar with the notions of convex analysis in infinite dimensional spaces as, for example, presented in [8].

In the classical formulation of detection theory (Bayesian hypothesis testing) it is desired to decide which of n possible hypotheses $H_1,\dots,H_n$ is true, based on observation of a random variable whose probability distribution depends on the several hypotheses. The decision entails certain costs that depend on which hypothesis is selected and which hypothesis corresponds to the true state of the system. A decision procedure or strategy prescribes which hypothesis is to be chosen for each possible outcome of the observed data; in general, it may be necessary to use a randomized strategy which specifies the probabilities with which each hypothesis should be chosen as a function of the observed data. The detection problem is to determine an optimal decision strategy.

In the quantum formulation of the detection problem, each hypothesis H_j corresponds to a possible ρ_j of the quantum system under consideration. Unlike the classical situation, however, it is not possible to measure all relevant variables associated with the state of the system and to specify meaningful probability distributions for the resulting values. For the quantum detection problem it is necessary to specify not only the procedure for processing the experimental data, but also what data to measure in the first place. Hence the quantum detection problem involves determining the entire measurement process, or, in mathematical terms, determining the probability operator measure corresponding to the measurement process.

2. OBSERVABLES, STATES AND MEASUREMENT IN QUANTUM SYSTEMS.

Let H be a complex Hilbert space. The real linear space of compact self-adjoint operators $\mathcal{K}_s(H)$ with the operator norm is a Banach space whose dual is isometrically isomorphic to the real Banach space $\tau_s(H)$ of self-adjoint trace-class operators with the trace norm, i.e.,
$\mathcal{K}_s(H)^* = \tau_s(H)$ under the duality

$$\langle A,B\rangle = \mathrm{tr}(AB) \leq |A|_{tr}|B| \qquad A \in \tau_s(H), \quad B \in \mathcal{K}_s(H) \quad .$$

Here, $|B|=\sup\{|B\phi| : \phi \in H, |\phi| \leq 1\} = \sup\{\mathrm{tr}(AB) : A \in \tau_s(H), |A|_{tr} \leq 1\}$

and $|A|_{tr}$ is the trace norm $\sum_i |\lambda_i| < +\infty$ where $A \in \tau_s(H)$ and $\{\lambda_i\}$ are the eigenvalues of A repeated according to multiplicity. The dual of $\tau_s(H)$ with the trace norm is isometrically isomorphic to the space of all linear bounded self-adjoint operators, i.e., $\tau_s(H)^* = L_s(H)$ under the duality

$$\langle A,B\rangle = \mathrm{tr}(AB) \quad A \in \tau_s(H), \quad B \in L_s(H) .$$

Moreover the orderings are compatible in the following sense. If $K_s(H)_+$, $\tau_s(H)_+$, and $L_s(H)_+$ denote the closed convex cones of nonnegative definite operators in $K_s(H)$, $\tau_s(H)$, and $L_s(H)$ respectively, then

$$[K_s(H)_+]^* = \tau_s(H)_+ \quad \text{and} \quad [\tau_s(H)_+]^* = L_s(H)_+$$

where the associated dual spaces are to be understood in the sense defined above.

In the classical formulation of Quantum Mechanics one is given a complex Hilbert space H and a measurement is identified with an element $A \in L_s(H)$. $L_s(H)$ is termed the algebra of observables on H. The a priori statistical information about the quantum system is incorporated in the "state" ρ of the system, where $\rho \in T_s(H)_+$ and is of unit trace. In Quantum Communication problems a more general concept of a measurement (observable) is needed. As we have mentioned before this is conveniently described in terms of an operator-valued measure. For a discussion on the need for going to generalized measurements see Davies [5] and Holevo [2].

In quantum mechanical measurement theory, it is nearly always the case that physical quantities have values in a locally compact Hausdorff space S, e.g. a subset of R^n, and we shall make this assumption. Let H be a complex Hilbert space. A (self-adjoint) <u>operator-valued regular Borel measure</u> on S is a map $m: \mathcal{B} \to L_s(H)$ such that $\langle m(\cdot)\phi|\psi\rangle$ is a regular Borel measure on S for every $\phi,\psi \in H$. In particular, since for a vector-valued measure countable additivity is equivalent to weak countable additivity $m(\cdot)\phi$ is a (norm-) countably additive H-valued measure for every $\phi \in H$; hence whenever $\{E_n\}$ is a countable collection of disjoint subsets in $\mathcal{B}$ then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n),$$

where the sum is convergent in the strong operator topology. We denote by $M(\mathcal{B}, L_s(H))$ the real linear space of all operator-valued regular Borel measures on S. We define <u>scalar semivariation</u> of $m \in M(\mathcal{B}, L_s(H))$ to be the norm

$$\bar{\bar{m}}(S) = \sup_{|\phi|\le 1} |\langle m(\cdot)\phi|\phi\rangle|(s)$$

where $|\langle m(\cdot)\phi|\phi\rangle|$ denotes the total variation measure of the real-valued Borel measure $E \mapsto \langle m(E)\phi|\phi\rangle$. It can be shown that scalar semivariation is always finite.

A <u>positive</u> operator-valued regular Borel measure is a measure $m \in M(\mathcal{B}, L_s(H))$ which satisfies

$$m(E) \ge 0 \quad \forall E \in \mathcal{B},$$

where by $m(E) \geq 0$ we mean $m(E)$ belongs to the positive cone $L_s(H)_+$ of all non-negative-definite operators. A probability operator measure (POM) is a positive operator-valued measure $m \in M(\mathcal{B}, L_s(H))$ which satisfies

$$m(S) = I.$$

If m is a POM then every $\langle m(\cdot)\phi|\phi\rangle$ is a probability measure on S and $\bar{\bar{m}}(S) = 1$. In particular, a resolution of the identity is an $m \in M(\mathcal{B}, L_s(H))$ which satisfies $m(S) = I$ and $m(E)m(F) = 0$ whenever $E \cap F = \emptyset$; it is then true that $m(\cdot)$ is projection-valued and satisfies

$$m(E \cap F) = m(E)m(F), \quad E,F \in \mathcal{B}.^{+}$$

3. INTEGRATION WITH RESPECT TO OPERATOR-VALUED MEASURES.

In treating quantum estimation problems it is necessary to have a theory of integration with respect to operator-valued measures. We outline this theory now. First, we consider integration of real-valued functions. Basically we identify the regular Borel operator-valued measures $m \in M(\mathcal{B}, L_s(H))$ with the bounded linear operators $L: C_o(S) \to L_s(H)$, to get a generalization of the Riesz Representation Theorem.

Theorem 3.1

Let S be a locally compact Huasdorff space with Borel sets $\mathcal{B}$. Let H be a Hilbert space. There is an isometric isomorphism $m \Leftrightarrow L$ between the operator-valued regular Borel measures $m \in M(\mathcal{B}, L_s(H))$ and the bounded linear maps $L \in \mathcal{L}(C_o(S), L_s(H))$. The correspondence $m \Leftrightarrow L$ is given by

$$L(g) = \int_S g(s)m(ds), \quad g \in C_o(S)$$

where the integral is well-defined for $g(\cdot) \in M(S)$ (bounded and totally measurable maps $g: S \to R$) and is convergent for the supremum norm on $M(S)$. If $m \Leftrightarrow L$, then $\bar{\bar{m}}(S) = |L|$ and $\langle L(g)\phi|\psi\rangle = \int g(s) \langle m(\cdot)\phi|\psi\rangle (ds)$ for every $\phi,\psi \in H$. Moreover L is positive (maps $C_o(S)_+$ into $L_S(H)_+$) iff m is a positive measure; L is positive and $L(1) = I$ iff m is a POM; and L is an algebra homomorphism with $L(1) = I$ iff m is a resolution of the identity, in which case L is actually an isometric algebra homomorphism of $C_o(S)$ onto a norm-closed subalgebra of $L_s(H)$. ▨

Remark

Since every real-linear map from a real-linear subspace of a complex space into another real-linear subspace of a complex space corresponds to a unique "Hermitian" complex-linear map on the complex linear spaces, we could just as easily identify the (self-adjoint) operator-valued regular measures $M(\mathcal{B}, L_s(H))$ with the complex-linear maps $L: C_o(S,C) \to L(H)$ which satisfy

$$L(g) = L(\bar{g})^*, \quad g \in C_o(S,C).$$

3.1 Integration of $\tau_s(H)$-valued functions.

We now consider $L(H)$ as a subspace of the "operations" $L(\tau(H),\tau(H))$, that is, bounded linear maps from $\tau(H)$ into $\tau(H)$. Every $B \in L(H)$ defines a bounded linear function $L_B: \tau(H) \to \tau(H)$ by

$$L_B(A) = AB, \quad A \in \tau(H)$$

with $|B| = |L_B|$. In particular, $A \to \mathrm{tr}AB$ defines a continuous (complex-) linear functional on $A \in \tau(H)$, and in fact every linear functional in $\tau(H)^*$ is of this form for some $B \in \mathcal{L}(H)$. We note that if A and B are selfadjoint then tr(AB) is real linear (although it is not necessarily true that AB is selfadjoint unless AB = BA). Thus, it is possible to identify the space $\tau_s(H)^*$ of real-linear continuous functionals on $\tau_s(H)$ with $\mathcal{L}_s(H)$, again under the pairing $\langle A,B\rangle = \mathrm{tr}AB$, $A \in \tau_s(H)$, $B \in \mathcal{L}_s(H)$. For our purposes we shall be especially interested in this latter duality between the spaces $\tau_s(H)$ and $\mathcal{L}_s(H)$, which we shall use to formulate a dual problem for the quantum estimation situation. However, we will also need to consider $\mathcal{L}_s(H)$ as a subspace of $\mathcal{L}(\tau(H),\tau(H))$ so that we may integrate $\tau_s(H)$-valued functions on S with respect to $\mathcal{L}_s(H)$-valued operator measures to get an element of $\tau(H)$.

Suppose $m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H))$ is an operator-valued regular Borel measure, and $f\colon\ S \to \tau_s(H)$ is a simple function with finite range of the form

$$f(s) = \sum_{j=1}^{n} 1_{E_j}(s)\rho_j$$

where $\rho_j \in \tau_s(H)$ and E_j are disjoint sets in $\mathcal{B}$, that is $f \in \mathcal{B} \otimes \tau_s(H)$. Then we may unambiguously (by finite additivity of m) define the integral

$$\int_S f(s)m(ds) = \sum_{j=1}^{n} m(E_j)\rho_j .$$

The question, of course, is to what class of functions can we properly extend the definition of the integral? Now if m has finite total variation $|m|(s)$, then the map $f \to \int f(s)m(ds)$ is continuous for the supremum norm $|f|_\infty = \sup|f(s)|_{tr}$ on $\mathcal{B} \otimes \tau_s(H)$, so that by continuity the integral map extends to a continuous linear map from the closure $M(S,\tau_s(H))$ of $\mathcal{B} \otimes \tau_s(H)$ with the $|\cdot|_\infty$ norm into $\tau(H)$. In particular, the integral $\int_S f(s)m(ds)$ is well-defined (as the limit of the integrals of uniformly convergent simple functions) for every bounded and continuous function $f\colon\ S \to \tau_s(H)$. Unfortunately, it is not the case that an arbitrary POM m has finite total variation. Since we wish to consider general quantum measurement processes as represented by POM's (in particular, resolutions of the identity), we can only assume that m has finite scalar semivariation $\bar{\bar{m}}(S) < +\infty$. Hence we must put stronger restrictions on the class of functions which we integrate. The answer is summarized in

Theorem 3.2

Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let H be a Hilbert space. There is an isometric isomorphism $L_1 \leftrightarrow m \leftrightarrow L_2$ between the bounded linear maps $L_1\colon\ C_o(S) \hat{\otimes}_\pi \tau(H) \to \tau(H)$,(1) the operator-valued regular Borel measures $m \in \mathcal{M}(\mathcal{B},\mathcal{L}(\tau(H),\tau(H)))$, and the bounded linear maps $L_2\colon\ C_o(S) \to \mathcal{L}(\tau(H),\tau(H))$. The correspondence $L_1 \leftrightarrow m \leftrightarrow L_2$ is given by the relations

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(1) For notation and facts regarding tensor products we follow Treves [7].

$$L_1(f) = \int_S f(s)m(ds), \quad f \in C_o(S) \hat{\otimes}_\pi \tau(H)$$

$$L_2(g)\rho = L_1(g(\cdot)\rho) = \rho \int g(s)m(ds), \; g \in C_o(S), \; \rho \in \tau(H)$$

and under this correspondence $L_1 = \bar{\bar{m}}(s) = L_2$. Moreover the integral $\int_S f(s)m(ds)$ is well-defined for every f $M(S) \hat{\otimes}_\pi \tau(H)$ and the map $f \mapsto \int_S f(s)m(ds)$ is bounded and linear from $M(S) \hat{\otimes}_\pi \tau(H)$ into $\tau(H)$. ▨

Corollary 3.3

If $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ then the integral $\int_S f(s)m(ds)$ is well-defined for every $f \in M(S) \hat{\otimes}_\pi \tau(H)$. ▨

In proving Theorem 3.2 we need the fact

Proposition 3.3

$M(S) \hat{\otimes}_\pi \tau(H)$ is a subspace of $M(S, \mathcal{T}(H))$. ▨

Remark

The above says that we may identify the tensor product space $M(S) \hat{\otimes}_\pi \tau_s(H)$ with a subspace of the totally measurable functions $f\colon S \to \tau_s(H)$ in a well-defined way. The reason why this is important is that the functions $f \in M(S) \hat{\otimes}_\pi \tau_s(H)$ are those for which we may legitimately define an integral $\int_S f(s)m(ds)$ for arbitrary operator-valued measures $m \in (\mathcal{B}, \mathcal{L}_s(H))$, since $f \mapsto \int_S f(s)m(ds)$ is a continuous linear map from $M(S) \hat{\otimes}_\pi \tau(H)$ into $\tau(H)$. In particular, it is obvious that $C_o(S) \hat{\otimes}_\pi \tau(H)$ may be identified with a subspace of continuous functions $f\colon S \mapsto \mathcal{T}_s(H)$ in a well-defined way, just as it is obvious how to define the integral $\int_S f(s)m(ds)$ for finite linear combinations

$$f(s) = \sum_{j=1}^{n} g_j(s)\rho_j \in C_o(S) \times \tau_s(H).$$

What is not obvious is that the completion of $C_o(S) \hat{\otimes} \tau_s(H)$ in the tensor product norm π may be identified with a subspace of continuous functions $f\colon S \to \tau_s(H)$.

4. A FUBINI THEOREM FOR THE BAYES POSTERIOR EXPECTED COST.

In the quantum estimation problem, a decision strategy corresponds to a probability operator measure $m \in \mathcal{M}(\mathcal{B}, \mathcal{L}_s(H))$ with posterior expected cost

$$R_m = \int_S tr[\rho(s) \int_S C(t,s)m(dt)]\mu(dt)$$

where for each s, $\rho(s)$ specifies a state of the quantum system, $C(t,s)$ is a cost function, and μ is a prior probability measure on S. We would like to show that the order of integration can be interchanged to yield

$$R_m = tr \int_S f(s)m(ds)$$

where

$$f(s) = \int_S C(t,s)\rho(t)\mu(dt)$$

is a map $f: S \to \tau_s(H)$ that belongs to the space $M(S) \hat{\otimes}_\pi \tau(H)$ of functions integrable against operator-valued measures.

Let $(S,\mathcal{B},\mu)$ be a finite nonnegative measure space, X a Banach space. A funtion $f: S \to X$ is <u>measurable</u> iff there is a sequence $\{f_n\}$ of simple measurable functions converging pointwise to f, i.e. $f_n(s) \to f(s)$ for every $s \in S$. A useful criterion for measurability is the following: f is measurable iff it is separably-valued and for every subset V of X, $f^{-1}(V) \in \mathcal{B}$. In particular, every $f \in C_o(S,X)$ is measurable, when S is a locally compact Hausdorff space with Borel sets $\mathcal{B}$. A function $f: S \to X$ is <u>integrable</u> iff it is measurable and $\int_S |f(s)|\cdot\mu(ds) < +\infty$, in which case the integral $\int f(s)\mu(ds)$ is well-defined as Bochner's integral; we denote by $L_1(S,\mathcal{B},\mu;X)$ the space of all integrable functions $f: S \to X$, a normed space under the L_1 norm $|f|_1 = \int_S |f(s)|\mu(ds)$. The uniform norm $|\cdot|_\infty$ on functions $f: S \to X$ is defined by $|f|_\infty = \sup_{s\in S}|f(s)|$; $M(S,X)$ denotes the Banach space of all uniform limits of simple X-valued functions, with norm $|\cdot|_\infty$, i.e. $M(S,X)$ is the closure of the simple X-valued functions with the uniform norm. We abbreviate $M(S,R)$ to $M(S)$.

<u>Proposition 4.1</u>

Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$, μ a probability measure on S, and H a Hilbert space. Suppose $\rho: S \to \tau_s(H)$ belongs to $M(S,\tau_s(H))$, and $C: S \times S \to R$ is a real-valued map satisfying

$$t \to C(t,\cdot) \in L_1(S,\mathcal{B},\mu;M(S)).$$

Then for every $s \in S$, $f(s)$ is well-defined as an element of $\tau_s(H)$ by the Bochner integral

$$f(s) = \int_S C(t,s)\rho(t)\mu(dt);$$

moreover $f \in M(S) \hat{\otimes}_\pi \tau_s(H)$ and for every operator-valued measure $m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H))$, we have

$$\int_S f(s)m(ds) = \int_S \rho(t)\left[\int_S C(t,s)m(ds)\right]\mu(dt)$$

Moreover if $t \to C(t,\cdot)$ in fact belongs to $L_1(S,\mathcal{B},\mu;C_o(S))$ then $f \in C_o(S) \hat{\otimes}_\pi \tau_s(H)$. ∎

5. THE QUANTUM ESTIMATION PROBLEM AND ITS DUAL.

We are now prepared to formulate the quantum detection problem in a duality framework and calculate the associated dual problem. Let S be a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let H be a Hilbert space associated with the physical variables of the system under consideration. For each parameter value $s \in S$ let $\rho(s)$ be a state or density operator for the quantum system, i.e. every $\rho(s)$ is a nonnegative-definite selfadjoint trace-class operator on H with trace 1;

we assume $\rho \in M(S,\tau_s(H))$. We assume that there is a cost function $C: S \times S \to R$, where $C(s,t)$ specifies the relative cost of an estimate t when the true parameter value is s. If the operator-valued measure $m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H))$ corresponds to a given measurement and decision strategy, then the posterior expected cost is

$$R_m = \mathrm{tr}\int_S \rho(t)[\int_S C(t,s)m(ds)]\mu,$$

where μ is a prior probability measure on $(S,\mathcal{B})$. By Proposition 4.1

this is well-defined whenever the map $t \to C(t,\cdot)$ belongs to $L_1(S,\mathcal{B},\mu;M(S))$, in which case we may interchange the order of integration to get

$$(5.1) \qquad R_m = \mathrm{tr}\int_S f(s)m(ds)$$

where $f \in M(S)\,\hat{\otimes}_\pi\,\tau_s(H)$ is defined by

$$f(s) = \int_S \rho(t)C(t,s)\mu(ds).$$

The quantum estimation problem is to minimize (5.1) over all operator-valued measures $m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H))$ which are POM's i.e. the constraints are that $m(E) \geq 0$ for every $E \in \mathcal{B}$ and $m(S) = I$.

We formulate the estimation problem in a duality framework. We take perturbations on the equality constraint $m(S) = I$. Define the convex function $F: \mathcal{M}(\mathcal{B},\mathcal{L}_s(H)) \to \bar{R}$ by

$$F(m) = \delta_{\geq 0}(m) + \mathrm{tr}\int_S f(s)m(ds), \quad m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H)),$$

where $\delta_{\geq 0}$ denotes the indicator function for the positive operator-valued measures, i.e. $\delta_{\geq 0}(m)$ is 0 if $m(\mathcal{B}) \subset \mathcal{L}_s(H)_+$ and $+\infty$ otherwise. Define the convex function $G: \mathcal{L}_s(H) \to \bar{R}$ by

$$G(x) = \delta_{\{0\}}(x), \quad x \in \mathcal{L}_s(H)$$

i.e. $G(x)$ is 0 if $x = 0$ and $G(x) = +\infty$ if $x \neq 0$. Then the quantum detection problem may be written

$$P_0 = \inf\{F(m) + G(I-Lm): \; m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H))\}$$

where $L: \mathcal{M}(\mathcal{B},\mathcal{L}_s(H)) \to \mathcal{L}_s(H)$ is the continuous linear operator

$$L(m) = m(S).$$

We consider a family of perturbed problems defined by

$$P(x) = \inf\{F(m) + G(x-Lm): \; m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H))\}, \quad x \in \mathcal{L}_s(H).$$

Thus we are taking perturbations in the equality constraint, i.e. the problem $P(x)$ requires that every feasible m be nonnegative and satisfy $m(S) = x$; of course, $P_0 = P(I)$. Since F and G are convex, $P(\cdot)$ is convex $\mathcal{L}_s(H) \to \bar{R}$.

In order to construct the dual problem corresponding to the family of perturbed problems $P(x)$, we must calculate the conjugate functions of F and G. We shall work in the norm topology of the constraint space $\mathcal{L}_s(H)$, so that the dual problem is posed in $\mathcal{L}_s(H)^*$. Clearly $G^* \equiv 0$. The adjoint of the operator L is given by

$$L^*: \mathcal{L}_s(H)^* \to \mathcal{M}(\mathcal{B},\mathcal{L}_s(H))^*: \quad y \to (m \to y \cdot m(S)).$$

To calculate $F^*(L^*y)$, we have the following lemma.

Lemma 5.1

Suppose $y \in \mathcal{L}_s(H)^*$ and $f \in M(S) \hat{\otimes}_\pi \tau_s(H)$ satisfy

(5.2) $\quad y \cdot m(S) \le \mathrm{tr}\int_S f(s)m(ds)$

for every positive operator-valued measure $m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H)_+)$. Then $y_{sg} \le 0$ and $y_{ac} \le f(s)$ for every $s \in S$, where $y = y_{ac} + y_{sg}$ is the unique decomposition of y into $y_{ac} \in \tau_s(H)$ and $y_{sg} \in \mathcal{K}_s(H)$. ▨

Proposition 5.2

The perturbation function $P(\cdot)$ is continuous at I, and hence $\partial P(I) \ne \emptyset$. In particular, $P_o = D_o$ and the dual problem D_o has optimal solutions. Moreover every solution $\hat{y} \in \mathcal{L}_s(H)^*$ of the dual problem D_o has 0 singular part, i.e. $\hat{y}_{sg} = 0$ and $\hat{y} = \hat{y}_{ac}$ belongs to the canonical image of $\tau_s(H)$ in $\tau_s(H)^{**}$. ▨

In order to show that the problem P_o has solutions, we could define a family of dual perturbed problems $D(v)$ for $v \in C_o(S) \hat{\otimes}_\pi \tau_s(H)$ and show that $D(\cdot)$ is continuous. Or we could take the alternative method of showing that the set of feasible POM's m is weak* compact and the cost function is weak*-lsc when $\mathcal{M}(\mathcal{B},\mathcal{L}_s(H)) \cong \mathcal{L}(C_o(S),\mathcal{L}_s(H))$ is identified as the normed dual of the space $C_o(S) \hat{\otimes}_\pi \tau_s(H)$ under the pairing

$$\langle f,m\rangle = \mathrm{tr}\int f(s)m(ds).$$

Note that both methods require that f belong to the predual $C_o(S) \hat{\otimes}_\pi \tau_s(H)$); it suffices to assume that $t \to C(t,\cdot)$ belongs to $L_1(S,\mathcal{B},\mu;C_o(S))$.

Proposition 5.3

The set of POM's is compact for the weak* $\equiv w(\mathcal{M}(\mathcal{B},\mathcal{L}_s(H)), C_o(S) \hat{\otimes}_\pi \tau_s(H))$ topology. If $t \to C(t,\cdot) \in L_1(S,\mathcal{B},\mu;C_o(S))$ then P_o has optimal solutions $\hat{m}$. ▨

The following theorem summarizes the results we have obtained so far, as well as providing a necessary and sufficient characterization of the optimal solution.

MAIN THEOREM.

Let H be a Hilbert space, S a locally compact Hausdorff space with Borel sets $\mathcal{B}$. Let $\rho \in M(S,\tau_s(H))$, $C: S \times S \to R$ a map satisfying $t \to C(t,\cdot) \in L_1(S,\mathcal{B},\mu;C_o(S))$, and μ a probability measure on $(S,\mathcal{B})$. Then for every $m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H))$,

$$\mathrm{tr}\int_S \rho(t)\left[\int_S C(t,s)m(ds)\right]\mu(dt) = \mathrm{tr}\int_S f(s)m(ds)$$

where $f \in C_o(S)\,\hat{\otimes}_\pi\,\tau_s(H)$ is defined by

$$f(s) = \int_S \rho(t)C(t,s)\mu(ds).$$

Define the optimization problems

$$P_o = \inf\{\mathrm{tr}\int_S f(s)m(ds) : m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H)),\ m(S) = I, m(E) \geq 0 \text{ for every } E \in \mathcal{B}\}$$

$$D_o = \sup\{\mathrm{tr}y: \ y \in \tau_s(H),\ y \leq f(s) \text{ for every } s \in S\}.$$

Then $P_o = D_o$, and both P_o and D_o have optimal solutions. Moreover the following statements are equivalent for $m \in \mathcal{M}(\mathcal{B},\mathcal{L}_s(H))$, assuming $m(S) = I$ and $m(E) \geq 0$ for every $E \in \mathcal{B}$:

1) m solves P_o

2) $\int_S f(s)m(ds) \leq f(t)$ for every $t \in S$

3) $\int_S m(ds)f(s) \leq f(t)$ for every $t \in S$.

Under any of the above conditions it follows that $y = \int_S f(s)m(ds) = \int_S m(ds)f(s)$ is selfadjoint and is the unique solution of D_o, with

$$P_o = D_o = \mathrm{tr}(y).$$

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