Instability of Bose-Einstein condensation into the one-particle ground state on quantum graphs under repulsive perturbations

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In this Note we investigate Bose-Einstein condensation into the one-particle ground state in interacting quantum many-particle systems on graphs. We extend previous results obtained for particles on an interval and show that even arbitrarily small repulsive two-particle interactions destroy the condensate in the one-particle ground state present in the non-interacting Bose gas. Our results also cover singular two-particle interactions, such as the well-known Lieb-Lininger model, in the thermodynamic limit.

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I. INTRODUCTION

Since its prediction by Einstein[1,2], Bose-Einstein condensation (BEC) has played an important role in many-particle quantum mechanics. From a physical point of view, this type of condensation refers to the fact that a finite fraction of the particles occupy the same one-particle state in the thermodynamic limit and that this macroscopic occupation of the same state induces a coherent behaviour in the many-particle system bearing some resemblance with a superconducting behaviour. Indeed, this observation led to the well-known Bogoliubov mean-field model of BEC (see, e.g., Ref. 3). On the other hand, from a mathematical point of view the difficulties to establish BEC in interacting systems were soon realised. This applies, in particular, to continuous systems and only recently rigorous results have been obtained for them[4–6]. We stress that there exist various generalised notions of Bose-Einstein condensation in the literature[7–10]. In this Note, however, we shall always refer to BEC as the macroscopic occupation of a one-particle state in the thermodynamic limit. In particular, we shall be concerned with the condensation into the one-particle ground state.

A related question to proving BEC in an interacting system is the following: Suppose that a non-interacting many-particle system shows BEC as, e.g., a free gas in a three-dimensional box. The question then arises, whether this condensation is stable under perturbing the free gas by introducing repulsive particle interactions. In a number of cases it had been shown that hardcore interactions destroy BEC[11–13], and this was recently confirmed for many-particle quantum systems on (finite, metric) graphs[14]. However, since hardcore interactions can be viewed as very strong one might wonder whether small repulsive interactions can be implemented such that the condensation survives. It appears that the answer strongly depends on the type of condensation in the free system. Whereas the results of Ref. 6 show that under some circumstances BEC is stable with respect to superstable repulsive two-particle interactions, it was shown in another example[15] that even small repulsive interactions destroy the condensate. The reason for this very different behaviour lies in the nature of the one-particle ground state of the free gas. Whereas in the model of Ref. 6 this ground state is a plane wave and, hence, completely delocalised, the ground state in the model of Ref. 15 is localised at the boundary of the system. Intuitively, this explains why the effect of repulsive interactions is much stronger in the latter system, leading to the destruction of the condensate.

It is our goal in this Note to prove that BEC of a free gas on a quantum graph into the one-particle ground state is unstable under the addition of a fairly general repulsive two-particle interaction. Quantum graphs are (ramified) one-dimensional models with a potentially complex topology. Although quantum graphs are studied in various areas of mathematics and physics (see, e.g., Ref. 16), they are particularly prominent in the field of quantum chaos[17]. This is due to the fact that the spectral correlations of sufficiently complex quantum graphs follow the Bohigas-Gianonni-Schmit conjecture[18] and are well described by random matrix theory. Contrary to what is often believed, BEC can occur in a free gas in one dimension when attractive boundary conditions are chosen[19]. In a similar spirit, a free Bose gas on a finite graph can show BEC at finite temperature. We previously identified the class of boundary conditions in the vertices of the graph that lead to BEC of a free gas and showed that any condensation (in terms of singularities of the free energy) is destroyed by adding hardcore two-particle interactions[14]. In this Note we now allow repulsive two-particle interactions to be generated by a potential in the same way as in Ref. 15, where the case of an interval (a graph with one edge in our language) with attractive boundary conditions at one end was investigated. However, here we consider arbitrary (finite) graphs and also include interactions approaching the Lieb-Liniger model[20] (constructed on graphs in Ref. 21) in the thermodynamic limit. In all cases we prove that BEC into the one-particle ground state is destroyed by the interactions at any finite temperature.

It is important to note that besides being one of the few explicitly solvable many-particle models, rigorous results concerning BEC in the Lieb-Liniger model are scarce. Recently, e.g., condensation in the Lieb-Liniger model on an interval with additional random potentials was discussed in Ref. 22, proving condensation at zero temperature in a Gross-Pitaevskii regime. The thermodynamic limit employed in Ref. 22 is a high-density limit since the volume of the one-particle configuration space is not changed. In this Note, however, we study BEC on general (finite) graphs at finite temperature in the standard thermodynamic limit with fixed density.

II. BACKGROUND

In this section we briefly summarise relevant concepts of one-particle and many-particle quantum graphs, as well as Bose-Einstein condensation. For more details on BEC see

Refs. 23–25, on quantum graphs see Refs. 26–29 and on many-particle quantum graphs see Refs. 21,30. For a discussion of BEC on quantum graphs see Ref. 14.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a finite graph with vertices $\mathcal{V} = \{v_1, \dots, v_V\}$ and edges $\mathcal{E} = \{e_1, \dots, e_E\}$ connecting the vertices. The graph is equipped with a metric structure by assigning a (finite) length $l_e > 0$ to each edge $e \in \mathcal{E}$. Hence, each edge e is associated with an interval $[0, l_e]$, and this allows us to define the one-particle Hilbert space,

$$\mathcal{H}_1 = \bigoplus_{e \in \mathcal{E}} L^2(0, l_e) \ . \tag{1}$$

In order to obtain a quantum graph one introduces a self-adjoint realisation of the Laplacian in $\mathcal{H}_1$. As a differential expression the Laplacian acts on smooth functions $F = (f_1, \dots, f_E) \in \mathcal{H}_1$ as

$$-\Delta_1 F = (-f_1'', \dots, -f_E'') \ . \tag{2}$$

Here the index 1 refers to the fact that (2) is a one-particle operator which, in our case, serves as the one-particle Hamiltonian.

There are several ways to characterise self-adjoint realisations of the differential expression (2), see Refs. 26,31. In the following we shall refer to the approach developed in Ref. 26. This characterises the domains $\mathcal{D}_1(P_1, L_1) \subset \mathcal{H}_1$ on which $-\Delta_1$ is self-adjoint in terms of two linear maps P_1, L_1 on $\mathbb{C}^{2E}$, where P_1 is a projection and L_1 is a self-adjoint endomorphism on $\ker P_1$. These maps act on the boundary values of functions and their derivatives on the edges and hence implement the connectivity of the graph. Any self-adjoint realisation of the one-particle Laplacian has compact resolvent. Its spectrum, therefore, is purely discrete, with an eigenvalue count following a Weyl asymptotics. Moreover, there are at most finitely many negative eigenvalues, whose number is bounded by the number of positive eigenvalues of L_1[27].

Following the usual construction, the bosonic N-particle Hilbert space is the symmetrised N-fold tensor product of the one-particle Hilbert space (1), i.e., $\mathcal{H}_B^N = \mathcal{H}_1 \otimes_s \cdots \otimes_s \mathcal{H}_1$. Accordingly, the N-particle Hamiltonian is given by

$$-\Delta_N = \sum_{j=1}^{N} \mathbf{1} \otimes \cdots \otimes (-\Delta_1) \otimes \cdots \otimes \mathbf{1} \ . \tag{3}$$

As a differential expression this is a Laplacian in N variables. A number of self-adjoint realisations of $-\Delta_N$ are discussed in Refs. 21,30, including non-interacting as well as interacting ones. Non-interacting realisations $(-\Delta_N, \mathcal{D}_N(P_1, L_1))$ follow from a tensor product construction, where the domain $\mathcal{D}_N(P_1, L_1)$ is an N-fold tensor power of the one-particle domain $\mathcal{D}_1(P_1, L_1)$. An important consequence is that the spectrum of $(-\Delta_N, \mathcal{D}_N(P_1, L_1))$ consists of eigenvalues that are sums of one-particle eigenvalues. For more details see Ref. 14.

Among the interacting realisations of the N-particle Laplacian developed in Ref. 21 is a rigorous version of

$$H_N = -\Delta_N + \alpha \sum_{i<j} \delta(x^{(i)} - x^{(j)}) \ , \tag{4}$$

defining a Lieb-Liniger model (see Ref. 20) on a graph. Here i, j are particle labels attached to coordinates on the same edge. In this context $-\Delta_N$ stands for a non-interacting realisation of the Laplacian; the interaction is a singular two-particle contact interaction, as indicated by the δ-potentials.

In bosonic many-particle systems, BEC refers to the macroscopic occupation of a one-particle state. We work in the canonical ensemble, i.e., with a fixed particle number N and Hilbert space $\mathcal{H}_B^N$. The expectation value of a (bounded) observable A_N in the Gibbs state ω_β at inverse temperature β therefore is

$$\omega_\beta(A_N) = \frac{1}{Z_N(\beta)} \operatorname{Tr}(A_N e^{-\beta H_N}) \ , \tag{5}$$

where $Z_N(\beta) = \operatorname{Tr}(e^{-\beta H_N})$ is the partition function and H_N is the N-particle Hamiltonian operator that is assumed to preserve the particle number. Condensation only takes place in the thermodynamic limit, which in the canonical ensemble is obtained by letting the volume of the one-particle configuration space tend to infinity while keeping the particle density constant[32]. In the context of quantum graphs, the volume is the total length $\mathcal{L} = \sum_{e=1}^{E} l_e$ of the graph.

Definition II.1. In a quantum graph the thermodynamic limit is obtained by rescaling each edge length l_e as nl_e, and letting $n \to \infty$. At the same time the number of particles N is increased such that the particle density $\rho = N/\mathcal{L}$ remains constant.

We denote the thermodynamic limit by writing $\mathcal{L} \to \infty$.

Although, strictly speaking, not needed in the canonical ensemble, some tools of second quantisation will be useful in the following. Hence, we let $\mathcal{F}_B$ be the bosonic Fock space over the one-particle Hilbert space $\mathcal{H}_1$ defined in (1). When $\Phi = (\varphi_e)_{e\in\mathcal{E}} \in \mathcal{H}_1$, the standard annihilation and creation operators in $\mathcal{F}_B$ are $a(\Phi)$ and $a^*(\Phi)$. They can be represented as

$$a(\Phi) = \sum_{e=1}^{E} \int_0^{l_e} \overline{\varphi_e}(x) a_e(x) \, \mathrm{d}x \ , \qquad a^*(\Phi) = \sum_{e=1}^{E} \int_0^{l_e} \varphi_e(x) a_e^*(x) \, \mathrm{d}x \ , \tag{6}$$

with operator-valued distributions a_e and a_e^* attached to each edge that satisfy the CCR

$$[a_e(x), a_{e'}(y)] = 0 = [a_e^*(x), a_{e'}^*(y)] \ , \qquad [a_e(x), a_{e'}^*(y)] = \delta_{ee'}\delta(x-y) \ . \tag{7}$$

The particle number operator $\hat{N}$ can be expressed as

$$\hat{N} = \sum_{e=1}^{E} \int_0^{l_e} a_e^*(x) a_e(x) \, \mathrm{d}x \ . \tag{8}$$

If the N-particle system is in the Gibbs state (5), the number of particles in a given one-particle state $\Phi \in \mathcal{H}_1$ is $\omega_\beta(a^*(\Phi)a(\Phi))$.

Definition II.2. Let $\Phi \in \mathcal{H}_1$ be any (normalised, pure) one-particle state. We say that Bose-Einstein condensation is exhibited in the state Φ at inverse temperature $\beta > 0$, if

$$\limsup_{\mathcal{L}\to\infty} \frac{\omega_\beta(a^*(\Phi)a(\Phi))}{\mathcal{L}} > 0 \ . \tag{9}$$

In Ref. 14 non-interacting Bose gases on quantum graphs were classified according to whether or not they show BEC:

Theorem II.3. *Let $(-\Delta_1, \mathcal{D}_1(P_1, L_1))$ be a one-particle Laplacian on a graph and denote the associated non-interacting N-particle Laplacian by $(-\Delta_N, \mathcal{D}_N(P_1, L_1))$. Then BEC occurs for $(-\Delta_N, \mathcal{D}_N(P_1, L_1))$ below some critical temperature, if and only if the map L_1 has at least one positive eigenvalue.*

The key mechanism that leads to condensation in the non-interacting Bose gas is a gap in the one-particle spectrum that separates a finite number of states, in particular the ground state, from the states of positive energy. It is known that there can only be negative eigenvalues of the one-particle Laplacian when L_1 is not negative semi-definite, hence the requirement for L_1 to possess a positive eigenvalue. Furthermore, according to Lemma 3.3 in Ref. 14, the ground state eigenvalue converges to $-L_{max}^2$ in the thermodynamic limit, where L_{max} is the largest eigenvalue of L_1. Therefore, the essential condition for BEC is a gap in the one-particle spectrum that persists in the thermodynamic limit.

It is generally assumed that a gap in the one-particle spectrum stabilises condensation, or even makes it possible at all[6]. This is true, in particular, for a one-dimensional Bose gas since no condensation is present for standard boundary conditions as expressed by the well-known result of Hohenberg[33]. In three dimensions, where condensation occurs without a gap, an additional gap stabilises the condensate. As an example, in Ref. 6 an artificial gap was introduced, and it was shown that a certain class of repulsive two-particle interactions does not destroy the condensate. However, although a gap in the one-particle spectrum generally stabilises condensation, it still may not be possible to implement repulsive two-particle interactions without destroying the condensate. An example for this was studied in Ref. 15. The reason for the differences in the examples of Ref. 6 and Ref. 15 lies in the strongly localised nature of the ground state of the model studied in Ref. 15. In the cases covered by Theorem II.3 where BEC occurs, the ground states are localised around the vertices of the graph. One thus expects a similar behaviour to the one found in Ref. 15.

III. RESULTS

In this section we start with non-interacting Bose gases that, according to Theorem II.3, show BEC and then investigate the effect of additional, repulsive two-particle interactions. Working in the canonical ensemble we shall define a restriction of the N-particle Hamiltonian to the (finite) one-particle configuration space[32] and then investigate the particle number density (9) of the one-particle ground state in the limit $\mathcal{L} \to \infty$. More explicitly, the N-particle Hamiltonian is given by

$$H_N = -\Delta_N + U_N^{\mathcal{L}} \ , \tag{10}$$

where $(-\Delta_N, \mathcal{D}_N(P_1, L_1))$ is such that L_1 has at least one positive eigenvalue. Therefore, according to Theorem II.3, the free Bose gas with Hamiltonian $-\Delta_N$ shows BEC below a critical temperature. The interaction potential $U_N^{\mathcal{L}}$ is defined in terms of a function $U_{\mathcal{L}} : \mathbb{R} \to \mathbb{R}_+$ such that, in the language of second quantisation,

$$U_N^{\mathcal{L}} = \frac{1}{2} \sum_e \int_0^{l_e} \int_0^{l_e} a_e^*(x) a_e^*(y) U_{\mathcal{L}}(x-y) a_e(x) a_e(y) \; \mathrm{d}x \, \mathrm{d}y \ , \tag{11}$$

see also (6) and (7). We require the function $U_{\mathcal{L}}$ to be (i) positive, in order to generate repulsive interactions, and (ii) to result in a Kato-Rellich perturbation $U_N^{\mathcal{L}}$ of $-\Delta_N$, so that H_N is self-adjoint on the domain $\mathcal{D}_N(P_1, L_1)$. Furthermore, we assume that

1. $\|U_{\mathcal{L}}\|_{L^1(\mathbb{R})}$ is finite and independent of $\mathcal{L}$,

2. For all $\mathcal{L} > 0$ there exists $A_{\mathcal{L}}, \epsilon_{\mathcal{L}} > 0$ such that $U_{\mathcal{L}}(x) \geq \epsilon_{\mathcal{L}}$ for all $x \in [-A_{\mathcal{L}}, +A_{\mathcal{L}}]$.

More specifically, we require that $A_{\mathcal{L}}$ is either independent of $\mathcal{L}$, or that $\lim_{\mathcal{L}\to\infty} A_{\mathcal{L}} = 0$. These families of potentials include δ-sequences, e.g., of the form $U_{\mathcal{L}}(x) := \mathcal{L}V(\mathcal{L}x)$ with $V \in C_0^\infty(\mathbb{R})$, $V \geq 0$ and $\|V\|_{L^1(\mathbb{R})} = \alpha > 0$, so that $\lim_{\mathcal{L}\to\infty} U_{\mathcal{L}}(x) = \alpha\delta(x)$. This implies that in the thermodynamic limit $\mathcal{L} \to \infty$ we may include Lieb-Liniger models, see (4) and Ref. 21.

The following result is adapted from Ref. 15.

Lemma III.1. *Let $U_{\mathcal{L}}$ be a sequence of potentials with the properties described above. Then the energy density remains finite in the thermodynamic limit, i.e.,*

$$\limsup_{\mathcal{L}\to\infty} \frac{\omega_\beta(H_N)}{\mathcal{L}} < \infty \ . \tag{12}$$

Proof. The proof follows the strategy outlined in Ref. 15. It uses a normalised one-particle vector $\Phi \in \mathcal{H}_1$ such that each component $(\Phi)_e = \varphi_e \in H^1(0, l_e)$ is supported in $(0, l_e)$, bounded in absolute value by $\frac{1}{\sqrt{El_e}}$, and equal to $\frac{d_e}{\sqrt{El_e}}$ on the interval $[a, l_e - a]$ for some $a > 0$ with $d_e \to 1$ as $l_e \to \infty$.

Furthermore, we require Φ to be such that there is a constant $c_1 > 0$ with

$$\|\nabla\varphi_e\|^2_{L^2(0,l_e)} \leq c_1 \ , \quad \forall e \in \mathcal{E} \ . \tag{13}$$

Due to the repulsive nature of the potential one has

$$f_{\mathcal{L}}(\beta) = \frac{1}{\beta\mathcal{L}} \log \mathrm{Tr}(e^{-\beta H_N}) \leq f^0_{\mathcal{L}}(\beta) = \frac{1}{\beta\mathcal{L}} \log \mathrm{Tr}(e^{\beta \Delta_N}) \ . \tag{14}$$

Defining $\Psi_N = \Phi \otimes \cdots \otimes \Phi$ one gets

$$\begin{aligned} \mathrm{Tr}(e^{-\beta H_N}) &\geq e^{-\beta\langle \Psi_N, H_N \Psi_N\rangle_{\mathcal{H}_N}} \\ &\geq e^{-\beta(NEc_1 + c_2\frac{N(N-1)}{2\mathcal{L}}\|U_{\mathcal{L}}\|_{L^1})} \ , \end{aligned} \tag{15}$$

where $c_2 > 0$ is a constant. As a consequence,

$$-\Big(Ec_1\rho + c_2\frac{\rho^2}{2}\Big) - \epsilon \leq f_{\mathcal{L}}(\beta) \leq f^0_{\mathcal{L}}(\beta) \tag{16}$$

for $\mathcal{L}$ large enough, with some $\epsilon > 0$. Finally, using the convexity of $f_{\mathcal{L}}(\beta)$,

$$-\frac{\omega_\beta(H_N)}{\mathcal{L}} = \frac{\mathrm{d}f_{\mathcal{L}}}{\mathrm{d}\beta}(\beta) \geq \frac{f_{\mathcal{L}}(\beta) - f_{\mathcal{L}}(\beta-\delta)}{\delta} , \tag{17}$$

the Lemma then follows using the bounds (16) and taking into account that $\limsup_{\mathcal{L}\to\infty} f^0_{\mathcal{L}}(\beta)$ exists. The latter property follows from a bracketing argument and the explicit knowledge of the eigenvalues for Dirichlet and Neumann vertex conditions (see, e.g., Ref. 30). □

Following Ref. 15, the general idea is to show that BEC into the one-particle ground state, after repulsive interactions are switched on, would contradict Lemma III.1.

Lemma III.2. *Let $\Phi = (\varphi_e)_{e\in\mathcal{E}} \in \mathcal{H}_1$ be a pure one-particle state. Define Φ_1 and Φ_2 as*

$$\begin{aligned}(\Phi_1)_e &:= \varphi_e \chi_{[0,l^\delta_{\min}]} , \\ (\Phi_2)_e &:= \varphi_e \chi_{[l_e - l^\delta_{\min}, l_e]} ,\end{aligned} \tag{18}$$

where $\delta < \frac{1}{3}$ is some constant, $l_{\min}$ is the shortest edge length and χ_I is the characteristic function of the interval I. Then, given that the potential $U_{\mathcal{L}}$ described above is such that $\epsilon_{\mathcal{L}}$ and $A_{\mathcal{L}}$ are both constant or $\epsilon_{\mathcal{L}} A^3_{\mathcal{L}} = O(\mathcal{L}^{3\delta+\gamma-1})$ with $\gamma < 1 - 3\delta$, one has

$$\limsup_{\mathcal{L}\to\infty} \frac{\omega_\beta(a^*(\Phi_j)a(\Phi_j))}{\mathcal{L}} = 0 , \quad j = 1, 2 . \tag{19}$$

Proof. We follow the strategy outlined in Ref. 15. For this, we partition the interval $[0, l^\delta_{\min}]$ into $\lceil l^\delta_{\min}/A_{\mathcal{L}} \rceil$ sub-intervals $I_j = [(j-1)A_{\mathcal{L}}, jA_{\mathcal{L}}]$, $0 < j < \lceil l^\delta_{\min}/A_{\mathcal{L}} \rceil$. In the same way, we partition $[l_e - l^\delta_{\min}, l_e]$ into $\lceil l^\delta_{\min}/A_{\mathcal{L}} \rceil$ sub-intervals $\tilde{I}_j = [l_e - jA_{\mathcal{L}}, l_e - (j-1)A_{\mathcal{L}}]$, $0 < j < \lceil l^\delta_{\min}/A_{\mathcal{L}} \rceil$. We then estimate:

$$\begin{aligned}\frac{\omega_\beta(H_N)}{\mathcal{L}} &\geq -|E_0|\frac{N}{\mathcal{L}} \\ &\quad + \frac{1}{2\mathcal{L}} \sum_e \int_0^{l_e}\int_0^{l_e} U_{\mathcal{L}}(x-y)\omega_\beta(a^*_e(x)a^*_e(y)a_e(x)a_e(y))\,\mathrm{d}x\,\mathrm{d}y \\ &\geq -|E_0|\frac{N}{\mathcal{L}} \\ &\quad + \frac{1}{2\mathcal{L}} \sum_e \sum_{j=1}^{\lceil l^\delta_{\min}/A_{\mathcal{L}}\rceil} \int_{I_j}\int_{I_j} U_{\mathcal{L}}(x-y)\omega_\beta(a^*_e(x)a^*_e(y)a_e(x)a_e(y))\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{1}{2\mathcal{L}} \sum_e \sum_{j=1}^{\lceil l^\delta_{\min}/A_{\mathcal{L}}\rceil} \int_{\tilde{I}_j}\int_{\tilde{I}_j} U_{\mathcal{L}}(x-y)\omega_\beta(a^*_e(x)a^*_e(y)a_e(x)a_e(y))\,\mathrm{d}x\,\mathrm{d}y ,\end{aligned} \tag{20}$$

where $E_0 < 0$ is the ground-state eigenvalue of the one-particle Laplacian $-\Delta_1$. Now, using the lower bound on the potential, we obtain

$$\begin{aligned}\frac{\omega_\beta(H_N)}{\mathcal{L}} \geq -|E_0|\frac{N}{\mathcal{L}} \\ &+ \frac{\epsilon_{\mathcal{L}}}{2\mathcal{L}}\sum_e \sum_{j=1}^{\lceil l_{\min}^{\delta}/A_{\mathcal{L}}\rceil} \int_{I_j}\int_{I_j} \omega_\beta(a_e^*(x)a_e^*(y)a_e(x)a_e(y))\,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{\epsilon_{\mathcal{L}}}{2\mathcal{L}}\sum_e \sum_{j=1}^{\lceil l_{\min}^{\delta}/A_{\mathcal{L}}\rceil} \int_{\tilde{I}_j}\int_{\tilde{I}_j} \omega_\beta(a_e^*(x)a_e^*(y)a_e(x)a_e(y))\,\mathrm{d}x\,\mathrm{d}y\ .\end{aligned} \tag{21}$$

We define $\varphi_e^{(i)} := \varphi_e\chi_{I_i}$ and $\tilde{\varphi}_e^{(i)} := \varphi_e\chi_{\tilde{I}_i}$ as the components of functions $\Phi_{ji}, \tilde{\Phi}_{ji} \in \mathcal{H}_1$, such that $(\Phi_{ji})_e = \delta_{ej}\varphi_j^{(i)}$ and $(\tilde{\Phi}_{ji})_e = \delta_{ej}\tilde{\varphi}_j^{(i)}$. However, for simplicity we restrict our attention in the following to Φ_{ji}. Using the Cauchy-Schwarz inequality for the Gibbs state[25] we then obtain,

$$\begin{aligned}|\omega_\beta(a^*(\Phi_{ji})a(\Phi_{lk}))|^4 &\leq \omega_\beta^2(a^*(\Phi_{ji})a(\Phi_{ji}))\omega_\beta^2(a^*(\Phi_{lk})a(\Phi_{lk})) \\ &\leq \omega_\beta(a^*(\Phi_{ji})a(\Phi_{ji})a^*(\Phi_{ji})a(\Phi_{ji})) \\ &\quad \cdot \omega_\beta(a^*(\Phi_{lk})a(\Phi_{lk})a^*(\Phi_{lk})a(\Phi_{lk}))\ ,\end{aligned} \tag{22}$$

and

$$\begin{aligned}\omega_\beta(a^*(\Phi_{ji})a(\Phi_{ji})a^*(\Phi_{ji})a(\Phi_{ji})) &= \omega_\beta(a^*(\Phi_{ji})a^*(\Phi_{ji})a(\Phi_{ji})a(\Phi_{ji})) \\ &\quad + \|\varphi_j^{(i)}\|^2_{L^2(0,l_j)}\ \omega_\beta(a^*(\Phi_{ji})a(\Phi_{ji}))\ .\end{aligned} \tag{23}$$

Next we establish two useful estimates. First, using the Hölder and then again the Cauchy-Schwarz inequality, yields,

$$\begin{aligned}\omega_\beta(a^*(\Phi_{ji})a(\Phi_{ji})) &= \int_0^{l_j}\int_0^{l_j} \varphi_j^{(i)}(x)\bar{\varphi}_j^{(i)}(y)\omega_\beta(a_j^*(x)a_j(y))\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \int_0^{l_j}\omega_\beta(a_j^*(x)a_j(x))\,\mathrm{d}x \int_0^{l_j}|\varphi_j^{(i)}(y)|^2\,\mathrm{d}y \\ &\leq N\int_0^{l_j}|\varphi_j^{(i)}(y)|^2\,\mathrm{d}y\ .\end{aligned} \tag{24}$$

Second, again using the Hölder and the Cauchy-Schwarz inequality,

$$\begin{aligned}\omega_\beta(a^*(\Phi_{ji})a^*(\Phi_{ji})a(\Phi_{ji})a(\Phi_{ji})) &\leq \int_{I_i}\int_{I_i}\omega_\beta(a_j^*(x)a_j^*(y)a_j(x)a_j(y))\,\mathrm{d}y\,\mathrm{d}x \\ &\quad \cdot \left(\int_{I_i}|\varphi_j^{(i)}(x)|^2\right)^2\,\mathrm{d}x \\ &\leq \int_{I_i}\int_{I_i}\omega_\beta(a_j^*(x)a_j^*(y)a_j(x)a_j(y))\,\mathrm{d}y\,\mathrm{d}x \\ &:= C_{ji}\ .\end{aligned} \tag{25}$$

Combining (22), (23), (24) and (25) we obtain

$$\sum_{i,k=1}^{\lceil l_{\min}^{\delta}/A_{\mathcal{L}}\rceil} |\omega_\beta(a^*(\Phi_{ji})a(\Phi_{lk}))|^4 \le \left(\sum_{i=1}^{\lceil l_{\min}^{\delta}/A_{\mathcal{L}}\rceil} C_{ji} + N\right)\left(\sum_{k=1}^{\lceil l_{\min}^{\delta}/A_{\mathcal{L}}\rceil} C_{lk} + N\right) . \tag{26}$$

Using the inequality $|\sum_{j=1}^n a_j|^4 \le n^3 \sum_{j=1}^n |a_j|^4$ then gives

$$\begin{aligned} |\omega_\beta(a^*(\Phi_1)a(\Phi_1))|^4 &= \left|\sum_{j,l=1}^{E}\sum_{i,k=1}^{\lceil l_{\min}^{\delta}/A_{\mathcal{L}}\rceil} \omega_\beta(a^*(\Phi_{ji})a(\Phi_{lk}))\right|^4 \\ &\le 2\frac{E^6 l_{\min}^{6\delta}}{A_{\mathcal{L}}^6} \sum_{j,l=1}^{E}\sum_{i,k=1}^{\lceil l_{\min}^{\delta}/A_{\mathcal{L}}\rceil} |\omega_\beta(a^*(\Phi_{ji})a(\Phi_{lk})|^4 \\ &\le 2\frac{E^6 l_{\min}^{6\delta}}{A_{\mathcal{L}}^6}\left(\sum_{j=1}^{E}\sum_{i=1}^{\lceil l_{\min}^{\delta}/A_{\mathcal{L}}\rceil} C_{ji} + EN\right)^2 . \end{aligned} \tag{27}$$

Hence,

$$\begin{aligned} \frac{\epsilon_{\mathcal{L}}}{2\mathcal{L}}\sum_{j=1}^{E}\sum_{i=1}^{\lceil l_{\min}^{\delta}/A_{\mathcal{L}}\rceil} C_{ji} &\ge \frac{\epsilon_{\mathcal{L}} A_{\mathcal{L}}^3}{2\sqrt{2}E^3 l_{\min}^{3\delta}}\frac{1}{\mathcal{L}}\omega_\beta^2(a^*(\Phi_1)a(\Phi_1)) - E\rho \\ &\ge \frac{\epsilon_{\mathcal{L}} A_{\mathcal{L}}^3}{2\sqrt{2}E^3 \mathcal{L}^{3\delta-1}}\left(\frac{\omega_\beta(a^*(\Phi_1)a(\Phi_1))}{\mathcal{L}}\right)^2 - E\rho . \end{aligned} \tag{28}$$

Defining

$$D_{ji} := \int_{\tilde{I}_i}\int_{\tilde{I}_i} \omega_\beta(a_j^*(x)a_j^*(y)a_j(x)a_j(y))\,\mathrm{d}y\,\mathrm{d}x , \tag{29}$$

one obtains in a similar way,

$$\frac{\epsilon_{\mathcal{L}}}{2\mathcal{L}}\sum_{j=1}^{E}\sum_{i=1}^{\lceil l_{\min}^{\delta}/A_{\mathcal{L}}\rceil} D_{ji} \ge \frac{\epsilon_{\mathcal{L}} A_{\mathcal{L}}^3}{2\sqrt{2}E^3 \mathcal{L}^{3\delta-1}}\left(\frac{\omega_\beta(a^*(\Phi_2)a(\Phi_2))}{\mathcal{L}}\right)^2 - E\rho . \tag{30}$$

The right-hand sides of (28) and (30), therefore, provide lower bounds to (20). We choose $A_{\mathcal{L}}, \epsilon_{\mathcal{L}}$ both either constant, or such that $\epsilon_{\mathcal{L}} A_{\mathcal{L}}^3 = O(\mathcal{L}^{3\delta+\gamma-1})$, where $0 < \gamma < 1 - 3\delta$. The latter choice is possible as $\delta < \frac{1}{3}$. Hence, the lower bounds in (28) and (30) tend to infinity in the thermodynamic limit, unless (19) is fulfilled. Lemma III.1, however, requires the energy density to remain finite, hence (19) follows. □

To prove the absence of condensation into the one-particle ground state we need the following statement.

Lemma III.3. *Let $\Phi_0 = (\varphi_e)_{e\in\mathcal{E}} \in \mathcal{H}_1$ be the normalised one-particle ground state with components $\varphi_e(x) = a_e e^{-\sqrt{|E_0|}x} + b_e e^{+\sqrt{|E_0|}x}$ and corresponding eigenvalue $E_0 < 0$. Then,*

$$\max_{e\in\mathcal{E}} \sup_{\mathcal{L}} \big(|\varphi_e(0)| + |\varphi_e(l_e)|\big) < \infty \ , \tag{31}$$

and the coefficients are such that $|a_e| = O(1)$ and $|b_e| = O(e^{-\sqrt{|E_0|}l_e})$.

Proof. The squared norm of the function $\Phi_0 = (\varphi_e)_{e\in\mathcal{E}}$ is

$$\|\Phi_0\|^2 = \sum_e \left(\frac{|a_e|^2}{2\sqrt{|E_0|}} \left(1 - e^{-2\sqrt{|E_0|}l_e}\right) + \frac{|b_e|^2}{2\sqrt{|E_0|}} \left(e^{2\sqrt{|E_0|}l_e} - 1\right) + 2|(\bar{a}_e b_e)|l_e \right) . \tag{32}$$

In order for this to equal one, as $l_e \to \infty$, one has to require that $|a_e| = O(1)$ and $|b_e| = O(e^{-\sqrt{|E_0|}l_e})$. Since $\varphi_e(0) = a_e + b_e$ and $\varphi_e(l_e) = a_e \mathrm{e}^{-\sqrt{|E_0|}l_e} + b_e \mathrm{e}^{\sqrt{|E_0|}l_e}$ the property (31) follows. □

As a consequence, the one-particle ground state is localised around the vertices of the graph. This is similar to the model in Ref. 15 and differs essentially from the model in Ref. 6.

We can now formulate the main result of this Note.

Theorem III.4. *Let $\Phi_0 \in \mathcal{H}_1$ be the ground state of the one-particle system. Furthermore, let H_N be given with interaction potential $U_{\mathcal{L}}$ as in Lemma III.2. Then,*

$$\limsup_{\mathcal{L}\to\infty} \frac{\omega_\beta(a^*(\Phi_0)a(\Phi_0))}{\mathcal{L}} = 0 \ . \tag{33}$$

Hence, in the interacting many-particle system there is no condensation into the one-particle ground state.

Proof. We use the cut-offs introduced in Lemma III.2 and write $\Phi_0 = \Phi_1 + \Phi_2 + \Phi_3$ where $(\Phi_3)_e := \varphi_{0,e}\chi_{[l^\delta_{\min}, l_e - l^\delta_{\min}]}$. This gives

$$\frac{\omega_\beta(a^*(\Phi_0)a(\Phi_0))}{\mathcal{L}} = \sum_{i,j=1}^{3} \frac{\omega_\beta(a^*(\Phi_i)a(\Phi_j))}{\mathcal{L}} \ . \tag{34}$$

For the diagonal terms, first Lemma III.2 implies that

$$\limsup_{\mathcal{L}\to\infty} \frac{\omega_\beta(a^*(\Phi_i)a(\Phi_i))}{\mathcal{L}} = 0 \ , \qquad i = 1, 2 \ . \tag{35}$$

Then, Lemma III.3 yields that $\lim_{\mathcal{L}\to\infty}\|\Phi_3\|_{\mathcal{H}_1}=0$. Using the Cauchy-Schwarz inequality we hence obtain

$$\limsup_{\mathcal{L}\to\infty}\frac{\omega_\beta(a^*(\Phi_3)a(\Phi_3))}{\mathcal{L}}\leq\limsup_{\mathcal{L}\to\infty}\frac{N}{\mathcal{L}}\|\Phi_3\|^2_{\mathcal{H}_1}=0\ . \tag{36}$$

Using the Cauchy-Schwarz inequality the off-diagonal terms can be bounded by the diagonal terms,

$$\frac{\omega_\beta(a^*(\Phi_i)a(\Phi_j))}{\mathcal{L}}\leq\frac{\omega_\beta(a^*(\Phi_i)a(\Phi_i))}{\mathcal{L}}+\frac{\omega_\beta(a^*(\Phi_j)a(\Phi_j))}{\mathcal{L}}\ , \tag{37}$$

which concludes the proof. □

Remark III.5. Theorem III.4 only proves that the condensation into the one-particle ground state is unstable with respect to additional repulsive interactions. As already indicated in the introduction, our result does not rule out other potential types of Bose-Einstein condensation (see, e.g., Refs. 34,35).

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