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Detecting Redundant CSS Rules in HTML5 Applications: A Tree Rewriting Approach

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HTML5 applications normally have a large set of CSS (Cascading Style Sheets) rules for data display. Each CSS rule consists of a node selector and a declaration block (which assigns values to selected nodes' display attributes). As web applications evolve, maintaining CSS files can easily become problematic. Some CSS rules will be replaced by new ones, but these obsolete (hence redundant) CSS rules often remain in the applications. Not only does this "bloat" the applications – increasing the bandwidth requirement – but it also significantly increases web browsers' processing time. Most works on detecting redundant CSS rules in HTML5 applications do not consider the dynamic behaviors of HTML5 (specified in JavaScript); in fact, the only proposed method that takes these into account is dynamic analysis, which cannot soundly prove redundancy of CSS rules. In this paper, we introduce an abstraction of HTML5 applications based on monotonic tree-rewriting and study its "redundancy problem". We establish the precise complexity of the problem and various subproblems of practical importance (ranging from P to EXP). In particular, our algorithm relies on an efficient reduction to an analysis of symbolic pushdown systems (for which highly optimised solvers are available), which yields a fast method for checking redundancy in practice. We implemented our algorithm and demonstrated its efficacy in detecting redundant CSS rules in HTML5 applications.

1 Introduction

HTML5 is the latest revision of the HTML standard of the World Wide Web Consortium (W3C), which has become a standard markup language of the Internet. HTML5 provides a uniform framework for designing a web application: (1) data content is given as a standard HTML tree, (2) rules for data display are given in Cascading Style Sheets (CSS), and (3) dynamic behaviors are specified through JavaScript.

An HTML5 application normally contains a large set of CSS rules for data display, each consisting of a *(node) selector* given in an XPath-like query language and a *declaration block* which assigns values to selected nodes' display attributes. However, many of these styling rules are often redundant (in the sense of unreachable code), which "bloat" the application. As a web application evolves, some rules will be replaced by new rules and developers often forget to remove obsolete rules. Another cause of redundant styling rules is the common use of HTML5 boilerplate (e.g. WordPress) since they include many rules that the application will not need. A recent case study [45] shows that in several industrial web applications on average 60% of the CSS rules are redundant. These bloated applications are not only harder to maintain, but they also increase the bandwidth requirement of the website and significantly increase web browsers' processing time. In fact, a recent study [46] reports that when web browsers are loading popular pages around 30% of the CPU time is spent on CSS selectors (18%) and parsing (11%). [These numbers are calculated *without* even including the extra 31% uncategorised operations of the total CPU time, which could include operations from these two

categories.] This suggests the importance of detecting and removing redundant CSS rules in an HTML5 application. Indeed, a sound and automatic redundancy checker would allow bloated CSS stylesheets to be streamlined during development, and generic stylesheets to be minimised before deployment.

There has been a lot of work on optimising CSS (e.g. [43, 46, 45, 21, 14]), which include merging "duplicated" CSS rules, refactoring CSS declaration blocks, and simplifying CSS selectors, to name a few. However, most of these works analyse the set of CSS rules *in isolation*. In fact, the only available methods (e.g. Cilla [45] and UnCSS [57]) that take into account the dynamic nature of HTML5 introduced by JavaScript are based on simple *dynamic analysis* (a.k.a. testing), which cannot soundly prove redundancy of CSS rules since such techniques cannot in general test all possible behaviors of the HTML5 application. For example, from the benchmarks of Mesbah and Mirshokraie [45] there are some non-redundant CSS rules that their tool Cilla falsely identifies as redundant, e.g., due to the use of JavaScript to compensate for browser-specific behavior under certain HTML5 tags like `<input/>` (see Section 6 for more details). Removing such rules can distort the presentation of HTML5 applications, which is undesirable.

Static Analysis of JavaScript A different approach to identifying redundant CSS rules by using *static analysis* for HTML5. Since JavaScript is a Turing-complete programming language, the best one can hope for is approximating the behaviors of HTML5 applications. Static analysis of JavaScript code is a challenging goal, especially in the presence of libraries like jQuery. The current state of the art is well surveyed by Andreasen and Møller [9], with the main tools in the field being WALA [50, 54] and TAJS [9, 28, 27]. These tools (and others) provide traditional static analysis frameworks encompassing features such as points-to [26, 54] and determinacy analysis [9, 50], type inference [33, 28] and security properties [23, 24]. The modelling of the HTML DOM is generally treated as part of the heap abstraction [27, 24] and thus the tree structure is not precisely tracked.

For the purpose of soundly identifying redundant CSS rules, we need a technique for computing a symbolic representation of an *overapproximation* of the set of all reachable HTML trees that is sufficiently precise for real-world applications. Currently there is no *clean* abstract model that captures *common* dynamics of the HTML (DOM) tree caused by the JavaScript component of an HTML5 application and at the same time is *amenable to algorithmic analysis*. Such a model is not only important from a theoretical viewpoint, but it can also serve as a useful *intermediate language* for the analysis of HTML5 applications which among others can be used to identify redundant CSS rules.

Tree rewriting as an intermediate language. The tree-rewriting paradigm — which is commonly used in databases (e.g. [39, 16, 7, 20, 19, 8]) and verification (e.g. [25, 36, 37, 38, 5, 35]) — offers a clean theoretical framework for modelling the dynamics of tree updates and usually lends itself to fully-algorithmic analysis. This makes tree-rewriting a suitable framework in which to model the dynamics of tree updates commonly performed by HTML5 applications. Surveying real-world HTML5 applications (including Nivo-Slider [48] and real-world examples from the benchmarks in Mesbah and Mirshokraie [45]), we were surprised to learn that one-step tree updates used in these applications are extremely simple, despite the complexity of the JavaScript code from the point of view of static analysers. That said, we found that these updates are *not* restricted to modifying only certain regions of the HTML tree. As a result, models such as *ground tree rewrite systems* [37] and their extensions [38, 36, 25, 42, 22] (where *only* the bottom part of the tree may be modified) are not appropriate. However, systems

with rules that may rewrite nodes in *any* region of a tree are problematic since they render the simplest problem of reachability undecidable. Recently, owing to the study of *active XML*, some restrictions that admit decidability of verification (e.g. [7, 19, 20, 8]) have been obtained. However, these models have very high complexity (ranging from double exponential time to nonelementary), which makes practical implementation difficult.

Contributions. The main contribution of the paper is to give a simple and clean tree-rewriting model which strikes a good balance between: (1) expressivity in capturing the dynamics of tree updates commonly performed in HTML5 applications (esp. insofar as detecting redundant CSS rules is concerned), and (2) decidability and complexity of rule redundancy analysis (i.e. whether a given rewrite rule can ever be fired in a reachable tree). We show that the complexity of the problem is EXP-complete[1], though under various practical restrictions the complexity becomes PSPACE or even P. This is substantially better than the complexity of the more powerful tree rewriting models studied in the context of active XML, which is at least double-exponential time. Moreover, our algorithm relies on an efficient reduction to a reachability analysis in *symbolic pushdown systems* for which highly optimised solvers (e.g. Bebop [12], Getafix [32], and Moped [47]) are available.

We have implemented our reduction, together with a proof-of-concept translation tool from HTML5 to our tree rewriting model. Our translation by no means captures the full feature-set of JavaScript and is simply a means of testing the underlying model and analysis we introduce[2]. We specifically focus on modelling standard features of *jQuery* [29] — a simple JavaScript library that makes HTML document traversal, manipulation, event handling, and animation easy from a web application developer's viewpoint. Since its use is so widespread in HTML5 applications nowadays (e.g., used in more than half of the top hundred thousand websites [30]) some authors [33] have advocated a study of jQuery as a language in its own right. Our experiments demonstrate the efficacy of our techniques in detecting redundant CSS rules in HTML5 applications. Furthermore, unlike dynamic analysis, our techniques will not falsely report CSS rules that *may* be invoked as redundant (at least within the fragment of HTML5 applications that our prototypical implementation can handle). We demonstrate this on a number of non-trivial examples (including a specific example from the benchmarks of Mesbah and Mirshokraie [45] and an HTML application using the image slider package Nivo-Slider [48]).

Connection with existing works on static analysis of JavaScript. As surveyed by Andreasen and Møller [9], the static analysis of JavaScript in the presence of the jQuery library — which is essential for analysing HTML5 applications — is currently a formidable task for existing static analysers. We consider our work to be complementary to these works: first, our tree-rewriting model may form part of a static analysis abstraction, and second, static analysis will be essential in translating HTML5 applications into our tree rewriting model by extracting accurate tree update operations. In our implementation, we provide an ad-hoc extraction of tree update operations that allows us to demonstrate the applicability of our approach. Creating a robust static analysis that achieves this is an interesting and worthwhile research challenge, which might benefit from the recent remarkable effort by Bodin *et al.* [13] of capturing the full JavaScript semantics and verifying it with Coq.

Organisation. We give a quick overview of HTML5 applications via a simple example in Section 2. We then introduce our tree rewriting model in Section 3. Since the general model

[1] These complexity classes are defined below and we describe the roles they play in our investigation.

[2] Handling JavaScript in its full generality is a difficult problem [9], which is beyond the scope of this paper.

is undecidable, we introduce a monotonic abstraction in Section 4. We provide an efficient reduction from an analysis of the monotonic abstraction to symbolic pushdown systems in Section 5. Experiments are reported in Section 6. We conclude with future work in Section 7. Missing proofs and further technical details can be found in the appendix.

Notes on computational complexity. In this paper, we study not only decidability but also the *complexity* of computational problems. We believe that pinpointing the precise complexity of verification problems is not only of fundamental importance, but also it often suggests algorithmic techniques that are most suitable for attacking the problem in practice. In this paper, we deal with the following computational complexity classes (see [53] for more details): P (problems solvable in polynomial-time), PSPACE (problems solvable in polynomial space), and EXP (problems solvable in exponential time). Verification problems that have complexity PSPACE and EXP or beyond — see [52, 31] for a few examples — have substantially benefited from techniques like symbolic model checking [44]. The redundancy problem that we study in this paper is another instance of a hard computational problem that can be efficiently solved by BDD-based symbolic techniques in practice.

2 HTML5: a quick overview

In this section, we provide a brief overview of a simple HTML application. We assume basic familiarity with the static elements of HTML5, i.e., HTML documents (a.k.a. HTML DOM document objects) and CSS rules (e.g. see [58]). We will discuss their formal models in Section 3. Our example (also available at the URL [2]) is a small modification of an example taken from an online tutorial [49], which is given in Figure 1. To better understand the application, we suggest the reader open it with a web browser and interact with it.

In this example the page displays a list of input text boxes contained in a `div` with class `input_wrap`. The user can add more input boxes by clicking the "add field" button, and can remove a text box by clicking its neighbouring "remove" button. The script, however, imposes a limit (i.e. 10) on the number of text boxes that can be added. If the user attempts to add another text box when this limit is reached, the `div` with ID `limit` displays the text "Limits reached" in red.

This dynamic behavior is specified within the second `<script/>` tag starting at Line 11 (the first simply loads the jQuery library). To understand the script, we will provide a quick overview of jQuery calls (see [29] for more detail). A simple jQuery call may take the form

```
$(selector).action(...);
```

where '$' denotes that it is a jQuery call, `selector` is a CSS selector, and `action` is a rule for modifying the subtree rooted at this node. For example, in Figure 1, Line 29, we have

```
$('#limit').addClass('warn');
```

The CSS selector `#limit` identifies the unique node in the tree with ID `limit`, while the `addClass()` call adds the class `warn` to this node. The CSS rule

```
.warn { color: red }
```

appearing in the head of the document at Line 2 will now match the node, and thus its contents will be displayed in red.

Another simple example of a jQuery call in Figure 1 is at Line 37

```
<html>
<head><style>.warn { color: red }</style></head>
<body>
    <div class="input_wrap">
        <button class="button">Add Fields</button>
        <div><input type="text" name="mytext[]"> </div>
    </div>
    <div>Total: </div><div id="counter">1</div>
    <div id="limit">Limits not reached</div>

    <script src="http://ajax.googleapis.com/ajax/libs/jquery/1.11.1/jquery.min.js">
    </script>
    <script type="text/javascript">
        $(document).ready(function() {
            var x = 1;

            $('.button').click(function(e){
                if(x < 10){
                    x++;
                    $('.input_wrap').append(
                      '<div>' +
                      '    <input type="text" name="mytext[]"/>' +
                      '    <a href="#" class="delete">Remove</a>' +
                      '</div>'
                    );
                    $('#counter').html(x);
                }
                else {
                    $('#limit').html('Limits reached');
                    $('#limit').addClass('warn');
                }
            });

            $('.input_wrap').on('click', '.delete', function(e){
                $(this).parent('div').remove();
                x--;
                $('#counter').html(x);
                $('.warn').removeClass('warn');
                $('#limit').html('Limits not reached');
            })
        });
    </script>
    </body>
</html>
```

Figure 1: A simple HTML5 application (see [2]).

```
$('.warn').removeClass('warn');
```

The selector `.warn` matches *all*[3] nodes in the tree with class `warn`. The call to `removeClass()` removes the class `warn` from these nodes. Observe that when this call is invoked, the CSS rule above will no longer be matched.

Some jQuery calls may contain an event listener. E.g. at Line 16 we have

```
$('.button').click(...);
```

in Figure 1. This specifies that the function in '`...`' should fire when a node with class `button` is clicked. Similarly at Line 33,

```
$('.input_wrap').on('click', '.delete', ...);
```

adds a click listener to any node within the `input_wrap` div that has the class `delete`.

In general, jQuery calls might form chains. E.g. at Line 34 we have

```
$(this).parent('div').remove();
```

In this line, the call `$(this)` selects the node which has been clicked. The call to `parent()` and then `remove()` moves one step up the tree and if it finds a `div` element, removes the entire subtree (which is of the form `<div><input/><a></a></div>`) from the document.

In addition to the action `remove()` which erases an entire subtree from the document, Figure 1 also contains other actions that potentially modify the shape of the HTML tree. The first such action is `append(string1)` (at Line 19), which simply appends `string1` at the *end* of the string inside the selected node tag. Of course, `string1` might represent an HTML tree; in our example, it is a tree with three nodes. So, in effect `append()` adds this tree as the right-most child of the selected node. The second such action is `html(string1)` (e.g. at Line 25), which first erases the string inside the selected node tag and then appends it with `string1`. In effect, this erases all the descendants of the selected node and adds a *forest* represented by `string1`.

Remark 1. An example where the CSS rule in Figure 1 becomes redundant is when the limit on the number of boxes is removed from the application (in effect, removing `x`), but the CSS is not updated to reflect the change (e.g. see [1]).

In general, CSS selectors are non-trivial. For example

```
.a.b .c { color: red }
```

matches all nodes with class `c` and some ancestor containing both classes `a` and `b` (the space indicates `c` appears on a descendant). Thus, detecting redundant CSS rules requires a good knowledge of the kind of trees constructed by the application. In practice, redundant CSS rules easily arise when one modifies a sufficiently complex HTML5 application (the size of the top 1000 websites has recently exceeded 1600K Bytes [10]). Some popular web pages are known to have an average of 60% redundant CSS rules, as suggested by recent case studies [45].

3 A tree-rewriting approach

In this section, we present our tree-rewriting model. Our design philosophy is to put a special emphasis on model simplicity and fully-algorithmic analysis with good complexity, while retaining adequate expressivity in modelling common tree updates in HTML5 applications (insofar as detecting redundant CSS rules is concerned). We will start by giving an informal description of our approach and then proceed to our formal model.

[3] Unlike node IDs, a single class might be associated with multiple nodes

3.1 An informal description of the approach

Data representation The data model of HTML5 applications is the standard HTML (DOM) tree. In designing our tree rewriting model, we will adopt a data representation consisting of a finite set $\mathcal{K}$ of *classes*, and an unordered, unranked tree with the set $2^{\mathcal{K}}$ of node labels. An unordered tree does not have a sibling ordering, and an unranked tree does not fix the number of children of its nodes. Since jQuery and CSS selectors may reason about adjacent node siblings, unordered trees are in general only an *overapproximation* of HTML trees. As we shall see later, the consequence of this approximation is that some CSS rules that we identify in our analysis as non-redundant might turn out to be redundant when sibling ordering is accounted for, though all CSS rules that we identify as redundant will *definitely* be redundant even with the sibling ordering (see Remark 2 in Section 4). Although it is possible in theory to extend our techniques to ordered unranked trees, we choose to use unordered trees in our model for the purpose of simplicity. Not only do unordered trees give a clean data model, but they turn out to be sufficient for analysing redundancy of CSS rules in most HTML5 applications. In the examples we studied, no false positives were reported as a consequence of the unordered approximation. The choice of tree labels is motivated by CSS and HTML5. Nodes in an HTML document are tagged by HTML elements (e.g. `div` or `a`) and associated with a set of classes, which can be added/removed by HTML5 scripts. Node IDs and data attributes are also often assigned to specific nodes, but they tend to remain unmodified throughout the execution of the application and so can conveniently be treated as classes.

An "event-driven" abstraction Our tree-rewriting model is an "event-driven" abstraction of the script component of HTML5 applications. The abstraction consists of a (finite) set of tree-rewrite rules that can be fired *any* time in *any* order (so long as they are enabled). In this abstraction, one can imagine that each rewrite rule is associated with an external event listener (e.g. listening for a mouse click, hover, etc.). Since these external events cannot be controlled by the system, it is standard to treat them (e.g. see [40]) as *nondeterministic* components, i.e., that they can occur concurrently and in any order[4] Incidentally, the case for event-driven abstractions has been made in the context of transformations of XML data [11].

A tree-rewrite rule σ in our rewrite systems is a tuple (g, χ) consisting of a node selector g (a.k.a. *guard*) and a rewrite operation χ.

To get a feel for our approach, we will construct an event-driven abstraction for the script component of the HTML5 example in Figure 1. For simplicity, we will now use jQuery calls as tree-rewrite rules. We will formalise them later.

The event-driven abstraction for the example in Figure 1 contains four rewrite rules as follows:

```
(1) $('#limit').addClass('warn');
(2) $('.warn').removeClass('warn');
(3) $('.input_wrap').append('<div>
      <input/><a class="delete"></a></div>');
(4) $('.input_wrap').find('.delete').
      parent('div').remove();
```

[4]Note, although, in our model, each rewrite rule is executed atomically, an event that leads to two or more tree updates will be modelled by several rewrite rules. Our analysis will be a "path-insensitive" over-approximation in that no ordering or connection is maintained between these individual update rules. Thus, an event leading to several tree updates is not assumed to be handled atomically. Indeed, the order of the updates is also not maintained. In our experiments this over-approximation did not lead to false positives in the analysis.

Note that we removed irrelevant attributes (e.g. `href`) and text contents since they do not affect our analysis of redundant CSS rules. Rules (1)–(3) were extracted directly from the script. However, the extraction of Rule (4) is more involved. First, the calls to `parent()` and `remove()` come directly from the script. Second, the other calls — which select all elements with class `delete` that are descendants of a node with class `input_wrap` — derive from the semantics of `on()`. The connection of the two parts arrives because the jQuery selection is passed to the event handler via the `this` variable. This connection may be inferred by a data-flow analysis that is sensitive to the behaviour of jQuery.

Detecting CSS Redundancy It can be shown that the set S_1 of all reachable HTML trees in the example in Figure 1 is a *subset* of the set S_2 of all HTML trees that can be reached by applying Rules (1)–(4) to the initial HTML document. We may detect whether

```
.warn { color: red }
```

is redundant by checking whether its selector may match some part of a tree in S_2. If not, then since $S_1 \subseteq S_2$ we can conclude that the rule is *definitely* redundant. In contrast, if the rule can be matched in S_2, we *cannot* conclude that the rule is redundant in the original application.

Let us test our abstraction. First, by applying Rule (1) to the initial HTML tree, we confirm that `warn` can appear in a tree in S_2 and hence the CSS rule *may* be fired. We now revisit the scenario in Remark 1 in Section 2 where the limit on the number of boxes is removed, but the CSS is not updated. In this case, the new event-driven abstraction for the modified script will not contain Rule (1) and the CSS rule can be seen to be redundant in S_2. This necessarily implies that the rule is *definitely* redundant in S_1.

Thus, we guarantee that redundancies will not be falsely identified, but may fail to identify some redundancies in the original application.

3.2 Notations for trees

Before defining our formal model, we briefly fix our notations for describing trees. In this paper we use unordered, unranked trees. A *tree domain* is a nonempty finite subset D of $\mathbb{N}^*$ (i.e. the set of all strings over the alphabet $\mathbb{N} = \{0, 1, \cdots\}$) satisfying prefix-closure, i.e., $w \cdot i \in D$ with $i \in \mathbb{N}$ implies $w \in D$. Note that the natural linear order of $\mathbb{N}$ is immaterial in this definition, i.e., we could use any countably infinite set in place of $\mathbb{N}$.

A *(labeled) tree* over the nonempty finite set (a.k.a. alphabet) Σ is a tuple $T = (D, \lambda)$ where D is a tree domain and λ is a mapping (a.k.a. *node labeling*) from D to Σ. We use standard terminologies for trees, e.g., parents, children, ancestors, descendants, and siblings. The *level of a node* $v \in D$ in T is $|v|$. Likewise, the *height of the tree* T is $\max\{|v| : v \in D\}$. Let $\textsc{Tree}(\Sigma)$ denote the set of trees over Σ. For every $k \in \mathbb{N}$, we define $\textsc{Tree}_k(\Sigma)$ to be the set of trees of height k.

If $T = (D, \lambda)$ and $v \in D$, the *subtree of* T *rooted at* v is the tree $T_{|_v} = (D', \lambda')$, where $D' := \{w \in \mathbb{N}^* : vw \in D\}$ and $\lambda'(w) := \lambda(vw)$.

We remark, for example, that in our definitions, the trees $T_1 = (\{\varepsilon, 1\}, \lambda_1)$ and $T_2 = (\{\varepsilon, 2\}, \lambda_2)$ with $\lambda_1(\varepsilon) = \lambda_2(\varepsilon)$ and $\lambda_1(1) = \lambda_2(2)$ define distinct trees, although both trees contain a root node with a single child with the same labels. It is easy to see that our guards discussed in the following sections cannot distinguish trees up to isomorphism. In Section 4 we discuss morphisms between trees in the context of a monotonicity property.

3.3 The formal model

We now formally define our tree-rewriting model TRS for HTML5 tree updates. A *rewrite system* $\mathcal{R}$ in TRS is a (finite) set of *rewrite rules*. Each rule σ is a tuple (g, χ) of a guard g and a (rewrite) operation χ. Let us define the notion of guards and rewrite operations in turn.

Our language for guards is simply modal logic with special types of modalities. It is a subset of *Tree Temporal Logic*, which is a formal model of the query language XPath for XML data [51, 41, 34]. More formally, a *guard* over the node labeling $\Sigma = 2^{\mathcal{K}}$ with $\mathcal{K} = \{c_1, \cdots, c_n\}$ can be defined by the following grammar:

$$g ::= \top \mid c \mid g \wedge g \mid g \vee g \mid \neg g \mid \langle d \rangle g$$

where c ranges over $\mathcal{K}$ and d ranges over $\{\uparrow, \uparrow^*, \downarrow, \downarrow^*\}$, standing for parent, ancestor, child, and descendant respectively. Note, we will also use $\langle \downarrow^+ \rangle g$ as shorthand for the formula $\langle \downarrow^* \rangle \langle \downarrow \rangle g$ and similarly for $\langle \uparrow^+ \rangle g$. The guard g is said to be *positive* if there is no occurrence of $\neg$ in g. Given a tree $T = (D, \lambda)$ and a node $v \in D$, we define whether v *matches* a guard g (written $v, T \models g$) below. Intuitively, we interpret $v, T \models c$ (for a class $c \in \mathcal{K}$) as $c \in \lambda(v)$, and each modality $\langle d \rangle$ (where $d \in \{\uparrow, \uparrow^*, \downarrow, \downarrow^*\}$) in accordance with the arrow orientation.

Given a tree $T = (D, \lambda)$ and a node $v \in D$, we define whether v of T *matches* a guard g (written $v, T \models g$) by induction over the following rules:

- $v, T \models \top$.
- $v, T \models c$ if $c \in \lambda(v)$.
- $v, T \models g \wedge g'$ if $v, T \models g$ and $v, T \models g'$.
- $v, T \models g \vee g'$ if $v, T \models g$ or $v, T \models g'$.
- $v, T \models \neg g$ if it is not the case that $v, T \models g$.
- $v, T \models \langle \uparrow \rangle g$ if $v = w.i$ (for some $i \in \mathbb{N}$), and $w, T \models g$.
- $v, T \models \langle \uparrow^* \rangle g$ if $v = w.w'$ (for some $w' \in \mathbb{N}^*$), and $w, T \models g$.
- $v, T \models \langle \downarrow \rangle g$ if there exists a node $v.i \in D$ (for some $i \in \mathbb{N}$) such that $v.i, T \models g$.
- $v, T \models \langle \downarrow^* \rangle g$ if there exists a node $v.w \in D$ (for some $w \in \mathbb{N}^*$) such that $v.w, T \models g$.

See the section below on encoding jQuery rules for some examples.

We say that g is *matched* in T if $v, T \models g$ for some node v in T. Likewise, we say that g is *matched* in a set S of Σ-trees if it is matched in some $T \in S$. In the sequel, we sometimes omit mention of the tree T from $v, T \models g$ whenever there is no possibility of confusion.

Having defined the notion of guards, we now define our rewrite operations, which can be one of the following: (1) `AddChild`(X), (2) `AddClass`(X), (3) `RemoveClass`(X), and (4) `RemoveNode`, where $X \subseteq \mathcal{K}$. Intuitively, the semantics of Operations (2)–(4) coincides with the semantics of the jQuery actions `addClass(.)`, `removeClass(.)`, and `remove()`, respectively. Similarly, the semantics of `AddChild`(X) coincides with the semantics of the jQuery action `append(str)` in the case when `str` represents a single node associated with classes X. By adding extra classes, appending a larger subtree can be easily simulated by several steps of `AddChild`(X) operations. This is demonstrated in the next section.

We now formally define the semantics of these rewrite operations. Given two trees $T = (D, \lambda)$ and $T' = (D', \lambda')$, we say that T *rewrites* to T' via $\sigma = (g, \chi)$ (written $T \rightarrow_\sigma T'$) if there exists a node $v \in D$ such that $v \models g$ and

- if $\chi = \texttt{AddClass}(X)$ then $D' = D$ and $\lambda' := \lambda[v \mapsto X \cup \lambda(v)]$. [5]
- if $\chi = \texttt{AddChild}(X)$ then $D' = D \cup \{v.i\}$ and $\lambda' := \lambda[v.i \mapsto X]$ and $v.i \notin D$
- if $\chi = \texttt{RemoveClass}(X)$ then $D' = D$ and $\lambda' := \lambda[v \mapsto \lambda(v) \setminus X]$
- if $\chi = \texttt{RemoveNode}$ and v is *not* the root node, $D' := D \setminus \{v.w : w \in \mathbb{N}^*\}$ and λ' is the restriction of λ to D'.

Note that the system cannot execute a `RemoveNode` operation on the root node of a tree. I.e. there is no transition $T \to_\sigma T'$ if σ would erase the root node of T.

Given a rewrite system $\mathcal{R}$ over Σ-trees, we define $\to_\mathcal{R}$ to be the union of $\to_\sigma$, for all $\sigma \in \mathcal{R}$. For every $k \in \mathbb{N}$, we define $\to_{\mathcal{R},k}$ to be the restriction of $\to_\mathcal{R}$ to $\textsc{Tree}_k(\Sigma)$. Given a set $\mathcal{C}$ of Σ-trees, we write $post^*_\mathcal{R}(\mathcal{C})$ (resp. $post^*_{\mathcal{R},k}(\mathcal{C})$) to be the set of trees T' satisfying $T \to^*_\mathcal{R} T'$ (resp. $T \to^*_{\mathcal{R},k} T'$) for some tree $T \in \mathcal{C}$.

Encoding jQuery Rewrite Rules Let us translate the four "jQuery rewrite rules" for the application in Figure 1 into our formalism. The first rule translates directly to the rule $(\texttt{\#limit}, \texttt{AddClass}(\{\texttt{.warn}\}))$, while the second rule translates to $(\texttt{.warn}, \texttt{RemoveClass}(\{\texttt{.warn}\}))$. The fourth rule identifies `div` nodes that have some child with class `delete` that in turn has some ancestor with class `input_wrap`. Thus, it translates to

$$(\texttt{div} \wedge \langle\downarrow\rangle(\texttt{.delete} \wedge \langle\uparrow^+\rangle\texttt{.input\_wrap}), \texttt{RemoveNode}).$$

Finally, the third rule requires the construction of a new sub-tree. We achieve this through several rules and a new class `tmp`. We first add the new `div` element as a child node, and use the class `tmp` to mark this new node:

$$(\texttt{.input\_wrap}, \texttt{AddChild}(\{\texttt{div}, \texttt{tmp}\}))\ .$$

Then, we add the children of the `div` node with the two rules

$$\begin{array}{ll} & (\texttt{tmp}, \texttt{AddChild}(\{\texttt{input}\})) \\ \text{and} & (\texttt{tmp}, \texttt{AddChild}(\{\texttt{a}, \texttt{.delete}\}))\ . \end{array}$$

We show in the appendix. how to encode a large number of jQuery tree traversals into our guard language. In general, some of these traversals have to be approximated. For example the `.next()` operation can be approximated by the modalities $\langle\uparrow\rangle\langle\downarrow\rangle$ which select a sibling of the current node.

The redundancy problem The *redundancy problem for* Trs is the problem that, given a rewrite system $\mathcal{R}$ over Σ-trees, a finite nonempty set S of guards over Σ, and an initial Σ-labeled tree T_0, compute the subset $S' \subseteq S$ of guards that are not matched in $post^*_\mathcal{R}(T_0)$. The decision version of the redundancy problem for Trs is simply to check if the aforementioned set S' is empty. Similarly, for each $k \in \mathbb{N}$, we define the *k-redundancy problem for* Trs to be the restriction of the redundancy problem for Trs to trees of height k (i.e. we use $post^*_{\mathcal{R},k}$ instead of $post^*_\mathcal{R}$).

The problem of identifying redundant CSS node selectors in a CSS file can be reduced to the problem of the redundancy problem for Trs. This is because CSS node selectors can easily

[5] Given a map $f : A \to B$, $a' \in A$ and $b' \in B$, we write $f[a' \mapsto b']$ to mean the map $(f \setminus \{(a', f(a'))\}) \cup \{(a', b')\}$

be translated into our guard language (e.g. using the translation given in [21]). Observe that the converse is false. E.g., $\langle\downarrow\rangle a \wedge \langle\downarrow\rangle b$ cannot be expressed as a CSS selector since a and b may appear on different children of the matched node. The guard in the fourth rule of our running example is also not expressible as a CSS selector. However, this increased expressivity is required to model tree traversals in jQuery. Note that the redundancy problem for TRS could also have potential applications beyond detecting redundant CSS rules, e.g., detecting redundant jQuery calls in HTML5.

Despite the simplicity of our rewrite rules, it turns out that the redundancy problem is in general undecidable (even restricted to trees of height at most two); see the appendix.

Proposition 1. *The 1-redundancy problem for $\mathcal{R}$ is undecidable.*

4 A monotonic abstraction

The undecidability proof of Proposition 1 in fact relies fundamentally on the power of negation in the guards. A natural question, therefore, is what happens in the case of positive guards. Not only is this an interesting theoretical question, such a restriction often suffices in practice. This is partly because the use of negations in CSS and jQuery selectors (i.e. `:not(...)`) is rather limited in practice. In particular, there was no use of negations in CSS selectors found in the benchmark in [45] containing 15 live web applications. In practice, negations are almost always limited to negating atomic formulas, i.e., $\neg c$ (for a class $c \in \mathcal{K}$) which can be overapproximated by $\top$ often without losing too much precision.

Note, in general it is not possible to express $\neg c$ via the formula $\bigvee_{c' \in \mathcal{K} \setminus \{c\}} c'$ since nodes may be labelled by multiple classes. That is, labelling a node by c' does not prevent the node also being labelled by c. However, when c is an HTML tag name (e.g. `div` or `img`), we can assert $\neg c$ by checking whether the node is labelled by some other tag name, since a node can only have one tag (e.g. a node cannot be both a `div` and an `img`).

A main result of the paper is that the "monotonic" abstraction that is obtained by restricting to positive guards gives us decidability with a good complexity. In this section, we prove the resulting tree rewriting class is "monotonic" in a technical sense of the word, and summarise the main technical results of the paper.

Notation. *Let us denote by TRS_0 the set of rewrite systems with positive guards. The guard databases in the input to redundancy and k-redudancy problems for TRS_0 will only contain positive guards as well. In the sequel, unless otherwise stated, a "guard" is understood to mean a positive guard.*

4.1 Formalising and proving "monotonicity"

Recall that a binary relation $R \subseteq S \times S$ is a *preorder* if it is transitive, i.e., if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. We start with a definition of a preorder $\preceq$ over $\text{TREE}(\Sigma)$, where $\Sigma = 2^{\mathcal{K}}$. Given two Σ-trees $T = (D, \lambda)$ and $T' = (D', \lambda')$, we write $T \preceq T'$ if there exists an *embedding* from T' to T, i.e., a function $f : D \to D'$ such that:

(H1) $f(\epsilon) = \epsilon$

(H2) For each $v \in D$, $\lambda(v) \subseteq \lambda'(f(v))$

(H3) For each $v.a \in D$ where $v \in \mathbb{N}^*$ and $a \in \mathbb{N}$, we have $f(v.a) = f(v).b$ for some $b \in \mathbb{N}$.

Note that this is equivalent to the standard notion of *homomorphisms* from database theory (e.g. see [6]) when each class $c \in \mathcal{K}$ is treated as a unary relation. The following is a basic property of $\preceq$, whose proof is easy and is left to the reader.

Fact. *$\preceq$ is a preorder on* $\textsc{Tree}(\Sigma)$.

The following lemma shows that embeddings preserve positive guards.

Lemma 1. *Given trees $T = (D, \lambda)$ and $T' = (D', \lambda') \in \textsc{Tree}(\Sigma)$ satisfying $T \preceq T'$ with a witnessing embedding $f : D \to D'$, and a positive guard g over Σ, then if $v, T \models g$ then $f(v), T' \models g$.*

We relegate to the appendix the proof of Lemma 1, which is similar to (part of) the proof of the homomorphism theorem for conjunctive queries (e.g. see [6, Theorem 6.2.3]). This lemma yields the following monotonicity property of $\textsc{Trs}_0$.

Lemma 2 (Monotonicity)**.** *For each $\sigma \in \mathcal{R}$, if $T_1 \preceq T_2$ and $T_1 \to_\sigma T_1'$, then either $T_1' \preceq T_2$ or $T_2 \to_\sigma T_2'$ for some T_2' satisfying $T_1' \preceq T_2'$.*

Intuitively, the property states that any rewriting step of the "smaller" tree either does not expand the tree beyond the "bigger tree", or, the step can be simulated by the "bigger" tree while still preserving the embedding relation. The proof of the lemma is easy (by considering all four possible rewrite operations), and is relegated to the appendix.

One consequence of this monotonicity property is that, when dealing with the redundancy problem, we can safely ignore rewrite rules that use one of the rewrite operations `RemoveNode` or $\texttt{RemoveClass}(X)$. This is formalised in the following lemma, whose proof is given in the appendix.

Lemma 3. *Given a rewrite system $\mathcal{R}$ over Σ-trees, a guard database S, and an initial tree T_0, let $\mathcal{R}^-$ be the set of $\mathcal{R}$-rules less those that use either* `RemoveNode` *or* $\texttt{RemoveClass}(X)$*. Then, for each $g \in S$, g is matched in $post^*_{\mathcal{R}}(T_0)$ iff g is matched in $post^*_{\mathcal{R}^-}(T_0)$.*

Convention. *In the sequel, we assume that there are only two possible rewrite operations, namely,* $\texttt{AddChild}(X)$*, and* $\texttt{AddClass}(X)$.

Remark 2. When choosing unordered trees for our CSS redundancy analysis in Section 3 (i.e. instead of ordered trees), we remarked that we have added a layer of *sound* approximation to our analysis. We will now explain why this is a sound approximation. We could consider the extension of our guard language with the left-sibling and right-sibling operators $\langle\leftarrow\rangle$ and $\langle\rightarrow\rangle$ (which would still be contained in Tree Temporal Logic, which as we already mentioned is a formal model of XPath [51, 41, 34]). The semantics of formulas of the form $\langle\leftarrow\rangle g$ and $\langle\rightarrow\rangle g$ (with respect ordered trees) can be defined in the same way as our guard language. Given an ordered tree T, let T' be the unordered version of T obtained by ignoring the sibling ordering from T. Given a formula g with left/right sibling operators, we could define its "unordered approximation" g' by replacing every occurrence of $\langle\leftarrow\rangle$ and $\langle\rightarrow\rangle$ by $\langle\uparrow\rangle\langle\downarrow\rangle$. For positive guards g, it is easy to show by induction on g that $v, T \models g$ implies $v, T' \models g'$. By the same token, we could also consider an extension of our tree rewriting to ordered trees that allows adding an immediate left/right sibling, e.g., by the operators $\texttt{AddLeftKin}(X)$ and $\texttt{AddRightKin}(X)$ (these are akin to the `.before()` and `.after()` jQuery methods). Given a tree rewriting $\mathcal{R}$ in this extended rewrite system, we could construct an approximated rewriting $\mathcal{R}'$ in $\textsc{Trs}_0$ as follows: (1) for every rewrite rule of the form $(g, \texttt{AddLeftKin}(X))$ or $(g, \texttt{AddRightKin}(X))$ in this extended tree rewriting, add the rewrite rule $(\langle\downarrow\rangle g', \texttt{AddChild}(X))$ in $\mathcal{R}'$, and (2) for

every other rewrite rule (g, χ) in $\mathcal{R}$, add the rewrite rule (g', χ) in $\mathcal{R}$. A consequence of this approximation is that a guard g can be matched in a reachable tree $post^*_{\mathcal{R}}(T_0)$ implies that the unordered approximation g' can be matched in a reachable tree $post^*_{\mathcal{R}'}(T'_0)$. That is, if g' is redundant in $\mathcal{R}'$, then g is redundant in $\mathcal{R}$, i.e., that g can be safely removed.

4.2 Summary of technical results

We have completely identified the computational complexity of the redundancy and k-redundancy problem for $\textsc{Trs}_0$. Our first result is:

Theorem 3. *The redundancy problem for* $\textsc{Trs}_0$ *is* EXP*-complete.*

Our upper bound was obtained via an efficient reduction to an analysis of symbolic pushdown systems (see Section 5), for which there are highly optimised tools (e.g. Bebop [12], Getafix [32], and Moped [47]). We have implemented our reduction and demonstrate its viability in detecting redundant CSS rules in HTML5 applications (see Section 6). The proof of the lower bound in Theorem 3 is provided in the appendix. In the case of k-redundancy problem, a better complexity can be obtained.

Theorem 4. *The k-redundancy problem* $\textsc{Trs}_0$ *is:*

(i) PSPACE*-complete if k is part of the input in unary.*

(ii) solvable in P*-time — $n^{O(k)}$ — for each fixed parameter k, but is* W[1]*-hard.*

Recall that PSPACE $\subseteq$ EXP. The second item of Theorem 4 contains the complexity class W[1] from *parameterised complexity theory* (e.g. see [18]), which provides a theory for answering whether a computational problem with multiple input parameters is *efficiently solvable* when certain parameters are *fixed*. In the case of the k-redundancy problem for $\textsc{Trs}_0$ a problem instance contains $k \in \mathbb{N}$ and $\mathcal{R} \in \textsc{Trs}_0$ as the input parameters. The problem can be solved in time $n^{O(k)}$, where n is the size of $\mathcal{R}$. We would like to know whether the parameter k can be removed from the exponent of n in the time-complexity. That is, whether the problem is solvable in time $f(k)n^c$ for some computable function $f : \mathbb{N} \to \mathbb{N}$ and a constant $c \in \mathbb{N}$ (a.k.a. *fixed-parameter-tractable (FPT) algorithms*). Observe that, asymptotically, $f(k)n^c$ is smaller than $n^{O(k)}$ for *every* fixed value of k. By showing that the problem is W[1]-hard, we have in effect shown that the parameter k cannot be removed from the exponent of n, i.e., that our $n^{O(k)}$-time algorithm is, in a sense, optimal. For space reasons, we relegate the proofs of Theorem 4 to the appendix.

Remark 5. Decidability for the k-redundancy problem for $\textsc{Trs}_0$ is immediate from the theory of well-structured transition systems (e.g. see [4, 17]). We have shown in Lemma 2 that the tree embedding relation $\preceq$ is monotonic. It can be shown that $\preceq$ is also *well-founded* on trees of height k (e.g. see [19]), i.e., there is no infinite descending chain $T_1 \succ T_2 \succ \cdots$ for trees $T_1, \ldots$ of height k. The theory of well-structured transition systems (e.g. see [4, 17]) would imply decidability for the k-redundancy problem. Unfortunately, this only gives a nonelementary upper bound complexity (i.e. an unbounded tower of exponential) for the k-redundancy and does yield decidability for the general redundancy problem.

5 The algorithm

In this section, we provide an efficient reduction to an analysis of symbolic pushdown systems, which will give an algorithm with an exponential-time (resp. polynomial-space) worst-case upper bound for the redundancy (resp. k-redundancy) problem for $\textsc{Trs}_0$. To this end, we first provide a preliminary background on symbolic pushdown systems. We will then provide a roadmap of our reduction to symbolic pushdown systems, which will consist of a sequence of three polynomial-time reductions described in the last three subsections.

5.1 Pushdown systems: a preliminary

Before describing our reduction, we will first provide a preliminary background on pushdown systems and their extensions to symbolic pushdown systems.

Pushdown systems are standard (nondeterministic) pushdown automata without input labels. Input labels are irrelevant since one mostly asks about their transition graphs (in our case, reachability). More formally, a *pushdown system (PDS)* is a tuple

$$\mathcal{P} = (\mathcal{Q}, \Gamma, \Delta)$$

where

- $\mathcal{Q}$ is a finite set of *control states*,
- Γ is a finite set of *stack symbols*, and
- Δ is a finite subset of $(\mathcal{Q} \times \Gamma) \times (\mathcal{Q} \times \Gamma^*)$ such that if $((q, a), (q', w)) \in \Delta$ then $|w| \leq 2$.

This PDS generates a transition relation $\to_{\mathcal{P}} \subseteq (\mathcal{Q} \times \Gamma^*) \times (\mathcal{Q} \times \Gamma^*)$ as follows: $(q, v) \to_{\mathcal{P}} (q', v')$ if there exists a rule $((q, a), (q', w)) \in \Delta$ such that $v = ua$ and $v' = uw$ for some word $u \in \Gamma^*$.

Symbolic pushdown systems are pushdown systems that are succinctly represented by boolean formulas. They are equivalent to *(recursive) boolean programs*. More precisely, a *symbolic pushdown system (sPDS)* is a tuple

$$(\mathcal{V}, \mathcal{W}, \Delta)$$

where

- $\mathcal{V} = \{x_1, \ldots, x_n\}$ and $\mathcal{W} = \{y_1, \ldots, y_m\}$ are two disjoint sets of boolean variables, and
- Δ is a finite set of pairs (i, φ) of number $i \in \{0, 1, 2\}$ and boolean formula φ over the set of variables $\mathcal{V} \cup \mathcal{W} \cup \mathcal{V}' \cup \mathcal{W}'$, where
  - $\mathcal{V}' := \{x'_1, \ldots, x'_n\}$, and
  - $\mathcal{W}' := \bigcup_{j=1}^{i} \mathcal{W}_j$ with $\mathcal{W}_j := \{y_1^j, \ldots, y_m^j\}$.

This sPDS generates a (exponentially bigger) pushdown system $\mathcal{P} = (\mathcal{Q}, \Gamma, \Delta')$, where $\mathcal{Q} = \{0,1\}^n$, $\Gamma = \{0,1\}^m$, and $((q, a), (q', w)) \in \Delta'$ iff there exists a pair $(i, \varphi) \in \Delta$ satisfying $i = |w|$, and φ is satisfied by the assignment that assigns[6] q to $\mathcal{V}$, a to $\mathcal{W}$, q' to $\mathcal{V}'$, and w to $\mathcal{W}'$ (i.e. assigning the jth letter of w to $\mathcal{W}_j$).

The *bit-toggling problem for sPDS* is a simple reachability problem over symbolic pushdown systems. Intuitively, we want to decide if we can toggle on the variable y_i from the initial configuration. More precisely, given an sPDS $\mathcal{P} = (\mathcal{V}, \mathcal{W}, \Delta)$ with $\mathcal{V} = \{x_1, \ldots, x_n\}$ and $\mathcal{W} =$

[6] Meaning that if $q = (q_1, \ldots, q_n)$, then q_i is assigned to x_i

$\{y_1, \ldots, y_m\}$, a variable $y_i \in \mathcal{W}$, and an initial configuration $I_0 = ((b_1, \ldots, b_n), (b'_1, \ldots, b'_m)) \in \{0,1\}^n \times \{0,1\}^m$, decide if $I_0 \to^*_{\mathcal{P}} (q, a)$ for some $q \in \{0,1\}^n$ and $a = (b''_1, \ldots, b''_m) \in \{0,1\}^m$ with $b''_i = 1$.

The *bounded bit-toggling problem* is the same as the bit-toggling problem but the stack height of the pushdown system cannot exceed some given input parameter $h \in \mathbb{N}$ (given in unary).

Proposition 2. *The bit-toggling (resp. bounded bit-toggling) problem for sPDS is solvable in* EXP *(resp.* PSPACE*).*

The proof of this is standard (e.g. see [52]), which for completeness we provide in the appendix.

Despite the relatively high complexity mentioned in Proposition 2, nowadays there are highly optimised sPDS and boolean program solvers (e.g. Moped [52, 47], Getafix [32], and Bebop [12]) that can solve sPDS bit-toggling problem efficiently using BDD (Binary Decision Diagram) representation of boolean formulas. In fact, the boolean formulas that we produce in our polynomial-time reductions below have straightforward small representations as BDDs.

5.2 Intuition/Roadmap of the reduction

The following theorem formalises our reduction claim.

Theorem 6. *The redundancy (resp. k-redundancy) problem for* $\textsc{Trs}_0$ *is polynomial-time reducible to the bit-toggling (resp. bounded bit-toggling) problem for sPDS.*

Together with Proposition 2, Theorem 6 implies an EXP (resp. PSPACE) upper bound for the redundancy (resp. k-redundancy) problem for sPDS. Moreover, as discussed in the appendix, it is straightforward to construct from our reduction a counterexample path in the rewrite system witnessing the non-redundancy of a given guard.

The actual reduction in Theorem 6 involves several intermediate polynomial-time reductions. Here is a roadmap.

- We first show that it suffices to consider "simple" rewrite systems. These systems are simple in the sense that guards only test direct parents or children of the nodes. Furthermore, these systems have the property that we only need to check redundancy at the root node. This is given in Section 5.3.
- We then show that the simplified problem can be solved by a "saturation" algorithm that uses a subroutine to check whether a given class can be added to a given node via a sequence of rewrite rules. This subroutine solves what we call the "class-adding problem": a simple reachability problem that checks whether a class can be added to a given node via a series of rewrite rules that do not change any other existing nodes in the tree (but may add new nodes). That is, the node is considered as the only node in a single node tree, possibly with some contextual (immutable) information about its parent. This is given in Section 5.4.
- Finally, we show that the class-adding problem is efficiently reducible to the bit-toggling problem for sPDS. This is given in Section 5.5.

The case of k-redundancy for $\textsc{Trs}_0$ is similar but each intermediate problem is relativised to the version with bounded height.

The reason we need to simplify the system is because, in the simplified system, we can test redundancy only by inspection of the root node, and all guards only refer to direct neighbours

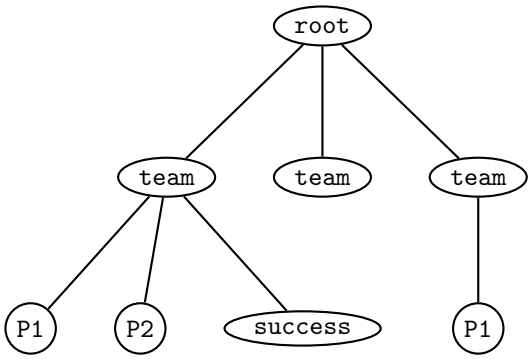

Figure 2: A reachable configuration

of each node. The latter simplification makes the reduction to sPDS possible. As explained in Section 5.5, the constructed sPDS performs a kind of depth-first search over the constructed trees. Since an sPDS can only see the top of its stack, it is important that it only needs to maintain local information about the node being inspected by the depth-first search. The former simplification justifies us only maintaining labelling information about the nodes in the original tree (in particular, the root node) without having to (explicitly) add new nodes.

Running Example. We provide a running example for our sequence of reductions. Imagine a tennis double tournament web page which can be used to keep track of a list of teams (containing exactly two players). The page allows a user to create a team, and add players to a team. The page will also indicate a success next to the team details after both players have been added. An overapproximation of the behavior of the page could be abstracted as a rewrite system $\mathcal{R}$ as follows:

- The initial tree is the single-node tree with label `root`.
- The set of node labels is $\{\texttt{root}, \texttt{team}, \texttt{P1}, \texttt{P2}, \texttt{success}\}$.
- A team can be added to the tournament, i.e., there is a rule $(\texttt{root}, \texttt{AddChild}(\texttt{team})) \in \mathcal{R}$.
- Player 1 can be added to a team, i.e., there is a rule $(\texttt{team}, \texttt{AddChild}(\texttt{P1})) \in \mathcal{R}$
- Player 2 can be added to a team, i.e., there is a rule $(\texttt{team}, \texttt{AddChild}(\texttt{P2})) \in \mathcal{R}$
- Success after both of the required players Player 1 & Player 2 are added, i.e., there is a rule $(\langle\downarrow\rangle\texttt{P1} \wedge \langle\downarrow\rangle\texttt{P2}, \texttt{AddChild}(\texttt{success})) \in \mathcal{R}$

The set S of guards that we want to check for redundancy is $\{\texttt{success}\}$. A snapshot of a reachable configuration is provided in Figure 2.

5.3 Simplifying the rewrite system

We will make the following two simplifications: (1) restricting the problem to only checking redundancy at the *root* node, (2) restrict the guards to be used.

To achieve simplification (1), one can simply define a new set of guards from S as $\{\langle\downarrow^*\rangle g : g \in S\}$. Then, for each tree $T = (D, \lambda) \in \textsc{Tree}(\Sigma)$ and guard g, it is the case that $(\exists v \in D : v, T \models g)$ iff $\epsilon, T \models \langle\downarrow^*\rangle g$.

We now proceed to simplification (2). A guard over the node labeling $\Sigma = 2^{\mathcal{K}}$ is said to be *simple* if it is of one of the following two forms

$$\bigwedge_{i=1}^{m} c_i \qquad \text{or} \qquad \langle d\rangle \bigwedge_{i=1}^{m} c_i$$

for some $m \in \mathbb{N}$, where each c_i ranges over $\mathcal{K}$ and d ranges over $\{\uparrow, \downarrow\}$. [Note: if $m = 0$, then $\bigwedge_{i=1}^{m} c_i \equiv \top$.] For notational convenience, if $X = \{c_1, \ldots, c_m\}$, we shall write X (resp. $\langle d\rangle X$) to mean $\bigwedge_{i=1}^{m} c_m$ (resp. $\langle d\rangle \bigwedge_{i=1}^{m} c_m$). A rewrite system $\mathcal{R} \in \textsc{Trs}_0$ is said to be *simple* if (i) all guards occurring in $\mathcal{R}$ are simple, and (ii) if $(\langle d\rangle X, \chi) \in \mathcal{R}$, then χ is of the form $\texttt{AddClass}(Y)$.

We define $\textsc{Trs}'_0 \subseteq \textsc{Trs}_0$ to be the set of simple rewrite systems. The redundancy (resp. k-redundancy) problem for $\textsc{Trs}'_0$ is defined in the same way as for $\textsc{Trs}_0$ *except that* all the guards in the set of guards are restricted to be a subset of $\mathcal{K}$.

The following lemma shows that the redundancy (resp. k-redundancy) problem for $\textsc{Trs}_0$ can be reduced in polynomial time to the redundancy (resp. k-redundancy) problem for $\textsc{Trs}'_0$. Essentially, the reduction works by introducing new classes representing (non-simple) subformulas of the guards. New rewrite rules are introduced that inductively calculate which subformulas are true. That is, if a subformula g is true at v, then the labelling of v will include a class representing g.

Note, in the lemma below, S' is a set of atomic guards. That is, each guard in S' is of the form c for some $c \in \mathcal{K}'$.

Lemma 4. *Given a $\mathcal{R} \in \textsc{Trs}_0$ over $\Sigma = 2^{\mathcal{K}}$ and a set S of guards over Σ, there exists $\mathcal{R}' \in \textsc{Trs}'_0$ over $\Sigma' = 2^{\mathcal{K}'}$ (where $\mathcal{K} \subseteq \mathcal{K}'$) and a set S' of atomic guards such that:*

(P1) *For each $k \in \mathbb{N}$, S is k-redundant for $\mathcal{R}$ iff S' is k-redundant for $\mathcal{R}'$.*

(P2) *S is redundant for $\mathcal{R}$ iff S' is redundant for $\mathcal{R}'$.*

Moreover, we can compute $\mathcal{R}'$ and S' in polynomial time.

We show how to compute $\mathcal{R}'$. The set $\mathcal{K}'$ is defined as the union of $\mathcal{K}$ with the set G of all non-atomic subformulas (i.e. occurring in the parse tree) of guard formulas in S and $\mathcal{R}$. In the sequel, to avoid potential confusion, we will often underline members of G in $\mathcal{K}'$, e.g., write $\underline{\langle\downarrow\rangle c}$ instead of $\langle\downarrow\rangle c$. Note that $\underline{c} = c$ for all $c \in \mathcal{K}$.

We now define the simple rewrite system $\mathcal{R}'$. Initially, we will define a rewrite system $\mathcal{R}_1$ that allows the operators $\langle\uparrow^*\rangle$ and $\langle\downarrow^*\rangle$; later we will show how to remove them. We first add the following "intermediate" rules to $\mathcal{R}_1$:

1. $(\{\underline{g}, \underline{g'}\}, \texttt{AddClass}(\underline{g \wedge g'}))$, for each $(g \wedge g') \in G$.
2. $(\underline{g}, \texttt{AddClass}(\underline{g \vee g'}))$ and $(\underline{g'}, \texttt{AddClass}(\underline{g \vee g'}))$, for each $(g \vee g') \in G$.
3. $(\langle d\rangle \underline{g}, \texttt{AddClass}(\underline{\langle d\rangle g}))$, for each $\langle d\rangle g \in G$.
4. $(\underline{g}, \chi)$, for each $(g, \chi) \in \mathcal{R}$.

Note that in Rule (3) the guard $\langle d \rangle \underline{g}$ is understood to mean a non-atomic guard over $2^{\mathcal{K}'}$ – that is, $\underline{g}$ is an atomic guard. Finally, we define $S' := \{\underline{g} : g \in S\}$. Notice that each guard in S' is atomic. The aforementioned algorithm computes $\mathcal{R}_1$ and S' in linear time.

We now show how to remove the operators $\langle \uparrow^* \rangle$ and $\langle \downarrow^* \rangle$ (i.e. rules of type (3)). To do this we will introduce new rules $\mathcal{R}_2$ that essentially compute the required transitive closures using the $\uparrow$ and $\downarrow$ operators. Our final rewrite system $\mathcal{R}'$ will be $\mathcal{R}_1 \cup \mathcal{R}_2$. We define $\mathcal{R}_2$ to be the set containing the following rules for each rule $(\langle d^* \rangle \underline{g}, \chi) \in \mathcal{R}_1$ where $d \in \{\uparrow, \downarrow\}$:

(a) $(\underline{g}, \texttt{AddClass}(\underline{\langle d^* \rangle g}))$.

(b) $(\langle d \rangle \underline{\langle d^* \rangle g}, \texttt{AddClass}(\underline{\langle d^* \rangle g}))$.

Note that from Rule (4) $\mathcal{R}_1$ contains the rule $(\underline{\langle d^* \rangle g}, \chi)$, where $\underline{\langle d^* \rangle g}$ is understood to mean an atomic guard over $2^{\mathcal{K}'}$. Intuitively, this simplification can be done because $v, T \models \langle d^* \rangle g$ iff at least one of the following cases holds: (i) $v, T \models g$, (ii) there exists a node w in T such that $w, T \models \langle d^* \rangle g$ and w can be reached from v by following the direction d for one step. The aforementioned computation step again can be done in linear time. The proof of correctness (i.e. **(P1)** and **(P2)**) is provided in the appendix. In particular, for all $g \in S$, it is the case that g is redundant in $\mathcal{R}$ iff $\underline{g}$ is redundant in $\mathcal{R}'$.

Running Example. In our example, initially, S is changed into $\{\underline{\langle \downarrow^* \rangle \texttt{success}}\}$ after the first simplification. To perform the second, we first obtain the rewrite system $\mathcal{R}_1$ containing the following rules (recall we equate $\underline{c}$ and c for each class):

- $(\texttt{success}, \texttt{AddClass}(\underline{\langle \downarrow^* \rangle \texttt{success}}))$
- $(\{\langle \downarrow \rangle \texttt{P1}, \langle \downarrow \rangle \texttt{P2}\}, \texttt{AddClass}(\underline{\langle \downarrow \rangle \texttt{P1} \wedge \langle \downarrow \rangle \texttt{P2}}))$
- $(\langle \downarrow \rangle \underline{P}, \texttt{AddClass}(\underline{\langle \downarrow \rangle P}))$ for each $P \in \{\texttt{P1}, \texttt{P2}\}$

and from Rule (4) we also have in $\mathcal{R}_1$.

- $(\texttt{root}, \texttt{AddChild}(\texttt{team}))$,
- $(\texttt{team}, \texttt{AddChild}(\texttt{P1}))$,
- $(\texttt{team}, \texttt{AddChild}(\texttt{P2}))$,
- $(\underline{\langle \downarrow \rangle \texttt{P1} \wedge \langle \downarrow \rangle \texttt{P2}}, \texttt{AddChild}(\texttt{success}))$.

Now, $\mathcal{R}_2$ contains the following rules:

- $(\texttt{success}, \texttt{AddClass}(\underline{\langle \downarrow^* \rangle \texttt{success}}))$
- $(\langle \downarrow \rangle \underline{\langle \downarrow^* \rangle \texttt{success}}, \texttt{AddClass}(\underline{\langle \downarrow^* \rangle \texttt{success}}))$

Finally, we define S' to be $\left\{\underline{\langle \downarrow^* \rangle \texttt{success}}\right\}$.

5.4 Redundancy → class-adding

We will show that redundancy for $\textsc{Trs}'_0$ can be solved in polynomial time assuming an oracle to the "class adding problem" for $\textsc{Trs}'_0$. The class-adding problem is a reachability problem for $\textsc{Trs}'_0$ involving only single-node input trees possibly with a parent node that only provides a "context" (i.e. cannot be modified). As we will see in the following subsection, the class-adding problem for $\textsc{Trs}'_0$ lends itself to a fast reduction to the bit-toggling problem for sPDS. Similarly, k-redundancy can be solved via the same routine, where intermediate problems are restricted to their bounded height equivalents.

High-level idea. We first provide the high-level idea the reduction. After simplifying the rewrite system in the previous step, we only need to check redundancy at the root node of the tree. Our approach is a "saturation" algorithm that exploits the monotonicity property: we begin with an initial tree and then repeatedly apply a "saturation step" to build the tree where each node is labelled by all classes that may label the node during any execution. The saturation step examines the tree built so far and the rules of the rewrite system. If it finds that it is possible to apply a sequence of rewrite rules to the tree to add a class c to some node v, it updates the tree by adding c to the label of v. In this way, larger and larger trees are built. Once it is no longer possible to add any new classes to the existing nodes of the tree, the algorithm terminates. In particular, we have all classes that could label the root node, and thus we can detect which classes are redundant by inspecting the labelling of the root.

Given a rewrite system $\mathcal{R} \in \textsc{Trs}_0$, an initial tree $T = (D, \lambda) \in \textsc{Tree}(\Sigma)$ with $\Sigma = 2^{\mathcal{K}}$, and a set S of guards, we try to "saturate" each node $v \in D$ with the classes that may be added to v. Our saturation step is able to reason about the addition of nodes when determining if a class c can be added to v, but does not need to remember which new nodes needed to be added to T to add c to v. This is due to monotonicity: since each saturation step begins with a larger tree than the previous step, additional nodes can be regenerated on-the-fly if needed.

Each saturation step proceeds as follows. Let $T_1 = (D, \lambda_1)$ be the tree before the saturation step, and $T_2 = (D, \lambda_2)$ be the tree after applying then saturation step. The tree domain does not change and there exists a node $v \in D$ such that $\lambda_1(v) \subset \lambda_2(v)$ (i.e. some classes are added to $\lambda_1(v)$). In particular, we have $T_1 \prec T_2$.

There are two saturation rules that are repeatedly applied until we reach a fixpoint.

- The first saturation rule corresponds to the application of a rewrite rule $(g, \texttt{AddClass}(X)) \in \mathcal{R}$ at v. We simply set $\lambda_2(v) = \lambda_1(v) \cup X$. Note, in this case we do not need to reason about the addition of nodes to the tree.

- The second saturation rule is a call to the class-adding subroutine and asks whether some class c can be added to node v. This step incorporates the behaviour of rewrite rules of the form $(g, \texttt{AddChild}(X))$. In this case we need to reason about whether the node added by these rules could lead to the addition of c to v. To do this, we construct a pushdown system $\mathcal{P}$ that explores the possible impact of these new nodes. This is discussed in the next section. If it is found that c could be added to v, we set $\lambda_2(v) = \lambda_1(v) \cup \{c\}$.

In sum, our "reduction" is in fact an algorithm for the (k-) redundancy problem for $\textsc{Trs}_0$ that runs in polynomial-time with an oracle to the bit-toggling (resp. for bounded stack height) for sPDS.

The formal reduction. Before formally defining the class-adding problem, we first need the definition of an "assumption function", which plays the role of the possible parent context node

but is treated as a *separate* entity from the input tree. As we shall see, this leads to a more natural formulation of the computational problem. More precisely, an *assumption function* f over the alphabet $\Sigma = 2^{\mathcal{K}}$ is a function mapping each element of $\{\mathtt{root}\} \cup \mathcal{K}$ to $\{0,1\}$. The boolean value of $f(\mathtt{root})$ is used to indicate whether the input single-node tree is a root node[7]. Given a tree $T = (D, \lambda) \in \textsc{Tree}(\Sigma)$, a node $v \in D$, and a simple guard g over Σ, we write $v, T \models_f g$ if one of the following three cases holds: (i) $v \neq \epsilon$ and $v, T \models g$, (ii) $v = \epsilon$, g is not of the form $\langle\uparrow\rangle X$, and $v, T \models g$, and (iii) $v = \epsilon$, $g = \langle\uparrow\rangle X$, $f(\mathtt{root}) = 0$, and $X \subseteq \{c \in \mathcal{K} : f(c) = 1\}$. In other words, $v, T \models_f g$ checks whether g is satisfied at node v assuming the assumption function f (in particular, if $f(\mathtt{root}) = 0$, then any guard referring to the parent of the root node of T is checked against f).

Given a simple rewrite system $\mathcal{R} \in \textsc{Trs}'_0$ over Σ and an assumption function f, we may define the rewriting relation $\to_{\mathcal{R},f} \subseteq \textsc{Tree}(\Sigma) \times \textsc{Tree}(\Sigma)$ in the same way as we define $\to_{\mathcal{R}}$, except that $\models_f$ is used to check guard satisfaction. The *class-adding (reachability) problem* is defined as follows: given a single-node tree $T_0 = (\{\epsilon\}, \lambda_0) \in \textsc{Tree}(\Sigma)$ with $\Sigma = 2^{\mathcal{K}}$, an assumption function $f : (\{\mathtt{root}\} \cup \mathcal{K}) \to \{0,1\}$, a class $c \in \mathcal{K}$, and a simple rewrite system $\mathcal{R}$, decide if there exists a tree $T = (D, \lambda)$ such that $T_0 \to^*_{\mathcal{R},f} T$ and $c \in \lambda(\epsilon)$. Similarly, the *k-class-adding* problem is defined in the same way as the class-adding problem except that the reachable trees are restricted to height k (k is part of the input).

Lemma 5. *The redundancy (resp. k-redundancy) problem for simple rewrite systems is* P*-time solvable assuming oracle calls to the class-adding (resp. k-class-adding) problem.*

Given a tree is $T_0 = (D_0, \lambda_0) \in \textsc{Tree}(\Sigma)$ with $\Sigma = 2^{\mathcal{K}}$, a simple rewrite system $\mathcal{R}$ over Σ, and a set $S \subseteq \mathcal{K}$, the task is to decide whether S is redundant (or k-redundant). We shall give the algorithm for the redundancy problem; the k-redundancy problem can be obtained by simply replacing oracle calls to the class-adding problem by the k-class-adding problem.

The algorithm is a fixpoint computation. Let $T = (D, \lambda)$
$:= T_0$. At each step, we can apply any of the following "saturation rules":

- if $(g, \mathtt{AddClass}(B))$ is applicable at a node $v \in D_0$ in T, then $\lambda(v) := \lambda(v) \cup B$.
- If the class-adding problem has a positive answer on input $\langle T_v, f, c, \mathcal{R}\rangle$, then $\lambda(v) := \lambda(v) \cup \{c\}$, where $v \in D$, $c \in \mathcal{K} \setminus \lambda(v)$, and $T_v := (\epsilon, \lambda_v)$ with $\lambda_v(\epsilon) = \lambda(v)$, where we define $f : (\{\mathtt{root}\} \cup \mathcal{K}) \to \{0,1\}$ with $f(\mathtt{root}) = 1 \Leftrightarrow v = \epsilon$ and, if $f(\mathtt{root}) = 0$ and u is the parent of v, then $f(a) = 1 \Leftrightarrow a \in \lambda(u)$.

Observe that saturation rules can be applied at most $\mathcal{K} \times |D|$ times. Therefore, when they can be applied no further, we check whether $S \cap \lambda(\epsilon) \neq \emptyset$ and terminate. Assuming constant-time oracle calls to the class-adding problem, the algorithm easily runs in polynomial time. Furthermore, since each saturation rule only adds new classes to a node label, the correctness of the algorithm can be easily proven using Lemma 1 and Lemma 2; see the appendix.

Running Example The application of the saturation algorithm to our running example is quite simple. Since the only node in the initial tree is the root node, we can solve the redundancy problem with a single call to the class-adding problem. That is, is it possible to add the class $\underline{\langle\downarrow^*\rangle\mathtt{success}}$ to the root node?

[7] Note that the guard $\langle\uparrow\rangle\top$ evaluates to false on the root node

5.5 Class-adding → bit-toggling

We now show how to solve the class-adding problem for a given node v and class c. Let $\lambda(v)$ be the labelling of node v. We reduce the class-adding problem to the bit-toggling problem as follows. The pushdown system performs a kind of "depth-first search" of the trees that could be built from the new node and the rewrite rules. It starts with an initial stack of height 1 containing the node being inspected. More precisely, the single item on this stack is the set $\lambda(v)$ (possibly with some extra "context" information). It then "simulates" each possible branch that is spawned from v by pushing items onto the stack when a new node is created (i.e. at each given moment, the stack contains a single branch in a reachable configuration). It pops these nodes from the stack when it wishes to backtrack and search other potential branches of the tree.

The pushdown system is an sPDS that keeps one boolean variable for each class $c \in \mathcal{K}$. If $\mathcal{P}$ reaches a single item stack (i.e. corresponding to the node v) where the item contains c then the answer to the class adding problem is positive. That is, the sPDS has explored the application of the rewrite rules to the possible children of v and determined that the label of v can be expanded to include c.

The reason why it suffices to only keep track of the labelling of v (instead of the entire subtree rooted at v that $\mathcal{P}$ explored) is monotonicity of $\textsc{Trs}_0$ (cf. Lemma 2), i.e., that the labelling of v contains sufficient information to "regrow" the destroyed subtree.

Lemma 6. *The class-adding (resp. k-class-adding) problem for $\textsc{Trs}'_0$ is polynomial-time reducible to the bit-toggling (resp. bounded bit-toggling) problem for sPDS.*

We prove the lemma above. Fix a simple rewrite system $\mathcal{R}$ over the node labeling $\Sigma = 2^{\mathcal{K}}$, an assumption function $f : (\{\texttt{root}\} \cup \mathcal{K}) \to \{0, 1\}$, a single node tree $T_0 = (\{\epsilon\}, \lambda_0) \in \textsc{Tree}(\Sigma)$, and a class $\alpha \in \mathcal{K}$.

We construct an sPDS $\mathcal{P} = (\mathcal{V}, \mathcal{W}, \Delta)$. Intuitively, the sPDS $\mathcal{P}$ will simulate $\mathcal{R}$ by exploring all branches in all trees reachable from T_0 while accumulating the classes that are satisfied at the root node. Define $\mathcal{V} := \{x_c : c \in \mathcal{K}\} \cup \{\texttt{pop}\}$, and $\mathcal{W} := \{y_c, z_c : c \in \mathcal{K}\} \cup \{\texttt{root}\}$. Roughly speaking, we will use the variable y_c (resp. z_c) to remember whether the class c is satisfied at the current (resp. parent of the current) node being explored. The variable x_c is needed to remember whether the class c is satisfied at a child of the current node (i.e. after a pop operation). The variable `root` signifies whether the current node is a root node, while the variable `pop` indicates whether the last operation that changed the stack height is a pop. We next define Δ:

- For each $(A, \texttt{AddClass}(B)) \in \mathcal{R}$, add the rule $(1, \varphi)$, where φ asserts
  (a) $\bigwedge_{a \in A} y_a$ (A satisfied), and
  (b) $\bigwedge_{b \in B} y_b^1$ (B set to true), and
  (c) all other variables remain unchanged, that is we assert $\texttt{pop} \leftrightarrow \texttt{pop}'$, $\texttt{root} \leftrightarrow \texttt{root}^1$, $\bigwedge_{c \in \mathcal{K} \setminus B}(y_c \leftrightarrow y_c^1)$, $\bigwedge_{c \in \mathcal{K}}(z_c \leftrightarrow z_c^1)$, and $\bigwedge_{c \in \mathcal{K}}(x_c \leftrightarrow x'_c)$.
- For each $(\langle\uparrow\rangle A, \texttt{AddClass}(B)) \in \mathcal{R}$, add the rule $(1, \varphi)$ where φ asserts
  (a) $\bigwedge_{a \in A} z_a$ (A satisfied in the parent), and
  (b) $\bigwedge_{b \in B} y_b^1$ (B set to true), and
  (c) all other variables remain unchanged, that is $\texttt{pop} \leftrightarrow \texttt{pop}'$, $\texttt{root} \leftrightarrow \texttt{root}^1$, $\bigwedge_{c \in \mathcal{K}}(x_c \leftrightarrow x'_c)$, $\bigwedge_{c \in \mathcal{K}}(z_c \leftrightarrow z_c^1)$, and $\bigwedge_{c \in \mathcal{K} \setminus B}(y_c \leftrightarrow y_c^1)$.

- For each $(\langle\downarrow\rangle A, \texttt{AddClass}(B)) \in \mathcal{R}$, add the rule $(1, \varphi)$ where φ asserts
  (a) $\texttt{pop}$ (we popped from a child), and
  (b) $\bigwedge_{a\in A} x_a$ (A satisfied in child), and
  (c) $\bigwedge_{b\in B} y_b^1$ (B set to true), and
  (d) all other variables remain unchanged, that is $\texttt{pop} \leftrightarrow \texttt{pop}'$, $\texttt{root} \leftrightarrow \texttt{root}^1$, $\bigwedge_{c\in\mathcal{K}}(x_c \leftrightarrow x'_c)$, $\bigwedge_{c\in\mathcal{K}}(z_c \leftrightarrow z_c^1)$, and $\bigwedge_{c\in\mathcal{K}\setminus B}(y_c \leftrightarrow y_c^1)$.
- For each $(A, \texttt{AddChild}(B)) \in \mathcal{R}$, add the rule $(2, \varphi)$, where φ asserts
  (a) $\bigwedge_{a\in A} y_a$ (A satisfied), and
  (b) $\bigwedge_{b\in B} y_b^2$ (B true in new child), and
  (c) $\bigwedge_{c\in\mathcal{K}\setminus B} \neg y_c^2$ (new child has no other classes), and
  (d) $\bigwedge_{c\in\mathcal{K}}(y_c \leftrightarrow z_c^2)$ (new child's parent classes), and
  (e) $\neg\texttt{root}^2$ (new child is not root), and
  (f) $\neg\texttt{pop}' \wedge \bigwedge_{c\in\mathcal{K}} \neg x'_c$ (new child does not have a child), and
  (g) the current node is unchanged, that is $\bigwedge_{c\in\mathcal{K}}(y_c \leftrightarrow y_c^1)$, $\bigwedge_{c\in\mathcal{K}}(z_c \leftrightarrow z_c^1)$, and $(\texttt{root} \leftrightarrow \texttt{root}')$
- Finally, add the rule $(0, \varphi)$, where φ asserts
  (a) $\neg\texttt{root}$ (can return to parent), and
  (b) $\texttt{pop}'$ (flag the return), and
  (c) $\bigwedge_{c\in\mathcal{K}}(y_c \leftrightarrow x'_c)$ (return classes to parent).

These boolean formulas can easily be represented as BDDs of linear size (see appendix).

Continuing with our translation, the bit that needs to be toggled on is y_α. We now construct the initial configuration for our bit-toggling problem. For each subset $X \subseteq \mathcal{K}$ and a function $q : \mathcal{V} \to \{0, 1\}$, define the function $I_{X,f,q} : (\mathcal{V} \cup \mathcal{W}) \to \{0, 1\}$ as follows: $I_{X,f,q}(\texttt{root}) := f(\texttt{root})$, $I_{X,f,q}(\texttt{pop}) := q(\texttt{pop})$, and for each $c \in \mathcal{K}$: (i) $I_{X,f,q}(x_c) := q(x_c)$, (ii) $I_{X,f,q}(y_c) = 1$ iff $c \in X$, and (iii) $I_{X,f,q}(z_c) := f(c)$. We shall write $I_{X,f}$ to mean $I_{X,f,q}$ with $q(x) = 0$ for each $x \in \mathcal{V}$. Define the initial configuration I_0 as the function $I_{\lambda_0(\epsilon),f}$.

Let us now analyse our translation. The translation is easily seen to run in polynomial time. In fact, with a more careful analysis, one can show that the output sPDS is of linear size and that the translation can be implemented in polynomial time. Correctness of our translation immediately follows from the following technical lemma:

Lemma 7. *For each subset $X \subseteq \mathcal{K}$, the following are equivalent:*

(A1) *There exists a tree $T = (D, \lambda) \in \textsc{Tree}(\Sigma)$ such that $T_0 \to^*_{\mathcal{R},f} T$ and $\lambda(\epsilon) = X$.*

(A2) *There exists $q' : \mathcal{V} \to \{0, 1\}$ such that $I_{\lambda_0(\epsilon),f} \to^*_{\mathcal{P}} I_{X,f,q'}$.*

This lemma intuitively states that the constructed sPDS performs a "faithful simulation" of $\mathcal{R}$. Moreover, the direction (A2)⇒(A1) gives *soundness* of our reduction, while (A1)⇒(A2) gives *completeness* of our reduction. The proof is very technical, which we relegate to the appendix.

Running Example We show how the sPDS constructed can determine that the required class $\underline{\langle\downarrow^*\rangle\texttt{success}}$ can be added to the root node of the tree, thus implying that there are no redundant selectors.

We write configurations of a symbolic pushdown system using the notation $(X, Y_1 \dots Y_m)$ where X is the set of classes c such that the variable x_c is true, and $Y_1 \dots Y_m$ is a stack of sets of classes (with the top on the right) such that each Y_i is a set of classes c such that y_c is true at stack position i. Note, we do not show the z_c variables' values since they are only needed to evaluate $\langle\uparrow\rangle$ modalities, which do not appear in our example.

First we apply the rule $(\texttt{root}, \texttt{AddChild}(\texttt{team}))$ and then $(\texttt{team}, \texttt{AddChild}(\texttt{P1}))$. Each step pushes a new item onto the stack. By executing these steps the sPDS is exploring a branch from `root` to `P1`.

$$\begin{array}{c}(\emptyset, \{\texttt{root}\}) \rightarrow \\ (\emptyset, \{\texttt{root}\}\,\{\texttt{team}\}) \rightarrow \\ (\emptyset, \{\texttt{root}\}\,\{\texttt{team}\}\,\{\texttt{P1}\})\end{array}$$

At this point there's nothing more we can do with the `P1` node. Hence, the sPDS backtracks, remembering in its control state the classes contained in the child it has returned from. Once this information is remembered in its control state, it knows that a child is labelled `P1` and hence $(\langle\downarrow\rangle\texttt{P1}, \texttt{AddClass}(\underline{\langle\downarrow\rangle\texttt{P1}}))$ can be applied.

$$\begin{array}{c}(\emptyset, \{\texttt{root}\}\,\{\texttt{team}\}\,\{\texttt{P1}\}) \rightarrow \\ (\{\texttt{P1}\}, \{\texttt{root}\}\,\{\texttt{team}\}) \rightarrow \\ \left(\{\texttt{P1}\}, \{\texttt{root}\}\left\{\texttt{team}, \underline{\langle\downarrow\rangle\texttt{P1}}\right\}\right)\end{array}$$

Now the sPDS can repeat the analogous sequence of actions to obtain that $\underline{\langle\downarrow\rangle\texttt{P2}}$ can also label the `team` node.

$$\begin{array}{c}\left(\{\texttt{P1}\}, \{\texttt{root}\}\left\{\texttt{team}, \underline{\langle\downarrow\rangle\texttt{P1}}\right\}\right) \rightarrow \\ \left(\emptyset, \{\texttt{root}\}\left\{\texttt{team}, \underline{\langle\downarrow\rangle\texttt{P1}}\right\}\{\texttt{P2}\}\right) \rightarrow \\ \left(\{\texttt{P2}\}, \{\texttt{root}\}\left\{\texttt{team}, \underline{\langle\downarrow\rangle\texttt{P1}}\right\}\right) \rightarrow \\ \left(\{\texttt{P2}\}, \{\texttt{root}\}\left\{\texttt{team}, \underline{\langle\downarrow\rangle\texttt{P1}}, \underline{\langle\downarrow\rangle\texttt{P2}}\right\}\right)\end{array}$$

Now we are able to deduce that `success` can also label the `team` node.

$$\begin{array}{c}\left(\{\texttt{P2}\}, \{\texttt{root}\}\left\{\texttt{team}, \underline{\langle\downarrow\rangle\texttt{P1}}, \underline{\langle\downarrow\rangle\texttt{P2}}\right\}\right) \rightarrow \\ \left(\{\texttt{P2}\}, \{\texttt{root}\}\left\{\dots, \underline{\langle\downarrow\rangle\texttt{P1} \wedge \langle\downarrow\rangle\texttt{P2}}\right\}\right) \rightarrow \\ (\emptyset, \{\texttt{root}\}\,\{\dots\}\,\{\texttt{success}\})\end{array}$$

We are then able to backtrack to the root node, accumulating the information that $\langle\downarrow^*\rangle\texttt{success}$ holds at the root node, and thus $(\underline{\langle\downarrow^*\rangle\texttt{success}})$ is not redundant.

$$\begin{array}{c}(\emptyset, \{\texttt{root}\}\,\{\dots\}\,\{\texttt{success}\}) \rightarrow \\ (\emptyset, \{\texttt{root}\}\,\{\dots\}\,\{\dots, \langle\downarrow^*\rangle\texttt{success}\}) \rightarrow \\ (\{\dots, \langle\downarrow^*\rangle\texttt{success}\}, \{\texttt{root}\}\,\{\dots\}) \rightarrow \\ (\{\dots, \langle\downarrow^*\rangle\texttt{success}\}, \{\texttt{root}\}\,\{\dots, \langle\downarrow^*\rangle\texttt{success}\}) \rightarrow \\ (\{\dots, \langle\downarrow^*\rangle\texttt{success}\}, \{\texttt{root}\}) \rightarrow \\ \left(\{\dots\}, \left\{\dots, \underline{\langle\downarrow^*\rangle\texttt{success}}\right\}\right)\end{array}$$

6 Experiments

We have implemented our approach in a new tool TreePed which is available for download [56]. We tested it on several case studies. Our implementation contains two main components: a proof-of-concept translation from HTML5 applications using jQuery to our model, and a redundancy checker (with non-redundancy witness generation) for our model. Both tools were developed in Java. The redundancy checker uses jMoped [55] to analyse symbolic pushdown systems. In the following sections we discuss the redundancy checker, translation from jQuery, and the results of our case studies.

6.1 The Redundancy Checker

The main component of our tool implements the redundancy checking algorithm for proving Theorem 3. Largely, the algorithm is implemented directly. The most interesting differences are in the use of jMoped to perform the analysis of sPDSs to answer class-adding checks. In the following, assume a tree T_v, rewrite rules $\mathcal{R}$, and a set $\mathcal{K}$ of classes.

Optimising the sPDS For each class $c \in \mathcal{K}$ we construct an sPDS. We can optimise by restricting the set of rules in $\mathcal{R}$ used to build the sPDS. In particular, we can safely ignore all rules in $\mathcal{R}$ that cannot appear in a sequence of rules leading to the addition of c. To do this, we begin with the set of all rules that either directly add the class c or add a child to the tree (since these may lead to new nodes matching other rules). We then add all rules that directly add a class c' that appears in the guard of any rules included so far. This is iterated until a fixed point is reached. The rules in the fixed point are the rules used to build the sPDS.

Reducing the number of calls. We re-implemented the global backwards reachability analysis (and witness generation) of Moped [52] in jMoped. This means that a single call to jMoped can allow us to obtain a BDD representation of all initial configurations of the sPDSs obtained from class-adding problems $\langle T_v, f, c, \mathcal{R}\rangle$ that have a positive answer to the class-adding problem for a given class $c \in \mathcal{K}$. Thus, we only call jMoped once per class.

6.2 Translation from HTML5

The second component of our tool provides a proof-of-concept prototypical translation from HTML5 using jQuery to our model. We provide a detailed description in the Appendix and provide a small example below. There are three main parts to the translation.

- The DOM tree of the HTML document is directly translated to a tree in our model. We use classes to encode element types (e.g. `div` or `a`), IDs and CSS classes.
- We support a subset of CSS covering the most common selectors and all selectors in our case studies. Each selector in the CSS stylesheet is translated to a guard to be analysed for redundancy. For pseudo-selectors such as `g:hover` and `g:before` we simply check for the redundancy of `g`.
- Dynamic rules are extracted from the JavaScript in the document by identifying jQuery calls and generating rules as outlined in Section 3.3. Developing a translation tool that covers all covers all aspects of such an extremely rich and complex language as JavaScript is a difficult problem [9]. Our proof-of-concept prototype covers many, but by no means

all, interesting features of the language. The implemented translation is described in more detail in the appendix.

Sites formed of multiple pages with common CSS files are supported by automatically collating the results of independent page analyses and reporting site-wide redundancies.

To give a flavour of the translation of a single line we recall one of the rules from the example in Figure 1, except we adjust it to a call `addClass()` instead of `remove()` (since calls to `remove()` are ignored by our abstraction).

```
$('.input_wrap').find('.delete')
                .parent('div')
                .addClass('deleted');
```

We can translate this rule inductively. From the initial jQuery call `$('.input_wrap')` we obtain a guard that is simply `.input_wrap` that matches any node with the class `input_wrap`. We can then extend this guard to handle the `find()` call. This looks for any child of the currently matched node that has the class `delete`. Thus, we build up the guard to

$$\texttt{.delete} \wedge \langle\uparrow^+\rangle \texttt{.input\_wrap}$$

Next, because of the call to `.parent()` we have to extend the guard further to match the parent of the currently selected node, and enforce that the parent node is a `div`.

$$\texttt{div} \wedge \langle\downarrow\rangle(\texttt{.delete} \wedge \langle\uparrow^+\rangle \texttt{.input\_wrap})$$

Finally, we encounter the call to `addClass()` which we translate to an `AddClass`({`deleted`}) rule.

$$(\texttt{div} \wedge \langle\downarrow\rangle(\texttt{.delete} \wedge \langle\uparrow^+\rangle \texttt{.input\_wrap}), \texttt{AddClass}(\{\texttt{deleted}\}))$$

6.3 Case Studies

We performed several case studies. One is based on the Igloo example from the benchmark suite of the dynamic CSS analyser Cilla [45], and is described in detail below. Another (and the largest) is based on the Nivo Slider plugin [48] for animating transitions between a series of images. The remaining examples are hand built and use jQuery to make frequent additions to and removals from the DOM tree. The first `bikes.html` allows a user to select different frames, wheels and groupsets to build a custom bike, `comments.html` displays a comments section that is loaded dynamically via an AJAX call, and `transactions.html` is a finance page where previous transactions are loaded via AJAX and new transactions may be added and removed via a form. The example in Figure 1 is `example.html` and `example-up.html` is the version without the limit on the number of input boxes. These examples are available in the `src/examples/html` directory of the tool distribution [56].

All case studies contained non-trivial CSS selectors whose redundancy depended on the dynamic behaviour of the system. In each case our tool constructed a rewrite system following the process outlined above, and identified all redundant rules correctly. Below we provide the answers provided by UnCSS [57] and Cilla [15] when they are available[8].

The experiments were run on a Dell Latitude e6320 laptop with 4Gb of RAM and four 2.7GHz Intel i7-2620M cores. We used OpenJDK 7, using the argument "-Xmx" to limit RAM

[8]We did not manage to successfully set up Cilla [45] and so only provide the output for the Igloo example that has been provided by the authors in [15].

Case Study	Ns	Ss	Ls	Rs	Time
`bikes.html`	22	18 (0)	97	37	3.6s
`comments.html`	5	13 (1)	43	26	2.9s
`example.html`	11	1 (0)	28	4	.6s
`example-up.html`	8	1 (1)	15	3	.6s
`igloo/`		261 (89)			3.4s
`index.html`	145		24	1	
`engineering.html`	236		24	1	
`Nivo-Slider/`					
`demo.html`	15	172 (131)	501	21	6.3s
`transactions.html`	19	9 (0)	37	6	1.6s

Table 1: Case study results.

usage to 2.5Gb. The results are shown in Table 1. *Ns* is the initial number of elements in the DOM tree, *Ss* is the number of CSS selectors (with the number of redundant selectors shown in brackets), *Ls* is the number of Javascript lines reported by `cloc`, and *Rs* is the number of rules in the rewrite system obtained from the JavaScript[9] after simplification (unsimplified rules may have arbitrarily complex guards). The figures for the Igloo example are reported per file or for the full analysis as appropriate.

We remark that the translation from JavaScript and jQuery is the main limitation of the tool in its application to industrial websites, since we do not support JavaScript and jQuery in its full generality. However, we also note that the number of rules required to model websites is often much smaller than the size of the code. For example, the Nivo-Slider example contains 501 lines of JavaScript, but its abstraction only requires 21 rules in our model. It is the number of these rules and the number of CSS selectors (and the number of classes appearing in the guards) that will have the main effect on the scalability of the tool.

Although we report the number of nodes in the webpages of our examples, checking CSS matching against these pre-existing nodes is no harder than standard CSS matching. The dynamic addition of nodes to the webpage is where symbolic pushdown analysis is required, hence the number of rewrite rules gives a better indication of the difficulty of an analysis instance.

Example from Figure 1 TreePed suggests that the selector `.warn` might be reachable and, therefore, is not deleted. We have run UnCSS on this example, which incorrectly identifies `.warn` as unreachable.

The Igloo example The Igloo example is a mock company website with a home page (`index.html`) and an engineering services page (`engineering.html`). There are a total of 261 CSS selectors, out of which 89 of them are actually redundant (we have verified this by hand). UnCSS reports 108 of them are redundant rules.

We mention one interesting selector which both Cilla and UnCSS have identified as redundant, but is actually reachable. The Igloo main page includes a search bar that contains some placeholder text which is present only when the search bar is empty and does not have focus. Placeholder text is supported by most modern browsers, but not all (e.g. IE9), and the page

[9]Not including the number of rules required to represent the CSS selectors.

```
(function($) {
    function supportsInputPlaceholder() {
      var el = document.createElement('input');
      return 'placeholder' in el;
    }
    $(function() {
      if (!supportsInputPlaceholder()) {
        var searchInput = $('#searchInput'),
          placeholder = searchInput.attr('placeholder');
        searchInput.val(placeholder).focus(function() {
           var $this = $(this);
           $this.addClass('touched');
             ...
        })
             ...
      });
    }
})(jQuery);
```

Figure 3: JavaScript code snippet from Igloo example

contains a small amount of JavaScript to simulate this functionality when it is not provided by the browser. The relevant code is shown in Figure 3. In particular, the CSS class `touched`, and rule

```
#search .touched { color: #333; }
```

are used for this purpose. Both Cilla and UnCSS incorrectly claim that the rule is redundant. Since we only identify genuinely redundant rules, our tool correctly does not report the rule as redundant.

In addition, TreePed identified a further unexpected mistake in Igloo's CSS. The rule

```
h2 a:hover, h2 a:active, h2 a:focus
h3 a:hover, h3 a:active, h2 a:focus { ... }
```

is missing a comma from the end of the first line. This results in the redundant selector "`h2 a:focus h3 a:hover`" rather than two separate selectors as was intended. Cilla did not report this redundancy as it appears to ignore all CSS rules with pseudo-selectors. Finally, we remark that the second line of the above rule contains a further error: "`h2 a:focus`" should in fact be "`h3 a:focus`". This raises the question of selector subsumption, on which there is already a lot of research work (e.g. see [14, 45]). These algorithms may be incorporated into our tool to obtain a further size reduction of CSS files.

The Nivo-Slider example Nivo-Slider [48] is an easy-to-use image slider JavaScript package that heavily employs jQuery. In particular, it provides some beautiful transition effects when displaying a gallery of images. In this case study, we use a demo file that displays a series of four images. The file contains a total of 172 CSS selectors, out of which 131 are actually redundant (we verified this by hand). UnCSS reports 152 redundant CSS selectors (i.e. about 50% of false positives).

```
...
<div id="slider" class="nivoSlider">
  ...
  <a href="http://dev7studios.com">
     <img src="images/up.jpg"
          data-thumb="images/up.jpg"
          alt=""
          title="This is an example of a caption"/>
  </a>
...
<div>
...
```

Figure 4: HTML code snippet for Nivo-Slider demo.

```
var slider = $('#slider');
slider.data('nivo:vars', vars)
    .addClass('nivoSlider');

// Find our slider children
var kids = slider.children();
kids.each(function() {
   var child = $(this);
   var link = '';
   if(!child.is('img')){
     if(child.is('a')){
       child.addClass('nivo-imageLink');
       link = child;
     }
     ...
   }
   ...
}
```

Figure 5: JavaScript code snippet for Nivo-Slider demo.

We mention the following interesting rule that UnCSS reports as redundant:

```
.nivoSlider a.nivo-imageLink {
    ...
    z-index:6;
    ...
}
```

Recall that the `z-index` of an HTML element specifies the vertical stack order; a greater stack order means that the element is *in front* of an element with a lower stack order. This CSS rule, among others, sets the `z-index` of an HTML element that matches the selector to a higher value. In effect, this allows a hyperlink that overlaps with the second image in the series to be clicked (which takes the user to a different web page). Removing this rule disables the hyperlink from the page. The code snippet in Figure 4 provides the relevant part of the HTML document.

In particular, the class `nivo-imageLink` will be added by the JavaScript bit of the page to the the depicted hyperlink element above, as is shown in the JavaScript code snippet in Figure 5. In contrast to UnCSS, TreePed correctly identifies that the above CSS rule may be used.

7 Conclusion and Future Work

At the moment our translation from HTML5 applications to our tree-rewriting model is prototypical and does not incorporate many features that such a rich and complex language as JavaScript has, especially in the presence of libraries. Therefore, an important research direction is to develop a more robust translation, perhaps by building on top of existing JavaScript static analysers like WALA [50, 54] and TAJS [9, 28, 27]. As Andreasen and Møller [9] describe, a static analysis of JavaScript in the presence of jQuery is presently a formidable task for existing static analysers for JavaScript. For this reason, we do not expect the task of building a more robust translation to our tree-rewriting model to be easy.

Our technique should not be seen as competing with dynamic analysis techniques for identifying redundant CSS rules (e.g. UnCSS and Cilla). In fact, they can be combined to obtain a more precise CSS redundancy checker. Our tool TreePed attempts to output the definitely redundant rules, while Uncss/Cilla attempts to output those that are definitely non-redundant. The complement of the union of these sets is the "dont know set", which can (for example) be checked manually by the developer. For future work, we would like to combine static and dynamic analysis to build a more precise and robust CSS redundancy checker.

Another direction is to find better ways of overapproximating redundancy problems for the undecidable class TRS of rewrite systems using our monotonic abstractions (other than replacing non-positive guards by $\top$). In particular, for a general guard, can we automatically construct a more precise positive guard that serves as an overapproximation? A more precise abstraction would be useful in identifying redundant CSS rules with negations in the node selectors, which can be found in real-world HTML5 applications (though not as common as those rules without negations).

We may also consider other ways of improving CSS performance using the results of our analysis. For example, a descendant selector

```
.a .b { ... }
```

is less efficient than a direct child selector

```
.a > .b { ... }
```

since the descendant selector must search through all descendants of a given node. However, it is very common for the former to be used in place of the latter. If we can identify that the guard $\mathtt{a} \wedge \langle\downarrow\rangle\langle\downarrow^+\rangle\mathtt{b}$ is never matched, while $\mathtt{a} \wedge \langle\downarrow\rangle\mathtt{b}$ is matched, then we can safely replace the descendant selector with the direct child selector.

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A Proof of Proposition 1

We prove the undecidability of the 1-redundancy problem when negation is allowed in the guards. The proof is by a reduction from the undecidable control-state reachability problem for

deterministic two counter machines (2-CM): given a 2-CM $\mathcal{M} = (\mathcal{Q}, \Delta, q_0, q_F)$ with counters X and Y, decide whether there exist a path from $(q_0, 0, 0)$ to some configuration in $\{q_F\} \times \mathbb{N} \times \mathbb{N}$. We will use X and Y to denote the two counters, and Z to range over $\{X, Y\}$. The two counter tests will be $=$ and $>$ and we use $\ominus$ to range over these tests. We may assume that

(i) counter tests and counter increments / decrements never take place in the same transition, i.e., rules are of form $(q, [Z \ominus 0]) \to q'$ or $q \to (q', \mathsf{inc}(Z))$ or $q \to (q', \mathsf{dec}(Z))$, and

(ii) there are no self-loops in the counter machine i.e. there is no transition of the form $(q, \cdot) \to q$.

The initial tree $T_0 = (D_0, \lambda_0)$ is defined to be $D_0 = \{\epsilon\}$ and $\lambda_0(\epsilon) = q_0$. We now construct a rewrite system $\mathcal{R} \in$ TRS. The state of the machine is always recorded in the root node. The values of the two counters X and Y will be encoded by the number of nodes at level 1 (i.e. children of the root node) that are associated with class X and Y, respectively. More precisely, for each transition rule σ of the machine, we introduce a class σ. In addition, we use the following four classes: X, Y, *new*, and *del*. The main problem that we have to overcome in our reduction is to simulate one transition of $\mathcal{M}$ by several simpler steps of T_0 owing to the simplicity of rewrite operations that we allow in TRS. This is in fact the reason for adding the last two classes *new* and *del*, which are just "flags" to indicate whether a node (at level 1) is "new" or is "scheduled to be deleted". Our rewrite system $\mathcal{R}$ operates over $\Sigma := 2^{\mathcal{K}}$ with $\mathcal{K} = \mathcal{Q} \cup \Delta \cup \{X, Y, new, del\}$. Next we introduce the guards: (a) $g(Z = 0) := \neg\langle\downarrow\rangle Z$, (b) $g(Z > 0) := \langle\downarrow\rangle Z$, (c) $\theta_1 := \neg\langle\downarrow\rangle new \wedge \neg\langle\downarrow\rangle del$, and (d) $\theta_2 := \bigwedge_{\gamma \in \Delta} \neg\gamma$. As we shall see, the guard $\theta_1 \wedge \theta_2$ means that there is no "pending job". The rewrite rules for $\mathcal{R}$ are as follows:

- If $\sigma = (q, [Z \ominus 0]) \to q'$, then add the following rules:
  (1) $(q \wedge g(Z \ominus 0) \wedge \theta_1 \wedge \theta_2, \texttt{AddClass}(\{\sigma, q'\}))$
  (2) $(\sigma, \texttt{RemoveClass}(\{q, \sigma\}))$
- If $\sigma = q \to (q', \mathsf{inc}(Z))$, then add:
  (1) $(q \wedge \theta_1 \wedge \theta_2, \texttt{AddClass}(\{\sigma, q'\})$
  (2) $(\sigma \wedge \theta_1, \texttt{AddChild}(\{Z, new\}))$
  (3) $(\sigma \wedge \langle\downarrow\rangle(Z \wedge new), \texttt{RemoveClass}(\{\sigma, q\}))$.
  (4) $(new \wedge \langle\uparrow\rangle\theta_2, \texttt{RemoveClass}(\{new\}))$
- If $\sigma = q \to (q', \mathsf{dec}(Z))$, then add:
  (1) $(q \wedge \theta_1 \wedge \theta_2, \texttt{AddClass}(\{\sigma, q'\}))$
  (2) $(Z \wedge \langle\uparrow\rangle(\sigma \wedge \neg\langle\downarrow\rangle del), \texttt{AddClass}(\{del\}))$
  (3) $(\sigma \wedge \langle\downarrow\rangle(Z \wedge del), \texttt{RemoveClass}(\sigma, q))$
  (4) $(del \wedge \langle\uparrow\rangle\theta_2, \texttt{RemoveClass}(\{Z, del\}))$

Observe that the trees reachable from the initial tree is of height at most 1. Possible labels of the root are only $q \in \mathcal{Q}$ and $\sigma \in \Delta$. Possible labels of a leaf node (necessarily a child of the root) are X, Y, new and *del*.

It is not difficult to see that $\mathcal{R}$ when initialised in T_0 performs a faithful simulation of the counter machine $\mathcal{M}$ from the configuration $(q, 0, 0)$. Our rewrite system $\mathcal{R}$ is defined in such a way that each group of rewrite rules introduced in $\mathcal{R}$ to simulate a transition σ of $\mathcal{M}$ has to be executed before the next transition of $\mathcal{M}$ can be simulated by $\mathcal{R}$. In particular, to simulate

a $\mathcal{M}$ transition σ from a given state $q \in \mathcal{Q}$, our rewrite system $\mathcal{R}$ needs to add σ to the root node, which can only be done if the following properties are satisfied: (1) the label of the root node is $\{q\}$ for some $q \in \mathcal{Q}$, and (2) no child of the root has *del* or *new* in the label. This is owing to the conjuncts $\theta_1 \wedge \theta_2$ in the guard. In particular, this means that the root contains *at most* one σ. In addition, if σ is a class of the form $q \to (q', \mathsf{inc}(Z))$ or $q \to (q', \mathsf{dec}(Z))$, then σ can only be removed only after proper simulation of counter increments/decrements has been performed.

In summary, we have that $\mathcal{M}$ reaches the control state q_F iff q_F is not redundant in $\mathcal{R}$. This concludes our reduction.

B Proof of Lemma 1

We prove that if $T \preceq T'$, then T' satisfies at least the same guards as T at each node v. The proof is by induction on g. For the base cases, $g = \top$ is trivial, and that $v, T \models X$ implies $f(v), T' \models X$ holds since, by **(H2)**, we have $X \subseteq \lambda(v) \subseteq \lambda'(f(v))$. We now proceed to the inductive case. The cases of conjunction and disjunction are obvious.

Assume that $v, T \models \langle d \rangle g$ for some $d \in \{\uparrow, \downarrow\}$. We will only show the case when $d = \downarrow$; the other case can be handled in the same way. This means that $v.i, T \models g$ for some $i \in \mathbb{N}$. By induction, we have $f(v.i), T' \models g$. By **(H3)**, we have that $f(v.i) = f(v).j$ for some $j \in \mathbb{N}$. Altogether, this implies that $f(v), T' \models \langle\downarrow\rangle g$ as desired.

Assume that $v, T \models \langle d^* \rangle g$ for some $d \in \{\uparrow^*, \downarrow^*\}$. We will only show the case when $d = \downarrow^*$; the other case can be handled in the same way. The proof is by induction on the length n of path from v to a descendant vw (with $w \neq \epsilon$) satisfying $vw, T \models g$. If $n = 0$, then by the previous paragraph we have $f(v), T' \models g$ and so $f(v), T' \models \langle\downarrow^*\rangle g$ as desired. If $n > 0$ and $w = iu'$ for some $i \in \mathbb{N}$ and $u' \in \mathbb{N}^*$, then $vi, T \models \langle\downarrow^*\rangle g$ with a witnessing path of length $n - 1$. Therefore, by induction, we have $f(vi), T' \models \langle\downarrow^*\rangle g$. Since $f(vi) = f(v).j$ for some $j \in \mathbb{N}$, it follows that $f(v), T' \models \langle\downarrow\rangle\langle\downarrow^*\rangle g$ and so $f(v), T' \models \langle\downarrow^*\rangle g$ as desired.

C Proof of Lemma 2

We prove that, whenever $T_1 \preceq T_2$ and $T_1 \to_\sigma T_1'$, then either $T_1' \preceq T_2$ or $T_2 \to_\sigma T_2'$ with $T_1' \preceq T_2'$.

Suppose $T_1 \to_\sigma T_2$ where $T_1 = (D_1, \lambda_1)$, $T_1' = (D_1', \lambda_1')$ and $\sigma = (g, \chi)$; and $T_1 \preceq T_2 = (D_2, \lambda_2)$ is witnessed by an embedding $f : D_1 \to D_2$. Assume $v, T_1 \vDash g$ for some $v \in D_1$. Then $f(v), T_2 \vDash g$ by Lemma 1. We consider each of the four cases of χ in turn.

Case: $\chi = \texttt{AddClass}(X)$. Then set $T_2' = (D_2', \lambda_2')$ with $D_2' := D_2$ and $\lambda_2' := \lambda_2[f(v) \mapsto \lambda_2(f(v)) \cup X)]$. Hence $T_2 \to_\sigma T_2'$. Moreover we have $T_1' \preceq T_2'$, as witnessed by $f : D_1' \to D_2'$ (note that $D_1' = D_1$ and $D_2' = D_2$).

Case: $\chi = \texttt{AddChild}(X)$. Assume $D_1' = D_1 \cup \{v.i\}$ and $\lambda_1' = \lambda_1[v.i \mapsto X]$ where i is the number of children of v in T_1. Let i' be the number of children of $f(v)$ in T_2. Then set $T_2' = (D_2', \lambda_2')$ with $D_2' := D_2 \cup \{f(v).i'\}$ and $\lambda_2' := \lambda_2[f(v).i' \mapsto X]$. Then $T_2 \to_\sigma T_2'$ and $T_1' \preceq T_2'$ is witnessed by $f' : D_1' \to D_2'$ such that $f' := f[v.i \mapsto f(v).i']$.

Case: $\chi = \texttt{RemoveClass}(X)$ or $\chi = \texttt{RemoveNode}$. In both cases, the function f (restricted to D_1') is still an embedding from T_1' to T_2 and so $T_1' \preceq T_2$.

D Proof of Lemma 3

We show that, when guards are monotonic, the $\texttt{RemoveClass}(B)$ and $\texttt{RemoveNode}$ operations do not affect the redundancy of the guards. The $\Leftarrow$-direction is trivial. For the other direction, assume $T_0 \to_{\sigma_0} T_1 \to_{\sigma_1} \cdots \to_{\sigma_{n-1}} T_n \to_{\sigma_n} T_{n+1} = (D, \lambda)$ where each $\sigma_i = (g_i, \chi_i)$, and $v, T_{n+1} \vDash g$ for some $v \in D$. It suffices to construct trees $T'_0, \cdots, T'_{n+1}$ with $T'_0 = T_0$ such that g matches T'_{n+1}, and for each $i \in [1..n]$, $T_i \preceq T'_i$ and $T'_i \to_{\sigma'_i} T'_{i+1}$ where each σ'_i is *not* a $\texttt{RemoveNode}$ or a $\texttt{RemoveClass}(X)$ operation but may be the identity operation. We shall construct such a sequence of trees by induction on i. The base case follows immediately from the assumptions. Let $i \geq 0$. Suppose T'_i has been constructed such that $T_i \preceq T'_i$. There are two cases. If $\chi_i = \texttt{RemoveNode}$ or $\chi_i = \texttt{RemoveClass}(X)$ then set $T'_{i+1} := T'_i$ and σ'_i is the identity operation; otherwise, thanks to Lemma 2, set $\sigma'_i := \sigma_i$ and we can construct T'_{i+1} such that $T'_i \to_{\sigma'_i} T'_{i+1}$ and $T_{i+1} \preceq T'_{i+1}$. Finally, since $T_{n+1} \preceq T'_{n+1}$ and g matches T_{n+1}, we have g also matches T'_{n+1} by Lemma 1 as required.

E Proof of Proposition 2

To show EXP membership for the bit-toggling problem, we simply compute an exponential-sized PDS from a given sPDS and apply the standard polynomial-time algorithm for solving control-state reachability for PDS (this is the same as emptiness for pushdown automata over a unary input alphabet $\{a\}$).

To show PSPACE membership for the bounded bit-toggling problem, suppose that the input to the problem is $\mathcal{P} = (\mathcal{V}, \mathcal{W}, \Delta)$ and $h \in \mathbb{N}$ with $\mathcal{V} = \{x_1, \ldots, x_n\}$ and $\mathcal{W} = \{y_1, \ldots, y_m\}$. Each configuration is a pair (q, w) of $q \in \{0,1\}^n$ and w is a word over the stack alphabet $\Gamma = \{0,1\}^m$. Since the stack height is always bounded by h, each configuration is of size at most $n + (h \times m)$. Also, checking whether $\sigma = ((q, a), (q', w))$ is a transition rule of $\mathcal{P}$ — where $q, q' \in \{0,1\}^n$, $a \in \{0,1\}^m$, and w is a word of length at most 2 over Γ — can be checked in linear time (and therefore polynomial space) by simply evaluating each symbolic rule of the form $(|w|, \varphi)$ with respect to the boolean assignment σ. [Evaluating BDDs, boolean formulas, and boolean circuits can all be done in linear time.] So, we can obtain a nondeterministic polynomial space algorithm (and therefore membership in PSPACE) by guessing the symbolic rule that needs to be applied at any given moment.

F Proof of Correctness of Lemma 4

We show correctness of the reduction to simple rewrite systems. To show **(P1)** and **(P2)**, it suffices to show the following four statements:

(P1a) For each $k \in \mathbb{N}$, S is k-redundant for $\mathcal{R}$ iff S' is k-redundant for $\mathcal{R}_1$.

(P1b) For each $k \in \mathbb{N}$, S is k-redundant for $\mathcal{R}_1$ iff S' is k-redundant for $\mathcal{R}'$.

(P2a) S is redundant for $\mathcal{R}$ iff S' is redundant for $\mathcal{R}_1$.

(P2a) S is redundant for $\mathcal{R}$ iff S_1 is redundant for $\mathcal{R}'$.

We first show **(P1a)** and **(P2a)**. To this end, we say that a tree $T = (D, \lambda) \in \textsc{Tree}(\Sigma')$ is *consistent* if, for each $g \in G$ and each node $v \in D$ with $\underline{g} \in \lambda(v)$, it is the case that $v, T \models g$. We say that T is *maximally consistent* if it is consistent and that, for each $g \in G$ and each node

$v \in D$, if $v, T \models g$ then $\underline{g} \in \lambda(v)$. Observe that each tree in $\textsc{Tree}(\Sigma)$ (in particular, the initial tree) is consistent and that each of the rules (1–3) preserves consistency. Therefore, for all trees $T \in \textsc{Tree}(\Sigma)$ and $T' \in \textsc{Tree}(\Sigma')$, if $T \to^*_{\mathcal{R}_1} T'$ then T' is consistent. In addition, for a tree $T \in \textsc{Tree}(\Sigma')$, we write $T \cap \Sigma$ to denote the the tree that is obtained from T by restricting to node labels from Σ. Let $\Theta \subseteq \mathcal{R}_1$ denote the set of intermediate rules added above. Since the class G is closed under taking subformulas, for each tree $T \in \textsc{Tree}(\Sigma)$, it is the case that $T \to^*_\Theta T'$ for some maximally consistent tree $T' \in \textsc{Tree}(\Sigma')$ satisfying $T' \cap \Sigma = T$. Now, by a simple induction on the length of paths, it follows that that $T_1 = (D_1, \lambda_1) \to^*_{\mathcal{R}} T_2 = (D_2, \lambda_2)$ iff $T_1 \to^*_{\mathcal{R}_1} T_2'$ for some maximally consistent tree T_2' such that $T_2' \cap \Sigma = T_2$. Since S' contains precisely all $\underline{g}$ for which $g \in S$, **(P1a)** and **(P2a)** immediately follow.

We now show **(P1b)** and **(P2b)**. Observe that both rules (a) and (b) preserve tree-consistency since $v, T \models \langle d^* \rangle g$ iff at least one of the following cases holds: (i) $v, T \models g$, (ii) there exists a node w in T such that $w, T \models \langle d^* \rangle g$ and w can be reached from v by following the direction d for one step. As before, by a simple induction on the length of paths, for all consistent trees $T_1, T_2 \in \textsc{Tree}(\Sigma')$, it is the case that $T_1 \to^*_{\mathcal{R}'} T_2$ iff $T_1 \to^*_{\mathcal{R}_1} T_2$. This implies implies **(P1b)** and **(P2b)**, i.e., the correctness of our entire construction.

Note, in particular, for all $g \in S$, it is the case that g is redundant in $\mathcal{R}$ iff $\underline{g}$ is redundant in $\mathcal{R}'$.

G Proof of Algorithm for Lemma 5

We argue the correctness of the fixpoint algorithm in the proof of Lemma 5, which shows that the redundancy (resp. k-redundancy) problem is polynomial time solvable assuming oracle calls to the class-adding (resp. k-class-adding) problem.

Soundness The proof is by induction over the number of applications of the fixpoint rules. Fix some sequence of rule applications reaching a fixpoint and let $T_i = (D_0, \lambda_i)$ be the tree obtained after ith application. We prove by induction that we have a tree T such that $T_0 \to^*_{\mathcal{R}} T = (D', \lambda')$ and $T_i \preceq T$ and thus our algorithm is sound.

In the base case, when $i = 0$, $T = T_0$ and the proof is trivial. Hence, assume by induction we have a tree T such that $T_0 \to^*_{\mathcal{R},f} T$ and $T_i \preceq T$. There are two cases.

- When we apply a rule $\sigma = (g, \texttt{AddClass}(B))$ at a node $v \in D_0$, then $T_i \to_\sigma T_{i+1}$ and by Lemma 2 (Monotonicity) we obtain T' satisfying the induction hypothesis.
- When the class-adding problem has a positive answer on input $\langle T_v, f, c, \mathcal{R} \rangle$ and we update $\lambda(v) := \lambda(v) \cup \{c\}$, then there is a sequence of rule applications $\sigma_1, \ldots, \sigma_n$ witnessing the positive solution to the class-adding problems. By n applications of Lemma 2 we easily obtain T' satisfying the induction hypothesis as required.

Completeness Take any sequence of rules $\sigma_1, \ldots, \sigma_n$ such that

$$T_0 \to_{\sigma_1} T_1 \to_{\sigma_2} \cdots \to_{\sigma_n} T_n$$

where for each i, $T_i = (D_i, \lambda_i)$. Note $D_0 \subseteq D_i$ for each i. Let $T_F = (D_0, \lambda)$ be the tree obtained as the conclusion of the fixpoint calculation. We show $\lambda_i(v) \subseteq \lambda(v)$ for all $v \in D_0$ and hence our algorithm is complete.

We induct over i. In the base case $i = 0$ and the result is trivial. For the induction, if σ_i was applied to $v \notin D_0$ or σ_i is an $\texttt{AddChild}(B)$ rule, the result is also trivial. Else σ_i is of the form $\sigma = (g, \texttt{AddClass}(B))$ and applied to $v \in D_0$. There are several cases.

- When $g = A$ let $v' = v$ and when $g = \langle\uparrow\rangle A$ let v' be the parent of v. Since $v \in D_0$ we also have $v' \in D_0$ and thus $\lambda_i(v') \subseteq \lambda(v)$ and thus σ is applicable at v' and the first fixpoint rule applies. Since T_F is a fixpoint, we necessarily have $B \subseteq \lambda(v)$.
- When $g = \langle\downarrow\rangle A$ there are two cases.
  - If the guard was satisfied by matching with $v' \in D_0$, then we have $B \subseteq \lambda(v)$ exactly as above.
  - Otherwise, the guard was satisfied in the run to T' by matching with $v' \notin D_0$. Thus we can pick out the sub-sequence of $\sigma_1, \ldots, \sigma_n$ beginning with the `AddChild`(C) rule (applied at v) that created v' and all rules applied at v' or a descendant. By Lemma 2 (Monotonicity) and the fact that all guards are simple, this sequence is a witness to the positive solution of the class-adding problem $\langle T_v, f, c, \mathcal{R}\rangle$ for all $c \in B$ and thus $B \subseteq \lambda(v)$ as required.

H BDD representation in the Proof of Lemma 6

We first recall the definition of BDDs. A *binary decision diagram (BDD)* is a symbolic representation of boolean functions. A *BDD over the variables* $\mathcal{V} = \{x_1, \ldots, x_n\}$ is a rooted directed acyclic graph $G = (V, E)$ together with a mapping $\lambda : (V \cup E) \to (\mathcal{V} \cup \{0, 1\})$ such that:

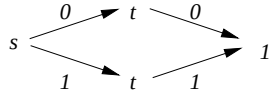
(i) if $v \in V$ is an internal node, then $\lambda(v) \in \mathcal{V}$ and it has *precisely two* outgoing edges e_1, e_2 labeled by $\lambda(e_1) = 0$ and $\lambda(e_2) = 1$, respectively, and

(ii) (ii) if $v \in V$ is a leaf node, then it is labeled by $\lambda(v) \in \{0, 1\}$.

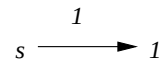
Given an assignment $\nu : \mathcal{V} \to \{0, 1\}$, we can determine the truth value of G under ν as follows: start at the root node of G; if the current node is v labeled by $x \in \mathcal{V}$, follow the outgoing $\nu(x)$-labeled from v to determine the next node; and if the current node is a leaf, then the truth value is given by the label of this leaf.

An *ordered BDD (oBDD)* is a BDD together with an ordering of $\mathcal{V}$ (say $x_1 < \cdots < x_n$) such that if v is a predecessor of u in G where both v and u are internal nodes, then $\lambda(v) < \lambda(u)$. So, no variable $x \in \mathcal{V}$ occurs more than once in each path from the root to a leaf in an oBDD. Finally, an oBDD is reduced if (i) it has no two isomorphic subgraphs, and (ii) it has no internal node whose two children are the same node. In the sequel, we use the term BDD to mean reduced oBDD.

We now demonstrate how to represent boolean formulas in the reduction as BDDs. Observe that each boolean formula in our $\mathcal{P}$-rules is a conjunction of formulas, in which *no two conjuncts share any common variable*. This allows an easy conversion to BDD representation. For example, the BDD $G = (V, E, \lambda)$ for the formula φ in the first type of rules in our sPDS construction in Section 5 can be defined as follows. Each conjunct of the form $s \leftrightarrow t$ is turned into the following BDD



where a missing edge means that it goes to a 0-labeled leaf (also omitted from diagram). Similarly, a conjunct of the form s is turned into the following BDD



where, again, a missing edge means that it goes to a 0-labeled leaf. Now, if G_1 and G_2 are BDDs for the conjuncts φ_1 and φ_2 with different sets of variables, then the BDD for $\varphi_1 \wedge \varphi_2$ can be constructed by gluing the 1-labeled leaf in G_1 by the root of G_2 (using the label of the root of G_2). Since no two conjuncts in φ share a common variable, the BDD for φ can be obtained by linking these simpler BDDs in this way. The resulting BDD is of size linear in the size of φ.

I Proof of Lemma 7

We first prove the direction (A2)⇒(A1). We will prove this by induction on the length of paths. The problem is that our induction hypothesis is not strong enough to get us off the ground (since the state component of $I_{\lambda_0(\epsilon),f}$ is always $(0,\ldots,0)$). Therefore, we will first prove this by induction with a stronger induction hypothesis:

Lemma 8. *Given $\mathcal{P}$-configurations (q,α) and (q',α') with $q,q' : \mathcal{V} \to \{0,1\}$ and $\alpha,\alpha' : \mathcal{W} \to \{0,1\}$ (i.e. stack of height 1), suppose that $(q,\alpha) \to_{\mathcal{P}}^* (q',\alpha')$. Define $T = (D,\lambda) \in \textsc{Tree}(\Sigma)$ as (1) $D = \{\epsilon\} \cup \{0 : q(\texttt{pop}) = 1\}$, (2) $\lambda(\epsilon) = \{c \in \mathcal{K} : \alpha(y_c) = 1\}$ and $\lambda(0) = \{c \in \mathcal{K} : q(x_c) = 1\}$. Define the assumption function $f_\alpha : (\mathcal{K} \cup \{\texttt{root}\}) \to \{0,1\}$ with $f_\alpha(c) = \alpha(z_c)$ for $c \in \mathcal{K}$ and $f_\alpha(\texttt{root}) = \alpha(\texttt{root})$. Then, there exists $T' = (D',\lambda') \in \textsc{Tree}(\Sigma)$ such that $\lambda'(\epsilon) = \{c \in \mathcal{K} : \alpha'(y_c) = 1\}$ and $T \to_{\mathcal{R},f_\alpha}^* T'$.*

Observe that the direction (A2)⇒(A1) is immediate from this Lemma: simply apply it with $(q,\alpha) = I_{\lambda_0(\epsilon),f}$ and $(q',\alpha') = I_{X,f,q'}$.

Proof. The proof is by induction on the length k of the witnessing path $\pi : (q,\alpha) \to^* (q',\alpha')$. When $k = 0$, we have that $(q,\alpha) = (q',\alpha')$. So, we may set $T' = T$ and the statement is satisfied.

Let us now consider the case when $k > 0$. There are two cases to consider depending on the second configuration in the path π:

(i) $\pi = (q,\alpha) \to (q_1,\alpha_1) \to^* (q',\alpha')$, for some $q_1 : \mathcal{V} \to \{0,1\}$ and $\alpha_1 : \mathcal{W} \to \{0,1\}$.

(ii) $\pi = (q,\alpha) \to (q_1,\alpha_1\beta_1) \to^* (q',\alpha')$, where $q_1 : \mathcal{V} \to \{0,1\}$ and $\alpha_1,\beta_1 : \mathcal{W} \to \{0,1\}$.

Note that the case when the first action is a pop (i.e. $(q,\alpha) \to (q_1,\epsilon)$) is not possible since (q_1,ϵ) is a deadend.

Let us consider Case (i). The configuration (q_1,α_1) is obtained by applying a rule $\sigma = (1,\varphi) \in \Delta$, which was generated by a rule $\gamma = (g, \texttt{AddClass}(B)) \in \mathcal{R}$. Suppose that $g = A \subseteq \mathcal{K}$; the other two cases (i.e. when g is of the form $(\langle d\rangle A, \texttt{AddClass}(B))$) can be handled in the same way. Since φ has a conjunct of the form y_a for each $a \in A$, it is the case that $A \subseteq \lambda(\epsilon)$. Since y_b^1 is a conjunct of φ for each $b \in B$, The rule σ toggles on the variables y_b for each $b \in B$ while preserving the value of all other variables in $\mathcal{V} \cup \mathcal{W}$. So, if $T_1 := (D,\lambda_1)$ with $\lambda_1(0) := \lambda(0)$ and $\lambda_1(\epsilon) := \lambda(\epsilon) \cup B$, then $T_0 \to_{\mathcal{R},f_\alpha} T_1$. We may now apply the induction hypothesis on the subpath $(q_1,\alpha_1) \to^* (q',\alpha')$ of π of shorter length and obtain $T_1 \to_{\mathcal{R},f_\alpha} T'$. Concatenating paths, we obtain a path $T \to_{\mathcal{R},f_\alpha} T'$ as desired.

Let us now consider Case (ii). The configuration $(q_1,\alpha_1\beta_1)$ is obtained by applying a rule $\sigma = (2,\varphi) \in \Delta$, which was generated by a rewrite rule $(A, \texttt{AddChild}(B)) \in \mathcal{R}$. Therefore, it must be the case that $(q_1,\beta_1) \to_{\mathcal{P}}^* (q_2,\beta_2) \to_{\sigma_2} (q_3,\epsilon)$ and $(q_3,\alpha_1) \to_{\mathcal{P}}^* (q',\alpha')$ for some configurations (q_2,β_2) and (q_3,ϵ) with stack height 1 and 0, respectively, and some $\mathcal{P}$-rule

$(0, \varphi')$. As in the previous case, we have $A \subseteq \lambda(\epsilon)$ and so, applying $(A, \texttt{AddChild}(B))$ at ϵ, we obtain $T_1 = (D_1, \lambda_1)$ with $D_1 := D \cup \{1\}$ and λ_1 is an extension of λ to D_1 with $\lambda_1(1) := B$. Let $T_1' = (\{\epsilon\}, \lambda_1')$ be the subtree of T_1 rooted at node 1. Applying induction on the path $(q_1, \beta_1) \to_{\mathcal{P}}^* (q_2, \beta_2)$, we obtain a tree $T_2' = (D_2', \lambda_2')$ such that: (1) $\nu : T_1' \to_{\mathcal{R},f}^* T_2'$ with assumption function $f := f_{\beta_1}$, and (2) $\lambda_2'(\epsilon) = \{c \in \mathcal{K} : \beta_2(y_c) = 1\}$. Similarly, if we let $T_3 = (\{\epsilon, 0\}, \lambda_3)$ be the tree with $\lambda_3(\epsilon) = \lambda(\epsilon)$ and $\lambda_3(0) = \lambda_2'(\epsilon)$, we apply induction hypothesis on the path $(q_3, \alpha_1) \to_{\mathcal{P}}^* (q', \alpha')$ and obtain a path $\mu : T_3 \to_{\mathcal{R},f_\alpha}^* T''$ (here $f_\alpha = f_{\alpha_1}$) for some tree $T'' = (D'', \lambda'')$ with $\lambda''(\epsilon) = \{c \in \mathcal{K} : \alpha'(y_c) = 1\}$. Now let T' be the tree obtained from T'' by: (1) removing the node 0 of T'' and replacing it by the tree T_2' (note: $T''(0) = T_2'(\epsilon)$), and (2) adding a node labeled by $\lambda(0)$ as a child of the root node of T''. We have $T'(\epsilon) = T''(\epsilon) = \{c \in \mathcal{K} : \alpha'(y_c) = 1\}$. Finally, we see that $T \to_{\mathcal{R},f_\alpha} T_1 \to_{\mathcal{R},f_\alpha}^* T'$, where the path $T_1 \to_{\mathcal{R},f_\alpha}^* T'$ is obtained by first applying the path ν on the subtree of T_1 rooted at 1 and then applying the path μ on the resulting tree (in the second step, we keep the subtree of T_1 rooted at the node 0 unchanged). □

We now prove the direction (A1)⇒(A2) of the above claim. The proof is by induction on the length k of the path $\pi : T_0 \to_{\mathcal{R},f}^* T$. The base case is when $k = 0$, in which case we set $q'(x) := 0$ for all $x \in \mathcal{V}$ and (A2) is satisfied since $I_{\lambda_0(\epsilon),f} = I_{X,f,q'}$.

We proceed to the induction case, i.e., when $k > 0$. We may assume that $\lambda(\epsilon)$ is a *strict* superset of $\lambda_0(\epsilon)$; otherwise, we may replace π with an empty path, which reduces us to the base case. Therefore, some rewrite rule $(g, \texttt{AddClass}(B))$ with $B \cap \lambda_0(\epsilon) \neq \emptyset$ is applied at ϵ at some point in π. We have two cases depending on the first application of such a rewrite rule $\sigma = (g, \texttt{AddClass}(B))$ at ϵ in π:

(Case 1) We have $g = A$ or $g = \langle\uparrow\rangle A$ for some $A \subseteq \mathcal{K}$. Since $\lambda_0(\epsilon) = \lambda'(\epsilon)$, the rule σ can already be applied at ϵ in T_0 yielding a tree $T_1 = (\{\epsilon\}, \lambda_1)$ with $\lambda_1(\epsilon) = \lambda_0(\epsilon) \cup B$. Owing to Lemma 2, we may apply the same sequence of rewrite rules witnessing π but with T_1 as an initial configuration. Furthermore, we may remove any application of σ at ϵ in this sequence (since B is always a subset of the label of ϵ), which gives a path $\pi' : T_1 \to_{\mathcal{R},f} T$ of length $\leq k-1$. Altogether, we obtain another path $T_0 \to_{\sigma,f} T_1 \to_{\mathcal{R},f}^* T$ of length $\leq k$. In addition, we have $I_{\lambda_0(\epsilon),f} \to_{\mathcal{P}} I_{\lambda_1(\epsilon),f}$ by applying the rule $(1, \varphi) \in \Delta$ generated from σ in the construction of $\mathcal{P}$. Now we may apply the induction hypothesis to the path $\pi' : T_1 \to_{\mathcal{R},f}^* T$ (of length $< k$) and obtain $q' : \mathcal{V} \to \{0, 1\}$ such that $I_{\lambda_1(\epsilon),f} \to_{\mathcal{P}}^* I_{\lambda(\epsilon),f,q'}$, which gives us a path from $I_{\lambda_0(\epsilon),f}$ to $I_{\lambda(\epsilon),f,q'}$, as desired.

(Case 2) We have $g = \langle\downarrow\rangle A$ for some $A \subseteq \mathcal{K}$. In this case, the first rule applied in π must be of the form $\gamma = (C, \texttt{AddChild}(Y))$ for some $C, Y \subseteq \mathcal{K}$. That is, we have $\pi : T_0 \to_{\gamma,f} T_2 \to_{\mathcal{R},f}^* T_3 \to_{\sigma,f} T_4 \to_{\mathcal{R},f}^* T$ for some trees $T_i = (D_i, \lambda_i)$ $(i \in \{2, 3, 4\})$ such that: (1) $\lambda_0(\epsilon) = \lambda_i(\epsilon)$ for all $i \in \{2, 3\}$, (2) $D_2 = \{\epsilon, 0\}$, (3) At T_3, the rule σ is applied at ϵ, and so $D_4 = D_3$ and $\lambda_4(v) = \lambda_3(v)$ for $v \neq \epsilon$ and $\lambda_4(\epsilon) = \lambda_3(\epsilon) \cup B \supset \lambda_3(\epsilon)$. So, this means that some child $i \in \mathbb{N}$ of the root node in T_3 satisfies $\lambda_3(i) \supseteq A$, which enables σ to be executed at the root node. Now recall that simple guards allow a node to inspect only its immediate neighbours (i.e. parent or children). For this reason, we may assume without a loss of generality that $i = 0$, i.e., that it was the child of ϵ that was spawned by the first transition γ. In fact, we may go one step further to assume that 0 is the only child of the root node in the subpath $T_2 \to_{\mathcal{R},f}^* T_3$! This is because that without modifying the label of ϵ, simple guards do not allow the node 0 to "detect" the presence of the other children.

To finish off the proof, we will apply induction hypotheses twice on two different paths. Firstly, we apply the induction hypothesis on the shorter path $(T_2)_{|0} \to_{\mathcal{R},f_\epsilon}^* (T_3)_{|0}$, where f_ϵ is an assumption function that captures the node label $\lambda_3(\epsilon)$, i.e., $f_\epsilon(\texttt{root}) = 0$ and $f_\epsilon(c) = 1$ iff $c \in \lambda_3(\epsilon)$; note that $\lambda_0(\epsilon) = \lambda_2(\epsilon) = \lambda_3(\epsilon)$. [Recall that $(T_2)_{|0}$ means the subtree of T_2 rooted at

the node 0.] This gives us the path $I_{\lambda_2(0),f_\epsilon} \to^*_{\mathcal{P}} I_{\lambda_3(0),f_\epsilon,q''}$ for some $q'' : \mathcal{V} \to \{0,1\}$. Therefore, we have the path $\nu_1 : I_{\lambda_0(\epsilon),f} \to_{\mathcal{P}} (\bar{0}, \mu_\epsilon\mu_0) \to^*_{\mathcal{P}} (q'', \mu_\epsilon\mu'_0) \to_{\mathcal{P}} (q', \mu_\epsilon) \to (q', \mu'_\epsilon)$, where

- $\mu_\epsilon(r) = 1$ iff $I_{\lambda_0(\epsilon),f}(r) = 1$ for each $r \in \mathcal{W} = \{y_c, z_c : c \in \mathcal{K}\} \cup \{\texttt{root}\}$.
- $\mu_0(\texttt{root}) = 0$; $\mu_0(z_c) := \mu_\epsilon(y_c)$; and $\mu_0(y_c) = 1$ iff $c \in B$, for each $c \in \mathcal{K}$.
- $\mu'_0(r) = I_{\lambda_3(0),f_\epsilon,q''}(r)$ for each $r \in \mathcal{W}$.
- $q'(\texttt{pop}) = 1$ and $q'(x_c) = \mu'_0(y_c)$ for each $c \in \mathcal{K}$.
- $\mu'_\epsilon(r) = 1$ iff $\mu_\epsilon(r) = 1$ or $r \in B$.

Note that (q', μ'_ϵ) is the same as $I_{\lambda_4(\epsilon),f,q'}$. We are now going to apply the induction hypothesis the second time. To this end, we let $T'_4 = (\{\epsilon\}, \lambda'_4)$ be the single-node tree defined by restricting T_4 to the root, i.e., $\lambda'_4(\epsilon) = \lambda_4(\epsilon) \supset \lambda_0(\epsilon)$. Since $T_0 \preceq T_4$, by the proof of Lemma 2, we have $T'_4 \to^*_{\mathcal{R},f} T$ by following the actions in the path π, except for the action σ. This gives us a path π' whose length is $|\pi| - 1$. So, by induction, we have a path $\nu_2 : I_{\lambda_4(\epsilon),f} \to^*_{\mathcal{P}} I_{\lambda(\epsilon),f,p}$ for some $p : \mathcal{V} \to \{0,1\}$. In order to connect ν_1 and ν_2, we need to use the following lemma, which completes the proof of Lemma 7.

Lemma 9. *Suppose $p, q : \mathcal{V} \to \{0,1\}$ such that $p(x) \leq q(x)$ for each $x \in \mathcal{V}$. Then, for all stack content $v = \alpha_1 \cdots \alpha_m$ where each $\alpha_i : \mathcal{W} \to \{0,1\}$, if $(p, v) \to_\sigma (p', w)$ with $\sigma \in \Delta$, then $(q, v) \to_\sigma (q', w)$ for some $q' : \mathcal{V} \to \{0,1\}$ such that $p'(x) \leq q'(x)$ for each $x \in \mathcal{V}$.*

Proof. Easy to verify by checking each $\mathcal{P}$-rule. □

J A polynomial-time fixed point algorithm for bounded height

We now provide a simple fixpoint algorithm for the k-redundancy problem that runs in time $n^{O(k)}$. We assume that we have simplified our rewrite systems using the simplification given in Section 5. The input to the algorithm is a simple rewrite system $\mathcal{R}$ over $\textsc{Tree}(\Sigma)$ with $\Sigma = 2^{\mathcal{K}}$, a guard database $S \subseteq \mathcal{K}$, and an initial Σ-labeled tree $T_0 = (D_0, \lambda_0) \in \textsc{Tree}(\Sigma)$. The number $k \in \mathbb{N}$ is fixed (i.e. not part of the input). Our algorithm will compute a tree $T_F = (D_F, \lambda_F)$ such that $T_F \in \mathit{post}^*_{\mathcal{R}}(T_0)$ and, for *all* trees $T \in \mathit{post}^*_{\mathcal{R}}(T_0)$, we have $T \preceq T_F$. By Lemma 1, if a guard $g \in S$ is matched in some $T \in \mathit{post}^*_{\mathcal{R}}(T_0)$, then g is matched in T_F. The converse also holds since $T_F \in \mathit{post}^*_{\mathcal{R}}(T_0)$. Furthermore, we shall see that T_F is of height at most k, each of whose nodes has at most $|T_0| + |\mathcal{R}|$ children (and so of size at most $O(|T_0| + |\mathcal{R}|)^{O(k)}$), and is computed in time $(|T_0| + |\mathcal{R}|)^{O(k)}$. Computing the rules in S that are matched in T_F can be easily done in time linear in $|S| \times |T_F|$.

The algorithm for computing T_F works as follows.

(1) Set $T = (D, \lambda) := T_0$;

(2) Set $W := \emptyset$;

(3) Add each pair (v, σ) to W, where $v \in D$ and $\sigma \in \mathcal{R}$ is a rule that is applicable at v, i.e. $v, T \models g$ where $\sigma = (g, \chi)$ for some χ; all pairs in W are initially "unmarked";

(4) Repeat the following for each unmarked pair $(v, \sigma) \in W$ with $\sigma = (g, \chi)$:

(a) if $\chi = \texttt{AddClass}(X)$, then $\lambda(v) := \lambda(v) \cup X$;

(b) if $\chi = \texttt{AddChild}(X)$, $|v| \leq k-1$, and $X \not\subseteq \lambda(v.i)$ for *each* $v.i \in D$ child of v, then set $D := D \cup \{v.(i+1)\}$ where $i := \max\{j \in \mathbb{N} : v.j \in D\} + 1$ (by convention, $\max \emptyset = 0$), and $\lambda(v.i) := X$;

(c) Mark (v, σ);

(d) For each $(w, \gamma) \in (D \times \mathcal{R}) \setminus W$ that is applicable, add (w, γ) to W unmarked;

(5) Output $T_F := T$;

Proof of Correctness. We will show that $T \preceq T_F$ for each $T \in post^*_{\mathcal{R}}(T_0)$. The proof is by induction on the length m of path from T_0 to T. We write $T_F = (D_F, \lambda_F)$. The base case is when $m = 0$, in which case $T = T_0 = (D_0, \lambda_0)$. Since our rewrite rules only add classes to nodes and add new nodes, it is immediate that $D_0 \subseteq D_F$ and that $\lambda_0(v) \subseteq \lambda_F(v)$ for each $v \in D_0$. That is, $T_0 \preceq T_F$. We now proceed to the inductive step and assume that $T = (D, \lambda) \preceq T_F$ with a witnessing embedding $f : D \to D_F$. Let $v \in D$ and $\sigma = (g, \chi)$ be a $\mathcal{R}$-rule that is applicable on the node v of T. By Lemma 1, it follows that $f(v), T_F \models g$. Let us suppose that after applying σ on v, we obtain $T' := (D', \lambda')$. We have two cases now:

1. $\chi = \texttt{AddClass}(X)$. In this case, $D' = D$, $\lambda'(u) = \lambda(u)$ whenever $u \neq v$, and $\lambda'(v) = \lambda(v) \cup X$. Since $T \preceq T_F$, it follows that $\lambda(v) \subseteq \lambda_F(f(v))$. Since $f(v), T_F \models g$, it follows that $(f(v), (g, \texttt{AddClass}(X)))$ is marked in W (by the end of the computation of T_F). This means that $X \subseteq \lambda_F(f(v))$. Altogether, $T' \preceq T_F$.

2. $\chi = \texttt{AddChild}(X)$ and $|v| \leq k-1$. In this case, $D' = D \cup \{v.i\}$ for some $i \in \mathbb{N}$ with $v.i \notin D$, $\lambda'(u) = \lambda(u)$ for each $u \in D$, and $\lambda'(v.i) = X$. Since $T \preceq T_F$, the properties (H1) and (H3) of embeddings imply that v and $f(v)$ have the same levels in T and T_F respectively. Also, since $v, T_F \models g$, it follows that $(f(v), (g, \texttt{AddChild}(X)))$ is marked in W (by the end of the computation of T_F). This means that there exists a child $f(v).j$ in D_F with $X \subseteq \lambda_F(f(v).j)$. Hence, the extension of f with $f(v.i) := f(v).j$ is a witness for $T' \preceq T_F$, as desired.

Running time analysis. We first analyse the time complexity of each individual step in the above algorithm. Checking whether a simple guard g is satisfied at a node v can be done by inspecting the node labels of v and its neighbours (i.e. parent and children), which can be performed in time $O(|\mathcal{K}| \times (\text{number of neighbours of } v))$. So, since each node in T_0 has most $|T_0|$ children, then step (3) can be achieved in time $O(|\mathcal{K}| \times |T_0|^2)$. Similarly, modifying a node label can be done in time $O(|\mathcal{K}|)$, which is the running time of Step (4a). Since no rewrite rule in $\mathcal{R}$ can be applied twice at a node, each node in T_F has at most $m := |T_0| + |\mathcal{R}|$ children, i.e., either that child was already in T_0 or that it was generated by a rewrite rule in $\mathcal{R}$. Therefore, Step (4b) takes time $O(|\mathcal{K}| \times m)$. Since Step (4d) needs to check only (w, γ) where w is a neighbor (i.e. children or parent) of v, then it can be done in time $O(|\mathcal{K}| \times m)$. Since the tree T_F has height at most k and each node has at most m children, the tree T_F has at most $O(m^k)$ nodes. So, the entire Step (4) runs in time $O(|\mathcal{K}| \times |T_0|^{k+1} \times |\mathcal{R}|^{k+1})$, which is also a time bound for the running time of the entire algorithm.

K W[1]-hardness for k-redundancy problem

Before proving our W[1]-hardness, we first briefly review some standard concepts from parameterised complexity theory [18]. A *parameterized problem* $\mathcal{C}$ is a computational problem with

input of the form $\langle v; k\rangle$, where $v \in \{0,1\}^*$ and $k \in \mathbb{N}$ (represented in unary) is called the *parameter*. An *fpt-algorithm* for $\mathcal{C}$ runs in time $O(f(k).n^c)$ for some constant c, and some computable function $f : \mathbb{N} \to \mathbb{N}$ (both independent of input). Here, n is the size of the input. So, an algorithm that runs in time $O(n^k)$ is not an fpt-algorithm. The problem $\mathcal{C}$ is said to be *fixed-parameter tractable* if it can be solved by an fpt-algorithm. There are classes of problems in parameterized complexity that are widely believed not to be solvable by fpt-algorithms, e.g., $\mathsf{W}[1]$ (and others in the W-hierarchy). To show that a problem $\mathcal{C}$ is $\mathsf{W}[1]$-hard, it suffices to give an fpt-reduction to $\mathcal{C}$ from the *short acceptance problem of nondeterministic Turing machines (NTM)*: given a tuple $\langle(\mathcal{M}, w); k\rangle$, where $\mathcal{M}$ is an NTM, $w \in \{0,1\}^*$ is an input word, and k is a positive integer represented in unary, decide whether $\mathcal{M}$ accepts w in k steps. Among others, standard reductions in complexity theory that satisfy the following two conditions are fpt-reductions: (1) fpt-algorithms, (2) if k is the input parameter, then the output parameter is $O(k)$.

We now show that k-redundancy is $\mathsf{W}[1]$-hard. To this end, we reduce from the short acceptance problem of nondeterministic Turing machines. The input to the problem is $\langle \mathcal{M}, w; k\rangle$, where $\mathcal{M}$ is an NTM, w is an input word for $\mathcal{M}$, and $k \in \mathbb{N}$ given in unary. Suppose that $\mathcal{M} = (\Gamma, \Sigma, \sqcup, \mathcal{Q}, \Delta, q_0, q_{acc})$, where $\Gamma = \{0,1\}$ is the input alphabet, $\Sigma = \{0,1,\sqcup\}$ is the tape alphabet, $\sqcup$ is the blank symbol, $\mathcal{Q}$ is the set of control states, $\Delta \subseteq (\mathcal{Q} \times \Sigma) \times (\mathcal{Q} \times \Sigma \times \{L, R\})$ is the transition relation (L/R signify go to left/right), $q_0 \in \mathcal{Q}$ is the initial state, and q_{acc} is the accepting state. Each configuration of $\mathcal{M}$ on w of length k can be represented as a word

$$(0, a_0, c_0)(1, a_1, c_1) \cdots (k, a_k, c_k)$$

where $a_i \in \Sigma$ for each $i \in [0, k]$, and for some $j \in [0, k]$ we have $c_j \in \mathcal{Q}$ and for all $r \neq j$ it is the case that $c_j = \star$. Note that this is a word (of a certain kind) over $\Omega := \Omega_k := [0, k] \times \Sigma \times (\mathcal{Q} \cup \{\star\})$. In the sequel, we denote by $\texttt{CONF}_k$ the set of all configurations of $\mathcal{M}$ of length k. Suppose that $w = a_1 \cdots a_n$. If $n < k$, we append some blank symbols at the end of w and assume that $|w| \geq k$. The initial configuration will be $I_0(w) := (0, \sqcup, q_0)(1, a_1, \star) \cdots (k, a_k, \star)$. [Notice that if $n > k$, then some part of the inputs will be irrelevant.] In the following, we will append the symbol $(-1, \sqcup, \star)$ (resp. $(k+1, \sqcup, \star)$) before the start (resp. after the end) of $\mathcal{M}$ configurations to aid our definitions. Therefore, define $\Omega_e := \Omega_{k,e} := [-1, k+1] \times \Sigma \times (\mathcal{Q} \cup \{\star\})$.

We now construct a tree-rewrite system $\mathcal{P}$ to simulate $\mathcal{M}$. First off, for each Δ-rule σ, define a relation $R_\sigma \subseteq \Omega_e^4$ such that $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}) \in R$ describes the update by rule σ of cell $\vec{v}_2$ to $\vec{w}$ given the neighbouring cells $\vec{v}_1$ and $\vec{v}_3$. That is, $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}) \in R$ iff a configuration $C_2 = \vec{u}_{-1} \cdots \vec{u}_{k+1}$ of $\mathcal{M}$, where $\vec{u}_{i+1} = \vec{w}$ for some i, can be reached in one step using σ from another configuration $C_1 = \vec{u}'_{-1} \cdots \vec{u}'_{k+1}$, where $\vec{u}'_i \vec{u}'_{i+1} \vec{u}'_{i+2} = \vec{v}_1 \vec{v}_2 \vec{v}_3$. Let $R = \bigcup_{\sigma \in \Delta} R_\sigma$. In this case, we say that rule σ is a witness for the tuple $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w})$. Notice that, given a rule σ of $\mathcal{P}$ and $\vec{v}_1, \vec{v}_2, \vec{v}_3$, there exists *at most one* $\vec{w}$ such that $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}) \in R$ and is witnessed by σ. In addition, it is easy to see that checking whether $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}) \in R$ can be done in polynomial time (independent of k) since this is a simple local check of $\mathcal{M}$ rules.

The rewrite system $\mathcal{P}$ works over trees of labels $\Xi = \Omega_e \cup \Delta$. The initial tree is a tree with a single node labeled by $I_0(w)$. We add the following rules to $\mathcal{P}$:

1. For each $\sigma \in \Delta$, we add $(\top, \texttt{AddChild}(\sigma))$ to $\mathcal{P}$.
2. For each $(\vec{v}_{i-1}, \vec{v}_i, \vec{v}_{i+1}, \vec{w}) \in R$ witnessed by $\sigma \in \Delta$, for $i \in [0, k]$, we add

   $$(\sigma \wedge \langle\uparrow\rangle\{\vec{v}_{i-1}, \vec{v}_i, \vec{v}_{i+1}\}, \texttt{AddClass}(\vec{w}))$$

   to $\mathcal{P}$.

3. For each $\sigma \in \Delta$, we add $(\sigma, \texttt{AddClass}((-1, \sqcup, \star)))$ and $(\sigma, \texttt{AddClass}((k+1, \sqcup, \star)))$ to $\mathcal{P}$.

Rule 1 first guesses the rule σ to be applied to obtain the next configuration C_2 of the NTM $\mathcal{M}$. This rule is added as a class in the child v of the current node u containing the current configuration C_1. Rule 3 is then applied to obtain the left and right delimiter. After that, configuration C_2 is written down in node v by applying Rule 2 for k times. In particular, the system will look up the rule σ that was guessed before and the content of the parent node u (i.e. C_1). If we let $S = [0, k] \times \Sigma \times \{q_{acc}\}$, then S is not k-redundant iff $\mathcal{M}$ accepts w within k steps. Notice that this is an fpt-preserving reduction. In particular, it runs in time $O(k^c n^d)$ for some constant c and d.

L Proof of PSPACE-hardness in Theorem 4

We reduce to the k-redundancy problem, when k forms part of the input, from membership of a linear bounded Turing machine. Note, since the runs of a linear bounded Turing machine may be of exponential length, the encoding in the previous section no longer works.

Suppose the input TM is $\mathcal{M} = (\Gamma, \Sigma, \sqcup, \mathcal{Q}, \Delta, q_0, q_{acc})$, where $\Gamma = \{a, b\}$ is an input alphabet, $\Sigma = \{a, b, \sqcup\}$ is the tape alphabet, $\sqcup$ is the blank symbol, $\mathcal{Q}$ is the set of control states, $\Delta \subseteq (\mathcal{Q} \times \Sigma) \times (\mathcal{Q} \times \Sigma \times \{L, R\})$ is the transition function (L/R signify go to left/right), $q_0 \in \mathcal{Q}$ is the initial state, and q_{acc} is the accepting state. We are only interested in runs of $\mathcal{M}$ that use $N := dn$ space for some constant d. Each configuration of $\mathcal{M}$ on w can be represented as a word of length $N+1$ in the alphabet $\Omega := \Omega_N$, which was defined in the proof of $\mathsf{W}[1]$-hardness. The initial configuration $I_0(w)$ is defined to be

$$(0, a_0, c_0)(1, a_1, c_1) \cdots (N, a_N, c_N)$$

where $a_0 a_1 \cdots a_N = \sqcup w \sqcup^{(d-1)n}$, $c_0 = q_0$, and $c_1 = \cdots = c_N = \star$. In the following, we will also use extension $\Omega_e := \Omega_{N,e}$ of Ω and the logspace-computability of the 4-ary relation $\Delta \subset \Omega_e^4$ defined in that proof.

We now construct our rewrite system $\mathcal{R}$, which works over the alphabet $\Xi := \{R, 0, 1\} \cup \Omega_e$. Let $M := 2(N + \log|\mathcal{Q}| + \log N)$. Firstly, it suffices to consider runs of $\mathcal{M}$ of at most length $3^N \times |\mathcal{Q}| \times N < 2^M$ since exceeding this length there must be a repeat of configurations by the pigeonhole principle. Our initial configuration is $T_0 = (D_0, \lambda_0)$, where $D = \{\epsilon\}$ and $\lambda_0(\epsilon) = R$.

We start by adding the following rules to $\mathcal{R}$: $(g, \texttt{AddChild}(0))$ and $(g, \texttt{AddChild}(1))$, where $g = (\langle\uparrow\rangle)^i R$ and $i \in [0, M-1]$. These rules simply build trees of height M, into which the binary tree of height M can be embedded. The label in any path from the root to a leaf node gives us a number i in $[0, 2^M]$ written in binary, which will in turn have a child representing the ith configuration of $\mathcal{M}$ from $I_0(w)$ (which is unique since $\mathcal{M}$ is deterministic).

We now add the rule $(g, \texttt{AddChild}(\{(-1, \sqcup, \star), (N+1, \sqcup, \star)\}))$, where $g = (\langle\uparrow\rangle)^M R$. This rule initialises the content of each configuration (of a given branch).

Notation. *For a guard g, a class c, and direction $d \in \{\uparrow, \downarrow\}$, we write $c\langle d\rangle g$ to mean $c \wedge \langle d\rangle g$.*

The following set of rules defines the first configuration $I_0(w)$ in the tree. For each $i \in [0, N]$, we add the rule $(g, \texttt{AddClass}((i, a_i, c_i)))$ $g = \langle\uparrow\rangle(0\langle\uparrow\rangle)^{M-1}R$.

We now add a set of rules for defining subsequent configurations in the tree. The guard will inspect a previous configuration of the current configuration. Based on this, it will add an appropriate content of each tape cell. Each of these rules will be of the form $(g, \texttt{AddChild}(\vec{v}))$, where $\vec{v} \in \Omega$. For each $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w})$ and $i \in [0, M-1]$, we add a rule $(g, \texttt{AddChild}(\vec{w}))$ where $g := \top\langle\uparrow\rangle\sigma_i\langle\downarrow\rangle\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where $\sigma_i = (0\langle\uparrow\rangle)^i 1\langle\uparrow\rangle\top\langle\downarrow\rangle 0(\langle\downarrow\rangle 1)^i$. The guard g selects precisely

the nodes containing the jth configuration of $\mathcal{M}$ (there can be at most one jth configuration of $\mathcal{M}$ since $\mathcal{M}$ is deterministic), where j is a number whose binary representation ends with 10^i. The guard traverses the tree upward trying to find the first bit that is turned on (i.e. of form 10^i) and then looks at the complement of this bit pattern (i.e. of form 01^i).

Finally, if we set $S = \{(i, a, q_F) : i \in [0, N], a \in \Sigma\}$, then S is not $(M+1)$-redundant iff $\mathcal{M}$ accepts w. The reduction is easily seen to run in polynomial time. This completes our reduction.

M Proof of EXP-hardness in Theorem 3

We show that the redundancy problem for $\mathcal{R}$ is EXP-hard. To this end, we reduce membership for alternating linear bounded Turing machines, which is EXP-complete. An alternating Turing machine is a tuple $\mathcal{M} = (\Gamma, \Sigma, \sqcup, \mathcal{Q}, \Delta, q_0, q_{acc})$, where $\Gamma = \{0, 1\}$ is the input alphabet, $\Sigma = \{0, 1, \sqcup\}$ is the tape alphabet, $\sqcup$ is the blank symbol, $\mathcal{Q}$ is the set of control states, $\Delta \subseteq (\mathcal{Q} \times \Sigma) \times (\mathcal{Q} \times \Sigma \times \{L, R\})^2 \times \{\wedge, \vee\}$ is the transition relation. Given an $\mathcal{M}$-configuration C, a transition $((q, a), (q_L, a_L, d_L), (q_R, a_R, d_R), s)$ spawns two new $\mathcal{M}$-configurations C_L and C_R such that C_L (resp. C_R) is obtained from C by applying a normal NTM transition $((q, a), (q_L, a_L, d_L))$ (resp. $((q, a), (q_R, a_R, d_R))$). In this way, a run π of $\mathcal{M}$ is a binary tree, whose nodes are labeled by $\mathcal{M}$-configurations and additionally each internal (i.e. non-leaf) node is labeled by a transition from Δ. To determine whether the run π is accepting, we construct a boolean circuit from π as follows: (1) each internal node labeled by a transition $((q, a), (q_L, a_L, d_L), (q_R, a_R, d_R), s)$ is replaced by a boolean operator s, and (2) assign 1 (resp. 0) to a leaf node if the configuration is (resp. is not) in the state q_{acc}. The run π is accepting iff the output of this circuit is 1.

We are only interested in tape configurations of size $d.n$ for a constant integer d, where n is the length of the input word w. Therefore, each configuration of $\mathcal{M}$ on input w is a word in $\texttt{CONF}_N$ defined in the proof of W[1]-hardness. As before, the initial configuration $I_0(w)$ is

$$(0, a_0, c_0)(1, a_1, c_1) \cdots (N, a_N, c_N)$$

where $a_0 a_1 \cdots a_N = \sqcup w \sqcup^{(d-1)n}$, $c_0 = q_0$, and $c_1 = \cdots = c_N = \star$. As before, we will also use extension $\Omega_e := \Omega_{N,e}$ of Ω and put a delimiter at the left/right end of each configuration. For each $\sigma = ((q, a), (q_L, a_L, d_L), (q_R, a_R, d_R), s) \in \Delta$ and $t \in \{L, R\}$, we define a relation $R_{\sigma,t} \subseteq \Omega_e^4$ as the log-computable relation $R_{\sigma'}$ defined in the previous W[1]-hardness proof, where σ' is the NTM transition rule $((q, a), (q_t, a_t, d_t))$.

Our rewrite system $\mathcal{R}$ will work over the set $\mathcal{K} := \{\texttt{root}, \texttt{success}, L, R\} \cup \Omega_e \cup \Delta$ of classes. The initial configuration is $T_0 = (D_0, \lambda_0)$ with $D_0 = \{\epsilon\}$ and $\lambda_0(\epsilon) = \{\texttt{root}, (-1, \sqcup, \star), (N+1, \sqcup, \star)\} \cup \{(i, a_i, c_i) : i \in [0, N]\}$. Define $g := \{\texttt{root}, \texttt{success}\}$, which is the guard we want to check for redundancy. Roughly speaking, our rewrite system $\mathcal{R}$ will guess an accepting run π of $\mathcal{M}$ on the input word w. Each accepting configuration (at the leaf) will then be labeled by `success`. This label `success` will be propagated to the root of the tree according to the $\mathcal{M}$-transition that produce a $\mathcal{M}$-configuration in the tree. More precisely, we define $\mathcal{R}$ by adding the following rewrite rules:

1. For each $t \in \{L, R\}$,

$$\sigma = ((q, a), (q_L, a_L, d_L), (q_R, a_R, d_R), s) \in \Delta$$

and $i \in [0, N]$, add the rule

$$((i, a, q), \texttt{AddChild}(\{\sigma, t, (-1, \sqcup, \star), (N+1, \sqcup, \star)\})$$

to $\mathcal{R}$.

2. For each $\sigma \in \Delta$, $t \in \{L, R\}$, and $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}) \in R_{\sigma,t}$, add the rule

$$(\{t, \sigma\}\langle\uparrow\rangle\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}, \texttt{AddClass}(\vec{w}))$$

to $\mathcal{R}$.

3. For each $i \in [0, N]$ and $a \in \Gamma$, add the rule

$$(\{(i, a, q_{acc}), \texttt{AddClass}(\texttt{success}))$$

to $\mathcal{R}$.

4. For each

$$\sigma = ((q, a), (q_L, a_L, d_L), (q_R, a_R, d_R), s) \in \Delta$$

if $s = \vee$, then add the rule

$$(\langle\downarrow\rangle\{\sigma, \texttt{success}\}, \texttt{AddClass}(\texttt{success}))\ .$$

If $s = \wedge$, then add the rule

$$(g, \texttt{AddClass}(\texttt{success}))$$

where

$$g = \langle\downarrow\rangle\{\sigma, \texttt{success}, L\} \wedge \langle\downarrow\rangle\{\sigma, \texttt{success}, R\}$$

The first rewrite rule spawns a child labeled by an $\mathcal{M}$-transition σ and a symbol $t \in \{L, R\}$ to indicate whether its a left/right child in the $\mathcal{M}$-path to be guessed. The second rewrite rule determines the configuration of the current node (because an $\mathcal{M}$-rule σ has been guessed). The third rewrite rule adds a label `success` to accepting configurations. The fourth rewrite rule propagates the label `success` upwards.

Our translation runs in polynomial-time. Furthermore, it is easy to see that $\mathcal{M}$ accepts w iff g is not redundant with respect to the rewrite system $\mathcal{R}$ and initial tree T_0. This completes our reduction.

N Implementation Optimisations

N.1 Limiting the Number of Rules

For each class $c \in \mathcal{K}$ we construct a symbolic pushdown system that is used to determine whether the class c can be added at a certain node. We improve the performance of the pushdown analysis by first removing all rules of the rewrite system with rules $\mathcal{R}$ that cannot contribute towards the addition of class c at a node. We then perform analysis with the reduced set of rules $\mathcal{R}'$. Since a rule is added for each CSS selector, with intermediate rules being introduced to simplify the guards on the rules, the removal of irrelevant rules to a particular class can give significant gains.

We identify rules which may contribute to the addition of class c by a backwards fixed point algorithm. We begin by including in $\mathcal{R}'$ all rules that either directly add the class c, or add a child to the tree. The reason we add all $\texttt{AddChild}(B)$ rules is because new nodes may lead to new guards being satisfied, even if the label of the new child does not appear in the guard (e.g. $\langle\uparrow\rangle a$ cannot be satisfied without a node labelled with a having a child).

Algorithm 1 Restricting $\mathcal{R}$

Require: A class $c \in \mathcal{K}$ and a set $\mathcal{R}$ of rewrite rules.
Ensure: $\mathcal{R}'$ contains all rules relevant to the addition of c.

$$\mathcal{R}' = \{\ \sigma \in \mathcal{R} \mid \sigma = (g, \texttt{AddClass}(B)) \wedge c \in B\ \} \cup \{\ \sigma \in \mathcal{R} \mid \sigma = (g, \texttt{AddChild}(B))\ \}$$

repeat
  for all $\sigma \in \mathcal{R}'$ and classes c' appearing in σ **do**
    $\mathcal{R}' = \mathcal{R}' \cup \{\ (g, \texttt{AddClass}(B)) \in \mathcal{R} \mid c' \in B\ \}$
  end for
until $\mathcal{R}'$ reaches a fixed point

We then search for any rules $\sigma = (g, \texttt{AddClass}(B)) \in \mathcal{R}$ such that there exists $c' \in B$ with c' appearing in the guard of some rule in $\mathcal{R}'$. We add all such σ to $\mathcal{R}'$ and iterate until a fixed point is reached. The full algorithm is given in Algorithm 1

Proposition 3. *Given a single-node tree T_0, assumption function f and a class $c \in \mathcal{K}$, the rewrite system $\mathcal{R}$ has a positive solution to the class-adding problem $\langle T_0, f, c, \mathcal{R}\rangle$ has a positive answer iff the class-adding problem $\langle T_0, f, c, \mathcal{R}'\rangle$ has a positive answer.*

Proof. The "if" direction is direct since $\mathcal{R}$ contains all rules in $\mathcal{R}'$. The "only-if" direction can be shown by induction over the length of a sequence of rule applications $\sigma_1, \ldots, \sigma_n$ where σ_n adds class c to the tree. Note, we relax the proposition for the induction hypothesis, allowing T_0 to be any arbitrary tree.

In the base case $n = 0$ and the proof is immediate. Hence, assume for the tree T_1 obtained by applying σ_1 to T_0 we have, by induction, a sequence of rule applications $\sigma'_1, \ldots, \sigma'_m$ which is a subsequence of $\sigma_1, \ldots, \sigma_m$, resulting in the addition of class c. We need to show there exists such a sequence beginning at T_0. There are several cases.

When $\sigma_1 = (g, \texttt{AddChild}(B))$ then $\sigma_1 \in \mathcal{R}'$ and the sequence $\sigma_1, \sigma'_1, \ldots, \sigma'_m$ suffices.

When $\sigma_1 = (g, \texttt{AddClass}(B))$ there are three cases. If $c \in B$ then $\sigma_1 \in \mathcal{R}'$ and the sequence σ_1 suffices. If there is some $c' \in B$ used in the matching of some guard g' of a rule in $\sigma'_1, \ldots, \sigma'_m$, then, since c' appears in a rule in $\mathcal{R}'$ we have also $\sigma_1 \in \mathcal{R}'$ and the sequence $\sigma_1, \sigma'_1, \ldots, \sigma'_m$ suffices. Otherwise, no $c' \in B$ is used in the matching of some guard g' in $\sigma'_1, \ldots, \sigma'_m$ and the sequence $\sigma'_1, \ldots, \sigma'_m$ suffices. □

N.2 Limiting the Number of jMoped Calls

By performing global backwards reachability analyses with jMoped we in fact only need to perform one pushdown analysis per class $c \in \mathcal{K}$. Given a set of target configurations, global backwards reachability analysis constructs a representation of the set of configurations from which a target configuration may be reached.

Thus, we begin with the set of target configurations (q, a) where $a = (b_1, \ldots, b_m) \in \{0, 1\}^m$ with $b_i = 1$ and $b_i = 1$ indicates the addition of class c to the root node. The global backwards reachability analysis of this target set then gives us a representation of all configurations that can eventually reach one of the targets, that is, add the class c. In particular, we obtain from jMoped a BDD representing all $\langle T_v, f, c, \mathcal{R}\rangle$ that have a positive answer to the class-adding problem. We can then use this BDD to determine whether a given class-adding problem for c represents a positive instance without having to use jMoped again.

N.3 Matching Guards Anywhere in the Tree

We can check whether a guard g is not matched anywhere in the tree by checking whether $\langle\downarrow^*\rangle g$ is redundant. However, during the simplification step this results in the addition of a new class for each guard that has to be propagated up the tree.

We avoid this cost by checking directly whether a guard may be matched anywhere in the tree. After simplification this amounts to checking whether a given class may be matched anywhere in the tree. Hence, after we have saturated T_v, we do a final check to detect whether each class c that does not appear in the saturated tree might still be able to appear in a node created by a sequence of `AddChild`(B) rules. To do this, we perform a similar class adding check as above, except we test reachability of a configuration of the form (q, w) where the top character of the stack w is $a = (b_1, \ldots, b_m) \in \{0, 1\}^m$ with $b_i = 1$. That is, the node where c was added does not have to be the root node. We obtain a single BDD per class as above, and check it against all assumption functions f that can represent nodes in the saturated tree.

O HTML5 Translation

We informally describe here our process for extracting rules from HTML5 documents that use jQuery. Note, this translation is for experimental purposes only and does not claim to be a rigorous, systematic, or sound approach. It is merely to demonstrate the potential for extracting interesting rewrite rules from real HTML5 and to obtain some representative benchmarks for our implementation. A robust dataflow analysis could in principle be built, and remains an interesting avenue of future work.

We begin by describing how to translate complete jQuery calls that do not depend on their surrounding code. E.g.

```
$('.a').next().append('<div class="b"/>');
```

does not depend on any other code in the document, whereas

```
$(x).next().append('<div class="b"/>');
```

depends on the value of the `x` variable. We will describe in a later section how we track variables, although we will make mention in when translating jQuery calls section when a particular variable receives a given value. First we describe the translation of CSS selectors and jQuery calls.

O.1 CSS Selectors

We translate the following subset of CSS selectors.

- Each class, element type, or ID is directly translated to a class in our model with the same name.
- Conjunctions of guards are translated to sets of classes. E.g. `div.selected.item` becomes $\{div, selected, item\}$.
- Descendant relations are encoded using $\langle\uparrow^+\rangle$. That is `x y`, where `x` and `y` are selectors with translations into guards g and g' respectively becomes $\langle\uparrow^+\rangle g \wedge g'$.
- Child relations are encoded using $\langle\uparrow\rangle$. That is `x > y`, where `x` and `y` are selectors with translations into guards g and g' respectively becomes $\langle\uparrow\rangle g \wedge g'$.

O.2 jQuery Calls

Given a jQuery call of the form

```
$(s).f(x)....g(y).h(z)
```

where s is a CSS selector string and `f(x)....g(y).h(z)` is a sequence of jQuery calls, we can extract new rules as follows. We recursively translate the sequence of calls `$(s).f(x)....g(y)` where `$(s)` is the base case, and obtain from the recursion a guard reflecting the nodes matched by the calls. Then we generate a rule depending on the effect of `h(z)`.

Note that the call to `$(s)` returns a jQuery object which contains within it a stack of sets of nodes matched. When `s` is just a CSS selector this stack will contain a single element that is the set of nodes matched by `s`. Each function in the sequence `f(x)....g(y).h(z)` receives as its `this` object the jQuery object and obtains a new set of nodes derived from the nodes on the top of the stack. This new set is then pushed onto the stack and passed to the next call in the sequence. There is a jQuery function `end()` that simply pops the top element from the stack and passes the remaining stack to the next call. We model this stack as a stack of guards during our translation.

We start with the base case `$(s)`. In this case we translate the CSS selector `s` into a guard g and return the single-element stack g. Note: this assumes `s` is a string constant. Otherwise we set g to be true (i.e. matches any node).

In the recursive case we have a call `l.f(x)` where `l` is a sequence of jQuery calls and `f(x)` is a jQuery call. We obtain a stack σ by recursive translation of `l` and then do a case split on `f`. In each case we obtain a new stack ρ which is returned by the recursive call. In some cases we also add new rules to our translated model. We describe the currently supported functions below. In all cases, let $\sigma = \sigma' g$. Note, our implementation directly supports both $\langle\downarrow^*\rangle\varphi$ and $\langle\downarrow^+\rangle\varphi$, and similarly for $\langle\uparrow^*\rangle\varphi$ and $\langle\uparrow^+\rangle\varphi$.

- `animate(...)`: we return $\rho = \sigma$.
- `addBack()`: let $\sigma = \sigma'' g' g$, we set $\rho = \sigma''(g \vee g')$.
- `addClass(`c`)`: set $\rho = \sigma$ and add the rule $(g, \texttt{AddClass}(\{c\}))$ when c is a string constant (otherwise add all classes).
- `ajax(...)`: we attempt to resolve all functions f in the argument to the call to constant functions (usually this coded directly in place by the programmer). If we are able to do this, we translate f with the `this` variable set to σ. We return $\rho = \sigma$. If we cannot resolve f we ignore it (the body of f will be translated elsewhere with `this` matching any node).
- `append(`s`)`: we attempt to obtain a constant string from s. If s is a concatenation of strings and variables, we attempt to resolve all variables to strings. If we fail, we make the simplifying assumption that variables do not contain HTML code (this can be controlled via the command line) and omit them from the concatenation. If we are able to obtain a constant HTML string in this way we use a sequence of new rules and classes to build that tree at the nodes selected by g. E.g. $(g, \texttt{AddChild}(\{x, a\}))$, $(\{x\}, \texttt{AddChild}(\{b\}))$, $(\{x\}, \texttt{AddChild}(\{c\}))$ where x is a fresh class adds a sub-tree containing a node with class "a" with two children with classes "b" and "c" respectively. In the case where we cannot find a constant string (e.g. if we assume variables may contain HTML strings), we simply add rules to construct all possible sub-trees.

- `bind(..., f)`: we attempt to resolve f to a constant function (usually this is provided directly by the programmer). If we are able to do this, we translate f with the `this` variable set to σ. We return $\rho = \sigma$. If we cannot resolve f we ignore it (the body of f will be translated elsewhere with `this` matching any node)
- `blur(f)`: we treat this like a call to `bind(..., f)`.
- `children(`s`)`: we first obtain from the CSS selector s a guard g' (or true if s is not constant), then we set $\rho = \sigma(\langle\uparrow\rangle g \wedge g')$.
- `click(`f`)`: same as `blur()`.
- `closest(`s`)`: we first obtain from the CSS selector s a guard g' (or true if s is not constant), then we set $\rho = \sigma(\langle\downarrow^*\rangle g \wedge g')$.
- `data(...)`: we return $\rho = \sigma$.
- `each(`f`)`: we attempt to resolve f to a constant, as with `click`. The function f may take 0 or 2 arguments (the second being the matched element). In both cases we translate f with `this` set to σ. If it has arguments, then the second argument is also set to σ. We return $\rho = \sigma$.
- `eq(...)`: we return $\rho = \sigma g$.
- `fadeIn(...)`: we return $\rho = \sigma$.
- `fadeOut(...)`: we return $\rho = \sigma$.
- `focus(f)`: same as `blur()`.
- `end()`: we return $\rho = \sigma'$.
- `find(`s`)`: we obtain from the CSS selector s a guard g' (or true if s is not constant), then we set $\rho = \sigma(\langle\uparrow^+\rangle g \wedge g')$.
- `filter(...)`: we simply return $\rho = \sigma g$.
- `first(...)`: we simply return $\rho = \sigma g$.
- `has(`s`)`: we obtain from the CSS selector s a guard g' (or true if s is not constant), then we set $\rho = \sigma(\langle\downarrow^+\rangle g' \wedge g)$.
- `hover(`f_1`,`f_2`)`: we attempt to resolve f_1 and f_2 to constant functions (else we ignore them to be translated elsewhere with less precise guards). Both f_1 and f_2 take a single argument, which is the element being hovered over. We translate both functions with the argument and the `this` variable set to σ.
- `html(`s`)`: this is handled like `append`.
- `next(`s`)`: we obtain from the CSS selector s a guard g' (or true if s is not constant or not supplied), then we set $\rho = \sigma(\langle\uparrow\rangle\langle\downarrow\rangle g \wedge g')$.
- `nextAll(`s`)`: we obtain from the CSS selector s a guard g' (or true if s is not constant or not supplied), then we set $\rho = \sigma(\langle\uparrow\rangle\langle\downarrow\rangle g \wedge g')$.
- `not(...)`: we ignore the negation and simply return $\rho = \sigma g$.

- `on(`e`,`s`,`f`)`: we obtain from the CSS selector s a guard g' (or true if s is not constant or not supplied), and we attempt to resolve f to a constant function (like `click`). We then translate f with `this` set to $\sigma(\langle\uparrow^+\rangle g \wedge g')$. We return with $\rho = \sigma$.
- `parent(`s`)`: we obtain from the CSS selector s a guard g' (or true if s is not constant or not supplied), then we set $\rho = \sigma(\langle\downarrow\rangle g \wedge g')$.
- `parents(`s`)`: we obtain from the CSS selector s a guard g' (or true if s is not constant or not supplied), then we set $\rho = \sigma(\langle\downarrow^+\rangle g \wedge g')$.
- `prepend(`s`)`: this is handled like `append`.
- `prev(`s`)`: we obtain from the CSS selector s a guard g' (or true if s is not constant or not supplied), then we set $\rho = \sigma(\langle\uparrow\rangle\langle\downarrow\rangle g \wedge g')$.
- `prevAll(`s`)`: we obtain from the CSS selector s a guard g' (or true if s is not constant or not supplied), then we set $\rho = \sigma(\langle\uparrow\rangle\langle\downarrow\rangle g \wedge g')$.
- `remove(`s`)`: this function is ignored and we return $\rho = \sigma$.
- `removeClass(`s`)`: this function is ignored and we return $\rho = \sigma$.
- `resize(f)`: same as `blur()`.
- `show(...)`: we return $\rho = \sigma$.
- `slideUp(...)`: we return $\rho = \sigma$.
- `slideDown(...)`: we return $\rho = \sigma$.
- `stop(...)`: we return $\rho = \sigma$.
- `val(...)`: we return $\rho = \sigma g$.

We also ignore the following functions that do not return a jQuery object or manipulate the DOM tree: `extend()`, `hasClass()`, `height()`, `is()`, `width()`.

O.3 More Complex JavaScript

In reality, it is rare for jQuery statements to be completely independent of the rest of the code. We extend our translation by identifying common jQuery programming patterns to enable the generation of more precise rules. This is, in effect, a kind of ad-hoc dataflow analysis over the syntax tree of the JavaScript.

We remark that current JavaScript static analysers do not track the kind of information needed to build meaningful guards from the jQuery calls. Indeed, implementing a systematic dataflow analysis for propagating jQuery guards remains an interesting avenue of future research.

Often a call will take the form

```
$(x).f(y)
```

for some variable `x`. In the worst case one can assume any value for guards derived from `x` by using the guard $\top$. In other cases it is possible to be more precise.

When translating a call such as

```
$('.a').each(function () {
    $(this).addClass('b');
));
```

for example, is it easy to infer that the selection returned by the `this` variable is all elements with the class "a". Hence, when we are translating constant functions such as these, we track the guards in the `this` variable as described in the previous section. Similarly, if we wrote

```
$('.a').each(function (i, e) {
    $(e).addClass('b');
));
```

we pass the stack of guards from the `$('a')` selector to the `e` variable, and are thus able to interpret `$(e)`.

Note, we are also able to translate `$(`s`, x)` where `x` is `this` or some variable known to hold a jQuery object with stack σg by using the guard

$$\langle\uparrow^+\rangle g \wedge g'$$

where g' is the guard obtained from the selector s. Similarly `$(document)` can be translated to the root node of the HTML.

We can further improve the analysis by tracking variables declared in the JavaScript. To this end, we identify all variables that are assigned outside of a loop or conditional variable (i.e. assigned only once) and remember their value for use in later translations. This captures common cases such as

```
var eles = $('myelements');
...
eles.addClass('b');
eles.append('<p>Some test.</p>');
```

Finally, we also provide some support for user-defined jQuery functions. In a preprocessing step, we look for lines of the form

```
$.fn.f = function (args) { ... };
```

and keep a map from `f` to the function definition. Thus, when we find elsewhere in the code a jQuery call of the form

```
...f(params)...
```

we can translate the function body of `f` with the `this` variable set to the current jQuery stack, any parameters can be passed to the arguments of the function. Note, if we discover a line of `f` which overwrites a parameter variable, we will forget the value passed from the call and use a top value instead.