

# Number fields without small generators

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## Abstract

Let  $D > 1$  be an integer, and let  $b = b(D) > 1$  be its smallest divisor. We show that there are infinitely many number fields of degree  $D$  whose primitive elements all have relatively large height in terms of  $b$ ,  $D$  and the discriminant of the number field. This provides a negative answer to a questions of W. Ruppert from 1998 in the case when  $D$  is composite. Conditional on a very weak form of a folk conjecture about the distribution of number fields, we negatively answer Ruppert's question for all  $D > 3$ .

## 1 Introduction

Let  $L$  be a number field of degree  $D$ , and for  $\alpha \in L$  let

$$H(\alpha) = \prod_{v \in M_L} \max\{1, |\alpha|_v\}^{\frac{d_v}{D}}$$

be the absolute multiplicative Weil height of  $\alpha$ . Here  $M_L$  denotes the set of places of  $L$  and for each place  $v$  we choose the unique representative  $|\cdot|_v$  that either extends the usual Archimedean absolute value on  $\mathbb{Q}$  or a usual  $p$ -adic absolute value on  $\mathbb{Q}$ , and  $d_v = [L_v : \mathbb{Q}_v]$  denotes the local degree at  $v$ . As is well-known  $H(\alpha)$  is independent of the number field  $L$  containing  $\alpha$ , and hence  $H(\cdot)$  extends to a function on the algebraic numbers  $\overline{\mathbb{Q}}$ .

From now on let  $L \subset \overline{\mathbb{Q}}$  be a number field of degree  $D > 1$ . We are interested in bounds, expressed in terms of the degree  $D$  and the absolute discriminant  $\Delta_L$  of  $L$ , for the smallest height of a generator. It is convenient to use the following invariant, introduced by Roy and Thunder [7],

$$\delta(L) = \inf\{H(\alpha); L = \mathbb{Q}(\alpha)\}.$$

By Northcott's Theorem [6, Theorem 1] the infimum is attained, and hence,  $\delta(L)$  denotes the smallest height of a generator of the extension  $L$  over  $\mathbb{Q}$ . Silverman

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[10, Theorem 1] has shown that

$$\delta(L) \geq D^{-\frac{1}{2(D-1)}} |\Delta_L|^{\frac{1}{2D(D-1)}}. \quad (1.1)$$

The following example due to Ruppert [8, p.18] and Masser [7, Proposition 1]) shows that in this general situation the exponent  $1/(2D(D-1))$  cannot be improved. Let  $p$  and  $q$  be primes that satisfy  $0 < p < q < 2p$ . Let  $\alpha = (p/q)^{1/D}$ , and let  $L = \mathbb{Q}(\alpha)$ . Then, by the Eisenstein criterion,  $L$  has degree  $D$ , and  $p$  and  $q$  are both totally ramified in  $L$ . Hence,  $(pq)^{D-1} | \Delta_L$ , and thus

$$H(\alpha) = q^{\frac{1}{D}} \leq (2pq)^{\frac{1}{2D}} \leq 2^{\frac{1}{2D}} |\Delta_L|^{\frac{1}{2D(D-1)}}. \quad (1.2)$$

Ruppert [8, Question 1] asked whether the exponent is always sharp, more precisely he proposed the following question.

**Question 1.1** (Ruppert, 1998). Is there a constant  $C_D$  such that for all number fields  $L$  of degree  $D$

$$\delta(L) \leq C_D |\Delta_L|^{\frac{1}{2D(D-1)}}?$$

In fact Ruppert used the naive height  $H_{naive}(\alpha)$  which is defined as the maximum norm of the coefficient vector of the minimal polynomial of  $\alpha$  over  $\mathbb{Z}$ . It is well-known [2, Lemma 1.6.7] that  $2^{-1}H(\alpha) \leq H_{naive}(\alpha)^{1/D} \leq 2H(\alpha)$ ; here  $D$  denotes the degree of  $\alpha$ . This shows that Ruppert's question is equivalent to the one stated above. Ruppert [8, Proposition 2] himself answered this question in the affirmative for  $D = 2$ . The aim of this note is to answer Ruppert's question in the negative for all composite  $D$ .

**Theorem 1.2.** *Let  $b = b(D) > 1$  be the smallest divisor of  $D$ , and suppose  $\gamma$  is a real number such that*

$$\gamma < \begin{cases} 1/(D(b+1)) : & \text{if } b \leq 3, \\ 1/(2D(b+1)) + 1/(Db^2(b+1)) : & \text{otherwise.} \end{cases}$$

*Then there exist infinitely many number fields  $L$  of degree  $D$  satisfying*

$$\delta(L) > |\Delta_L|^\gamma.$$

Note that for composite  $D > 4$  we have

$$\frac{1}{2D(b+1)} + \frac{1}{Db^2(b+1)} > \frac{1}{2D(\sqrt{D}+1)} > \frac{1}{2D(D-1)}.$$

Thus, Theorem 1.2 provides a negative answer to Question 1.1 for all composite  $D$ . In fact, we prove a stronger result, namely: let  $F$  be any number field of degree  $D/b$ ; when enumerated by the modulus of the discriminant, the subset of all degree  $b$  extensions  $L$  of  $F$ , defined by  $\delta(L) > |\Delta_L|^\gamma$ , has density 1 (for the precise statement we refer to Corollary 4.1).

Our proof strategy requires a good lower bound for the number of degree  $D$  fields with bounded modulus of the discriminant. Essentially optimal bounds are available when  $D$  is even or divisible by 3, and if  $D$  is composite we still have some useful bounds. However, a folk conjecture (sometimes attributed to Linnik) predicts the asymptotics  $c_D T$  as  $T$  goes to infinity, for some constant  $c_D > 0$ . Unfortunately, the best general lower bounds are only of order  $T^{1/2+1/D^2}$  which is just slightly weaker than what we need. Therefore, our next result is conditional.

**Theorem 1.3.** *Suppose that  $D > 3$ , and suppose that there exist constants  $c_D > 0$  and  $\nu_D > 1/2 + 1/(D - 1)$  such that the number of degree  $D$  fields  $L \subset \overline{\mathbb{Q}}$  with absolute value of the discriminant no larger than  $T$  is at least  $c_D T^{\nu_D}$  for all  $T$  large enough. Then there exists  $\gamma > 1/(2D(D - 1))$  such that there are infinitely many number fields  $L$  of degree  $D$  with*

$$\delta(L) > |\Delta_L|^\gamma.$$

Thanks to [1], the hypothesis of Theorem 1.3 is satisfied for  $D = 5$ , and hence we get an unconditional negative answer to Question 1.1 for  $D = 5$ . Furthermore, Theorem 1.3 shows that most likely the answer to Question 1.1 is “no” for all  $D > 3$ . However, our method sheds no light on the case  $D = 3$ .

In this article we use Vinogradov’s notation  $\ll$  and  $\gg$  at various places. The involved constants are allowed to depend on all quantities except on the parameter  $T$ , which is introduced in the next section.

## 2 Enumerating fields: discriminant versus delta-invariant

Any number field is considered a subfield of the fixed algebraic closure  $\overline{\mathbb{Q}}$ . Let  $k$  be a number field, let  $m = [k : \mathbb{Q}]$ , let  $L/k$  be a finite extension of degree  $d = [L : k] > 1$ , and put  $D = [L : \mathbb{Q}] = md$ . For the remainder of this paper we set

$$\mathcal{C} = \mathcal{C}_d(k) = \{L \subset \overline{\mathbb{Q}}; [L : k] = d\}, \quad (2.1)$$

and for a subset  $S \subset \mathcal{C}$ , and  $\gamma \geq 0$  we set

$$S_\gamma = \{L \in S; \delta(L) > |\Delta_L|^\gamma\}. \quad (2.2)$$

We want to enumerate the fields in  $S$  in two different ways: once by the discriminant (more precisely, the modulus thereof), and once by the delta invariant  $\delta(\cdot)$ . Thus we introduce the counting functions

$$\begin{aligned} N_\Delta(S, T) &= |\{L \in S; |\Delta_L| \leq T\}|, \\ N_\delta(S, T) &= |\{L \in S; \delta(L) \leq T\}|. \end{aligned}$$

Note that both cardinalities are finite; the first one by Hermite's Theorem, the second one by Northcott's Theorem. Next we introduce the set of generators of fields of  $S$

$$P_S = \{\alpha \in \overline{\mathbb{Q}}; \mathbb{Q}(\alpha) \in S\},$$

and its counting function

$$N_H(P_S, T) = |\{\alpha \in P_S; H(\alpha) \leq T\}|.$$

Again, the above cardinality is finite by Northcott's Theorem.

The proof of Theorem 1.2 is based on two simple observations, the first of which, is presented as the following proposition.

**Proposition 2.1.** *Suppose there are positive reals  $\eta$ ,  $\theta$ , and  $\gamma < \eta/\theta$  such that  $N_\Delta(S, T) \gg T^\eta$  and  $N_H(P_S, T) \ll T^\theta$  for all  $T$  large enough. Then*

$$\lim_{T \rightarrow \infty} \frac{N_\Delta(S_\gamma, T)}{N_\Delta(S, T)} = 1.$$

*Proof.* Directly from the definitions we get

$$N_\Delta(S \setminus S_\gamma, T) \leq N_\delta(S \setminus S_\gamma, T^\gamma) \leq N_\delta(S, T^\gamma).$$

The map  $\alpha \rightarrow \mathbb{Q}(\alpha)$  yields a surjection from  $\{\alpha \in P_S; H(\alpha) \leq T^\gamma\}$  to  $\{L \in S; \delta(L) \leq T^\gamma\}$ . Hence, we have

$$N_\delta(S, T^\gamma) \leq N_H(P_S, T^\gamma).$$

On the other hand, by the hypothesis,

$$N_H(P_S, T^\gamma) \ll T^{\gamma\theta},$$

and

$$N_\Delta(S, T) \gg T^\eta,$$

provided  $T$  is large enough. We conclude

$$\lim_{T \rightarrow \infty} \frac{N_\Delta(S \setminus S_\gamma, T)}{N_\Delta(S, T)} = 0,$$

whenever  $\gamma < \frac{\eta}{\theta}$  which proves the proposition.  $\square$

### 3 Bounds for the counting functions

In view of Proposition 2.1 we want to find a set  $S \subset \mathcal{C}$  that maximizes the ratio  $\eta/\theta$ . Taking  $S = \mathcal{C}_b(F) \subset \mathcal{C}$  as the set of fields that contain a fixed extension  $F/k$  of degree  $d/b$  does not affect  $\eta$  in a negative way as we shall see in Lemma 3.1, but it positively affects  $\theta$  as we shall see in Lemma 3.2. This is the second simple but important observation for the proof of Theorem 1.2.

We start with lower bounds for  $\eta$ , that is, lower bounds for  $N_\Delta(\mathcal{C}_b(F), T)$ .

**Lemma 3.1.** *Let  $b = b(d) > 1$  be the smallest divisor of  $d$ , and let  $F$  be an extension of  $k$  of degree  $d/b$ . Then we have*

$$N_{\Delta}(\mathcal{C}_b(F), T) \gg T^{1/2+1/b^2} \quad (3.1)$$

for all  $T$  large enough. If  $d$  is even or divisible by 3 then we even have

$$N_{\Delta}(\mathcal{C}_b(F), T) \gg T, \quad (3.2)$$

for all  $T$  large enough.

*Proof.* First we recall that for  $L \in \mathcal{C}_b(F)$  we have  $|\Delta_L| = |\Delta_F|^b N_{F/\mathbb{Q}}(\mathfrak{D}_{L/F})$ , where  $N_{F/\mathbb{Q}}(\cdot)$  is the absolute norm, and  $\mathfrak{D}_{L/F}$  is the relative discriminant. Thus, counting fields in  $\mathcal{C}_b(F)$  with  $|\Delta_L| \leq T$  is the same as counting fields in  $\mathcal{C}_b(F)$  with  $N_{F/\mathbb{Q}}(\mathfrak{D}_{L/F}) \leq T/|\Delta_F|^b$ . Therefore, Ellenberg and Venkatesh's [5, Theorem 1.1] shows that

$$N_{\Delta}(\mathcal{C}_b(F), T) \geq c' T^{1/2+1/b^2}$$

for some  $c' = c'(b, F) > 0$  and all  $T$  large enough. This yields (3.1). For (3.2) we note that the conjectured asymptotic formula

$$N_{\Delta}(\mathcal{C}_b(F), T) = cT + o(T),$$

where  $c = c(b, F) > 0$ , has been proven by Datskovsky and Wright for  $b = 2$  [3, Theorem 4.2] (see also [4, Corollary 1.2]) and for  $b = 3$  [3, Theorem 1.1]. This proves the lemma.  $\square$

Next we establish an upper bound for  $N_H(P_{\mathcal{C}_b(F)}, T)$ . Recall that  $k$  is a number field of degree  $m$ , and also recall the notation  $\mathcal{C} = \mathcal{C}_d(k)$  from (2.1).

**Lemma 3.2.** *We have for all  $T > 0$*

$$N_H(P_{\mathcal{C}}, T) \ll T^{md(d+1)}. \quad (3.3)$$

With the notation of Lemma 3.1, in particular,

$$N_H(P_{\mathcal{C}_b(F)}, T) \ll T^{md(b+1)}. \quad (3.4)$$

*Proof.* First we note that  $\mathbb{Q}(\alpha) \in \mathcal{C} = \mathcal{C}_d(k)$  implies  $[k(\alpha) : k] = d$ . Therefore,

$$N_H(P_{\mathcal{C}}, T) \leq |\{\alpha \in \overline{\mathbb{Q}}; [k(\alpha) : k] = d, H(\alpha) \leq T\}|.$$

Now Schmidt [9, Theorem] has shown that

$$|\{\alpha \in \overline{\mathbb{Q}}; [k(\alpha) : k] = d, H(\alpha) \leq T\}| \leq C(m, d) T^{md(d+1)}.$$

Therefore  $N_H(P_{\mathcal{C}}, T) \leq C(m, d) T^{md(d+1)}$ , which proves (3.3).  $\square$

## 4 Density results

Recall the notation in (2.2).

**Corollary 4.1.** *Let  $b = b(d) > 1$  be the smallest divisor of  $d$ , and suppose  $\gamma$  is a real number such that*

$$\gamma < \begin{cases} 1/(md(b+1)) : & \text{if } b \leq 3, \\ 1/(2md(b+1)) + 1/(mdb^2(b+1)) : & \text{otherwise.} \end{cases}$$

*Let  $F$  be an extension of  $k$  of degree  $d/b$  and let  $B = \{L \in \mathcal{C}; F \subset L\}$ . Then*

$$\lim_{T \rightarrow \infty} \frac{N_{\Delta}(B_{\gamma}, T)}{N_{\Delta}(B, T)} = 1.$$

*Proof.* First note that  $B = \mathcal{C}_b(F)$ . Thus (3.1) yields  $N_{\Delta}(B, T) \gg T^{1/2+1/b^2}$  for  $T$  large enough. If  $d$  is even or divisible by 3 then by (3.2) we even have  $N_{\Delta}(B, T) \gg T$  for  $T$  large enough. On the other hand (3.4) gives  $N_H(P_B, T) \ll T^{md(b+1)}$ . Applying Proposition 2.1 with  $S = B$  yields the statement.  $\square$

So almost all fields in  $B = \mathcal{C}_b(F)$  satisfy  $\delta(L) > |\Delta|^{\gamma}$ . Note that, of course,  $B$  is an infinite set, and so Theorem 1.2 follows from Corollary 4.1 by taking  $k = \mathbb{Q}$ .

**Corollary 4.2.** *Suppose  $\gamma < 1/(md(d+1))$  and suppose  $d$  is even or divisible by 3 then*

$$\lim_{T \rightarrow \infty} \frac{N_{\Delta}(\mathcal{C}_{\gamma}, T)}{N_{\Delta}(\mathcal{C}, T)} = 1.$$

*Proof.* Let  $F$  be an extension of  $k$  of degree  $d/2$  if  $d$  is even, and of degree  $d/3$  otherwise. Hence,  $\mathcal{C} \supset \mathcal{C}_2(F)$  or  $\mathcal{C} \supset \mathcal{C}_3(F)$  respectively, and so we conclude from (3.2) that  $N_{\Delta}(\mathcal{C}, T) \gg T$ . Furthermore, by (3.3) we have  $N_H(P_{\mathcal{C}}, T) \ll T^{md(d+1)}$ . Applying Proposition 2.1 with  $S = \mathcal{C}$  yields the statement.  $\square$

Finally, to prove Theorem 1.3 we apply Proposition 2.1 with  $S = \mathcal{C}$ ,  $k = \mathbb{Q}$ ,  $\eta = \nu_D > 1/2 + 1/(D-1)$  and  $\theta = D(D+1)$  (for the latter we have applied (3.3)). As  $\eta/\theta > 1/(2D(D-1))$  we conclude that there exists  $\gamma > 1/(2D(D-1))$  such that there exist infinitely many number fields  $L$  of degree  $D$  that satisfy

$$\delta(L) > |\Delta_L|^{\gamma}.$$

This proves Theorem 1.3.

## 5 Cluster points

We consider the set of values

$$\frac{\log \delta(L)}{\log |\Delta_L|}$$

as  $L$  runs over all number fields of fixed degree  $D > 1$ . What are the cluster points of this set? Combining (1.1) and (1.2) gives the smallest cluster point

$$\liminf_{[L:\mathbb{Q}]=D} \frac{\log \delta(L)}{\log |\Delta_L|} = \frac{1}{2D(D-1)}.$$

What about the largest cluster point? With  $b = b(D)$  as in Theorem 1.2 the latter implies that

$$\limsup_{[L:\mathbb{Q}]=D} \frac{\log \delta(L)}{\log |\Delta_L|} \geq \begin{cases} 1/(D(b+1)) & \text{if } b \leq 3, \\ 1/(2D(b+1)) + 1/(b^2(b+1)D) & \text{otherwise.} \end{cases}$$

If  $D$  is odd [11, Theorem 1.2] or if the Dedekind zeta-function associated to the Galois closure of  $L$  satisfies the Generalized Riemann Hypothesis for all number fields  $L$  of degree  $D$ , then [11, Theorem 1.3]

$$\limsup_{[L:\mathbb{Q}]=D} \frac{\log \delta(L)}{\log |\Delta_L|} \leq 1/(2D).$$

However, the best known unconditional general upper bound for the largest cluster point is  $1/D$ , see, e.g., [12, Lemma 4.5]. It might be an interesting problem to study the distribution of the cluster points, and to locate new cluster points.

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