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Long term behaviour of locally interacting birth-and-death processes

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Abstract

In this paper we study long-term evolution of a finite system of locally interacting birth-and-death processes labelled by vertices of a finite connected graph. A detailed description of the asymptotic behaviour is obtained in the case of both constant vertex degree graphs and star graphs. The model is motivated by modelling interactions between populations, adsorption-desorption processes and is related to interacting particle systems, Gibbs models with unbounded spins, as well as urn models with interaction.

1 The model

Let Λ be a finite connected graph. If two vertices $x, y \in \Lambda$ are connected by an edge, call them *neighbours* and write $x \sim y$. Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}_+$ be the set of all non-negative integers including zero. Consider a continuous time Markov chain (CTMC) $\xi(t) = \{\xi_x(t), x \in \Lambda\} \in \mathbb{Z}_+^\Lambda$ with the following transition rates: given $\xi(t) = \xi \in \mathbb{Z}_+^\Lambda$ a component (a spin) ξ_x increases by 1 at the rate $e^{\alpha\xi_x+\beta\phi(x,\xi)}$, where $\alpha, \beta \in \mathbb{R}$,

$$\phi(x, \xi) = \sum_{y:y\sim x} \xi_y \tag{1}$$

and at the same time each positive component ξ_x decreases by 1 at constant rate 1.

This birth-and-death dynamics belongs to a class of stochastic dynamics which is used in statistical physics to describe the time evolution of a system of interacting spins. Our particular dynamics is motivated by adsorption-desorption processes, where adsorption rates depend on a local environment and an adsorbed particle can depart at a non-zero rate ([3]). It is closely related to a particle deposition on a discrete substrate and urn models with interaction (e.g., [7], [12], and [13]). Recall also that a birth-and-death process on the non-negative integer half-line is a classic probabilistic model for the population size so that the Markov chain can be used for modelling different types of interaction between populations, where a component $\xi_x(t)$ can be interpreted as the size of a population which is located at $x \in \Lambda$ at time t.

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If we assume that spins are bounded and consider the same birth-and-death dynamics then we will get a finite irreducible Markov chain whose stationary probability distribution is a Gibbs measure (see Remark 1). A particular case of the model with bounded spins, where $\alpha = \beta$, $\Lambda \subset \mathbb{Z}^d$, was studied in [14]. For instance, if a spin takes values 0 and 1 only, and, in addition, $\alpha = \beta > 0$, then we obtain a finite Markov chain whose stationary probability distribution is a Gibbs measure on $\{0,1\}^\Lambda$ which is equivalent to a particular case of the famous Ising model on $\{-1,1\}^\Lambda$. The main goal in [14] was to study the asymptotic behaviour of the stationary distribution as $\Lambda \uparrow \mathbb{Z}^d$. In general, the asymptotic behaviour of such equilibrium distributions in thermodynamic limit, i.e. as graph Λ expands, is of main interest in statistical physics.

The aim of this paper, on the other hand, is to describe the asymptotic behaviour of the Markov chain with *unbounded* spins as time tends to infinity while the underlying graph remains fixed. In this case we deal with a countable Markov chain that can be either recurrent (or even positive recurrent) or non-recurrent (e.g., transient, or even explosive) depending both on the graph Λ and the values of parameters α, β.

It is easy to see that if $\beta = 0$ then the structure of graph Λ is irrelevant and the components of CTMC $\xi(t)$ are independent identically distributed birth-and-death processes with values in $\mathbb{Z}_+$. The well known results for birth-and-death processes (e.g. see [4] or [11]) yield that if $\alpha > 0, \beta = 0$, then each component is an explosive Markov chain. In turn, it implies that CTMC $\xi(t)$ is explosive. Moreover, independence of spins imply that their times to explosion are also independent and this allows to repeat the well known Rubin's argument (used in [2] in the case of classic Pólya urn scheme) in order to obtain that with probability 1 only a single component of $\xi(t)$ explodes. Notice that this fact can be also inferred from our Theorem 2. A non-zero interaction does not change the explosive behaviour of the Markov chain in the case $\alpha > 0$ but escape to infinity can happen in various ways which depend on both β and Λ.

If $\alpha < 0, \beta = 0$, then CTMC $\xi(t)$ is formed by a collection of independent positive recurrent Markov chains. It is quite obvious that if both $\alpha < 0$ and $\beta < 0$ then the Markov chain remains to be positive recurrent. If $\beta > 0$, then one could intuitively expect that given $\alpha < 0$ there exists some critical value β_{cr} such that if $\beta < \beta_{cr}$, then the stable ergodic evolution of the system is still observed, and, in contrast, if $\beta > \beta_{cr}$, then the system becomes unstable, i.e. transient or even explosive. We compute this critical value explicitly in some cases. It turns out that $\beta_{cr} = -\alpha c(\Lambda)$, where $c(\Lambda) = \nu^{-1}$ in the case of a graph Λ with the constant vertex degree ν and $c(\Lambda) = n^{-\frac{1}{2}}$ in the case of a star graph Λ with $n+1$ vertices.

The Markov chain under consideration is reversible, therefore the computation of its invariant measure is straightforward. Stationary probability distributions arising in positive recurrent cases are Gibbs measures with unbounded positive spins on a finite graph with empty boundary conditions. Consequently the model in positive recurrent cases is closely related to Gibbs random fields with unbounded spins on graphs (see [6], [8], and references therein).

We give a detailed description of how the Markov chain escapes to infinity in all the transient cases that we consider. We show that due to a rapid increase of birth rates in explosive cases, there are no death events in the system after some finite random moment of time, and the dynamics of the Markov chain is that of a pure birth process, obtained by setting the death rates to zero.

We will start with results that are valid in the case of an arbitrary finite connected graph Λ; they are presented in Theorems 1, 2 and 3. We also study two special cases in more detail, namely constant vertex degree graphs and star graphs. The results for these two cases are found in Theorems 4, 5 and 6. Graphs with the constant vertex degree and star graphs are particular examples of homogeneous graphs and of inhomogeneous graphs, respectively. Despite the obvious difference in the structure of these graphs the long term behaviour of the corresponding Markov chains is similar to each other. The main features of the model dynamics are illustrated in Section 3 by a model with graph Λ formed by just two neighbouring vertices. Proofs are given in Section 4.

Finally, we denote by C_i, $i = 1, 2, ...$, or just C various constants whose exact values are immaterial.

2 Results

Let Λ be an arbitrary finite graph. Given $\xi \in \mathbb{Z}_+^\Lambda$ define *potential* $U(x, \xi)$ of a vertex $x \in \Lambda$ as the following quantity

$$U(x, \xi) = \alpha\xi_x + \beta\phi(x, \xi). \tag{2}$$

Notice the following identity

$$\sum_{x\in\Lambda} U(x, \xi) = \sum_{x\in\Lambda}(\alpha + \beta\nu(x))\xi_x, \tag{3}$$

where $\nu(x)$ is the degree of vertex $x \in \Lambda$, i.e. the number of edges incident to the vertex. Throughout the paper we will also denote by 1_A the indicator of a set (or event) A. In these notations, given $\xi(t) = \xi \in \mathbb{Z}_+^\Lambda$ a component ξ_x jumps up by 1 with intensity $e^{U(x,\xi)}$ and the generator of the Markov chain is therefore

$$\mathsf{L}f(\xi) = \sum_{x\in\Lambda} \left(f\left(\xi + \mathsf{e}^{(x)}\right) - f(\xi)\right) e^{U(x,\xi)} + \left(f\left(\xi - \mathsf{e}^{(x)}\right) - f(\xi)\right) 1_{\{\xi_x>0\}}, \tag{4}$$

where $\mathsf{e}^{(x)}$ is a configuration such that $\mathsf{e}_x^{(x)} = 1$ and $\mathsf{e}_y^{(x)} = 0$ for all $y \neq x$ (addition of configurations is understood component-wise).

Let us define the following function

$$W(\xi) = \frac{\alpha}{2}\sum_x \xi_x(\xi_x - 1) + \beta\sum_{x\sim y}\xi_x\xi_y, \quad \xi \in \mathbb{Z}_+^\Lambda. \tag{5}$$

It is easy to see that

$$e^{U(x,\xi)}e^{W(\xi)} = e^{W\left(\xi+\mathsf{e}^{(x)}\right)}$$

for all $x \in \Lambda$ and $\xi \in \mathbb{Z}_+^\Lambda$. This equation is a detailed balance condition which implies that CTMC $\xi(t)$ is time-reversible with invariant measure $e^{W(\xi)}$, $\xi \in \mathbb{Z}_+^\Lambda$. If

$$Z_{\alpha,\beta,\Lambda} = \sum_{\xi\in\mathbb{Z}_+^\Lambda} e^{W(\xi)} < \infty, \tag{6}$$

then CTMC $\xi(t)$ has a stationary probability distribution given by

$$\mu_{\alpha,\beta,\Lambda}(\xi) = \frac{e^{W(\xi)}}{Z_{\alpha,\beta,\Lambda}}, \quad \xi \in \mathbb{Z}_+^{\Lambda}. \tag{7}$$

It is well known (e.g., [1] or [9]) that existence of a stationary probability distribution of an irreducible CTMC is equivalent to the CTMC being positive recurrent. Moreover, an irreducible positive recurrent CTMC is ergodic in a sense that it converges (in distribution) to its stationary probability distribution as time goes to infinity.

Remark 1 If a component of the Markov chain takes values in $\{0, 1, \dots, N\}$, where $N \geq 1$, then the invariant probability distribution of the Markov chain is defined similar to measure (7). Namely, it is a probability measure on $\{0, 1, \dots, N\}^{\Lambda}$ that is equal, up to a normalizing constant, to function $e^{W(\xi)}$, where, in turn, function W is defined, as before, by (5).

We are ready now to formulate the findings of our paper. We start with the results that are valid for all finite connected graphs.

Theorem 1 *Let Λ be a finite connected graph.*

1) If $\alpha < 0$ and $\alpha + \beta \max_{x\in\Lambda} \nu(x) \leq 0$ then CTMC $\xi(t)$ is not explosive. Moreover, if $\alpha < 0$ and $\alpha + \beta \max_{x\in\Lambda} \nu(x) < 0$ then CTMC $\xi(t)$ is positive recurrent.

2) If $\alpha \geq 0$ then CTMC $\xi(t)$ is not positive recurrent.

3) If $\alpha + \beta \min_{x\in\Lambda} \nu(x) > 0$ then CTMC $\xi(t)$ is explosive.

Recall that the embedded Markov chain, corresponding to a continuous time Markov chain, is a discrete time Markov chain (DTMC) with the same state space, and that makes the same jumps as the continuous time Markov chain with probabilities proportional to the corresponding jump rates. Let $\zeta(t)$ be the DTMC corresponding to CTMC $\xi(t)$. The states of the embedded Markov chain will be denoted by ζ and we will use the same symbol $t = 0, 1, 2, \dots,$ to denote the discrete time.

The non-recurrent behaviour of the Markov chain in Part 2) of Theorem 1 can now be described more precisely under certain additional assumptions. In order to do so, define the following event related to DTMC $\zeta(t)$:

$$\begin{aligned} B = \{&\exists \tau \in \mathbb{Z}_+ \text{ and a vertex } x \in \Lambda \text{ such that} \\ &\zeta_y(\tau + s + 1) = \zeta_y(\tau + s) + 1_{\{y=x\}}, \ \forall s \geq 1\big\}, \end{aligned} \tag{8}$$

in other words, the process grows only at point x after time τ.

Theorem 2 *Let Λ be a finite, not necessarily connected, graph. If $\alpha > \max\{0, \beta\}$ then with probability 1 event B defined by (8) occurs, and a single component of CTMC $\xi(t)$ explodes.*

Furthermore, given $x_1, x_2 \in \Lambda$ define the following event

$$B_{x_1,x_2} = \Big\{\exists s \in \mathbb{Z}_+ : \zeta_y(t) = \zeta_y(s) \text{ for all } y \notin \{x_1, x_2\} \text{ and all } t \geq s; \quad \lim_{t\to\infty} \frac{\zeta_{x_1}(t)}{t} = \lim_{t\to\infty} \frac{\zeta_{x_2}(t)}{t} = \frac{1}{2}\Big\}, \tag{9}$$

in other words, the process grows only at two points x_1 and x_2 after time s and the speed of growth is approximately the same.

Theorem 3 *Let Λ be a finite connected graph without triangles, i.e. such that there are no three distinct vertices $x, y, z \in \Lambda$ such that $x \sim y$, $y \sim z$ and $z \sim x$. If $0 < \alpha < \beta$ then with probability 1 there are two random adjacent vertices x_1 and x_2 such that the event (9) occurs. This implies that with probability 1 only a pair of adjacent components of the CTMC explodes.*

Theorem 4 *Let Λ be a graph with the constant vertex degree $\nu(x) \equiv \nu$.*

1) CTMC $\xi(t)$ is positive recurrent if and only if $\alpha < 0$ and $\alpha + \beta\nu < 0$.

2) If $\alpha < 0$ and $\alpha + \beta\nu = 0$ then CTMC $\xi(t)$ is transient.

3) If $\alpha \leq 0$ and $\alpha + \beta\nu > 0$ then CTMC $\xi(t)$ is explosive.

4) If $\alpha > 0$ then CTMC $\xi(t)$ is explosive. Moreover,

i) if $\beta < \alpha$ then with probability 1 the event (8) occurs, so that with probability 1 a single component of CTMC $\xi(t)$ explodes;

ii) if $\alpha < \beta$ and the graph Λ is without triangles (as explained in Theorem 3) then with probability 1 the event B_{x_1,x_2} occurs for some adjacent vertices $x_1, x_2 \in \Lambda$, so that with probability 1 a pair of adjacent components of the CTMC explodes.

Let us mention two examples of constant vertex degree graphs, both with and without triangles.

a) *Lattice models with local interaction.* Let $\mathbb{Z}$ be the set of all integers. Given integers $L > 0, d \geq 1$, let $\Lambda = \{0, ..., L-1\}^d \in \mathbb{Z}^d$ be a lattice cube with periodic boundary conditions. Call $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \Lambda$ neighbours if there is $j \in \{1, 2, \dots, d\}$ such that $x_i = y_i$ for all $i \neq j$ and $x_j - y_j = \pm 1 \mod L$.

In this case $\nu(x) \equiv 2d$ and the graph does not have triangles.

b) *Mean-field model.* Given $n \geq 2$ let Λ be a complete graph with n vertices. By construction, $\nu(x) \equiv n - 1$ in this example and the graph *does have* triangles.

The following statement complements Theorem 4 in the mean field case.

Theorem 5 *Let Λ be a complete graph with n vertices labelled by $1, \dots, n$, where $n \geq 1$. If either $0 < \alpha < \beta$ or $\alpha < 0 < \alpha + \beta\nu$ where $\nu = n - 1$, then*

1) $\zeta_k(t)/t \to 1/n$ for all $k = 1, \dots, n$ a.s.;

2) all components of CTMC $\xi(t)$ explode simultaneously a.s.;

3) a process of differences $(\zeta_1(t)-\zeta_n(t), \dots, \zeta_{n-1}(t)-\zeta_n(t)) \in \mathbb{Z}^{n-1}$ converges in distribution as $t \to \infty$.

Finally, Theorem 6 below describes the long-term behaviour of the Markov chain in the case of a star graph.

Theorem 6 *Given $n \geq 1$ let Λ be a star graph with $(n+1)$ vertices, i.e. where there is a central vertex x and its neighbouring vertices $y_1, \dots, y_n$, so that x is the only neighbour for each of y_i, $i = 1, \dots, n$, and $x \sim y_i$, $i = 1, \dots, n$. Then*

1) CTMC $\xi(t)$ is positive recurrent if and only if $\alpha < 0$ and $\alpha + \beta\sqrt{n} < 0$;

2) if $\alpha < 0$ and $\alpha + \beta\sqrt{n} = 0$ then CTMC $\xi(t)$ is transient;

3) if $\alpha < 0$ and $\alpha + \beta\sqrt{n} > 0$ then with probability 1

$$\frac{\zeta_x(t)}{t} \to \frac{n\beta + |\alpha|}{(n+1)\beta + 2|\alpha|}, \quad \frac{\zeta_{y_i}(t)}{t} \to \frac{\beta + |\alpha|}{(n+1)\beta + 2|\alpha|}, \ i = 1, 2, \dots, n,$$

as $t \to \infty$; moreover with probability 1 all components of CTMC $\xi(t)$ explode simultaneously.

4) If $\alpha > 0$ then CTMC $\xi(t)$ is explosive. Moreover,

i) if $\beta < \alpha$ then with probability 1 the event (8) occurs, so that with probability 1 a single component of CTMC $\xi(t)$ explodes;

ii) if $\alpha < \beta$ then with probability 1 the event B_{x,y_i} occurs for some $i = 1, \dots, n$, so that with probability 1 only a pair of adjacent components of CTMC $\xi(t)$ explodes.

Remark 2 It is easy to see that some parts of Theorems 4 and 6 are direct corollaries of Theorems 1, 2 and 3 and we formulate them mostly in order to have a complete stand-alone description of the asymptotic behaviour of the Markov chain in the case of both constant degree graphs and star graphs.

We would like also to comment on the asymptotic behaviour of the Markov chain in the case $\alpha = 0$. If $\alpha = 0$, $\beta > 0$ then CTMC $\xi(t)$ is explosive and the corresponding DTMC is transient (see Part 3) of Theorem 1 and its proof) in the case of an arbitrary finite connected graph Λ. On the other hand, we do not know a complete answer in the case $\alpha = 0$, $\beta < 0$, which seems to be more interesting in the following sense. We anticipate that either both Markov chains are null recurrent or transient and a particular behaviour depends on the structure of the underlying graph. We show rigorously in Section 3 that if $\alpha = 0$, $\beta < 0$ and graph Λ is formed by two vertices then both the DTMC and the CTMC are null recurrent. An intuitive argument supporting this fact is the following. If, say, both components of the Markov chain are large then they most likely will drift almost deterministically towards the origin. If one of them is zero and another one is sufficiently large then the latter evolves as a symmetric simple random walk which is null recurrent and the zero component has very small chances to increase. The same intuition suggests that if $\alpha = 0$, $\beta < 0$ and Λ is a star graph with three vertices ($n = 2$ in Theorem 6), then both Markov chains are null recurrent as well, but if Λ is a star graph with 4 vertices ($n = 3$), then both Markov chains are transient. We do not consider the case $\alpha = 0$, $\beta < 0$ in more detail here and hope to address it in our subsequent publications.

3 Random walk in the quarter plane

Let graph Λ be formed by two adjacent vertices. In this case the corresponding Markov chain is equivalent to an inhomogeneous random walk on the positive quarter plane. We will briefly comment on this particular case to illustrate some distinctive features of the model dynamics, which can be also observed in more general situations.

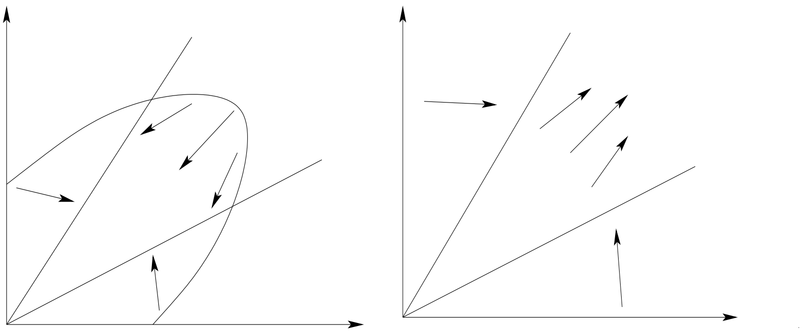

Figure 1: The vector field of mean jumps of Markov chains, $\alpha < 0$, $\beta > 0$. The vertical axis is y axis and the horizontal axis is x axis. Left: $\alpha + \beta < 0$; the upper line is $y = -\frac{\alpha}{\beta}x$, the lower line is $y = -\frac{\beta}{\alpha}x$, the curve is $Q(x, y) = C$, for some $C > 0$. Right: $\alpha + \beta > 0$; the upper line is $y = -\frac{\beta}{\alpha}x$, the lower line is $y = -\frac{\alpha}{\beta}x$.

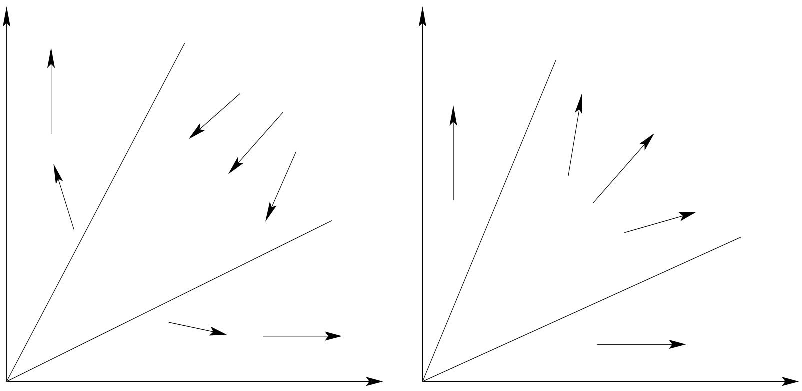

Figure 2: The vector field of mean jumps of Markov chains, $\alpha > 0$, $\beta < 0$. The vertical axis is y axis and the horizontal axis is x axis. Left: $\alpha + \beta < 0$; the upper line is $y = -\frac{\beta}{\alpha}x$, the lower line is $y = -\frac{\alpha}{\beta}x$. Right: $\alpha + \beta > 0$; the upper line is $y = -\frac{\alpha}{\beta}x$, the lower line is $y = -\frac{\beta}{\alpha}x$.

The theorems in Section 2 and Remark 3 imply the following results for the two-dimensional case.

1) If $\alpha < 0$ and $\beta < |\alpha|$ then both CTMC $\xi(t)$ and DTMC $\zeta(t)$ are positive recurrent. Left part of Figure 1 sketches the vector field of mean jumps of the Markov chain and level curves of Lyapunov function $Q(x, y) = -\alpha x^2 - \alpha y^2 - 2\beta xy$ in the positive recurrent case $0 < \beta < -\alpha$.

2) If $\alpha < 0$ and $\alpha + \beta \geq 0$ then DTMC $\zeta(t)$ is transient, CTMC $\xi(t)$ is explosive; moreover $\mathsf{P}(\zeta_1(t) = \zeta_2(t) \text{ infinitely often}) = 1$. The vector field of mean jumps in the case $\alpha < 0, \alpha+\beta > 0$ is illustrated by the right part of Figure 1.

3) If $\alpha > 0$ then CTMC $\xi(t)$ is explosive. If, in addition, $\beta < \alpha$ then with probability 1 a single component of DTMC will eventually grow (event (8) occurs). We illustrate this by the left part of Figure 2 in the case $\beta < -\alpha < 0$. The right part of the same figure corresponds to the transient/explosive case $-\alpha < \beta < 0$. If $\alpha < \beta$ then both components grow and

$$\mathsf{P}(\zeta_1(t) = \zeta_2(t) \text{ infinitely often}) = 1.$$

4) In the two-dimensional case we also deal with the case $\alpha = 0$ and $\beta < 0$ and show that both CTMC $\xi(t)$ and DTMC $\zeta(t)$ are null recurrent. Indeed, let us show recurrence of DTMC $\zeta(t)$. By the well-known criterion for recurrence (e.g., Theorem 2.2.1 in [5]) to show recurrence of DTMC $\zeta(t)$ it suffices to find a positive function $f(x, y)$ such that $f(x, y) \to \infty$ as $\sqrt{x^2 + y^2} \to \infty$ and for which the following inequality

$$\mathsf{E}(f(\zeta_1(t+1), \zeta_2(t+1)) | \zeta(t) = (x, y)) - f(x, y) \leq 0, \tag{10}$$

holds for all but finitely many (x, y). The above inequality is equivalent to the following one

$$\mathsf{L}f(x, y) \leq 0, \tag{11}$$

where L is the generator of the corresponding continuous Markov chain (see (4)). Consider a function $f(x, y) = \log(x + y + 1)$. We will show that if the sum $x + y$ is sufficiently large, then the inequality (11) holds. Indeed, if $y = 0$ then

$$\begin{aligned} \mathsf{L}f(x, 0) &= (\log(x+2) - \log(x+1)) \left(1 + e^{\beta x}\right) + (\log(x) - \log(x+1)) \\ &= \log\frac{x+2}{x+1} + \log\frac{x}{x+1} + e^{\beta x}\log\frac{x+2}{x+1} \leq \log\left(1 - \frac{1}{(x+1)^2}\right) + e^{\beta x} \leq 0, \end{aligned}$$

where the last inequality holds for sufficiently large $x > 0$. If both $x > 0$ and $y > 0$ then assuming that $C = x + y$ is large enough, we have the following bound:

$$\begin{aligned} \mathsf{L}f(x, y) &= (\log(C+2) - \log(C+1)) \left(e^{\beta y} + e^{\beta x}\right) \\ &\quad + (\log(C) - \log(C+1)) \left(1_{\{x>0\}} + 1_{\{y>0\}}\right) \\ &\leq 2\left(\log(C+2) - 2\log(C+1) + \log(C)\right) = 2\log\frac{C(C+2)}{(C+1)^2} \leq 0. \end{aligned}$$

It is well known (e.g. [9]) that either both an irreducible CTMC and its corresponding DTMC are recurrent or both chains are transient. Therefore, CTMC $\xi(t)$ is also recurrent. It is easy to see that CTMC $\xi(t)$ cannot be positive recurrent in this case. Indeed, had it been recurrent, then its stationary distribution would be given by formula (7), but the latter is impossible, since

$$\sum_{x,y\in Z_+} e^{\beta xy} = \infty$$

for all $\beta < 0$. Since all transition rates are uniformly bounded below and above, this yields that DTMC $\zeta(t)$ cannot be positive recurrent either.

4 Proofs

4.1 Proof of Theorem 1

We start with introduction of the following quantities

$$Q(\xi) = -\alpha\sum_x \xi_x^2 - 2\beta\sum_{x\sim y}\xi_x\xi_y, \tag{12}$$

$$S(\xi) = \sum_x \xi_x. \tag{13}$$

This allows us to rewrite function (5) as

$$W(\xi) = -\frac{1}{2}(Q(\xi) + \alpha S(\xi)). \tag{14}$$

Recall that $\nu(x)$ denotes the degree of vertex $x \in \Lambda$ and notice the following useful representations of the quadratic part of W

$$\begin{aligned} Q(\xi) &= \sum_x(-\alpha - \beta\nu(x))\xi_x^2 + \beta\sum_{x\sim y}(\xi_x - \xi_y)^2 \\ &= \sum_{x\in\Lambda}\big(-\alpha\xi_x^2 - \beta\xi_x\phi(x,\xi)\big) = -\sum_{x\in\Lambda}\xi_x U(x,\xi). \end{aligned} \tag{15}$$

Proof of Part 1) of Theorem 1. Notice first that if $\alpha < 0$ and $\beta \le 0$ then the stationary distribution (7) is well-defined and the Markov chain $\xi(t)$ is positive recurrent.

We will show now that CTMC does not explode, if $\alpha < 0$, $\beta > 0$ and $\alpha + \beta\max_{x\in\Lambda}\nu(x) \le 0$. Define

$$\tau_N = \min\left\{t : \max_{x\in\Lambda}\xi_x(t) = N\right\}.$$

It is obvious that the Markov chain is explosive if and only if

$$\mathsf{P}\left(\lim_{N\to\infty}\tau_N < \infty\right) > 0,$$

but the latter cannot happen. Indeed, given $\xi(t) = \xi$ let $x \in \Lambda$ be such that $\xi_x = \max_{y\in\Lambda}\xi_y$. Then

$$U(x,\xi) = \alpha\xi_x + \beta\phi(x,\xi) \le (\alpha + \beta\nu(x))\xi_x \le \left(\alpha + \beta\max_{x\in\Lambda}\nu(x)\right)\xi_x \le 0.$$

Therefore the waiting times $\tau_{N+1} - \tau_N$ are stochastically larger than exponentially distributed independent random variables with parameter $(2|\Lambda|)^{-1}$; as a result, the limit $\lim\limits_{N\to\infty} \tau_N$ is infinite with probability 1 and thus the chain does not explode.

Let us finally show that if

$$\alpha < 0,\ \beta > 0, \alpha + \beta \max_{x\in\Lambda} \nu(x) < 0, \tag{16}$$

then $Z_{\alpha,\beta,\Lambda} < \infty$ and consequently the stationary probability distribution is well defined. Since the quadratic part in $W(\xi)$ (see (5)) dominates the linear part, it is easy to see that $Z_{\alpha,\beta,\Lambda} < \infty$ if

$$\sum_{\xi\in\mathbb{Z}_+^\Lambda} \exp(-Q(\xi)/2) < \infty, \tag{17}$$

where $Q(\xi)$ is defined by (12). Consider a symmetric matrix $A_Q = (a_{xy})_{x,y\in\Lambda}$ determining the quadratic form Q, i.e.

$$Q(u) = (A_Q u, u), \quad u \in \mathbb{R}^{|\Lambda|}. \tag{18}$$

It is easy to see that $a_{xx} = -\alpha$, $a_{xy} = -\beta$, if $y \sim x$ and $a_{xy} = 0$ otherwise. Therefore for all $x \in \Lambda$

$$|a_{xx}| - \sum_{y\neq x} |a_{xy}| = -\alpha - \beta\nu(x) \geq -\alpha - \beta \max_{x\in\Lambda} \nu(x) > 0,$$

because of (16). In other words, matrix A_Q is strictly diagonally dominant with positive diagonal entries and hence, by standard algebra, is positive definite. In the case under consideration one can also observe positive definiteness of A_Q from representation (15). Positive definiteness of A_Q implies that

$$\int\limits_{\mathbb{R}^{|\Lambda|}} e^{-(A_Q u,u)/2} du = \frac{(2\pi)^{\frac{|\Lambda|}{2}}}{\sqrt{\det(A_Q)}} < \infty,$$

which, in turn, implies (17), so the stationary probability distribution is well defined as claimed. ■

Proof of Part 2) of Theorem 1. The Markov chain $\xi(t)$ does not have a stationary probability distribution if $\alpha \geq 0$. Indeed, fix $x \in \Lambda$ and define a set of configurations $D_x = \{\xi : \xi_x \geq 0,\ \xi_y = 0,\ y \neq x\}$. It is easy to see that

$$Z_{\alpha,\beta,\Lambda} \geq \sum_{\xi\in D_x} e^{W(\xi)} = \sum_{k=0}^{\infty} e^{\alpha k(k-1)/2} = \infty,$$

and the stationary distribution does not exist. ■

Proof of Part 3) of Theorem 1. Denote for short $\nu_{\min} = \min_{x\in\Lambda} \nu(x)$ throughout the proof of this part of the theorem. We start with proving a statement (Lemma 1) that implies transience of DTMC $\zeta(t)$.

Lemma 1 *Let Λ be a finite connected graph. If $\alpha + \beta\nu_{\min} > 0$ then with probability 1 there exists a time moment $\tau < \infty$ such that for all $t \geq \tau$ none of the components of DTMC $\zeta(t)$ decreases.*

Proof of Lemma 1. Recall that $U(x,\zeta)$ is the quantity defined by equation (2) and the quantity $S(\zeta)$ is defined by (13). Since $\alpha + \beta\nu_{\min} > 0$ equation (3) implies that for all ζ

$$\max_{x\in\Lambda} U(x,\zeta) \geq C_1 S(\zeta), \tag{19}$$

where $C_1 = (\alpha + \beta\nu_{\min})/|\Lambda|$. Using this bound for the maximal potential we get the following inequality

$$\begin{aligned}\mathsf{P}(S(\zeta(t+1)) = S(\zeta(t)) + 1|\zeta(t) = \zeta) &= 1 - \frac{\sum_{x\in\Lambda} 1_{\{\zeta_x>0\}}}{\sum_{x\in\Lambda} e^{U(x,\zeta)} + \sum_{x\in\Lambda} 1_{\{\zeta_x>0\}}} \\ &\geq 1 - \frac{|\Lambda|}{\max\limits_{x\in\Lambda} e^{U(x,\zeta)}} \geq 1 - |\Lambda| e^{-C_1 S(\zeta)}.\end{aligned}$$

Therefore, if $D_s = \{$none of the components ever decreases after time $s\}$, then

$$\mathsf{P}(D_s|\zeta(s) = \zeta) \geq \prod_{t=s}^{\infty} \left(1 - C_2(\zeta) e^{-C_1(t-s)}\right) = 1 - o(S(\zeta)) \tag{20}$$

where $C_2(\zeta) = |\Lambda| e^{-C_1 S(\zeta)}$ and $o(S(\zeta)) \to 0$ as $S(\zeta) \to \infty$. Since for each $N = 1, 2, \ldots,$ the set of configurations $\{\zeta :\ S(\zeta) \geq N\}$ is finite and the Markov chain is irreducible, we can define $\tau_N = \min\{t :\ S(\zeta(t)) = N\} < \infty$. As $\mathsf{P}(D_{\tau_N}) \to 1$, by continuity of probability $\mathsf{P}(\cup_N D_{\tau_N}) = 1$ and hence there exists N such that after time $\tau = \tau_N$ the only changes in the system are increases of the components. ■

It is quite obvious that Lemma 1 implies transience of the DTMC in the case $\alpha + \beta\nu_{\min} > 0$. Nevertheless we would like to provide another lemma (Lemma 2 below) that also ensures transience in this case. The main reason is that this lemma takes into account the geometry of mean jumps and formalizes intuition which can be inferred from, for example, the images located on the right hand side of Figures 1 and 2. Besides, it provides an idea for proving transience in Part 2) of Theorem 4 (see Lemma 5 below).

Lemma 2 *Let Λ be a finite connected graph. If $\alpha + \beta\nu_{\min} > 0$, then for any $0 < \varepsilon < 1$ the following bound holds*

$$\mathsf{E}(S(\zeta(t+1)) - S(\zeta(t))|\zeta(t) = \zeta) \geq \varepsilon, \tag{21}$$

provided that $S(\zeta) > C_1 = C_1(\varepsilon)$.

Proof of Lemma 2. It is easy to see that inequality (21) is equivalent to the following one

$$\mathrm{J}(\zeta,\varepsilon) := \sum_{x\in\Lambda} \left(\delta(\varepsilon) e^{U(x,\zeta)} - 1_{\{\zeta_x>0\}}\right) \geq 0, \tag{22}$$

where

$$\delta(\varepsilon) = \frac{1-\varepsilon}{1+\varepsilon}. \tag{23}$$

Using subsequently inequality $1_{\{\zeta_x>0\}} \leq 1$, equation (3) and inequality $e^u > 1+u$, $u \in \mathbb{R}$, we obtain

$$\begin{aligned}
\mathrm{J}(\zeta,\varepsilon) &\geq \sum_{x\in\Lambda}\left(\delta(\varepsilon)e^{U(x,\zeta)}-1\right) \\
&\geq \sum_{x\in\Lambda}\left(\delta(\varepsilon)(1+U(x,\zeta))-1\right) \\
&\geq \delta(\varepsilon)(\alpha+\beta\nu_{\min})S(\zeta)-(1-\delta(\varepsilon))|\Lambda| > 0,
\end{aligned}$$

provided that $S(\zeta) > C_1 := \frac{(1-\delta(\varepsilon))|\Lambda|}{\delta(\varepsilon)(\alpha+\beta\nu_{\min})}$.

Observe that it is also possible to use inequality between the arithmetical and geometric means and equation (3) in order to obtain that

$$\sum_{x\in\Lambda} e^{U(x,\zeta)} \geq |\Lambda| e^{\frac{(\alpha+\beta\nu_{\min})S(\zeta)}{|\Lambda|}}$$

and to arrive to a similar result (provided that $S(\zeta) > C_2$, where C_2 is another constant). ■

Lemma 2 means that conditions of Theorem 2.2.7 in [5] are satisfied with the linear function $f(\zeta) = S(\zeta)$ and set $A = \{\zeta \in \mathbb{Z}_+^\Lambda : S(\zeta) \geq C_1\}$ and the embedded Markov chain $\zeta(t)$ is transient in the case $\alpha + \beta\nu_{\min} > 0$.

Let us finish the proof of Part 3) of Theorem 1. If $\alpha + \beta\nu_{\min} > 0$, then transience of CTMC $\xi(t)$ is implied (regardless of the sign of α) by Lemma 1 (or by Lemma 2). Let us show that CTMC $\xi(t)$ is explosive. Indeed, given a configuration ξ bound (19) implies the following lower bound for the total transition rate

$$\sum_{x\in\Lambda}\left(e^{U(x,\xi)} + 1_{\{\xi_x>0\}}\right) \geq \max_{x\in\Lambda} e^{U(x,\xi)} \geq e^{C_1 S(\xi)},$$

where, as before, $C_1 = (\alpha+\beta\nu_{\min})/|\Lambda|$. Besides, none of the components decrease after τ steps of the embedded process (recall that τ is defined in Lemma 1). Therefore the only changes in the systems are jumps up and these jumps happen with exponentially increasing rates whose inverses are summable. This yields explosion.

■

Function Q as the Lyapunov function for Foster criterion. Observe that positive recurrence of the Markov chain in Part 1) of Theorem 1 can be shown by using Foster criterion for positive recurrence of a countable CTMC (see e.g. [10]). We skip the easy case, when both $\alpha < 0$ and $\beta < 0$ and show that if $\alpha < 0, \beta > 0$ and $\alpha + \beta \max_{x\in\Lambda}\nu(x) < 0$ the function Q serves as the corresponding Lyapunov function. Indeed, the equation (15) yields that $Q(\xi) > 0$ for all $\xi \in \mathbb{Z}_+^\Lambda$ outside the origin (i.e., $\xi \neq 0$) and that $Q(\xi) \to \infty$ as $\sum_{x\in\Lambda}\xi_x^2 \to \infty$. Recall that L is the generator (defined by (4)) of the Markov chain. We fix some $\varepsilon > 0$ and show that

$$\mathsf{L}Q(\xi) \leq -\varepsilon, \tag{24}$$

provided that $S(\xi) = \sum_{x\in\Lambda} \xi_x \geq C$, where $C = C(\varepsilon)$ is sufficiently large. It is easy to see that

$$\mathsf{L}Q(\xi) = \sum_{x\in\Lambda}(-\alpha - 2U(x,\xi))e^{U(x,\xi)} + \sum_{x\in\Lambda}(-\alpha + 2U(x,\xi))1_{\{\xi_x>0\}}, \quad (25)$$

where $U(x,\xi)$ is defined by equation (2). Sums in (25) can be respectively bounded as follows

$$\sum_{x\in\Lambda}(-\alpha - 2U(x,\xi))e^{U(x,\xi)} \leq |\Lambda| \max_{u\in\mathbb{R}}(-\alpha + 2u)e^{-u} = 2|\Lambda|e^{\frac{-\alpha-2}{2}},$$

and

$$\begin{aligned}\sum_{x\in\Lambda}(-\alpha + 2U(x,\xi))1_{\{\xi_x>0\}} &\leq \sum_{x\in\Lambda}(-\alpha + 2U(x,\xi))\\ &= -\alpha|\Lambda| + 2\sum_{x\in\Lambda}(\alpha + \beta\nu(x))\xi_x\\ &\leq -\alpha|\Lambda| + 2(\alpha + \beta \max_{x\in\Lambda}\nu(x))S(\xi)\\ &\leq -\alpha|\Lambda| + 2C(\alpha + \beta \max_{x\in\Lambda}\nu(x)),\end{aligned}$$

where we used the equation (3) to get the equality. Thus the LHS of (24) is bounded by the following quantity

$$2|\Lambda|e^{\frac{-\alpha-2}{2}} - \alpha|\Lambda| + 2C(\alpha + \beta \max_{x\in\Lambda}\nu(x)),$$

which is less than $-\varepsilon$, if $C > 0$ is sufficiently large. The inequality (24) allows to apply Foster criterion of positive recurrence (Theorem 1.7 in [10]) of a countable CTMC.

Remark 3 It should be noticed that DTMC $\zeta(t)$ is also positive recurrent under conditions of Part 1) of Theorem 1. This can be proved by applying the Foster criterion for positive recurrence of a countable DTMC (e.g. Theorem 2.2.3 in [5]) with the same function Q as the Lyapunov function for the criterion. Modifications of the above calculations are straightforward and we skip the details.

4.2 Proof of Theorem 2

Given $\zeta \in \mathbb{Z}_+^\Lambda$ define

$$M_\zeta = \max_{x\in\Lambda} U(x,\zeta) \quad\text{and}\quad D_\zeta = \{x \in \Lambda : U(x,\zeta) = M_\zeta\}.$$

Depending on the values of $\{\alpha, \beta\}$ there can be two different cases.

1) A finite connected graph Λ is such that

$$M_\zeta \geq 0 \text{ for all } \zeta \in \mathbb{Z}_+^\Lambda. \quad (26)$$

We say in this case that Λ is a **type I graph**.

2) The set of configurations

$$\mathcal{K} = \{\zeta : M_\zeta < 0\}, \tag{27}$$

is not empty, then we say that Λ is a **type II graph**.

Let us consider some examples before proceeding further. It is obvious that if both α and β are positive, then any graph is a type I graph. Also, if $\alpha > 0 > \beta$ and $\alpha + \beta \max_{x\in\Lambda} \nu(x) \geq 0$ then for every $x \in \Lambda$ such that $\zeta_x = N = \max_{y\in\Lambda} \zeta_y$ the following inequality holds

$$U(x,\zeta) = \alpha N + \beta\phi(x,\zeta) \geq N\left(\alpha + \beta \max_{x\in\Lambda} \nu(x)\right) \geq 0,$$

hence, Λ is a type I graph.

Consider two main examples of type II graphs. In both examples $\alpha > 0$ and $\beta < 0$.

(i) Suppose that $\alpha + \beta\nu < 0$, and let Λ be a constant vertex degree graph with $\nu(x) \equiv \nu$. In this case $\mathcal{K}$ is a non-empty since it contains all the points where $\zeta(x) \equiv c\ \forall x \in \Lambda$ for some $c \in \mathbb{Z}_+$.

(ii) Suppose $\alpha + \beta\sqrt{n} < 0$ and let Λ be a star graph with $n+1$ vertices. In this case the set $\mathcal{K}$ contains all the points where $\zeta(x) = c_1$, $\zeta(y_i) = c_2\ \forall i = 1, 2, \ldots, n$ (see the statement of Theorem 6) and $c_1, c_2 \in \mathbb{Z}_+$ solve the system of inequalities

$$\begin{cases} \alpha c_1 + \beta n c_2 < 0, \\ \alpha c_2 + \beta c_1 < 0 \end{cases}$$

(one can easily check that under the above conditions on α and β the solution is non-empty).

We start the proof with the following lemma.

Lemma 3 *There is a $\delta' > 0$ such that*

$$\mathsf{P}\left(B|\zeta(t) = \zeta\right) > \delta', \tag{28}$$

for all t and ζ.

Proof of Lemma 3 for type I graph. For a given $x \in \Lambda$ define the following event

$$B_x(t) = \{\zeta_x(s+1) = \zeta_x(s) + 1,\ \zeta_y(s) = \zeta_y(t),\ \text{for } y \neq x \text{ and } s \geq t\}.$$

Trivially, $B_x(t) \subset B$. We are going to show that for any ζ and $x \in D_\zeta$

$$\mathsf{P}(B_x(t)|\zeta(t) = \zeta) > \delta' > 0,$$

where δ' might depend only on parameters α, β and graph Λ. Given $x \in \Lambda$ and $\zeta \in \mathbb{Z}_+^\Lambda$ denote

$$R(x,\zeta) = \sum_{y\in\Lambda} e^{U(y,\zeta)} - \left(e^{U(x,\zeta)} + \sum_{y\sim x} e^{U(y,\zeta)}\right).$$

If $x \in D_\zeta$ then

$$R(x,\zeta)e^{-U(x,\zeta)} = R(x,\zeta)e^{-M_\zeta} \le (|\Lambda| - \nu(x) - 1) < |\Lambda|, \tag{29}$$

for all $\zeta \in \mathbb{Z}_+^\Lambda$. Given $x \in D_\zeta$ we have that

$$\begin{aligned}\mathsf{P}\left(B_x(t)|\zeta(t)=\zeta\right) &= \prod_{k=0}^{\infty} \frac{e^{M_\zeta+\alpha k}}{e^{M_\zeta+\alpha k} + \sum\limits_{y\sim x} e^{U(y,\zeta)+\beta k} + R(x,\zeta) + \sum_{y\in\Lambda} 1_{\{\zeta_y>0\}}} \\ &= \prod_{k=0}^{\infty} \frac{1}{1 + \sum\limits_{y\sim x} e^{U(y,\zeta)-M_\zeta-(\alpha-\beta)k} + \left(R(x,\zeta) + \sum\limits_{y\in\Lambda} 1_{\{\zeta_y>0\}}\right) e^{-M_\zeta-\alpha k}},\end{aligned}$$

for all $\zeta \in \mathbb{Z}_+^\Lambda$. It is easy to see that by choice of x we have

$$\sum_{y\sim x} e^{U(y,\zeta)-M_\zeta-(\alpha-\beta)k} \le e^{-(\alpha-\beta)k} \max_{y\in\Lambda} \nu(y).$$

Also, using (26) and (29) we get that

$$\left(R(x,\zeta) + \sum_{y\in\Lambda} 1_{\{\zeta_y>0\}}\right) e^{-M_\zeta-\alpha k} \le 2|\Lambda|e^{-\alpha k}. \tag{30}$$

Therefore, we obtain the following bound

$$\mathsf{P}\left(B_x(t)|\zeta(t)=\zeta\right) \ge \prod_{k=0}^{\infty} \frac{1}{1 + e^{-(\alpha-\beta)k}\max_{y\in\Lambda}\nu(y) + 2|\Lambda|e^{-\alpha k}} = \delta' > 0. \tag{31}$$

The preceding display implies bound (28) in the case of type I graph. ■

Proof of Lemma 3 for type II graph. Fix some $\varepsilon > 0$ and suppose that $\zeta \in \mathcal{K}_\varepsilon = \{\zeta : M_\zeta \ge -\varepsilon\}$. Given $x \in D_\zeta$ one can repeat, with a minor change, the same argument which led to bound (31). The only difference now is that the inequality $M_\zeta \ge -\varepsilon$ yields constant $(1+e^\varepsilon)|\Lambda|e^{-\alpha k}$ in the right side of (30) (instead of $2|\Lambda|e^{-\alpha k}$) and it results in a different $\delta'' \neq \delta'$ such that

$$\mathsf{P}(B_x(t)|\zeta(t)=\zeta) > \delta'' > 0.$$

Consider the opposite case, when $\zeta \in \mathcal{K}_\varepsilon^c = \{\zeta : M_\zeta < -\varepsilon\}$. Define a stopping time

$$\tau = \min\{t : \zeta(t) \in \mathcal{K}_\varepsilon\}.$$

We will now show that $\mathsf{P}(\tau < \infty|\zeta(0)=\zeta) = 1$ for all $\zeta \in \mathcal{K}_\varepsilon^c$, which means that the results of the previous paragraph apply. This would complete the proof of bound (28).

Indeed, define $F(\zeta) = |\,\zeta\,|^2$, where $|\,\zeta\,|$ is Euclidean norm in $\mathbb{R}^{|\Lambda|}$. A direct computation gives that there exists some $\varepsilon' > 0$ such that

$$\mathsf{E}(F(\zeta(t+1)) - F(\zeta(t))|\zeta(t)=\zeta) \le -\varepsilon'$$

for all $\zeta \in \mathcal{K}_\varepsilon^c$. Let $Y_t = F(X_{t\wedge\tau})$ then

$$\mathsf{E}(Y_{t+1} - Y_t | \zeta(0), \ldots, \zeta(t)) \le -\varepsilon' 1_{\{t<\tau\}} \tag{32}$$

so that Y_t is a non-negative supermartingale which converges a.s. Taking the expectation of in (32) yields $\mathsf{P}(\tau > t) \to 0$ as $t \to \infty$ thus $\tau < \infty$ a.s. ■

Proof of Theorem 2. By Lemma 3

$$\mathsf{E}\left(1_B | \mathcal{F}_t\right) > \delta', \tag{33}$$

where $\mathcal{F}_t = \sigma\{\zeta_0, \ldots, \zeta_t\}$ is the $\sigma-$algebra of events generated by DTMC up to time moment t. Since $B \in \mathcal{F}_\infty = \sigma\{\mathcal{F}_t,\, t \ge 0\}$ we get by Lévy's $0-1$ law that

$$\mathsf{E}(1_B | \mathcal{F}_t) \to \mathsf{E}(1_B | \mathcal{F}_\infty) = 1_B, \text{ as } t \to \infty.$$

By (33) the right hand side of the preceding display is positive. Therefore, it must be equal to 1, hence, $\mathsf{P}(B) = 1$.

Thus, eventually only a single component of the embedded chain continues to evolve by jumping up without jumping down. In the continuous time setting the only growing component evolves eventually as a pure birth process with exponentially growing birth rates. The latter process is explosive and, hence, CTMC $\xi(t)$ is explosive, where with probability 1 only a single component explodes. ■

Remark 4 Under the assumptions of the theorem with probability one a typical trajectory of DTMC $\zeta(t)$ returns to set $\mathcal{K}_\varepsilon^c$ only a finite number of times in the case of type II graph.

4.3 Proof of Theorem 3

We start with the following lemma.

Lemma 4 *Let $0 < \alpha < \beta$. Suppose that x_1 and x_2 are two vertices of Λ such that (1) $x_1 \sim x_2$; (2) there is no y such that $y \sim x_1$ and $y \sim x_2$ at the same time; (3) at some time s the configuration of the DTMC is such that $u_1 = U(x_1, \zeta(s))$ is the largest potential on the whole graph and $u_2 = U(x_2, \zeta(s))$ is the largest potential among all the neighbours of x_1. Then, with a positive probability depending on α, β and Λ only, the following events simultaneously occur*

$$\zeta_y(t) = \zeta_y(s) \text{ for all } y \notin \{x_1, x_2\} \text{ and all } t = s, s+1, s+2, \ldots;$$

$$\lim_{t\to\infty} \frac{\zeta_{x_1}(t)}{t} = \lim_{t\to\infty} \frac{\zeta_{x_2}(t)}{t} = \frac{1}{2}.$$

Proof of Lemma 4. Observe that every time when the component at x_1 increases by 1, the potential at x_1 increases by α while the potential at each of the neighbours of x_1 increases by β, therefore the potential at x_2 remains the largest among the neighbours of x_1. At the same time the difference between the potentials at x_1 and x_2 decreases by $\delta := \beta - \alpha > 0$.

Let $k = \left\lfloor \frac{u_1-u_2}{\delta} \right\rfloor$ where $\lfloor a \rfloor$ denotes the integer part of $a \in \mathbb{R}$. Assume that k is even; the case with odd k can be handled very similarly. Denoting by $\nu_1 = \nu(x_1)$ the degree of vertex x_1 (and $\nu_2 = \nu(x_2)$ respectively), we obtain that the probability that during the times $t = s, s+1, ...s+k$ only the component at x_1 increases is larger than

$$\begin{aligned}
p_1 &= \prod_{i=0}^{k} \frac{e^{u_1+i\alpha}}{e^{u_1+i\alpha} + \nu_1 e^{u_2+i\beta} + (|\Lambda| - \nu_1)e^{u_1}} \\
&= \prod_{i=0}^{k} \frac{1}{1 + \nu_1 e^{-(u_1-u_2)+i(\beta-\alpha)} + (|\Lambda| - \nu_1)e^{-i\alpha}} \\
&\geq \prod_{i=0}^{k} \frac{1}{1 + \nu_1 e^{-(k-i)\delta} + |\Lambda|e^{-i\alpha}} \\
&\geq \prod_{i=0}^{k/2} \frac{1}{1 + \nu_1 e^{-k\delta/2} + |\Lambda|e^{-i\alpha}} \times \prod_{j=0}^{k/2} \frac{1}{1 + \nu_1 e^{-j\delta} + |\Lambda|e^{-k\alpha/2}} \\
&\geq \left(\prod_{i=0}^{k/2} \frac{1}{1 + (\nu_1 + |\Lambda|)(e^{-i\delta} + e^{-i\alpha})} \right)^2 \\
&\geq \left(\prod_{i=0}^{\infty} \frac{1}{1 + (\nu_1 + |\Lambda|)(e^{-i\delta} + e^{-i\alpha})} \right)^2 = C_1(|\Lambda|, \alpha, \beta) > 0.
\end{aligned}$$

Consequently, by time $s+k$ we have $-\delta < U(x_2, \zeta(s+k)) - U(x_1, \zeta(s+k)) \leq 0$ with probability at least p_1.

From now on for simplicity of notations assume that the state where $u_2 \in (u_1 - \delta, u_1]$ has already been reached at time s (as opposed to a later time). Let $m_i(t)$, $i = 1, 2$ be the number of times x_i was chosen during the times $s+1, s+2 \ldots, s+t$. Define the events

$$\begin{aligned}
A'_k &= \{\zeta_y(s+i) = \zeta_y(s) \text{ for all } y \notin \{x_1, x_2\},\ i = 1, 2, \ldots, 2k^2\}, \\
A''_k &= \left\{ |m_1(2k^2) - m_2(2k^2)| \leq 2k \right\}, \\
A_k &= A'_k \cap A''_k.
\end{aligned}$$

Then under A_k we have $m_1(2k^2) + m_2(2k^2) = 2k^2$ and $|m_i(2k^2) - k^2| \leq k$ for $i = 1, 2$. So, denoting $s_{k,i} = s + 2k^2 + i$ we get that $\mathsf{P}(A'_{k+1}|A_k)$ is no less than

$$\begin{aligned}
&\prod_{i=0}^{4k+1} \frac{e^{U(x_1,\zeta(s_{k,i}))} + e^{U(x_2,\zeta(s_{k,i}))}}{e^{U(x_1,\zeta(s_{k,i}))} + e^{U(x_2,\zeta(s_{k,i}))} + (\nu_1 + \nu_2)e^{u_2+\beta(k^2+6k)} + (|\Lambda| - \nu_1 - \nu_2)e^{u_2}} \\
&\geq \prod_{i=0}^{4k+1} \frac{1}{1 + |\Lambda|e^{(7\beta+\alpha)k-\alpha k^2}} \geq 1 - C_2(|\Lambda|, \alpha)e^{-k},
\end{aligned}$$

since $U(x_1, \zeta(s_{k,i})) \geq u_1 + \alpha(k^2 - k) + \beta(k^2 - k)$, and the potential at any vertex y adjacent either to x_1 or to x_2 is bounded by

$$u_2 + \beta(k^2 + k + (4k + 1)) \leq u_2 + \beta(k^2 + 6k).$$

To estimate $\mathsf{P}(A''_{k+1}|A_k)$ observe that whenever $m_1(j) > m_2(j)+1$ the potential at x_2 is larger, and the similar statement holds if one swaps 1 and 2. Now, there are two possibilities at time $j = s+2k^2$: (a) $|m_1(2k^2) - m_2(2k^2)| \le 1.5k$ and (b) $|m_1(2k^2) - m_2(2k^2)| > 1.5k$.

In case (a), the difference $|m_1(j) - m_2(j)|$ can be majorized by the distance to the origin of the simple symmetric random walk on $\mathbb{Z}^1$. In particular, the probability that during $4k+2$ steps it is further than $k^{2/3}$ from the starting point is bounded by $c_3 e^{-k^{1/6}}$ where c_3 is some constant. As a result,, with probability at least $1 - c_3 e^{-k^{1/6}}$ we have

$$|m_1(2(k+1)^2) - m_2(2(k+1)^2)| < 1.5k + k^{2/3} < 2(k+1)$$

and A''_{k+1} occurs.

On the other hand, in case (b) we have $1.5k < |m_1(2k^2) - m_2(2k^2)| \le 2k$, hence the potential at the larger x_i in the pair $\{x_1, x_2\}$ is much smaller than the potential at the smaller x in this pair. Consequently, for the next k steps the probability to increase the larger component, divided by the probability to increase the smaller component, is bounded above by $e^{-\delta k/2}$, so we can couple $|m_1(j) - m_2(j)|$ with an asymmetric simple random walk on $\mathbb{Z}^1$ with the drift towards the origin. As a result, we obtain that with probability at least $1 - e^{-c_4 k}$ during the times $t = s + 2k^2 + i$, $i = 1, \ldots, k$, the distance between m_1 and m_2 decreases at least by $k/2$, bringing it to the value less than $2k - (k/2) = 1.5k$, and thus to case (a). Therefore,

$$\mathsf{P}(A''_{k+1}|A_k) \ge 1 - C_3 e^{-k^{1/6}} - e^{-C_4 k}.$$

Combining the above inequalities yields

$$\mathsf{P}(A_{k+1}|A_k) \ge 1 - C_3 e^{-k^{1/6}} - e^{-C_4 k} - C_2(|\Lambda|, \alpha) e^{-k}. \tag{34}$$

Since the product of the terms on the RHS of (34) over all large enough k is positive, the statement of the lemma follows. ∎

Now note that at any moment of time s there is a vertex x_1 with the largest potential. Because of our assumption it satisfies the conditions of Lemma 4 for *some* neighbour x_2. Hence, Theorem 3 follows from the Lévy's 0–1 law. ∎

4.4 Proof of Theorem 4

Proof of Part 1) of Theorem 4. Positive recurrence in the case $\alpha < 0$, $\alpha + \beta\nu < 0$ and absence of positive recurrence in the case $\alpha \ge 0$ are implied by Theorem 1. If $\alpha < 0$, $\alpha + \beta\nu \ge 0$ then, using equations (14) and (15), we get the following bound

$$Z_{\alpha,\beta,\Lambda} \ge \sum_{\xi \in Z_+^\Lambda} e^{W(\xi)} 1_{\{\xi: \xi_x = \xi_y, \forall x,y \in \Lambda\}} = \sum_{k=1}^{\infty} e^{|\Lambda|((\alpha+\beta\nu)k^2 - \alpha k)/2} = \infty,$$

which means that the stationary distribution does not exist in this case and, hence, the CTMC is not positive recurrence. ∎

We want to remark that if Λ is a constant vertex degree graph then $(-\alpha-\beta\nu)$ is the eigenvalue of A_Q with the corresponding eigenvector $(1,\ldots,1)$ and, hence, the function $\exp(-Q(\xi)/2)$ is not summable in the direction of this eigenvector, provided that $-\alpha-\beta\nu\geq 0$. Furthermore, if $\alpha<0,\beta>0$ then $-\alpha-\beta\nu$ is the minimal eigenvalue of A_Q, since all eigenvalues of matrix A_Q lie, by Gershgorin circle theorem (see e.g. [15]), within the closed interval $[-\alpha-|\beta|\nu,-\alpha+|\beta|\nu]$. Also, in the case of the mean-field model with n vertices (complete graph with n vertices) one can easily compute the characteristic polynomial of matrix A_Q:

$$(-1)^{n-1}(\alpha-\beta+\mu)^{n-1}(-\alpha-(n-1)\beta-\mu),$$

and analysis of the eigenvalues yields the same results.

Proof of Part 2) of Theorem 4 We start with showing transience of DTMC $\zeta(t)$. Transience in the discrete time case is implied by the following lemma (which based on the same intuition as Lemma 2).

Lemma 5 *Let Λ be a finite connected graph with the constant vertex degree $\nu(x)\equiv\nu$. If $\alpha+\beta\nu=0$, then there exist $\varepsilon>0$ and $C>0$ such that the following bound holds*

$$\mathsf{E}(S(\zeta(t+k(\zeta(t)))-S(\zeta(t))|\zeta(t)=\zeta)\geq\varepsilon, \tag{35}$$

provided that $S(\zeta)\geq C$ and where

$$k(\zeta)=\begin{cases}1, & \text{if } U(x,\zeta)\neq 0 \text{ for at least one } x\in\Lambda,\\ 2, & \text{if } U(x,\zeta)=0 \text{ for all } x\in\Lambda.\end{cases}$$

Proof of Lemma 5. As we already noted in the proof of Lemma 2 inequality (35) with $k\equiv 1$ is equivalent to the following one

$$\mathrm{J}(\zeta,\varepsilon)=\delta(\varepsilon)\sum_{x\in\Lambda}e^{U(x,\zeta)}-\sum_{x\in\Lambda}1_{\{\zeta_x>0\}}\geq 0, \tag{36}$$

where $\delta(\varepsilon)$ is defined by (23) and (36) would be implied by

$$\delta(\varepsilon)\sum_{x\in\Lambda}e^{U(x,\zeta)}-|\Lambda|\geq 0.$$

Notice that by inequality between geometric and arithmetic means we have that for all ζ

$$\sum_{x\in\Lambda}e^{U(x,\zeta)}-|\Lambda|\geq 0, \tag{37}$$

since by equation (3)

$$\sum_{x\in\Lambda}U(x,\zeta)=(\alpha+\beta\nu)S(\zeta)=0. \tag{38}$$

It is well known that given numbers $a_1,\ldots,a_m$ geometric and arithmetic means of these numbers are equal to each other if and only if $a_1=\ldots=a_m$. Therefore, equation (38) also implies that identity $\sum_{x\in\Lambda} e^{U(x,\zeta)} - |\Lambda| = 0$ holds if and only if $U(x,\zeta) = 0$, for all $x \in \Lambda$ otherwise we have got a strict inequality in (37). Thus, if there are exactly $0 < m \leq |\Lambda|$ vertices with non zero potentials then

$$\sum_{x\in\Lambda} e^{U(x,\zeta)} - |\Lambda| \geq \sum_{x\in\Lambda:U(x,\zeta)\neq 0} e^{U(x,\zeta)} - m > 0.$$

It is easy to see that since the inequality in the preceding display is strict there exists $\delta_m \in (0,1)$ such that

$$\delta_m \sum_{x\in\Lambda} e^{U(x,\zeta)} - |\Lambda| \geq \delta_m \sum_{x\in\Lambda:U(x,\zeta)\neq 0} e^{U(x,\zeta)} - m > 0,$$

because values of potentials U belong to a discrete set $\{\alpha(k - j/\nu),\ k, j \in \mathbb{Z}_+\}$ (where we used that $\beta = -\alpha/\nu$) which is bounded away from zero. Thus, given $0 < m \leq |\Lambda|$ we claim existence of δ_m and, hence, existence of the corresponding $\varepsilon = \varepsilon(\delta_m)$ (using equation (23)). The required in Lemma 4 ε is obtained thus by setting $\varepsilon = \min_m \varepsilon_m$.

It is easy to see that all potentials cannot stay zero for two steps in a row, hence

$$\mathsf{E}\left(S(\zeta(t+2)) - S(\zeta(t))|\zeta(t) = \zeta\right) = \mathsf{E}(S(\zeta(t+2)) - S(\zeta(t+1))|\zeta(t) = \zeta) \geq \varepsilon.$$

Thus inequality (35) is proven. ∎

Lemma 5 implies that the conditions of Theorem 2.2.7 in [5] are fulfilled and hence the embedded Markov chain is transient.

We are ready now to finish the proof of Part 2) of Theorem 4. If $\alpha + \beta\nu = 0$ then transience of DTMC $\zeta(t)$ implies at least transience of CTMC $\xi(t)$. By Theorem 1 CTMC $\xi(t)$ does not explode if $\alpha < 0$, $\alpha + \beta\nu = 0$. Hence, CTMC $\xi(t)$ is transient if $\alpha < 0$, $\alpha + \beta\nu = 0$. ∎

Remark 5 Let us notice how the sign of parameter α influences the process dynamics in the case $\alpha + \beta\nu = 0$. If $\alpha > 0$, $\alpha + \beta\nu = 0$, then Theorem 2 applies (since $\beta < 0$) and, eventually, a single component of the Markov chain explodes. A set of configurations $\{\xi : \xi_x = \xi_y,\ x, y \in \Lambda\}$ is "unstable" in the sense that the process tends to leave it and to never return. In contrast, if $\alpha < 0$, $\alpha + \beta\nu = 0$, then the process tends to stay in a neighbourhood of the same set of configurations (with equal components) while escaping to infinity. It is easy to see that vertex potentials are bounded around this set of configurations and this is why no explosion happens in this case.

Proof of Part 3) of Theorem 4. Part 3) of Theorem 4 is covered by Part 3) of Theorem 1. ∎

Proof of Part 4) of Theorem 4. If both $\alpha > 0$ and $\beta > 0$ then transience of DTMC $\zeta(t)$ and explosiveness of CTMC $\xi(t)$ are obvious. On the other hand, if $\alpha > \max\{0, \beta\}$ then Theorem 2 applies; if $0 < \alpha < \beta$ and the graph Λ is without triangles then Theorem 3 applies. ∎

4.5 Proof of Theorem 5

Let $\zeta(t) = (\zeta_1(t), \ldots, \zeta_n(t))$ be DTMC corresponding to a complete graph with n vertices. It is easy to see that the potential of a component at vertex i at time t is equal to

$$U(i, \zeta(t)) = \alpha\zeta_i(t) + \beta(S(\zeta(t)) - \zeta_i(t)) = (\alpha - \beta)\zeta_i(t) + \beta S(\zeta(t)).$$

First, we present an intuitive argument justifying the theorem, which is made rigorous later. In both cases described in the theorem, $\alpha + \beta\nu > 0$ hence by Lemma 1 there exists a moment of time τ after which none of the components decrease. Also, it is easy to see that in both cases of the theorem β must be positive. So, for $t > \tau$ the probability that it is the $i-$th component that increases is equal to

$$\frac{e^{(\alpha-\beta)\zeta_i(t)+\beta S(\zeta(t))}}{\sum_{k=1}^{n} e^{(\alpha-\beta)\zeta_i(t)+\beta S(\zeta(t))}} = \frac{e^{(\alpha-\beta)\zeta_i(t)}}{\sum_{k=1}^{n} e^{(\alpha-\beta)\zeta_k(t)}}$$

Therefore, in the long run DTMC evolves as a generalized Pólya urn model with weight function $g(x) = e^{(\alpha-\beta)x}$. Now the well-known results for a generalized Pólya urn scheme and Theorem 1 in [12] implies Parts 1) and 3) of Theorem 5. Finally, the explosiveness of the process $\xi(t)$ follows from Parts 3) and 4) of Theorem 4. (One can compare this and the following calculations with the argument presented in the proof of Part 3) of Theorem 6.)

The problem with the above argument is that, strictly speaking, the events $\{\zeta_{i+1}(t) = \zeta_i(t) + 1\}$, $i = 1, 2, \ldots, n$, are *not* independent of the event $\{\tau < t\}$, since the behaviour of the Pólya urn *may* affect the probability of decreasing of a component. Thus, to make the argument rigorous, we construct the following coupling.

Let Y_t, $t = 1, 2, \ldots$, be a sequence of i.i.d. uniform $[0, 1]$ random variables. At time t split the interval $[0, 1]$ into $2n$ intervals with lengths proportional to

$$[e^{U(1,\zeta(t))}, e^{U(2,\zeta(t))}, \ldots, e^{U(n,\zeta(t))}, 1, 1, \ldots, 1]$$

where U is defined by (2). If Y_t falls into the $i-$th subinterval with $1 \leq i \leq n$ then we set $\zeta_i(t + 1) = \zeta_i(t) + 1$; if $n + 1 \leq i \leq 2n$ then we set $\zeta_i(t + 1) = \max\{0, \zeta_i(t) - 1\}$. In both cases we leave the remaining components unchanged. It is easy to see that the process $\zeta(t)$, $t \geq 1$, has exactly the same distribution as the DTMC defined above. At the same time for a fixed $N \in \mathbb{Z}^+$ define the process $\zeta^{(N)}(t)$, $t = N, N + 1, \ldots$, such that $\zeta^{(N)}(N) := \zeta(N)$ and the transition rules of $\zeta^{(N)}(t)$ are exactly the same as that of $\zeta(t)$ with the only exception that when Y_t falls in the interval with index $\geq n + 1$ the process $\zeta^{(N)}(t)$ remains unchanged (i.e., "no deaths"). Let B_N be the event "none of Y_t falls in the intervals indexed $n+1, n+2, \ldots, 2n$ for all $t \geq N$", then on B_N we have $\zeta^{(N)}(t) \equiv \zeta(t)$, $t \geq N$, consequently $\zeta(t)$ has the behaviour of the above Pólya urn with weight function g. Let A be the event $\{\lim_{t\to\infty} \zeta_k(t)/t = 1/n\}$. Since $\zeta_k^{(N)}(t)/t \to 1/n$ a.s., we have

$$\mathsf{P}(A) \geq \mathsf{P}(A|B_N)\mathsf{P}(B_N) = \mathsf{P}(B_N).$$

On the other hand, Lemma 1 implies that $\mathsf{P}(B_N) \to 1$ as $N \to \infty$, which finishes the proof. ■

4.6 Proof of Theorem 6

Proof of Part 1) of Theorem 6. Throughout the proof, denote the center of the star graph by $n+1$ and all other vertices $1, 2, \dots, n$. We skip the trivial case, where $\alpha < 0$ and $\beta \le 0$. We will show that if

$$\alpha < 0 < \beta, \text{ and } \alpha + \beta\sqrt{n} < 0, \tag{39}$$

then the stationary distribution is well defined. Let A_Q be the matrix determined by equation (18) in the case of the star graph with $n+1$ vertices. Denote by $D_n(\mu)$ be the characteristic polynomial of matrix A_Q. A direct computation gives the following recursive equation

$$D_n(\mu) = (-\alpha - \mu)D_{n-1}(\mu) - \beta^2(-\alpha - \mu)^{n-1}, \quad n \ge 1,$$

which yields that

$$D_n(\mu) = (-1)^{n+1}(\mu + \alpha)^{n-1}(\mu + \alpha + \beta\sqrt{n})(\mu + \alpha - \beta\sqrt{n}).$$

Thus, $-\alpha > 0$ is the matrix eigenvalue of multiplicity $n-1$ and $-\alpha \pm \beta\sqrt{n}$ are eigenvalues of multiplicity 1. The eigenvalue $-\alpha - \beta\sqrt{n} > 0$ is the minimal one (since $\beta > 0$), hence A_Q is positive definite provided conditions (39) are satisfied. Positive definiteness of A_Q implies that $Z_{\alpha,\beta,\Lambda} < \infty$ (as in the proof of Part 1) of Theorem 1). Therefore, the stationary distribution is well defined and the CTMC $\xi(t)$ is ergodic.

We are going to show that if $\alpha < 0 < \beta$ and $\alpha + \beta\sqrt{n} > 0$ then $Z_{\alpha,\beta,\Lambda} = \infty$ and the stationary distribution is not defined. Start with noticing that $(1, \dots, 1, \sqrt{n}) \in \mathbb{Z}_+^{n+1}$ is the eigenvector corresponding to the eigenvalue $(-\alpha - \beta\sqrt{n})$. Therefore, if $\alpha + \beta\sqrt{n} \ge 0$ then the function $\exp(-Q(\xi)/2)$ is not summable along the direction of this eigenvalue and, hence, the CTMC $\xi(t)$ is not ergodic. Indeed, in this case, since $\alpha < 0$,

$$Z_{\alpha,\beta,\Lambda} = \sum_{\xi \in \mathbb{Z}_+^{n+1}} e^{-Q(\xi)/2 - \frac{\alpha}{2}\sum_{i=1}^{n+1}\xi_i} \ge \sum_{\xi \in \mathbb{Z}_+^{n+1} \cap G} e^{-Q(\xi)/2}$$

where $G = \{\xi : \xi_i = [\beta\xi_{n+1}/|\alpha|],\ i = 1, 2, \dots, n\}$ and $[x]$ denotes the closest integer to $x \in \mathbb{R}$, so that $|x - [x]| \le 1/2$. Using the expression (12) for $Q(\xi)$ and the fact that $\beta > \sqrt{n}|\alpha|$ we have

$$\begin{aligned} Z_{\alpha,\beta,\Lambda} &\ge \sum_{\xi \in \mathbb{Z}_+^{n+1} \cap G} \exp\left(-\frac{n|\alpha|}{8} + \frac{n\beta^2 - \alpha^2}{2|\alpha|}\,\xi_{n+1}^2\right) \\ &= e^{-\frac{n|\alpha|}{8}} \sum_{k=0}^{\infty} \exp\left(\frac{n\beta^2 - \alpha^2}{2|\alpha|}\,k^2\right) = \infty. \end{aligned}$$

■

Proof of Part 2) of Theorem 6. Observe that

$$\begin{aligned} U(i, \zeta) &= -|\alpha|\zeta_i + \beta\zeta_{n+1},\ i = 1, 2, \dots, n; \\ U(n+1, \zeta) &= -|\alpha|\zeta_{n+1} + \beta\sum_{i=1}^{n}\zeta_i. \end{aligned}$$

An easy calculation gives the following identity

$$(n\beta+|\alpha|)U(n+1,\zeta)+(\beta+|\alpha|)\sum_{i=1}^{n}U(i,\zeta)=\left(n\beta^2-\alpha^2\right)S(\zeta) \tag{40}$$

where S is defined by (13), valid in the case of any star graph. Therefore, if $\alpha+\sqrt{n}\beta=0$, then

$$\begin{aligned}&(n\beta+|\alpha|)U(n+1,\zeta)+(\beta+|\alpha|)\sum_{i=1}^{n}U(i,\zeta)\\&=\beta(1+\sqrt{n})\left(\sqrt{n}U(n+1,\zeta)+\sum_{i=1}^{n}U(i,\zeta)\right)=0\end{aligned}$$

which is equivalent to

$$\sqrt{n}U(n+1,\zeta)+\sum_{i=1}^{n}U(i,\zeta)=0. \tag{41}$$

Given ξ denote $m_n=m_n(\xi)=\max_{i=1,\dots,n}\xi_i$, and

$$\tau_N=\min\{t:\max(\xi_{n+1}(t),\lfloor\sqrt{n}m_n(\xi(t))\rfloor=N\}$$

where $\lfloor a\rfloor\le a$ denotes the integer part of a. It is obvious that the Markov chain is explosive if and only if

$$\mathsf{P}\left(\lim_{N\to\infty}\tau_N<\infty\right)>0,$$

but this cannot happen. Indeed, if $\xi_{n+1}\ge\lfloor\sqrt{n}m_n\rfloor$ then

$$U_{n+1}=\beta(-\sqrt{n}\xi_{n+1})+\xi_1+\cdots+\xi_n\le\sqrt{n}\beta(-\xi_{n+1}+\sqrt{n}m_n)\le 0,$$

and, on the other hand, if $\xi_{n+1}<\lfloor\sqrt{n}m_n\rfloor$ then

$$\begin{aligned}U_k&=\beta(-\sqrt{n}m_n+\xi_{n+1})\\&=\beta\left[(-\sqrt{n}m_n+\lfloor\sqrt{n}m_n\rfloor)-(\lfloor\sqrt{n}m_n\rfloor-\xi_{n+1})\right]<-\beta\end{aligned}$$

for all k such that $\xi_k=m_n$. Therefore the waiting time $\tau_{N+1}-\tau_N$ is stochastically larger than a certain exponentially distributed random variable which parameters depend only on n and β and hence the limit $\lim_{N\to\infty}\tau_N$ is infinite with probability 1. Thus CTMC $\xi(t)$ is not explosive.

We are now going to prove transience of DTMC $\zeta(t)$ and thereby transience of CTMC $\xi(t)$. Recall that $\upsilon=(1,\dots,1,\sqrt{n})\in\mathbb{Z}_+^{n+1}$ is the eigenvector corresponding to the eigenvalue $(-\alpha-\beta\sqrt{n})$. Define a function f as the scalar product (in $\mathbb{R}^{n+1}$) of vectors ζ and υ, i.e.

$$f(\zeta)=\zeta_1+\ldots+\zeta_n+\sqrt{n}\zeta_{n+1}.$$

For simplicity, denote $f_t=f(\zeta(t))$. We will show that there exists $\varepsilon>0$ such that for all ζ

$$\mathsf{E}\left[f_{t+2}-f_t|\zeta(t)=\zeta\right]\ge\varepsilon. \tag{42}$$

Since the function f is non-negative and has uniformly bounded jumps (as $|f_{t+1} - f_t| \le \sqrt{n}$) transience of $\zeta(t)$ will follow from Theorem 2.2.7 in [5] with $k(\alpha) \equiv 2$.

To establish (42), observe that for $\varepsilon \in [0, 1)$

$$\begin{aligned}
&\mathsf{E}\left[f_{t+1} - f_t | \zeta(t) = \zeta\right] - \varepsilon \\
&= \frac{\sum_{i=1}^n e^{U(i,\zeta)} + \sqrt{n} e^{U(n+1,\zeta)} - \sum_{i=1}^n 1_{\{\zeta_i > 0\}} - \sqrt{n} 1_{\{\zeta_{n+1} > 0\}}}{\sum_{i=1}^{n+1} \left[e^{U(i,\zeta)} + 1_{\{\zeta_i > 0\}}\right]} - \epsilon \\
&= \frac{H(\zeta, \varepsilon)}{\sum_{i=1}^{n+1} \left[e^{U(i,\zeta)} + 1_{\{\zeta_i > 0\}}\right]} \qquad (43)
\end{aligned}$$

where

$$\begin{aligned}
H(\zeta, \varepsilon) = (1 - \varepsilon) \sum_{i=1}^n e^{U(i,\zeta)} + \left(\sqrt{n} - \varepsilon\right) e^{U(n+1,\zeta)} \\
- (1 + \varepsilon) \sum_{i=1}^n 1_{\{\zeta_i > 0\}} - \left(\sqrt{n} + \varepsilon\right) 1_{\{\zeta_{n+1} > 0\}}.
\end{aligned}$$

From (41) and the inequality between the arithmetical and geometric means we have

$$\sum_{i=1}^n e^{U(i,\zeta)} \ge n \left[\prod_{i=1}^n e^{U(i,\zeta)}\right]^{1/n} = n e^{-\frac{U(n+1,\zeta)}{\sqrt{n}}}$$

hence

$$\begin{aligned}
\frac{H(\zeta, \varepsilon)}{1 - \varepsilon} &> \sum_{i=1}^n e^{U(i,\zeta)} + \frac{(\sqrt{n} - \sqrt{n}\varepsilon)}{1 - \varepsilon} e^{U(n+1,\zeta)} - \frac{1 + \varepsilon}{1 - \varepsilon} \sum_{i=1}^n 1_{\{\zeta_i > 0\}} \\
&- \frac{\sqrt{n} + \sqrt{n}\varepsilon}{1 + \varepsilon} 1_{\{\zeta_{n+1} > 0\}} \\
&= \sum_{i=1}^n e^{U(i,\zeta)} + \sqrt{n} e^{U(n+1,\zeta)} - \frac{1 + \varepsilon}{1 - \varepsilon}(n + \sqrt{n}) =: \varphi_\varepsilon(u)
\end{aligned}$$

where

$$\varphi_\varepsilon(u) = n e^{-u/\sqrt{n}} + \sqrt{n} e^u - \frac{1 + \varepsilon}{1 - \varepsilon}(n + \sqrt{n})$$

and $u = U(n + 1, \zeta) \in \mathbb{R}$.

One can easily check that $\varphi_\varepsilon'(0) = 0$ and $\varphi_\varepsilon''(u) = e^{-u/\sqrt{n}} + \sqrt{n} e^u > 0$ for all u, therefore $\varphi_\varepsilon(\cdot)$ attains its unique minimum at $u = 0$. If we set $\varepsilon = 0$ we also have $\varphi_0(0) = 0$ hence $\varphi_0(u) \ge 0$, $u \in \mathbb{R}$ implying that when $\varepsilon = 0$ the LHS of (43) is always non-negative and f_t is thus a submartingale.

To show that it actually increases on average by at least $\varepsilon > 0$ in *two* steps, note that $|U(n + 1, \zeta(t + 1)) - U(n + 1, \zeta(t))| \ge \beta > 0$ since $\zeta(t + 1)$ differs from $\zeta(t)$ in one of the coordinates, and $|\alpha| > \beta$. Therefore,

$$\min\left\{\left|U\left(n + 1, \zeta(t)\right)\right|, \left|U\left(n + 1, \zeta(t + 1)\right)\right|\right\} \ge \frac{\beta}{2}.$$

Without loss of generality, assume that it is $u = U(n+1, \zeta(t))$ which has the property $|u| \geq \beta/2$. To guarantee that the LHS (43) is non-negative for some small $\varepsilon > 0$ we will establish that

$$\inf_{u:\ |u|\geq\beta/2} \varphi_\varepsilon(u) = \min\{\phi_\varepsilon(-\beta/2), \phi_\varepsilon(\beta/2)\} > 0 \tag{44}$$

where the equality follows from the fact that $\varphi_\varepsilon(u)$ is increasing for $u > 0$ and decreasing for $u < 0$. However, since $\varphi_0(\pm\beta/2)$ is strictly positive, as we established before, and $\varphi_\varepsilon(u)$ is continuous in ε, by choosing $\varepsilon > 0$ sufficiently small we can ensure (44) and hence (42) and transience. ■

Proof of Part 3) of Theorem 6. Recall from (40) that

$$(n\beta + |\alpha|)U(n+1, \zeta) + (\beta + |\alpha|) \sum_{i=1}^{n} U(i, \zeta) = \left(n\beta^2 - \alpha^2\right) S(\zeta),$$

where now $n\beta^2 - \alpha^2 > 0$, due to our assumption $\beta > |\alpha|/\sqrt{n}$. Hence, using the elementary fact that if $a_1 + \cdots + a_{n+1} = x$ then $\max_i a_i \geq x/(n+1)$ we get that

$$\max_{i=1,\ldots,n+1} U(i, \zeta(k)) \geq CS(\zeta(k))$$

and $C > 0$ is some constant depending on n, α and β.

At the same time, whenever any of the component of ζ increases, $S(\zeta(k))$ also increases by 1. For a positive integer y define $\tau_y = \min\{t :\ S(\zeta(t)) \geq y\}$. For each $y \in \{1, 2, \ldots\}$ the set of configurations of ζ where $S(\zeta) < y$ is finite, so with probability one at some point of time k the system will reach the state where $S(\zeta(k)) \geq y$, consequently $\tau_y < \infty$ a.s. for all y. Hence we can define the events $A_y =$"there exists $t \geq \tau_y$ such that some component decreases at time t". Then one can easily obtain the following bound

$$\mathsf{P}(A_y) \leq 1 - \prod_{k=y}^{\infty} \left(1 - \frac{n}{e^{\max_i U(\zeta(k), i)}}\right) \leq 1 - \prod_{k=y}^{\infty} \left(1 - \frac{n}{e^{Ck}}\right) \sim \frac{n}{1 - e^{-C}} \cdot e^{-Cy}$$

for large enough y. Since $\sum_y e^{-Cy} < \infty$ by Borel-Cantelli lemma there will be a.s. a time y' for which no A_y $(y \geq \tau_{y'})$ occurs and thus the only changes in the system are increases of the components; this also implies that for any integer $k > \tau_{y'}$ we have $\max_i U(i, \zeta(k)) \geq Ck$, thus ensuring that the CTMC $\xi(t)$ explodes a.s., since the rates of jumps are bounded below by e^{Ck}, the inverses of which are again summable.

Let us now observe the DTMC after time y' thus assuming only increases of the components, i.e. $S(\zeta(k+1)) - S(\zeta(k)) = 1$ for all $k \geq y'$. Denote

$$z(k) = \sum_{i=1}^{n} \zeta_i(k) = S(\zeta(k)) - \zeta_{n+1}(k).$$

Since the probability that only the component at $n+1$ increases after time k equals

$$\prod_{l=k}^{\infty} \frac{e^{U(n+1,\zeta(k)) - |\alpha|(l-k)}}{e^{U(n+1,\zeta(k)) - |\alpha|(l-k)} + \sum_{i=1}^{n} e^{U(i,\zeta(k))}} = 0$$

on one hand, and the probability that the component at $n+1$ never increases after time k is equal to

$$\begin{aligned}
&\prod_{l=k}^{\infty}\left(1-\frac{e^{U(n+1,\zeta(k))}}{e^{U(n+1,\zeta(k))}+\sum_{i=1}^{n} e^{U(i,\zeta(l))}}\right)\\
&\le \prod_{l=k}^{\infty}\left(1-\frac{e^{U(n+1,\zeta(k))}}{e^{U(n+1,\zeta(k))}+ne^{\max_{i=1,\dots,n} U(i,\zeta(l))}}\right)\\
&= \prod_{l=k}^{\infty}\left(1-\frac{1}{1+ne^{(|\alpha|+2\beta)\zeta_{n+1}(k)-\beta S(\zeta(l))-|\alpha|\min_{i=1,\dots,n}\zeta_i(l)}}\right)\\
&\le \prod_{l=k}^{\infty} C\cdot e^{-\beta l}=0
\end{aligned}$$

on the other hand, we conclude that both $\zeta_{n+1}(k)\to\infty$ and $z(k)\to\infty$.

Now consider the process $\zeta(k)$ at those times $k_1<k_2<\dots$ when one of the components in $\{1,2,\dots,n\}$ increases. It is easy to see that $z(k_{i+1})-z(k_i)=1$ for all i and that one can couple the process

$$(\zeta_1(k_i),\zeta_2(k_i),\dots,\zeta_n(k_i)),\ i=1,2,\dots,$$

with the generalized Pólya urn with n types of balls and the weight function $g(x)=e^{\alpha x}$. Since $\alpha<0$, from, for example, a trivial comparison with the Friedman urn, we conclude that all $\zeta_j(k_i)$, $j=1,\dots,n$ grow at the same speed, resulting in $\zeta_j(k)/z(k)\to 1/n$. Therefore, for any $\epsilon>0$ there is a (random) time $k_1\ge y'$ such that

$$\frac{1-\epsilon}{n}\le \min_{j=1,\dots,n}\frac{\zeta_j(k)}{z(k)}\le \max_{j=1,\dots,n}\frac{\zeta_j(k)}{z(k)}\le\frac{1+\epsilon}{n}\ \text{for all } k\ge k_1.$$

Once this being the case, the odds that at time k the component at $n+1$ grows (as opposed to a component at i, $i\in\{1,\dots,n\}$) lies in the interval

$$\left[\frac{e^{-|\alpha|\zeta_{n+1}+\beta z}}{ne^{-|\alpha|(1-\varepsilon)\frac{z}{n}+\beta\zeta_{n+1}}},\frac{e^{-|\alpha|\zeta_{n+1}+\beta z}}{ne^{-|\alpha|(1+\varepsilon)\frac{z}{n}+\beta\zeta_{n+1}}}\right] = \left[e^{zR_{-\epsilon}-L\zeta_{n+1}-\log(n)},e^{zR_{+\epsilon}-L\zeta_{n+1}-\log(n)}\right]$$

where

$$R_{\pm\epsilon}=\beta+\frac{|\alpha|(1\pm\varepsilon)}{n},\ L=|\alpha|+\beta.$$

Let $X(k)=z(k)R_{-\epsilon}-\zeta_{n+1}(k)L$, $k=k_1,k_1+1,\dots$. Then $X(k)$ can be coupled with random walk $Y(k)$ on $[\log(np/(1-p)),+\infty)$ with the transitional probabilities

$$Y(k+1)=\begin{cases} Y(k)+R_{-\epsilon}, & \text{with probability } 1-p;\\ \max\left\{Y(k)-L,\log\left(\frac{np}{1-p}\right)\right\}, & \text{with probability } p,\end{cases}$$

in such a way that $X(k)\le Y(k)$. By choosing $p\in(0,1)$ such that

$$\mathsf{E}(Y(k+1)-Y(k))=R_{-\epsilon}(1-p)Lp<0$$

(provided $Y(k) \geq L + \log(np/(1-p))$) we ensure that $\lim_{k\to\infty} Y(k)/k = 0$, implying in turn that

$$\limsup_{k\to\infty} \frac{X(k)}{k} = \limsup_{k\to\infty} \frac{z(k)R_{-\epsilon} - \zeta_{n+1}(k)L}{k} \leq 0.$$

By the completely symmetric argument we also obtain

$$\liminf_{k\to\infty} \frac{z(k)R_{+\epsilon} - \zeta_{n+1}(k)L}{k} \geq 0.$$

Now, using the fact that $z(k) + \zeta_{n+1}(k) = k + \mathit{const}$ for large k,

$$\frac{R_{-\epsilon}}{L + R_{-\epsilon}} \leq \liminf_{k\to\infty} \frac{\zeta_{n+1}(k)}{k} \leq \limsup_{k\to\infty} \frac{\zeta_{n+1}(k)}{k} \leq \frac{R_{+\epsilon}}{L + R_{+\epsilon}}$$

Since $\epsilon > 0$ is arbitrary and $R_{+\epsilon} - R_{-\epsilon} \to 0$ as $\epsilon \to 0$, we get

$$\lim_{k\to\infty} \frac{\zeta_{n+1}(k)}{k} = \frac{\beta + |\alpha|/n}{\beta + |\alpha|/n + \beta + |\alpha|} = \frac{n\beta + |\alpha|}{2n\beta + (n+1)|\alpha|}$$

and, as a consequence,

$$\lim_{k\to\infty} \frac{\zeta_i(k)}{k} = \frac{\beta + |\alpha|}{2n\beta + (n+1)|\alpha|} \text{ for } i = 1, 2, \ldots, n.$$

Finally, we also conclude that all the components of the CTMC ξ actually explode simultaneously.

Proof of Part 4) of Theorem 6 The case *i*) of the theorem is covered by Theorem 2, and the case *ii*) is covered by Theorem 3, since a star graph does not have triangles.

■

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