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Dependent Possibilistic Arithmetic using Copulas

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Abstract

We describe two functions on possibility distributions which allow one to compute binary operations with dependence either specified by a copula or partially defined by an imprecise copula. We use the fact that possibility distributions are consonant belief functions to aggregate two possibility distributions into a bivariate belief function using a version of Sklar's theorem for minitive belief functions, i.e. necessity measures. The results generalise previously published independent and Fréchet methods, allowing for any stochastic dependence to be specified in the form of a (imprecise) copula. This new method produces tighter extensions than previous methods when a precise copula is used. These latest additions to possibilistic arithmetic give it the same capabilities as p-box arithmetic, and provides a basis for a p-box/possibility hybrid arithmetic. This combined arithmetic provides tighter bounds on the exact upper and lower probabilities than either method alone for the propagation of general belief functions.

Keywords: Possibility Theory, P-box, Copulas, Probabilistic Arithmetic, Probability Bounds Analysis, Imprecise Probabilities

1. Introduction

Due to its simplicity in formulation and calculus, possibility theory [9] is a popular model for bounding sets of probability measures. Early in the formulation of the theory, many authors [8] argued that Zadeh's classical min aggregation [26], and its implied levelwise interval arithmetic, was sufficient for computing functions of sets of probabilities, and that it corresponds to a *non-interactivity* between variables. More recently however [2], it has been shown that such a direct application of fuzzy set theory is not consistent with probability theory in most circumstances, with modifications being proposed [12]. These modifications are of the form of combination operations for constructing multivariate possibility distributions from univariate marginals, allowing one to make such constructions with stochastic independence and unknown interaction (Fréchet). It has been shown in [12] that when these combination methods are used with the extension principle, probability measures with the specified dependence are correctly propagated.

Probability box arithmetic, or probability bounds analysis, is based on three convolutions which were originally introduced by Schweizer and Sklar [21] as triangle functions, or solutions to the triangle inequality in probabilistic metric spaces. Williamson and Downs [24] describes how these convolutions may be used in an arithmetic of random variables with a partially defined dependence structure, given as a lower bound of a copula. They also describe a general numerical method for computing robust outer solutions to these convolutions, in the form of an upper and lower discrete approximation of quantile functions. These structures were originally named *dependency bounds*, but are now labeled probability boxes, or p-boxes, and have been generalised to include uncertainties other than dependency errors [11]. The three convolutions from [24] are the following. For a non-decreasing binary operator $f:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$, two random variables X and Y with distribution functions (df) F_X and F_Y and copula C_{XY}:

$$\sigma_{C_{XY},f}(F_X,F_Y)(z)=\int_{f\{z\}} dC_{XY}(F_X(x),F_Y(y)), \qquad (1)$$

$$\tau_{\underline{C}_{XY},f}(F_X,F_Y)(z)=\sup_{f(x,y)=z}[\underline{C}_{XY}(F_X(x),F_Y(y))], \qquad (2)$$

$$\rho_{\underline{C}_{XY},f}(F_X,F_Y)(z)=\inf_{f(x,y)=z}[\underline{C}^d_{XY}(F_X(x),F_Y(y))], \qquad (3)$$

where the set $f\{z\}=\{(x,y)|x,y\in\mathbb{R}^+, f(x,y)<z\}$, and where C^d is the dual of a copula: $C^d(u,v)=u+v-C(u,v)$. The σ convolution is a Lebesgue-Stieltjes integral which gives the df resulting from a binary operation f between

two dfs with a known copula. The τ and ρ convolutions compute the lower and the upper cdf of a p-box respectively when only the copula's lower bound is known. That is, $\tau_{\underline{C}_{XY},f}$ and $\rho_{\underline{C}_{XY},f}$ propagate all copulas more positive than $\underline{C}_{XY}$. For example $\tau_{W,f}$ and $\rho_{W,f}$ compute a p-box which bounds all stochastic dependencies, where W is the lower Fréchet bound. $\tau_{uv,f}$ and $\rho_{uv,f}$ propagate all dependencies more positive than independence, that is, all copulas which are *positive quadrant dependent* [16], where the product copula $C(u,v) = uv$ gives stochastic independence.

Convolutions (1)-(3) are defined for non-decreasing binary operations, but may be extended to non-increasing operations by first performing an appropriate unary operation to one of the variables and then evaluating the convolutions with a non-decreasing operator. For example subtraction may be performed by negating one of the variables and evaluating (1)-(3) with sum: i.e. $\sigma_{C_{X,-Y},+}(F_X, F_{-Y})$. The copula $C_{X,-Y}$ may be found from a simple transformation of $C_{X,Y}$. Note that if X and Y are positive quadrant dependent, then X and $-Y$ will be *negative* quadrant dependent [16], and therefore $\tau_{uv,-}$ and $\rho_{uv,-}$ give p-boxes which bound negative quadrant dependence [10]. P-box arithmetic has been extended to p-boxes defined on $\mathbb{R}$ and many of the base binary and unary operations that are required in a programming language [11].

In this contribution we present analogous functions to (1)-(3) for possibility distributions, giving a dependent possibilistic arithmetic which allows for any stochastic dependence to be precisely defined as a copula, or imprecisely as a copula's lower bound. This generalisation has perfect dependence (Zadeh), independence, and Fréchet as special cases. The results of this paper suggest that the propagation methods of [12] correspond to a τ and ρ convolution in p-box arithmetic (an imprecise propagation of dependencies), as opposed to σ convolution (a precise dependence). We further show how precise copulas may be propagated, giving tighter results than when only the lower bounds are used. In the context of propagating (imprecise) dependencies, the results of this paper bring possibilistic arithmetic in line with the capabilities of p-box arithmetic, and is the motivation of this work.

In the following section we discuss possibility theory in the context of imprecise probabilities.

2. Possibility theory

A possibility distribution is any measurable function $\pi_X : \mathbb{R} \to [0,1]$ which satisfies the normality condition

$$\sup_{x\in\mathbb{R}} \pi_X(x) = 1. \qquad (4)$$

In this paper we consider only continuous π_X. The measurability is required for the super and sub level sets of π_X to be measurable, and for the probability of these sets to be well defined; which will be discussed in the following.

Bounds on a probability measure $\mathbb{P}_X$ on the measurable space $(\mathbb{R}, \mathscr{B})$ where $\mathscr{B}$ is the Borel σ-algebra can be obtained from π_X by

$$\mathrm{Nec}_X(U) \leq \mathbb{P}_X(U) \leq \Pi_X(U) \qquad \forall U \in \mathscr{B}, \qquad (5)$$

where Π_X is the possibility measure given by

$$\Pi_X(U) = \sup_{x\in U} \pi_X(x), \qquad (6)$$

and where the necessity measure Nec_X is

$$\mathrm{Nec}_X(U) = 1 - \sup_{x\notin U}(\pi_X(x)) = 1 - \Pi_X(U^C). \qquad (7)$$

A probability measure $\mathbb{P}_X$ and a possibility distribution π_X are said to be *consistent* [5] if inequality (5) holds, that is if the probability measure is a bounded above by the possibility measure, and below by the necessity measure. This defines a bounded set of probability measures, a credal set $\mathfrak{C}(\pi_X)$, which are consistent with π_X:

$$\mathfrak{C}(\pi_X) = \{\mathbb{P}_X : \mathbb{P}_X(U) \leq \Pi_X(U) \forall U \in \mathscr{B}\}. \qquad (8)$$

Alternatively, consistency may be defined in terms of the level sets of π_X. In particular, a probability and a possibility measure are consistent iff the probabilities of the superlevel sets, the so-called α-cuts, $\mathbb{C}^{\alpha}_{\pi_X} = \{x \in \mathbb{R} : \pi_X(x) > \alpha\}$ for $\alpha \in [0,1]$ are bounded from below by $1-\alpha$ [4]:

$$\mathbb{P}_X\left(\mathbb{C}^{\alpha}_{\pi_X}\right) \geq 1 - \alpha \qquad \forall \alpha \in [0,1]. \qquad (9)$$

The propagation of possibility distributions is usually defined in terms of the extension principle. However it has been argued by several authors [1, 2, 12] that the propagated possibility distributions are only consistent with probability theory in limited circumstances. Therefore a variant of the extension principle has been defined in order to preserve consistency [12]: given N random input variables $X_1, \ldots, X_N$ with possibility distributions $\pi_{X_1}, \ldots, \pi_{X_N}$ and a function $f : \mathbb{R}^N \to \mathbb{R}$, then the possibility distribution of the output variable $Y = f(X_1, \ldots, X_N)$ is given by

$$\pi_Y(y) = \sup_{\substack{(x_1,\ldots,x_N)\in\mathbb{R}^N : \\ y=f(x_1,\ldots,x_N)}} \mathbb{J}\left(\pi_{X_1}(x_1), \ldots, \pi_{X_N}(x_N)\right) \qquad (10)$$

for all $y \in \mathbb{R}$. Therein, the operator $\mathbb{J}$ accounts for the joint dependency structure of the marginal input variables. Hose and Hanss [12] describe three possible choices:

Zadeh: $\mathbb{J}^{\min}(\alpha_1, \ldots, \alpha_N) = \min(\alpha_1, \ldots, \alpha_N)$,

Strong Independence:
$\mathbb{J}^{\mathrm{SI}}(\alpha_1, \ldots, \alpha_N) = 1 - (1 - \min(\alpha_1, \ldots, \alpha_N))^N$,

Unknown Interaction:
$\mathbb{J}^{\mathrm{UI}}(\alpha_1, \ldots, \alpha_N) = \min(1, N \cdot \alpha_1, \ldots, N \cdot \alpha_N)$,

with $\mathbb{J}^{\min}$ corresponding to the original extension principle, and maximally correlated inputs, $\mathbb{J}^{\mathrm{SI}}$ corresponding to inputs which are stochastically independent, and $\mathbb{J}^{\mathrm{UI}}$ corresponding to inputs which make no assumption about stochastic dependence (Fréchet).

The possibility measure induced by a possibility distribution may be viewed as a special case of a plausibility function [22], likewise the necessity is a special case of a belief function, and where the α-cuts correspond to the nested focal sets. In particular the belief of an α-cut is simply $1-\alpha$ [4]:

$$\mathrm{Nec}_X(\mathbb{C}^{\alpha}_{\pi_X}) = \mathrm{Bel}_X(\mathbb{C}^{\alpha}_{\pi_X}) = 1-\alpha. \tag{11}$$

Conversely, not every belief function Bel_X is consonant and can, therefore, not usually be precisely described by a possibility measure [7]. That is, if $\mathfrak{P}(\mathrm{Bel}_X) = \{\mathbb{P}_X : \mathrm{Bel}_X(U) \leq \mathbb{P}_X(U) \forall U \in \mathscr{B}\}$ describes the credal set of a belief function Bel_X (similar to the credal set of a possibility distribution), then it is not generally possible to find a possibility distribution π_X such that $\mathfrak{P}(\mathrm{Bel}_X) = \mathfrak{C}(\pi_X)$. However, in such situations, it is possible to find a consonant outer approximation achieving $\mathfrak{P}(\mathrm{Bel}_X) \subseteq \mathfrak{C}(\pi_X)$ via the imprecise probability-to-possibility transform introduced by Hose and Hanss [13].

Given an arbitrarily chosen candidate[1] possibility distribution q_X of X, Hose and Hanss show that the possibility distribution π_X given by

$$\begin{aligned}\pi_X(x) &= \sup_{\mathbb{P}_X \in \mathfrak{P}(\mathrm{Bel}_X)} \mathbb{P}_X(\{\xi \in \mathbb{R} : q_X(\xi) \leq q_X(x)\}) \\ &= 1 - \mathrm{Bel}_X(\{\xi \in \mathbb{R} : q_X(\xi) > q_X(x)\}) \\ &= 1 - \mathrm{Bel}_X(\mathbb{C}^{q_X(x)}_{q_X}),\end{aligned} \tag{12}$$

for $x \in \mathbb{R}$ is consistent with all $\mathbb{P}_X \in \mathfrak{P}_X$, i.e. it provides an outer approximation of $\mathfrak{P}_X$. Most importantly, it is also plausibility-conform to q_X, i.e. from $q_X(x_1) \leq q_X(x_2)$ it follows that $\pi_X(x_1) \leq \pi_X(x_2)$ for all $x_1, x_2 \in \mathbb{R}$ (but not vice versa), and it is maximally specific[2]. That is, even though π_X does not generally exactly describe $\mathfrak{P}_X$, one may not find a 'better' plausibility-conform possibility distribution that is also an outer approximation. Of course, much of the quality of the resulting π_X depends on the candidate possibility distribution q_X, the obvious degree of freedom in this approach, but Hose and Hanss provide several reasonable options for how to choose it. For instance, for the outer approximation of belief functions, they suggest the 'melting transform' given by the pointwise plausibilities

$$q_X(x) = \mathrm{Pl}_X(\{x\}) = 1 - \mathrm{Bel}_X(\mathbb{R} \setminus \{x\}), \tag{13}$$

for $x \in \mathbb{R}$. Below, a different (more constructive) way of finding q_Z is pursued.

1. Hose and Hanss call this a (subjective) plausibility function.
2. The concept of specificity is not discussed here. Refer, e.g., to [6] for further details.

3. Joint possibilities using copulas

In this paper, we study dependent binary operations and therefore restrict ourselves to bivariate copulas (2-copulas), however the generalisation to n-copulas is straightforward. A 2-copula is a function $C : [0,1]^2 \to [0,1]$ with the following properties [16]:

1. Grounded: $C(0,v) = C(u,0) = 0$,
2. Uniform margins: $C(u,1) = u$; $C(1,v) = v$,
3. 2-increasing: $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$ for all $0 \leq u_1 \leq u_2 \leq 1$ and $0 \leq v_1 \leq v_2 \leq 1$.

Three important 2-copulas are[3]

$$\begin{aligned} W(u,v) &= \max(u+v-1, 0), \\ \Psi(u,v) &= uv, \\ M(u,v) &= \min(u,v), \end{aligned}$$

with W and M being bounds on all 2-copulas: $W \leq C \leq M$. Copulas are mainly used in dependence modelling [14, 16], and can be used to construct multivariate dfs given their univariate marginals. This is enabled by a theorem from Sklar [23]:

Theorem 1 (Sklar's theorem) *Let X and Y be random variables with joint df H and univariate marginals F_X and F_Y. Then there exists a copula C such that for all $(x,y) \in \mathbb{R}^2$:*

$$H(x,y) = C(F_X(x), F_Y(y)). \tag{14}$$

If F_X and F_Y are continuous, then C is unique; otherwise C is uniquely determined on support $F_X \times$ support F_Y.

Moreover, X and Y are stochastically independent iff $C_{XY} = \Psi$, perfectly positively dependent (perfect) iff $C_{XY} = M$ and perfectly negatively dependent (opposite) iff $C_{XY} = W$.

Sklar's theorem may be applied in reverse, that is a copula may be constructed from any multivariate df with continuous marginals [16]:

Theorem 2 *Let H be a bivariate df with continuous marginals F and G, with quasi-inverses F^{-1} and G^{-1}. Then there exits a copula C such that for all $(u,v) \in [0,1]^2$:*

$$C(u,v) = H(F^{-1}(u), G^{-1}(v)). \tag{15}$$

The Gaussian copula is a well known copula family which is defined using (15):

$$C^{\Phi}_r(u,v) = \Phi_r(\Phi^{-1}(u), \Phi^{-1}(v)), \tag{16}$$

where Φ_r is a zero mean bivariate Gaussian cdf with unitary variance and correlation coefficient r, and Φ^{-1} is

3. The standard notation for the product copula is $\Pi(u,v) = uv$, however we reserve this for possibility measures

the inverse cdf of a standard normal Gaussian. Note that $C^{\Phi}_{-1} = W$, $C^{\Phi}_{0} = \Psi$ and $C^{\Phi}_{1} = M$.

The inequality $W \leq C \leq W$ suggests a partial ordering on the set of all copulas [16]. This ordering is useful in the preceding sections:

Definition 1 (Concordance ordering) *If C_1 and C_2 are copulas, we say that C_1 is smaller than C_2 (or C_2 is larger than C_1), and write $C_1 \prec C_2$ (or $C_2 \succ C_1$) if $C_1(u,v) \leq C_2(u,v)$ for all $(u,v) \in [0,1]^2$.*

Furthermore, two random variables X and Y are positive quadrant dependent if $\Psi \prec C_{XY}$ and are negative quadrant dependent if $\Psi \succ C_{XY}$.

Sklar's theorem has been extended to an imprecise setting by several authors. Montes et al. [15] describe an imprecise version of Sklar's theorem for the construction of multivariate p-boxes from two marginal p-boxes and an imprecise copula (bounded set of copulas). Schmelzer [18] investigates when copulas can be used to describe or model the dependence of belief functions. He demonstrates that in the general case a joint belief measure cannot be related to its margins by a single copula; except in the special cases that the belief function is a p-box (cumulative belief function) or a possibility distribution (minitive belief function). He further describes a version Sklar's theorem for minitive belief functions [19], stated in the language of possibility theory:

Theorem 3 (Sklar's theorem for minitive beliefs) *Let $N_{XY} : \mathscr{B}^2 \to [0,1]$ be a bivariate necessity measure with marginal necessity measures Nec_X and Nec_Y. There exists a copula C such that for all $U_X \in \mathscr{B}_X$, $U_Y \in \mathscr{B}_Y$:*

$$\mathrm{Nec}_{XY}(U_X, U_Y) = C(\mathrm{Nec}_X(U_X), \mathrm{Nec}_Y(U_Y)). \tag{17}$$

The copula is uniquely determined on support $\mathrm{Nec}_X \times$ support Nec_Y.

The copula C in Theorem 3 characterises the joint distribution of the alpha cuts (focal elements) of two marginal possibility distributions (consonant belief functions). For example, $C = \Psi$ would give random set independence between the belief functions corresponding to π_X and π_Y. In that respect, Theorem 3 gives a form of *random set dependence* for consonant random sets. Section 7 discusses some implications of this.

Theorem 3 may be used in an α-cut-based implementation of possibilistic arithmetic under a given copula, where interval arithmetic is performed on α-cuts correlated by copula C. In Section 4 we show how arithmetic operations may be performed when a copula C is known, giving a possibilistic analogue to (1). Section 5 describes how arithmetic may be performed when only bounds to C are known, giving an analogue to (2)-(3).

4. Arithmetic with known dependence

Because a possibility distribution specifies a consonant belief function with nested focal elements, i.e. a necessity measure: given π_X, π_Y with α-cuts $\mathbb{C}^{\alpha_x}_{\pi_X}$ and $\mathbb{C}^{\alpha_y}_{\pi_Y}$ correlated by copula C_{XY}, the basic mass assignment on the cartesian product of two cuts is computed by the Möbius inverse:

$$m(\mathbb{C}^{\alpha_x}_{\pi_X} \times \mathbb{C}^{\alpha_y}_{\pi_Y}) = \sum_{U_X \subseteq \mathbb{C}^{\alpha_x}_{\pi_X}, U_Y \subseteq \mathbb{C}^{\alpha_y}_{\pi_Y}} (-1)^{|\mathbb{C}^{\alpha_y}_{\pi_X} \setminus U_X| + |\mathbb{C}^{\alpha_y}_{\pi_Y} \setminus U_Y|} \mathrm{Nec}_{XY}(U_X, U_Y), \tag{18}$$

where Nec_{XY} is the bivariate necessity, which may be defined in terms of C_{XY} using Theorem 3. Proof of (18) may be found in [17]. This mass assignment may be thought of as the joint probability density of the α-cuts under copula C_{XY}. A non-decreasing function with an interval extension $f : \mathscr{B}^+ \times \mathscr{B}^+ \to \mathscr{B}^+$ may then be performed on the α-cuts:

$$\mathbb{K}_Z = f(\mathbb{C}^{\alpha_x}_{\pi_X}, \mathbb{C}^{\alpha_y}_{\pi_Y}), \tag{19}$$

where $\mathbb{K}_Z$ are the focal elements of the output belief function Bel_Z of π_X and π_Y under operation f. The mass assignment m of these focal elements is retained by the images [25]:

$$m(\mathbb{K}_Z) = m(f(\mathbb{C}^{\alpha_x}_{\pi_X} \times \mathbb{C}^{\alpha_y}_{\pi_Y})) = m(\mathbb{C}^{\alpha_x}_{\pi_X} \times \mathbb{C}^{\alpha_y}_{\pi_Y}). \tag{20}$$

The necessity of the resulting random set may be found by computing the belief:

$$\mathrm{Nec}(\mathbb{K}_Z) = \sum_{U \subseteq \mathbb{K}_Z} m(U). \tag{21}$$

In the case where Bel_Z is consonant, the focal elements $\mathbb{K}_Z$ are α-cuts of the corresponding possibility distribution, with the α-level found using (11):

$$\alpha_Z = 1 - \sum_{U \subseteq \mathbb{K}_Z} m(U). \tag{22}$$

Therefore, for a non-decreasing function f with an interval extension, a dependent possibilistic arithmetic with a precise dependence may be defined in terms of α-cuts:

$$\mathbb{C}^{\alpha_z}_{\pi_Z} = f(\mathbb{C}^{\alpha_x}_{\pi_X}, \mathbb{C}^{\alpha_y}_{\pi_Y}), \tag{23}$$

for $z = f(x,y)$ and with α_z given by:

$$\alpha_Z = 1 - \sum_{U_Z \subseteq f(\mathbb{C}^{\alpha_x}_{\pi_X}, \mathbb{C}^{\alpha_y}_{\pi_Y})} m(U_Z), \tag{24}$$

with the mass assignment given by (18) and Theorem 3. Notice in the above that the α-levels α_x, α_y may be distinct, so therefore $f(\mathbb{C}^{\alpha_x}_{\pi_X}, \mathbb{C}^{\alpha_y}_{\pi_Y})$ may lead to a non-consonant belief function, for which an outer approximation may be

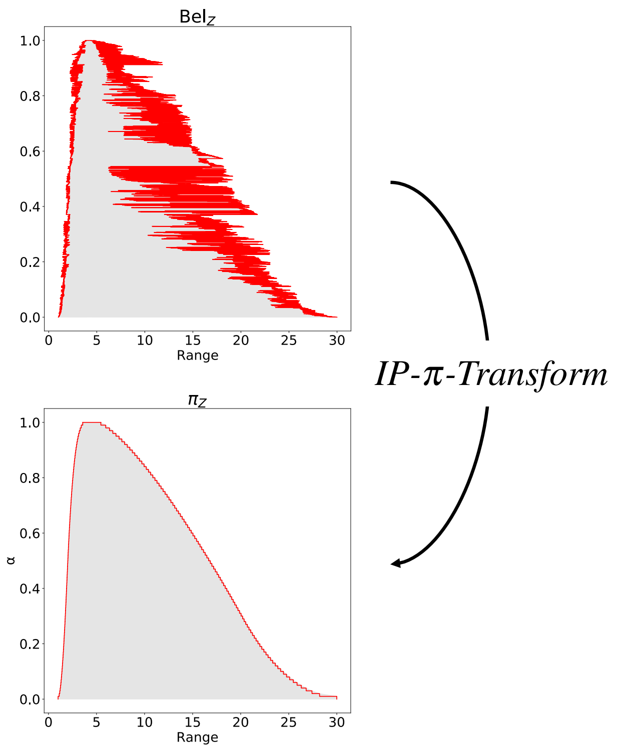

Figure 1: An example of the imprecise probability-to-possibility transformation. Above shows the focal elements (of equal mass) of a non-consonant belief function. Below shows the possibility distribution retrieved from the transformation.

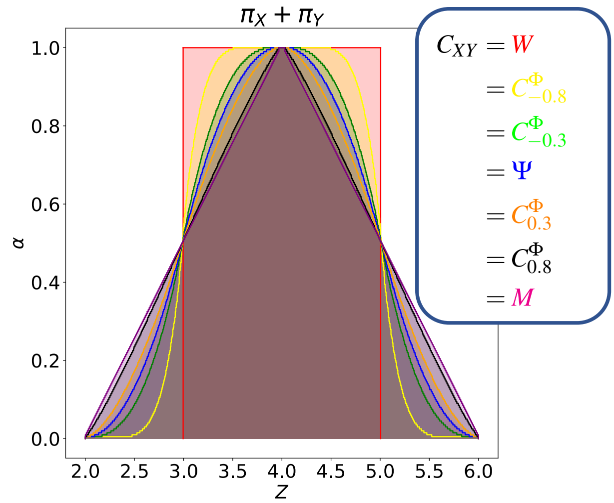

Figure 2: Sum of the two triangular fuzzy numbers $\pi_x, \pi_y = (1,2,3)$ with various precise copulas specified.

found via the imprecise probability-to-possibility transform. Figure 1 shows an example of this. The above shows the focal elements of the belief function Bel_Z found from the product of two triangular fuzzy numbers $\pi_X = (1,2,3)$ (range = $[1,3]$ and core = 2) and $\pi_Y = (1,2,10)$ using (23) with $C_{XY} = \Psi$ (random set independence). Below in Figure 1 shows the possibility distribution π_Z retrieved from the imprecise probability-to-possibility transformation of Bel_Z. The credal set induced by π_Z is an outer approximation to the credal set from Bel_Z, i.e. $\mathfrak{P}(\mathrm{Bel}_Z) \subseteq \mathfrak{C}(\pi_Z)$. Due to (9) it is sufficient to check that the necessity of π_Z is a lower bound on Bel_Z on the α-cuts of π_Z, i.e. $\mathrm{Nec}_X(\mathbb{C}^{\alpha_z}_{\pi_Z}) \leq \mathrm{Bel}_Z(\mathbb{C}^{\alpha_z}_{\pi_Z})$, which is a construct of the transformation.

Figure 2 shows (23) evaluated for the sum of two identical triangular $\pi_x, \pi_y = (1,2,3)$ with various C_{XY}. A Gaussian copula has been used for C_{XY}, varying its parameter from -1 to 1. Note that any copula may be used for C_{XY}, the Gaussian copula family is only used as a convenient example, since it contains the copulas W, Ψ, and M. The possibility distributions π_X and π_Y also do not necessarily have to be identical, triangular, nor uni-modal.

Figure 2 has some interesting special cases. $C_{XY} = M$ (perfect dependence) matches a levelwise interval arithmetic:

$$\mathbb{C}^{\alpha}_{\pi_Z} = f(\mathbb{C}^{\alpha}_{\pi_X}, \mathbb{C}^{\alpha}_{\pi_Y}), \tag{25}$$

and gives the same result as (10) with $\mathbb{J}^{\min}$, which is Zadeh's classical extension principle. $C_{XY} = W$ (opposite dependence) matches an opposite levelwise arithmetic:

$$\mathbb{C}^{\alpha}_{\pi_Z} = f(\mathbb{C}^{\alpha}_{\pi_X}, \mathbb{C}^{1-\alpha}_{\pi_Y}). \tag{26}$$

Although in this paper we provide no proof of (25) and (26), and are conjectured based on the observed possibility distributions produced by (23), the rationale is as follows. Theorem 3 characterises the bivariate distribution of the α-cuts of π_X and π_Y as copula C_{XY}, with probability mass given by (18). The copula $C_{XY} = W$ only has non-zero probability mass on the opposing diagonal of the unit square, i.e. when $u = 1 - v$. Therefore only opposing α-cuts will be assigned a positive (and equal) mass by (18) under W, and so are the only α-cuts needed to be considered in an evaluation of (23) when $C_{XY} = W$. A similar argument can be made for $C_{XY} = M$, which only has positive mass on the diagonal $u = v$. A proof may be constructed by showing that for $C_{XY} = W$ $(= M)$, the Möbius inverse (18) only gives a non-zero mass on α-cuts $\alpha_X = 1 - \alpha_Y$ $(= \alpha_Y)$.

Note that although (25) will always produce a consonant structure, (26) may require an imprecise probability-to-possibility transformation. In the case two identical $\pi_X = \pi_Y$, (26) with $f = sum$ produces an interval, shown in red in Figure 2. It would be interesting to find for which inputs, operators and copulas (23) produces intervals.

Like for the convolutions of p-box arithmetic, we extend (23) to non-increasing binary operations by first applying

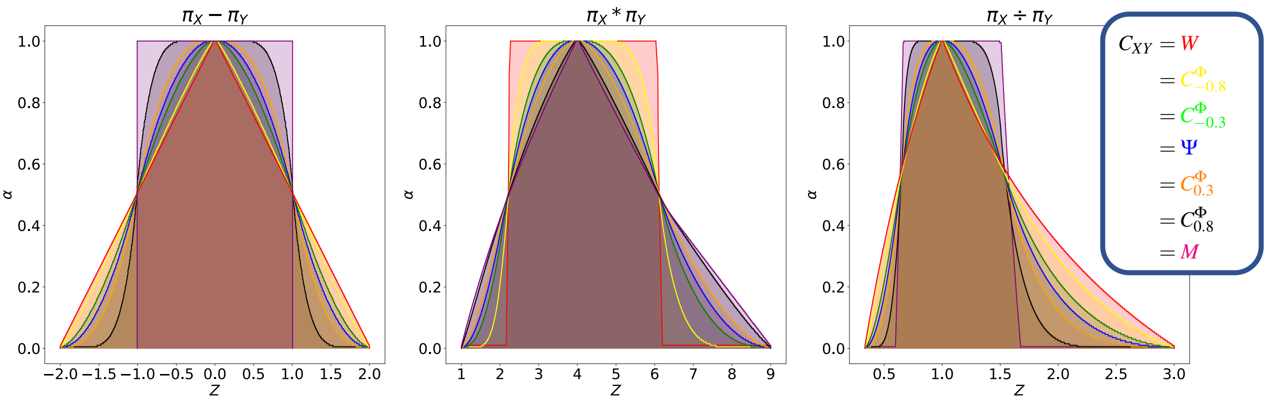

Figure 3: Subtraction, multiplication, and division of $\pi_x, \pi_y = (1,2,3)$ with various precise copulas specified.

an appropriate unary operation to one of the inputs and evaluating the function with a non-decreasing binary operation. For example a dependent subtraction between π_X and π_Y may be evaluated as

$$\mathbb{C}^{\alpha_x}_{\pi_X} + (-\mathbb{C}^{\alpha_y}_{\pi_Y}), \tag{27}$$

where the negation is performed levelwise on α-cuts of π_Y. Similarly division may be evaluated with a reciprocate and product:

$$\mathbb{C}^{\alpha_x}_{\pi_X} * (1/\mathbb{C}^{\alpha_y}_{\pi_Y}). \tag{28}$$

However when evaluating these non-increasing operations, $C_{X,-Y}$ and $C_{X,1/Y}$ must be used, which may be found by a simple transformation of C_{XY} [16]:

$$C_{X\beta(Y)}(u,v) = u - C_{XY}(u, 1-v), \tag{29}$$

where β is a strictly decreasing function on the support of Y. This gives the following mapping between copulas:

$$M \mapsto W,$$
$$W \mapsto M,$$
$$\Psi \mapsto \Psi,$$
$$C^{\Phi}_{r} \mapsto C^{\Phi}_{-r}.$$

The parameter of the Gaussian copula is negated due the symmetry of the Gaussian copula. Figure 3 shows the dependent subtraction, multiplication and division between $\pi_X, \pi_Y = (1,2,3)$ for various Gaussian copulas. Note that for $f = \{-, /\}$ transformation (29) has been applied, and so $C_{XY} = W$ gives levelwise arithmetic and $C_{XY} = M$ gives opposite levelwise arithmetic.

Equation (18) with Theorem 3 also implies a correlated random α-cut slicing strategy for propagating π_X, π_Y under C_{XY}, where random values of α_X and α_Y may be simulated from C_{XY}:

$$(\alpha_X, \alpha_Y) \sim C_{XY}, \tag{30}$$

followed by an evaluation of (23). Again the resulting belief function may be made consonant with an imprecise probability-to-possibility transformation. This correlated random α-cut slicing method produces an inner approximation to the results of Figures 2 and 3 when finite samples are used.

5. Arithmetic with partially known dependence

Sklar's theorem for minitive beliefs may be used in a dependent possibilistic arithmetic where only the bounds to copulas are known, whereby interval arithmetic may be performed on α-cuts of two possibility distributions, with resulting α-level of the output being found using Theorem 3. Because the necessity measure of an α-cut is: $N_X(\mathbb{C}^{\alpha}_{\pi_X}) = 1 - \alpha$, the belief of the image $f(\mathbb{C}^{\alpha}_{\pi_X}, \mathbb{C}^{\alpha}_{\pi_Y})$ can be bounded by

$$\begin{aligned} \mathrm{Bel}_Z(f(\mathbb{C}^{\alpha}_{\pi_X}, \mathbb{C}^{\alpha}_{\pi_Y})) &\geq \mathrm{Nec}_{XY}(\mathbb{C}^{\alpha}_{\pi_X}, \mathbb{C}^{\alpha}_{\pi_Y}) \\ &= C(1-\alpha, 1-\alpha). \end{aligned} \tag{31}$$

Choosing the candidate possibility distribution q_Z such that its α-cuts are given by these images, i.e. $\mathbb{C}^{\alpha}_{q_Z} = f(\mathbb{C}^{\alpha}_{\pi_X}, \mathbb{C}^{\alpha}_{\pi_Y})$, immediately implies that they are also the $1 - C(1-\alpha, 1-\alpha)$-cuts of a robust outer approximation π_Z of Bel_Z under the imprecise probability-to-possibility transform in Equation (12). So, for some non-decreasing function with an interval extension $f : \mathscr{B}^+ \times \mathscr{B}^+ \to \mathscr{B}^+$, two possibility distributions π_X and π_Y with alpha cuts $\mathbb{C}^{\alpha}_{\pi_X}$ and $\mathbb{C}^{\alpha}_{\pi_Y}$ correlated by copula C_{XY}, an outer approximation of $z = f(x,y)$ may be found as:

$$\mathbb{C}^{1-C_{XY}(1-\alpha,1-\alpha)}_{\pi_Z} = f(\mathbb{C}^{\alpha}_{\pi_X}, \mathbb{C}^{\alpha}_{\pi_Y}). \tag{32}$$

The image of the marginal input α-cuts of X and Y constitutes the output $1 - C_{XY}(1-\alpha, 1-\alpha)$-cut of π_Z. However,

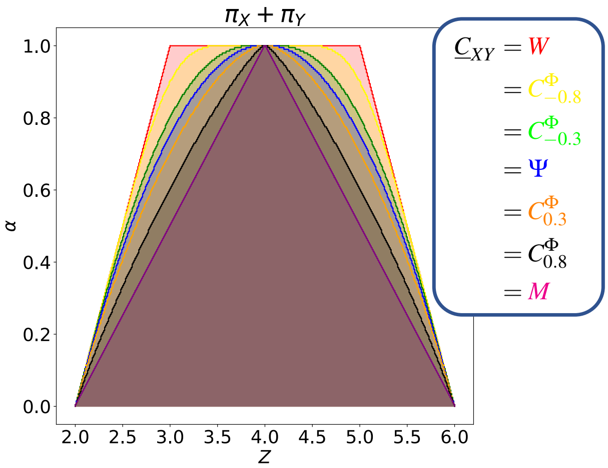

Figure 4: Sum of $\pi_x, \pi_y = (1,2,3)$ with various copula lower bounds specified.

if $C_1 \prec C_2$ then

$$C_1(1-\alpha, 1-\alpha) \leq C_2(1-\alpha, 1-\alpha),$$

and using Sklar's theorem for minitive beliefs:

$$\mathrm{Nec}^{C_1}_{XY} \leq \mathrm{Nec}^{C_2}_{XY}, \tag{33}$$

and which also from (32) gives

$$\pi^{C_1}_Z \geq \pi^{C_2}_Z. \tag{34}$$

Moreover, from (31) if $\mathrm{Nec}^{C_2}_{XY}$ is a robust approximation of $\mathrm{Bel}^{C_2}_Z$, then so is $\mathrm{Nec}^{C_1}_{XY}$; i.e.

$$\mathrm{Nec}^{C_1}_{XY} \leq \mathrm{Nec}^{C_2}_{XY} \leq \mathrm{Bel}^{C_2}_Z, \tag{35}$$

for any $C_1 \prec C_2$. The copula in (32) may therefore be replaced by a lower bound:

$$\mathbb{C}^{1-\underline{C}_{XY}(1-\alpha,1-\alpha)}_{\pi_Z} = f(\mathbb{C}^{\alpha}_{\pi_X}, \mathbb{C}^{\alpha}_{\pi_Y}), \tag{36}$$

since (32) will compute a robust outer approximation belief functions propagated with all copulas $C_{XY} \prec C$. The above equation gives, for a non-decreasing function, a levelwise α-cut based arithmetic for possibility distributions when only a lower bound on a copula is known. The α-level of the images is a simple scaling of α using $\underline{C}_{XY}$.

Figure 4 shows (36) evaluated for the sum of $\pi_x, \pi_y = (1,2,3)$ and various $\underline{C}_{XY}$. A Gaussian copula has been used for $\underline{C}_{XY}$, varying its parameter from -1 to 1. When $\underline{C}_{XY} = W$ (the lower bound on all copulas), (36) gives Fréchet, corresponding to the sum under unknown dependence between π_X and π_Y. The computed possibility distribution π^W_Z is greater than (i.e. it encloses) the possibility distributions calculated under any other copula π^C_Z, since $W \prec C$ for any C. It also encloses all those shown in Figure 2. It does not tightly enclose all those from Figure 2 since the example only shows several Gaussian copulas, whilst π^W_Z bounds those propagated using any copula (including non-Gaussian). $\underline{C}_{XY} = \Psi$ corresponds to positive quadrant dependent arithmetic, and propagates all copulas greater than Ψ. As expected, the computed π^{Ψ}_Z is greater than or equal to those in Figure 2 propagated with the precise copulas $C_{XY} = \{\Psi, C^{\Phi}_{0.3}, C^{\Phi}_{0.8}, M\}$, but not those propagated with $C_{XY} = \{C^{\Phi}_{-0.3}, C^{\Phi}_{-0.8}, W\}$. $\underline{C}_{XY} = M$ specifies a precise copula $C_{XY} = M$, and levelwise arithmetic is retrieved.

Only the lower bound on C_{XY} plays a role in (36). This is because $\underline{C}_{XY} \prec \overline{C}_{XY}$ gives $\pi^{\underline{C}_{XY}}_Z \geq \pi^{\overline{C}_{XY}}_Z$. That is, the result computed with the lower bound will always enclose the result from the upper bound. A similar situation occurs with the τ and ρ convolutions in p-box arithmetic.

Note that (36) with $\underline{C}_{XY} = M$ gives the same results as (10) using $\mathbb{J}^{\min}$, $\underline{C}_{XY} = \Psi$ gives (10) using $\mathbb{J}^{\mathrm{SI}}$, and $\underline{C}_{XY} = W$ gives (10) using $\mathbb{J}^{\mathrm{UI}}$. This suggests the generalised extension principle with aggregation operations of [12] corresponds to an imprecise propagation of dependencies, as opposed to a precise one.

Like in Section 4, and Williamson and Downs for the τ and ρ convolutions in p-box arithmetic, we extend (36) to non-increasing binary operations by first transforming one of the inputs and performing a non-decreasing binary operation. The copula must also be appropriately transformed. Note that if X and Y are positive quadrant dependent, then X and $-Y$ (and X and $1/Y$) will be negative quadrant dependent [16]. The outcome of this is, like for the τ and ρ convolutions, that the copula's upper bound is used instead of the lower bound, i.e. $\tau_{\Psi,-}$ and $\rho_{\Psi,-}$ compute a p-box which bounds negative quadrant dependence and $\tau_{M,-}$ and $\rho_{M,-}$ give Fréchet for subtraction. Therefore for a non-increasing binary operations, (36) is computed as:

$$\mathbb{C}^{1-\overline{C}^{*}_{XY}(1-\alpha,1-\alpha)}_{\pi_Z} = f(\mathbb{C}^{\alpha}_{\pi_X}, \mathbb{C}^{\alpha}_{\pi_Y}), \tag{37}$$

where $\overline{C}^{*}_{XY}$ is the appropriate transformation of the copula's upper bound. For $f = \{-, /\}$, this transformation is (29):

$$\overline{C}^{*}_{XY}(u,v) = u - \overline{C}_{XY}(u, 1-v),$$

$$\overline{C}^{*}_{XY}(1-\alpha, 1-\alpha) = 1-\alpha-\overline{C}_{XY}(1-\alpha, \alpha),$$

and (37) becomes:

$$\mathbb{C}^{\alpha+\overline{C}_{XY}(1-\alpha,\alpha)}_{\pi_Z} = f(\mathbb{C}^{\alpha}_{\pi_X}, \mathbb{C}^{\alpha}_{\pi_Y}). \tag{38}$$

As opposed to (36), for π_Z produced by (38) $C_1 \prec C_2$ gives $\pi^{C_1}_Z \leq \pi^{C_2}_Z$.

Figure 5 shows subtraction and division evaluated with (38) and multiplication with (36) for $\pi_X, \pi_Y = (1,2,3)$ with various Gaussian copulas as bounds. For $f = \{-, /\}$: $\overline{C}_{XY} = M$ gives Fréchet, $\overline{C}_{XY} = \Psi$ gives negative quadrant dependence, and $\overline{C}_{XY} = W$ gives levelwise arithmetic. When comparing the results from the precise copula calculation in Figure 3, the computed π^{Ψ}_Z using $\overline{C}_{XY}$

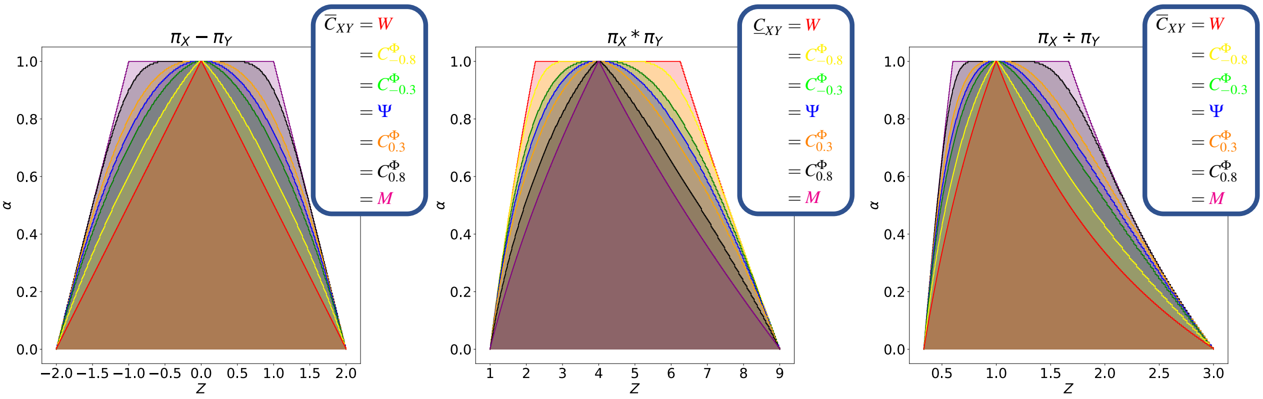

Figure 5: Subtraction, multiplication, and division of $\pi_x, \pi_y = (1,2,3)$ with various copulas copula bounds specified.

for $f = \{-, /\}$ encloses those propagated with $C_{XY} = \{\Psi, C^{\Phi}_{-0.3}, C^{\Phi}_{-0.8}, W\}$, but not those propagated with $C_{XY} = \{C^{\Phi}_{0.3}, C^{\Phi}_{0.8}, M\}$, as expected for negative quadrant dependence.

6. Relationship to p-box arithmetic

In this section we compare our dependent possibilistic arithmetic using precise C_{XY} and imprecise $\underline{C}_{XY}$ with the σ, τ, and ρ convolutions of p-box arithmetic. The method outlined in Section 4 may be considered a possibilistic analogue to a σ convolution, and the methods of Section 5 are a possibilistic analogue to the combination of a τ and ρ convolution (one for each p-box bound).

Consider performing similar calculations with p-box arithmetic. A possibility distribution π_X may be converted to a p-box by accumulating Π_X and N_X:

$$\overline{F}_X(x) = \overline{P}_X(X \leq x) = \Pi_X((-\infty, x]),$$
$$\underline{F}_X(x) = \underline{P}_X(X \leq x) = N_X((-\infty, x]), \qquad (39)$$

with the credal set defined by π_X, $\mathfrak{C}(\pi_X)$, being contained in credal set defined by the p-box: $\mathfrak{C}(\pi_X) \subseteq \mathfrak{C}([\underline{F}_X, \overline{F}_X])$ [1]. Figure 6 is an illustration of the similarity between the dependent arithmetic operations presented in this paper to those from [24]. Beginning with the fuzzy numbers $\pi_X, \pi_Y = (1,2,3)$ in the centre left, they may be converted into the p-box in the centre right. Several evaluations of τ and ρ with different copula lower bounds will yield the enclosing set of p-boxes on the top right, with Fréchet in red and perfect in purple. A similar set of σ convolutions yield the non-enclosing p-boxes on the bottom left, with the purple being perfect and matching the purple p-box from the τ-ρ convolution. Red (opposite) is an interval. The τ-ρ p-boxes calculated with copula $\underline{C}_{XY}$ enclose all p-boxes produced with copulas C which are greater $\underline{C}_{XY} \prec C$, for both σ and τ-ρ calculations. An identical behaviour can be seen in the dependent possibilistic arithmetic. Arithmetic with

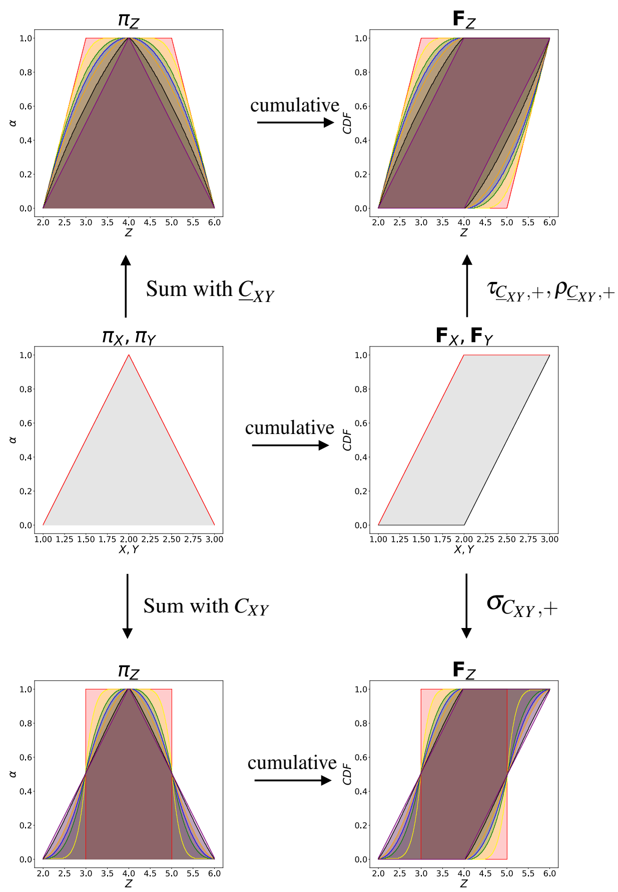

Figure 6: Illustration of the similarity between the dependent possibilistic operations derived in this paper to those from p-box arithmetic. The colours and copulas are the same as Figures 2 and 4.

$\underline{C}_{XY}$ produces a set of enclosing fuzzy numbers. Precise propagation produces results which are not self enclosing, but match the imprecise propagation when $C_{XY} = M$ and give an interval when $C_{XY} = W$. The $\underline{C}_{XY}$ calculation also encloses all possibility distributions calculated with copulas C which are greater $\underline{C}_{XY} \prec C$, for both the precise and imprecise cases.

Furthermore, when the $\pi_Z^{C_{XY}}$ and $\pi_Z^{\underline{C}_{XY}}$ possibility distributions are converted to p-boxes, the σ and τ-ρ p-boxes are retrieved. Note that here we are not stating that the p-boxes produced both ways are equivalent, nor that dependent arithmetic and 'building a p-box' commute. Figure 6 only serves as a comparison. However from the conducted numerical example the p-box bounds from both approaches are very similar. Performing a more rigorous mathematical or numerical study about the agreement of the two methods should be pursued. Particularly a comparison of the credal sets produced by both methods is interesting. A situation similar to Figure 6 can be observed for the other binary operations.

The numerical approaches for computing dependent π and p-box operations are also quite similar. For example in both approaches it is computationally simpler to calculate with an imprecise copula. A σ convolution requires a cumulative integration of a joint probability measure over an increasing domain defined by f. As outlined in [24], this may be done in terms of a number of discrete quasi-inverses of the quantiles of a p-box. The probability measure induced by a copula on this discretisation is found, followed by an operation f of these inverses and an outer approximation of the integral. Similarly, our precise C_{XY} operation may be performed by finding the probability measure induced by a copula on some discrete number of α-cuts, followed by an interval operation on these cuts and a outer approximation of the resulting belief function, which involves large sums over mass assignments. The τ and ρ convolutions on the other had only require the infimum and supremum of the joint cdf to be found in some domain, which may also be outer approximated in term of the quasi-inverses. Our imprecise propagation of $\underline{C}_{XY}$ only requires a scaled levelwise arithmetic, and thus requires less interval operations and no (explicit) imprecise probability-to-possibility transformation.

In the context of propagating dependencies, these latest additions to possibilistic arithmetic give it the same capabilities as p-box arithmetic, allowing for any dependence or partial dependence to be specified as a copula. This provides a basis for a p-box/possibility hybrid arithmetic which gives tighter bounds on the exact upper and lower probabilities than either method independently for the propagation of general belief functions. Baudrit and Dubois [1] discuss how well possibility distributions and p-boxes bound general belief functions, and on which events do these two imprecise representations return a tight bound on the belief and plausibility. P-boxes return tight probability bounds in their tail regions, whilst return vacuous probability intervals $[0,1]$ in their central regions. Conversely possibility distributions return vacuous intervals in their tails and tight probability bounds in their centre. Since these two imprecise structures bound a belief function in opposing ways, their arithmetics may be combined to give a tighter propagation of a general belief function than either method individually. The results of this paper provide a means to perform the dependent binary operations on possibility distributions previously only available to p-boxes.

7. Discussion

Theorem 3 gives a form of random set dependence between possibility distributions. It should be noted however that it is unclear how the dependence between the α-cuts relates to the dependence of the distributions in the credal set $\mathfrak{C}(\pi_{XY})$. That is to say, Theorem 3 does not imply that

$$C(F_X, F_Y) \in \mathfrak{C}(\pi_{XY}) \tag{40}$$

for all $F_X \in \mathfrak{C}(\pi_X)$ and $F_Y \in \mathfrak{C}(\pi_X)$, and where $\mathfrak{C}(\pi_{XY})$ is the credal set of bivariate distributions induced by Theorem 3. Further investigation is required to find if (40) holds, and its implication for dependent possibilistic arithmetic. Couso et al. [3] do however show that for the case of independence, a set of joint dfs for variables that are strongly independent (i.e. $H(u,v) = F(u)G(v)$) is a subset of the distributions under random set independence [10].

Schmelzer further makes the distinction between *bivariate* minitive belief functions and *joint* minitive belief functions [20], where the later are belief functions defined on product spaces of focal elements, and also account for the shape of the joint focal elements. He argues that joint belief functions provide an upper bound on a set of joint probability measures, and studies the relationship between bivariate and joint belief functions, and their relation to copulas.

Note that we only show examples of $f = \{+,-,*,/\}$, however the described methods may be ready expanded to any non-increasing and non-decreasing binary operation.

Software

The methods developed in this paper are made available in an open-source Julia package for performing dependent possibilistic arithmetic: https://github.com/AnderGray/PossibilisticArithmetic.jl. The figures of this paper may be reproduced by running the scripts found in *examples/ISIPTA2021*. The p-box arithmetic used in this paper was performed using the open-source Julia package: https://github.com/AnderGray/ProbabilityBoundsAnalysis.jl

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