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A NEW FRACTIONAL MODEL OF SINGLE DEGREE OF FREEDOM SYSTEM, BY USING GENERALIZED DIFFERENTIAL TRANSFORM METHOD

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Abstract. Generalized differential transform method (GDTM) is a powerful method to solve the fractional differential equations. In this paper, a new fractional model for systems with single degree of freedom (SDOF) is presented, by using the GDTM. The advantage of this method compared with some other numerical methods has been shown. The analysis of new approximations, damping and acceleration of systems are also described. Finally, by reducing damping and analysis of the errors, in one of the fractional cases, we have shown that in addition to having a suitable solution for the displacement close to the exact one, the system enjoys acceleration once crossing the equilibrium point.

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1. Introduction

In recent years, study of systems and fractional equations by various methods, helped physics and engineering, especially mechanics to improve a lot (see [3]-[5], [11], [16], [18], [20], [23], [26]). A system with one degree of freedom, which presented in vibration, is an example of these equations. Degree of freedom means the direction of body motion (see [13], [14], [19], [29]). We consider the Newton's equation of motion. A second order differential equation corresponding to a single degree of freedom system is as follows

$$m\ddot{x} + c\dot{x} + kx = 0, \tag{1.1}$$

where m, c, and k are mass, damping coefficient, and stiffness coefficient of spring, respectively. (1.1) is solved as follows

$$\begin{aligned} ms^2 + cs + k &= 0, && (1.2)\\ \Delta &= c^2 - 4mk. && (1.3) \end{aligned}$$

Now, considering the value of Δ, there will be three different answers. If $\Delta > 0$, then we have

$$x(t) = Ae^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + Be^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}, \tag{1.4}$$

where $\omega_n = \sqrt{\frac{k}{m}}, \zeta = \frac{c}{c_c}$, and $c_c = \sqrt{4mk}$ are the natural frequency, the ratio of damping coefficient, and the value of c in the critical damping, respectively. Furthermore, A and B are constants to be evaluated

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from initial conditions of equation. In this case, the motion is a time reducing function and is known as a non-periodic motion. This case is called a fast damped or beyond-damped case. If $\Delta = 0$, then we have

$$(A + tB)e^{-\omega_n t}. \tag{1.5}$$

In this case, the motion is a function of time decreasing and the solution tends to zero, faster than the beyond-damped.

Finally, if $\Delta < 0$, vibration shows the nature of oscillating motion and the equation of motion is obtained as follows

$$x(t) = \quad Ae^{(-\zeta+i\sqrt{1-\zeta^2})\omega_n t} + Be^{(-\zeta-i\sqrt{1-\zeta^2})\omega_n t}. \tag{1.6}$$

For knowing that how the solutions were obtained refer to Sec.2.3 in [29]. The natural frequency of the damped oscillation is shown as

$$\omega_d = \omega_n\sqrt{1-\zeta^2}. \tag{1.7}$$

Also, by using (1.7), damped period is calculated as follow

$$\tau_d = \frac{2\pi}{\omega_d}. \tag{1.8}$$

To solve (1.1), two types of the fractional models of differential orders have been considered by the researchers (see [3], [11], [27]). They were able to study only one fractional order of the system, but Gomez-Aguilar et al. and Zaillua et al. succeeded to introduce both fractional orders of the system (see [12], [31]). In the model below, it can be seen that the orders are dependent

$$\frac{m}{\sigma^{2(1-\gamma)}}D^{2\gamma}x(t) + \frac{c}{\sigma^{1-\gamma}}D^{\gamma}x(t) + kx(t) = 0. \tag{1.9}$$

In the next section we will see that the proposed model includes degrees, which are independent. Another point that must be reminded is that recently two or more order fractional differential equations have been considered by the GDTM (see [8]). They could study two or multi-order fractional differential equations by zeroing one of orders. But recently, even without considering zero order, and by using both fractional orders approximate solutions were obtained (see [25]). In pursuit of our discussion, some definitions and theorems will be considered. Then, we will analysis the considered system.

2. Definitions and theorems

In this section we provide some important definitions and theorems.

Definition 2.1. The Riemann-Liouville integral operator is defined by

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \tag{2.1}$$

where $\alpha \in \mathbb{R}^+, a \geq 0$, and $t > a$.

Definition 2.2. The Caputo fractional derivative of order α is defined by

$$D^{\alpha}f(t) = \frac{1}{\Gamma(-\alpha+m)}\int_a^t (t-\tau)^{-\alpha+m-1}f^{(m)}(\tau)d\tau, \tag{2.2}$$

where $m-1 < \alpha \leq m, m \in \mathbb{Z}^+$.

For more information one may refer to [4], [5], [17],[24].

Definition 2.3. We define the generalized differential transform for the i-th derivative of a function $f(t)$ as follows

$$F_\alpha(i) = \frac{1}{\Gamma(\alpha i+1)}[(D^\alpha)^i f(t)]_{t=t_0}, \tag{2.3}$$

where $0<\alpha\leq 1$ and $(D^\alpha)^i = D^\alpha.D^\alpha.\cdots.D^\alpha$ $\{\mathrm{i-times}\}$ (see [22]).

Also, the differential inverse transform of $F_\alpha(i)$ is defined as

$$f(t) = \sum_{i=0}^{\infty} F_\alpha(i)(t-t_0)^{\alpha i}. \tag{2.4}$$

By substituting (2.3) into (2.4) and by using the generalized Taylor's formula in [21], we obtain

$$f(t) = \sum_{i=0}^{\infty} \frac{(t-t_0)^{\alpha i}}{\Gamma(\alpha i+1)}((D^\alpha)^i f)(t_0). \tag{2.5}$$

Using Theorem 4 in [21], we will obtain an approximate function $f(t)$ by the finite series as (see [22])

$$f(t) \cong \sum_{i=0}^{n} F_\alpha(i)(t-t_0)^{\alpha i}. \tag{2.6}$$

The following theorems help us to solve fractional differential equations.

Theorem 2.4. *If $f(t)=g(t)\pm h(t)$, then $F_\alpha(i)=G_\alpha(i)\pm H_\alpha(i)$.*

Theorem 2.5. *If $f(t)=\lambda g(t)$ and $\lambda\in\mathbb{R}$, then $F_\alpha(i)=\lambda G_\alpha(i)$.*

Theorem 2.6. *If $f(t)=g(t)h(t)$, then $F_\alpha(i)=\sum_{j=0}^{i} G_\alpha(j)H_\alpha(i-j)$.*

Theorem 2.7. *If $f(t)=D^\alpha g(t)$, then $F_\alpha(i)=\frac{\Gamma(\alpha(i+1)+1)}{\Gamma(\alpha i+1)}G_\alpha(i+1)$.*

Theorem 2.8. *If $f(t)=D^\beta g(t), m-1<\beta\leq m$ and the function $g(t)$ satisfies the conditions of Theorem 2-5 in [22], then $F_\alpha(i)=\frac{\Gamma(\alpha i+\beta+1)}{\Gamma(\alpha i+1)}G_\alpha(i+\frac{\beta}{\alpha})$.*

The proofs may be found in [22].

3. Discussion

In this section, first we introduce the new model of SDOF system. Then, by the above definitions and theorems we compare accuracy of GDTM with other numerical methods. In addition, we study the new displacement and acceleration diagrams in any cases.

The new model of fractional differential equation corresponding to SDOF system is as follows

$$mx^\beta(t)+cx^\alpha(t)+kx(t)=0, \tag{3.1}$$

where $0<\alpha\leq 1$ and $1<\beta\leq 2$. We consider the following initial conditions

$$x(0)=0, \dot{x}(0)=1. \tag{3.2}$$

In the first case, we suppose $\alpha=1$ and $\beta=2$. The exact solution of the equation with the initial conditions will be

$$x(t) = \frac{me^{\frac{1}{2}(-c+\sqrt{c^2-4m})t}}{\sqrt{c^2-4m}} - \frac{me^{-\frac{1}{2}(c+\sqrt{c^2-4m})t}}{\sqrt{c^2-4m}}. \tag{3.3}$$

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Now, we assume $m = c = k = 1$. By the generalized differential transform and Theorems 2.8, 2.7, and 2.5, respectively, the terms in (3.1) are transformed as

$$\frac{\Gamma(\alpha i+\beta+1)}{\Gamma(\alpha i+1)}X_\alpha(i+\frac{\beta}{\alpha})+\frac{\Gamma(\alpha(i+1)+1)}{\Gamma(\alpha i+1)}X_\alpha(i+1)+X_\alpha(i)=0. \tag{3.4}$$

Setting values of α and β in above equation and arranging, respectively, (3.1) is transformed as

$$\frac{\Gamma(i+3)}{\Gamma(i+1)}X_1(i+2)+\frac{\Gamma(i+2)}{\Gamma(i+1)}X_1(i+1)+X_1(i)=0, \tag{3.5}$$

$$\frac{\Gamma(i+3)}{\Gamma(i+1)}X_1(i+2)=-\frac{\Gamma(i+2)}{\Gamma(i+1)}X_1(i+1)-X_1(i), \tag{3.6}$$

$$X_1(i+2)=-\frac{\Gamma(i+2)}{\Gamma(i+3)}X_1(i+1)-\frac{\Gamma(i+1)}{\Gamma(i+3)}X_1(i). \tag{3.7}$$

Also, according to Definition 2.3 and (3.2), the generalized differential transform of the initial conditions can be obtained as

$$X_1(0)=0, X_1(1)=1. \tag{3.8}$$

Considering $i = 0, 1, 2, \cdots, n$ in (3.7), other components are as follow

$$\begin{gathered}X_1(2)=-0.5, X_1(3)=0, X_1(4)=0.04166666667,\\ X_1(5)=-0.008333333334,\cdots,X_1(20)=-4.110317620\ 10^{-19}.\end{gathered} \tag{3.9}$$

By substituting (3.8), and (3.9) into (2.6) instead of $F_\alpha(i)$ for $i = 0, 1, 2\cdots n$, we obtain $x(t)$ up to $O(t^{20})$

$$\begin{gathered}x(t)\cong t+(-0.5)t^2+(0.04166666667)t^4\\ +(-0.008333333334)t^5+(0.0001984126984)t^7\\ +(-0.00002480158730)t^8+(2.755731922\ 10^{-7})t^{10}\\ +(2.505210838\ 10^{-8})t^{11}+(1.605904383\ 10^{-10})t^{13}\\ +(-1.147074559\ 10^{-11})t^{14}+(-2.\ 10^{-22})t^{15}\\ +(4.779477330\ 10^{-14})t^{16}+(-2.811457252\ 10^{-15})t^{17}\\ +(8.220635240\ 10^{-18})t^{19}+(-4.110317620\ 10^{-19})t^{20}.\end{gathered} \tag{3.10}$$

Table 3.1 shows a comparison of GDTM results with the exact solution and other numerical methods such as Taylor's series method in the interval $[0, 1]$. The Taylor's method can be used for high accuracy solutions (see [1]). The other methods find the numerical solutions by using Runge-Kutta method. RKF45 and CK45 methods are a Fehlberg and a Cash-Karp fourth-fifth order, respectively (see [2], [6], [9], [10], [28]). Also, dverk78 and Rosenbrock obtain approximate solutions by using a seventh-eight order and third-fourth order, respectively (see [7], [15], [28], [30]). Although, we can evaluate the approximation of GDTM for more terms of series solution, but according to the large values of calculations and the possible errors, it will not be efficient. As seen, the results of GDTM is as well as the results of the Taylor and dverk78's methods and more accurate than the other methods.

Table 3.1: Comparison of GDTM with other methods for (3.1) and (3.2).

t	Exact	GDTM	Taylor	dverk78	CK45	RKF45	Rosenbrock
0	0	0	0	0	0	0	0
0.1	0.09500408342	0.09500408336	0.09500408335	0.09500408335	0.09500408466	0.09500408321	0.09500415174
0.2	0.1800640025	0.1800640024	0.1800640025	0.1800640025	0.1800640029	0.1800640016	0.1800640947
0.3	0.2553172919	0.2553172918	0.2553172918	0.2553172918	0.2553172918	0.2553172914	0.2553173953
0.4	0.3209816424	0.3209816422	0.3209816422	0.3209816422	0.3209816420	0.3209816418	0.3209817546
0.5	0.3773452037	0.3773452035	0.3773452035	0.3773452035	0.3773452033	0.3773452021	0.3773453037
0.6	0.4247571395	0.4247571393	0.4247571393	0.4247571393	0.4247571390	0.4247571381	0.4247572282
0.7	0.4636185012	0.4636185010	0.4636185010	0.4636185010	0.4636185006	0.4636185002	0.4636185798
0.8	0.4943734769	0.4943734766	0.4943734766	0.4943734766	0.4943734762	0.4943734754	0.4943735463
0.9	0.5175010623	0.5175010621	0.5175010622	0.5175010621	0.5175010621	0.5175010600	0.5175011233
1.0	0.5335071953	0.5335071952	0.5335071951	0.5335071951	0.5335071960	0.5335071927	0.5335072481

In the second case, we suppose $\alpha = 0.5$ and $\beta = 2$. As before, by above mentioned theorems, (3.1) is transformed to (3.4). Now, by setting values of α and β in (3.4) and arranging, respectively, we have

$$\frac{\Gamma(0.5i+3)}{\Gamma(0.5i+1)}X_{0.5}(i+4) + \frac{\Gamma(0.5i+1.5)}{\Gamma(0.5i+1)}X_{0.5}(i+1) + X_{0.5}(i) = 0, \tag{3.11}$$

$$X_{0.5}(i+4) = -\frac{\Gamma(0.5i+1.5)}{\Gamma(0.5i+3)}X_{0.5}(i+1) - \frac{\Gamma(0.5i+1)}{\Gamma(0.5i+3)}X_{0.5}(i). \tag{3.12}$$

Also, according to Definition 2.3 and (3.2), the generalized differential transform of the initial conditions can be obtained as

$$X_{0.5}(0) = X_{0.5}(1) = X_{0.5}(3) = 0, X_{0.5}(2) = 1. \tag{3.13}$$

Since $\alpha = 0.5$, we obtain $X_{0.5}(i) = 1$ when $i = 2$. So, by considering $i = 0, 1, \cdots, n$, in (3.12), the other components are as follow

$$X_{0.5}(4) = 0, X_{0.5}(5) = -0.3009011113, \cdots, X_{0.5}(80) = -1.131502271\ 10^{-43}. \tag{3.14}$$

By Substituting (3.13), and (3.14) into (2.6) instead of $F_\alpha(i)$ for $i = 0, 1, 2 \cdots n$, we have the displacement of $x(t)$ up to $O(t^{40})$ as

$$\begin{aligned} x(t) &\cong t + (-0.3009011113)t^{2.5} + (-0.1666666667)t^3 \\ &+(0.04166666668)t^4 + (0.03820966493)t^{4.5} \\ &+(0.00833333333)t^5 + \cdots + (1.105571111\ 10^{-42})t^{39.5} \\ &+(-1.131502271\ 10^{-43})t^{40}. \end{aligned} \tag{3.15}$$

Finally, in the last case, we decrease both orders of the equation. That is, we suppose $\alpha = 0.5$ and $\beta = 1.5$. Once again, by setting values of α and β in (3.4) and arranging, respectively, the generalized differential transform of (3.1) is as follows

$$\frac{\Gamma(0.5i+2.5)}{\Gamma(0.5i+1)}X_{0.5}(i+3) + \frac{\Gamma(0.5i+1.5)}{\Gamma(0.5i+1)}X_{0.5}(i+1) + X_{0.5}(i) = 0, \tag{3.16}$$

$$X_{0.5}(i+3) = -\frac{\Gamma(0.5i+1.5)}{\Gamma(0.5i+2.5)}X_{0.5}(i+1) - \frac{\Gamma(0.5i+1)}{\Gamma(0.5i+2.5)}X_{0.5}(i). \tag{3.17}$$

Also, according to Definition 2.3 and (3.2), we can obtain the generalized differential transform of the initial conditions

$$X_{0.5}(0) = X_{0.5}(1) = X_{0.5}(2) = 0. \tag{3.18}$$

Considering $i = 0, 1, 2, \cdots, n$ in (3.17), other components are as follow

$$X_{0.5}(3) = 0, X_{0.5}(4) = -0.5, X_{0.5}(5) = 0, \cdots, X_{0.5}(80) = 2.947475100\ 10^{-42}. \tag{3.19}$$

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Hence, the displacement of $x(t)$ is as follows

$$x(t) \cong t + (-0.5)t^2 + (-0.3009011113)t^{2.5} \\ +(0.166666666)t^3 + (0.1719434921)t^{3.5} \\ +(0.00833333333)t^4 + \cdots + (2.247478671\ 10^{-42})t^{39.5} \\ +(2.947475100\ 10^{-42})t^{40}. \tag{3.20}$$

Fig.3.1(a) shows the comparison of the last three cases with the exact solution. When we fix β and decrease α, since the value of c is high, the damping of the system is reduced. Therefore, the diagram shows the higher peak. Using (1.7) and (1.8), according to natural frequency of ω_n, and the ratio of damping coefficient of ζ we obtained the damped period of this case, that is $\tau_{d,1} = 7.25$. On the other hand, by reducing both orders, SDOF system achieves high damping. The mass-spring moves slowly after the first seconds. Fig.3.1(b) shows the two cases errors. We see that the first case has a very low error. Now, we suppose $c = 0.001$ and compare both displacements of the first and second cases with each other, up to $O(t^{40})$, again. Fig.3.2 shows the results. The curve of the second case is compatible with the first case and with the exact solution. Clearly, both the error and damping are decreased, in comparison with when c was 1. Notice that, we can obtain the closer approximations with the reduction of c. The damped period is $\tau_{d,0.001} = 6.28$.

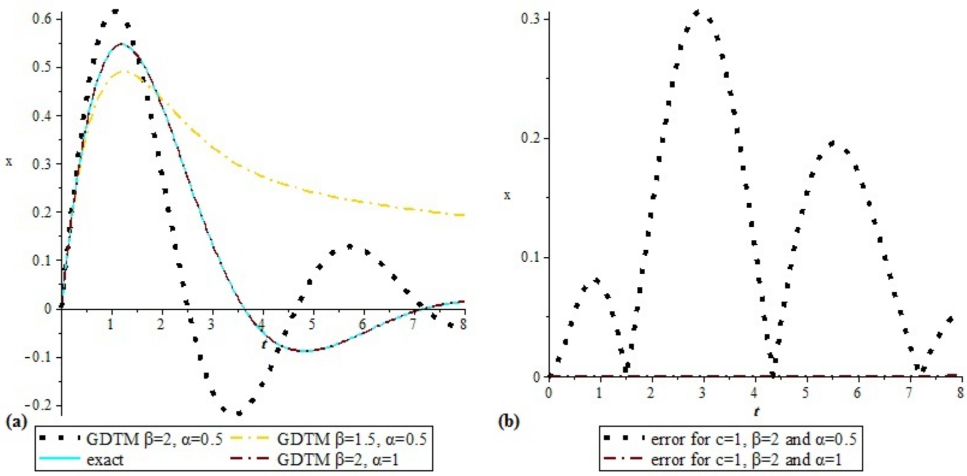

Fig. 3.1: Comparison of the displacement of mass-spring for three cases of GDTM with the exact solution when $c = 1$: (a) The displacement (b) GDTM errors.

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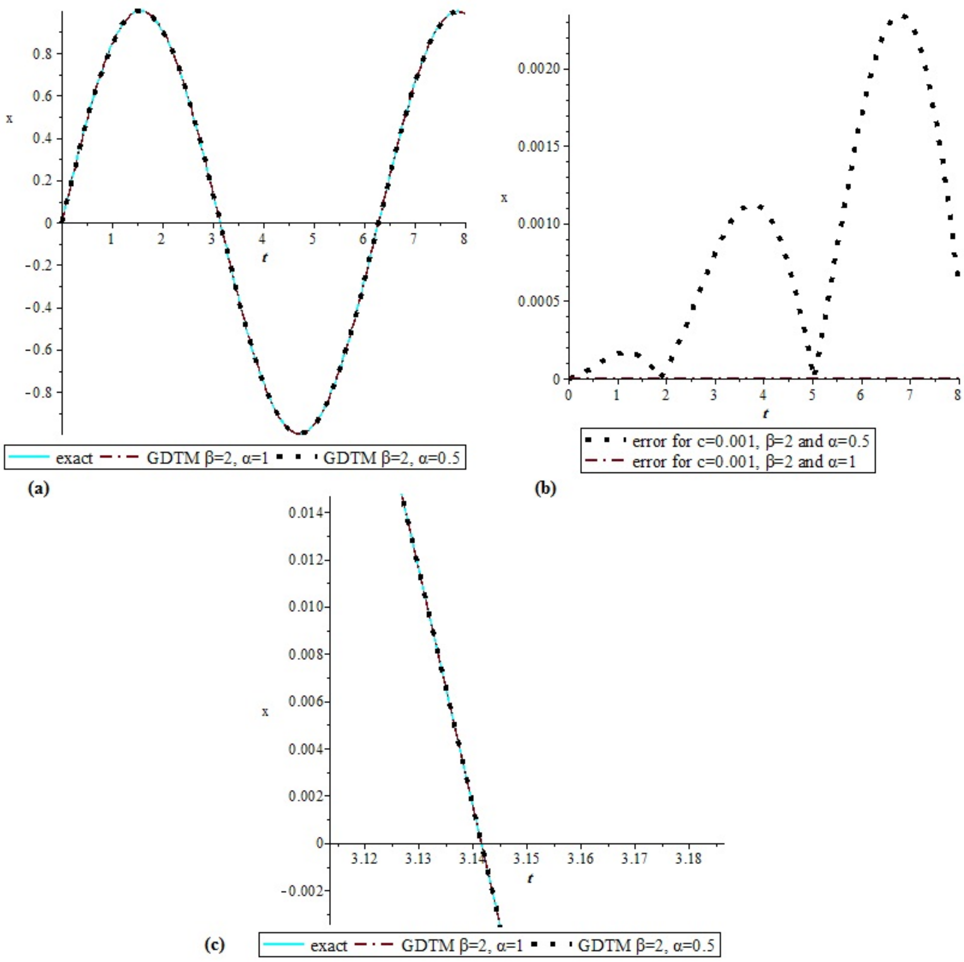

Fig. 3.2: Comparison of the displacement of mass-spring for two cases of GDTM with the exact solution when $c = 0.001$: (a) The displacement (b) GDTM errors (c) Equilibrium point.

Now, we suppose c to be a variable and $m = k = 1$. Considering to the mentioned theorems, we obtain

$$\frac{\Gamma(\alpha i+\beta+1)}{\Gamma(\alpha i+1)}X_\alpha(i+\frac{\beta}{\alpha})+c\frac{\Gamma(\alpha(i+1)+1)}{\Gamma(\alpha i+1)}X_\alpha(i+1)+X_\alpha(i)=0. \tag{3.21}$$

The generalized differential transform of the first case, by setting $\alpha = 1$ and $\beta = 2$, will be

$$X_1(i+2)=-c\frac{\Gamma(i+2)}{\Gamma(i+3)}X_1(i+1)-\frac{\Gamma(i+1)}{\Gamma(i+3)}X_1(i). \tag{3.22}$$

Regarding to the transformed initial conditions (3.8) and (3.22), we consider $i = 0, 1, 2, \cdots, n$, the first components of the generalized differential transform can be obtained as

$$X_1(1)=0, X_1(1)=1, X_1(2)=-0.5c, X_1(3)=-0.1666666667c^2-0.1666666667. \tag{3.23}$$

We would have the displacement of $x(t)$ by the same initial conditions (3.8)

$$\begin{aligned}
x(t) \cong\ & t + (-0.5c)t^2 + (-0.1666666667c^2 - 0.1666666667)t^3 + \cdots \\
& +(-1.225617439\ 10^{-48}c^{39} + 4.657346272\ 10^{-47}c^{37} \\
& -8.162612143\ 10^{-46}c^{35} + 8.750908516\ 10^{-45}c^{33} \\
& -6.417332911\ 10^{-44}c^{31} + 3.410354061\ 10^{-43}c^{29} \\
& -1.357454655\ 10^{-42}c^{27} + 4.125251811\ 10^{-42}c^{25} \\
& -9.668558933\ 10^{-42}c^{23} + 1.753509254\ 10^{-41}c^{21} \\
& -2.454912955\ 10^{-41}c^{19} + 2.631912949\ 10^{-41}c^{17} \\
& -2.130596197\ 10^{-41}c^{15} + 1.274715673\ 10^{-41}c^{13} \\
& -5.463067172\ 10^{-42}c^{11} + 1.602499704\ 10^{-42}c^{9} \\
& -3.004686946\ 10^{-43}c^{7} + 3.227540966\ 10^{-44}c^{5} \\
& -1.630071194\ 10^{-45}c^{3} + 2.451234881\ 10^{-47}c)t^{40}.
\end{aligned} \tag{3.24}$$

Also, in the second case, with regard to (3.21), by setting $\alpha = 0.5$ and $\beta = 2$, we obtain

$$X_{0.5}(i+4) = -c\frac{\Gamma(0.5i+1.5)}{\Gamma(0.5i+3)}X_{0.5}(i+1) - \frac{\Gamma(0.5i+1)}{\Gamma(0.5i+3)}X_{0.5}(i). \tag{3.25}$$

Considering the transformed initial conditions (3.18) and $i = 0, 1, 2 \cdots, n$, the first components of the generalized differential transform can be obtained as

$$\begin{aligned}
& X_{0.5}(0) = X_{0.5}(1) = X_{0.5}(3) = X_{0.5}(4) = X_{0.5}(7) = 0, X_{0.5}(2) = 1, \\
& X_{0.5}(5) = -0.3009011113c, X_{0.5}(6) = -0.1666666667.
\end{aligned} \tag{3.26}$$

The displacement of $x(t)$ will be obtained up to $O(t^{40})$

$$\begin{aligned}
x(t) \cong\ & t + (-0.3009011113c)t^{2.5} + (-0.1666666667)t^3 \\
& +(0.04166666668c^2)t^4 + (0.03820966493)t^{4.5} + \cdots \\
& +(1.225617440\ 10^{-48}c^{26} - 2.818920110\ 10^{-45}c^{22} \\
& +1.649632048\ 10^{-43}c^{18} - 1.001562316\ 10^{-42}c^{14} \\
& +7.925406148\ 10^{-43}c^{10} - 6.650690470\ 10^{-44}c^{6} \\
& +2.328673135\ 10^{-46}c^{2})t^{40}.
\end{aligned} \tag{3.27}$$

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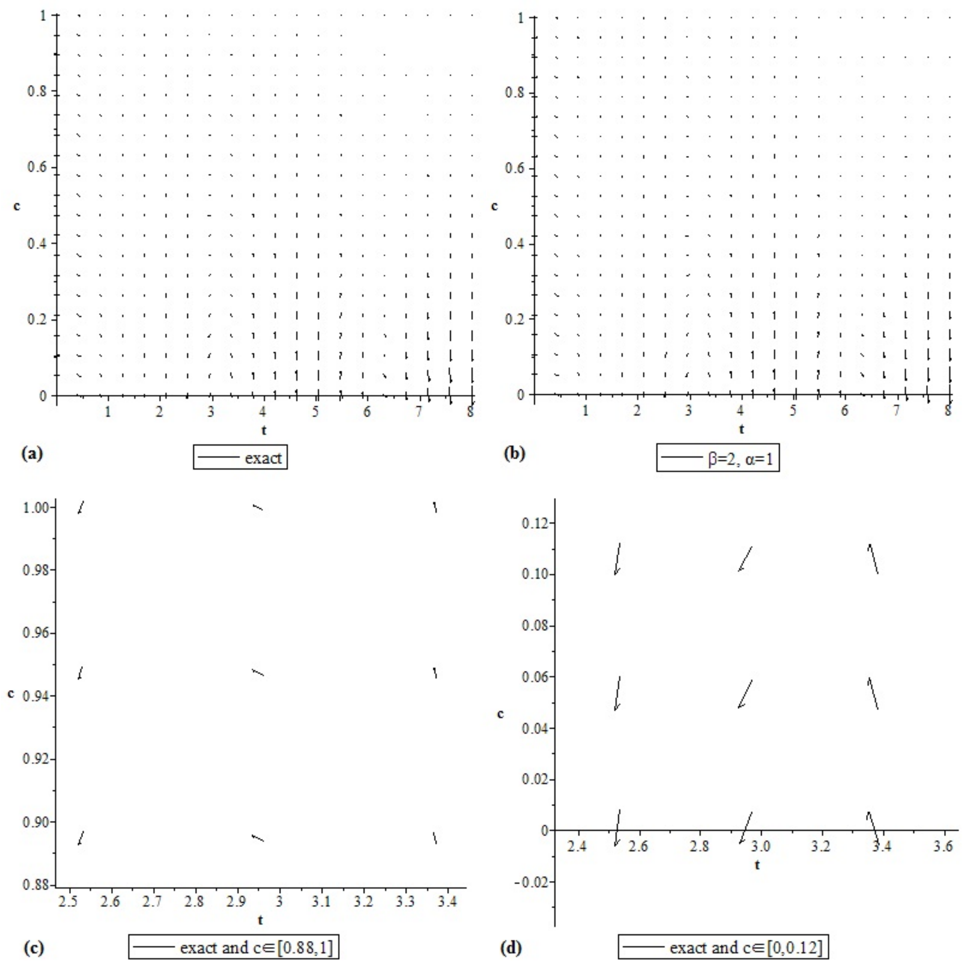

Fig. 3.3: Vector of acceleration: (a) Exact solution and $c \in [0.001, 1]$ (b) GDTM for $\beta = 2, \alpha = 1$ and $c \in [0.001, 1]$ (c) Exact solution and $c \in [0.88, 1]$ (d) Exact solution and $c \in [0, 0.12]$.

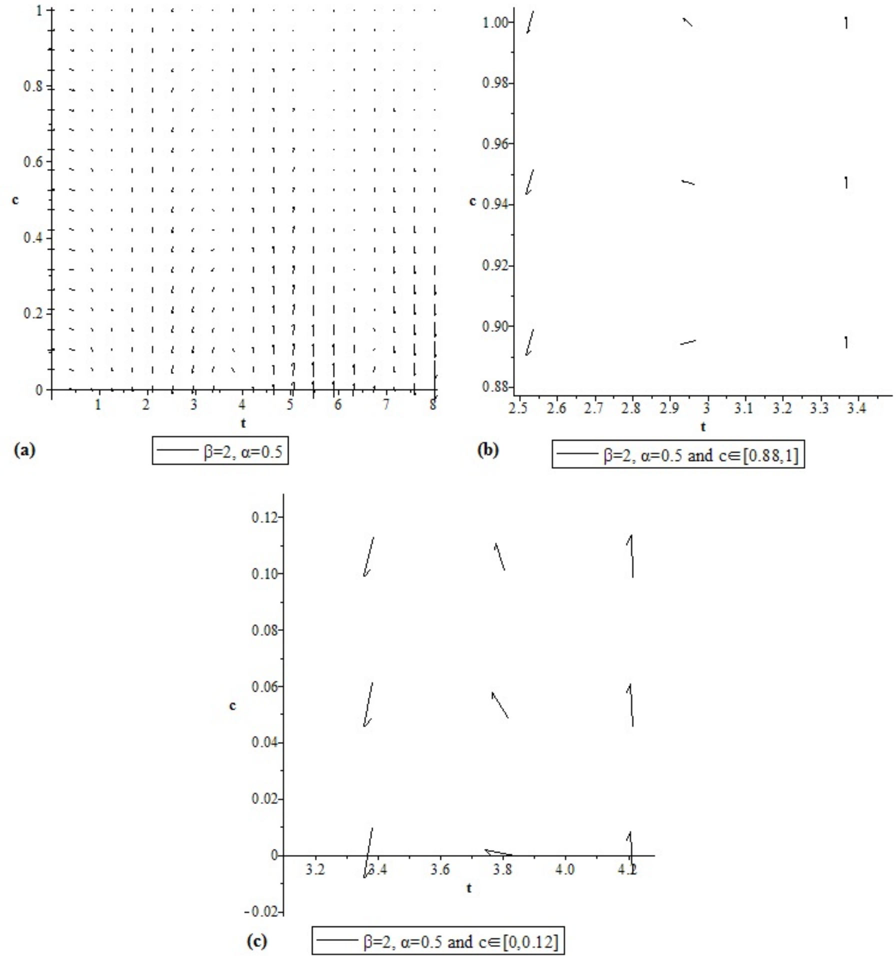

Fig. 3.4: Vector of acceleration for GDTM (a) $\beta = 2, \alpha = 0.5$ and $c \in [0.001, 1]$ (b) $\beta = 2, \alpha = 0.5$ and $c \in [0.88, 1]$ (c) $\beta = 2, \alpha = 0.5$ and $c \in [0, 0.12]$.

Fig.3.3 shows the acceleration-time diagram of (3.3), (3.24), and (3.27). c is assumed to be in the interval $[0.001, 1]$. According to the exact curve of Fig.3.1(a), the mass-spring reaches the peak at $t = 1.2$. Accordingly, the vector changes its direction and the acceleration reaches to zero in the interval $[2.5, 3]$. It is the equilibrium point. According to the Fig.3.3(d), this time would be $[2.94, 3.37]$ for $c = 0.001$. Also, it is clear that the value of the acceleration vector will be larger than when the damping reduces. The process continues until the mass-spring stops. By comparing Fig.3.3(a) and (b), there would be no significant difference. As seen, other acceleration approximates in Fig.3.4, as to this difference, the mass-spring still has an acceleration when it crosses the equilibrium point. The acceleration reaches to zero after few seconds, with a delay compared to the two other cases. The reason for occurrence is using the fractional order, and particularly reducing of c. Finally, the acceleration reaches to zero in the interval $[3.35, 3.75]$. But the time of equilibrium point is $t = 3.14$ in Fig.3.2(c). It's noticed that the acceleration will be zero in the equilibrium point when c increases, that is c reaches to 1. Also, Fig.3.5 shows the displacement of mass-spring, assuming the different damping coefficients in the space of $x(t) - c - t$. The difference in the solution surface in Fig.3.5(c) is considerable.

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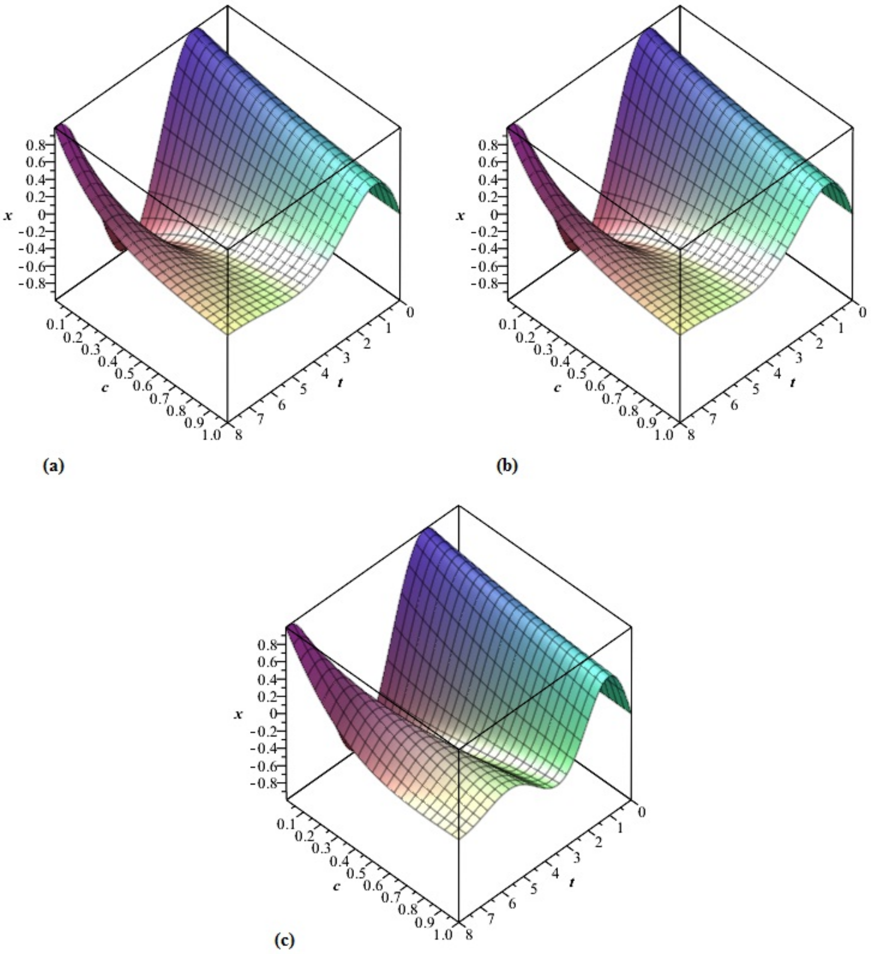

Fig. 3.5: The surface shows the solution of (3.24) and (3.27) in $x(t) - c - t$ space: (a) Exact solution (b) GDTM for $\beta = 2, \alpha = 1$ (c) GDTM for $\beta = 2, \alpha = 0.5$.

4. Conclusions

This paper tried to presented a new model of SDOF system by GDTM. It uses in door closer and shock absorber. The results have been compared with the Taylor's series and some other methods and show that our model works well and more accurate than other methods. The damping of the system increases when we decrease both orders. By considering fractional order of α and the reducing of damping coefficient we found that there would be an acceleration in the equilibrium point. But it becomes zero when damping coefficient increases.

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