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PLANE LANGUAGES AND THEIR PROPERTIES

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Abstract

In this paper we study the Generalized Parikh Vector of words over three letter alphabet. For Σ = {a, b, c} the GPVs of words lie in the tetrahedron whose vertices are (1, 0, 0), (0, 1, 0), (0, 0, 1) and (0, 0, 0). All GPVs of words of equal length lie on the same plane. Plane languages and their language theoretical properties are studied. Further, the GPVs of words lying on surfaces are discussed. The concepts of surface language, language surface and their properties are also studied in this paper.

Keywords:

Generalized Parikh Vectors, Plane Languages, Language Planes, Space Curves, Surface Language

1. INTRODUCTION

The concept of Generalized Parikh Vector (GPV) introduced by Siromoney et al. [4] proves to be a more powerful tool than the classical Parikh vector introduced by Parikh [1] as it gives the positions of the letters in a word w. It has been proved that the GPVs of the words of the same length lie on a hyper plane [3]. In the case of a binary alphabet, the GPVs of words of same length lie on a straight line. The concept of line languages introduced by Sasikala et al. [2] is the motivation for this study of plane languages. In this paper we extend the concept of line languages to define plane languages and study some of its language theoretical properties. In particular, we study the closure properties of quasi-plane languages and compare the languages in the quasi-plane family. We further investigate a new class of languages, namely Surface languages associated with curves on surfaces.

2. PRELIMINARIES

Σ, a non-empty finite set of symbols is an alphabet. The symbols in Σ are called letters. Any finite string over Σ is called a word over Σ. The length of a word w is the number of symbols present in the word and it is denoted by $|w|$. The empty word is denoted by λ and the set of all words over Σ is denoted by Σ^*. If $|w| = \infty$ then w is called an infinite word.

The collection of all infinite words is denoted by Σ^ω and $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$.

Let $u, v \in \Sigma^+$. Then u is a factor or a subword of v, if $v = w_1uw_2$ for some $w_1, w_2 \in \Sigma^*$. u is a prefix of v if $v = uw$ for some $u \in \Sigma^*$ and u is a suffix of v if $v = wu$ for some $u \in \Sigma^*$. The catenation of two words $u = a_1 a_2 \dots a_n$ and $w = b_1 b_2 \dots b_n$ denoted by $u{\cdot}w$ is $a_1 a_2 \dots a_n b_1 b_2 \dots b_n$. If $w = a_1 a_2 \dots a_n$ with $a_i \in \Sigma$ then $w^R = a_n \dots a_1$ is called the reversal of w.

The subsets of Σ^* are called languages over Σ. Upon languages, the operations like union, intersection, catenation, reversal, complement and Kleene star are defined respectively as follows. If L_1 and L_2 are two languages over Σ then,

(i) $L_1 \cup L_2 = \{w \in \Sigma^* / w \in L_1 \text{ or } w \in L_2\}$.

(ii) $L_1 \cap L_2 = \{w \in \Sigma^* / w \in L_1 \text{ and } w \in L_2\}$.

(iii) $L_1 \circ L_2 = \{w = w_1 w_2 \in \Sigma^* / w_1 \in L_1 \text{ or } w_1 \in L_2\}$.

(iv) $L^R = \{w^R \in \Sigma^* / w \in L\}$.

(v) $L^c = \Sigma^* - L$.

(vi) $L^* = \bigcup_{i\geq 0} L^i$ where $L^i = LL^{i-1}$ for $i \geq 1$ and $L_0 = \lambda$.

A partial word of length n over Σ is a partial function u: $\{1, 2,\dots, n\} \to \Sigma$. For $0 < i \leq n$ if $u(i)$ is defined we say that i belongs to the domain of u denoted by $i \in D(u)$. Otherwise we say that i belongs to the set of holes of u denoted by $i \in H(u)$. A hole is denoted by $\Diamond$.

If x is a word over Σ^+, then $x^2 \in \Sigma^+$ is a strictly square word and $x^3 \in \Sigma^+$ is a strictly cube word. $w = w_1^* w_2^2 w_3^* \in \Sigma^+$ for w_1, w_2, $w_3 \in \Sigma^+$ is a word with a square and $w = w_1^* w_2^3 w_3^* \in \Sigma^+$ is a word with a cube, for $w_1, w_2, w_3 \in \Sigma^+$.

Let $\Sigma = \{a_1, a_2,\dots, a_n\}$. Then the Parikh vector of a word u is given by $\pi(u) = \left(|u|_{a_1}, |u|_{a_2}, \dots, |u|_{a_i}, \dots, |u|_{a_n}\right)$ where $|u|_{a_i}$ represents the number of times a_i occurs in u [1].

Let $\Sigma = \{a_1, a_2, a_3\}$ and $x \in \Sigma^\infty$. The Generalized Parikh Vector (GPV) of x denoted by $p(x)$ is $(p_1, p_2, p_3) \in [0, 1]^3$, where, $p_i = \sum_{j\in A_i} \frac{1}{2^j}$, $A_i \subset N$ (set of natural numbers) and A_i contains the positions of a_i (i = 1, 2, 3) in the word x.

3. LINE LANGUAGES AND QUASI- LINE LANGUAGES

We define the GPV for Partial words as follows.

Definition 3.1: Let $\Sigma = \{a_1, a_2, a_3\}$ and u be a partial word over Σ. The Generalized Parikh Vector (GPV) of u denoted by $p(u)$ is $(p_1, p_2, p_3) \in [0, 1]^3$, where $\mathrm{p_i} = \sum_{\mathrm{j}\in \mathrm{A_i}} \frac{1}{2^{\mathrm{j}}}$, $A_i \subset N$ (set of natural numbers) and A_i contains the positions of a_i (i = 1, 2, 3) in u, where the positions representing the holes are neglected.

The following are a few examples for GPV of words over three letter alphabet.

Example 3.1: Let $\Sigma = \{a, b, c\}$.

(i) For $w = ababc$,

$$p(w) = \left(\frac{1}{2}+\frac{1}{2^3}, \frac{1}{2^2}+\frac{1}{2^4}, \frac{1}{2^5}\right) = \left(\frac{5}{8}, \frac{5}{16}, \frac{1}{32}\right).$$

(ii) For $v = (cab)^{\omega}$,

$$p(v) = \left(\frac{1}{2^2}+\frac{1}{2^5}+\frac{1}{2^8}+\cdots, \frac{1}{2^3}+\frac{1}{2^6}+\frac{1}{2^9}+\cdots, \frac{1}{2}+\frac{1}{2^6}+\frac{1}{2^9}+\ldots\right)$$

$$= \left(\frac{2}{7}, \frac{1}{7}, \frac{4}{7}\right)$$

(iii) For $u = \Diamond a \Diamond b \Diamond c$,

$$p(u) = \left(\frac{1}{2^2}, \frac{1}{2^4}, \frac{1}{2^6}\right)$$

Proposition 3.1: A point $P(x, y, z)$ in space represents the GPV of

(i) A finite word over $\Sigma = \{a, b, c\}$ if and only if $x+y+z = \frac{2^n-1}{2^n}$.

(ii) An infinite word over $\Sigma=\{a,b,c\}$ if and only if $x + y + z = 1$.

Definition 3.2: A language $L \subset \Sigma^{\infty}$ is defined as a point language if $|L| = 1$.

Note: Every singleton set is a point language.

Definition 3.3: For $\Sigma = \{a, b\}$, a language $L \subset \Sigma^{\infty}$ is called a line language if there exists a line l in $\mathbb{R}^2$ such that $L = \{x \in \Sigma^{\infty}: p(x)$ lies on l }. Then, l is said to be the language line of L.

Definition 3.4: For $\Sigma = \{a, b\}$, *a* language $L \subset \Sigma^{\infty}$ is called a quasi-line language if $p(L)$ lies on a line in $\mathbb{R}^2$. We call the line l as a quasi- language line.

Example 3.2: Let $L_1 = \{ab^2c\}$. For $x = ab^2c$,

$$p(x) = \left(\frac{1}{2}, \frac{1}{2^2}+\frac{1}{2^3}, \frac{1}{2^4}\right)$$

L_1 is a point language and $P\left(\frac{1}{2}, \frac{3}{8}, \frac{1}{16}\right)$ is the language point corresponding to L_1.

Example 3.3: Let $L_2 = \{ab^{\omega}\}$. For $y = ab^{\omega}$

$$p(y) = \left(\frac{1}{2}, \frac{1}{2^2}+\frac{1}{2^3}+\ldots+\infty\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

L_2 is a point language and $P\left(\frac{1}{2}, \frac{1}{2}\right)$ is the language point.

Theorem 3.1: The intersection of two language lines is a language point if they intersect.

Proof: Let ℓ_1 and ℓ_2 be language lines. The intersection of two lines is a point. Thus the intersection of ℓ_1 and ℓ_2 gives a point which will correspond to the GPV of the word common to line languages corresponding to ℓ_1 and ℓ_2.

Example 3.3: Let, ℓ_1: $x + y = \frac{3}{4}$ and ℓ_2: $2y = x$ be two language lines. Their intersection is the point $\left(\frac{1}{2}, \frac{1}{4}\right)$.

The line languages of ℓ_1 and ℓ_2 are $L_1 = \{a^2, b^2, ab, ba\}$ and $L_2 = \{ab, (ab)^2, \ldots, (ab)^n, \ldots\}$ i.e., $\{(ab)^n \,/\, n \geq 1\}$ respectively. The point of intersection of ℓ_1 and ℓ_2 is $\left(\frac{1}{2}, \frac{1}{4}\right)$ which, is the GPV of the word *ab* and *ab* is common to both L_1 and L_2.

4. PLANE LANGUAGES AND QUASI-PLANE LANGUAGES

The following are some observations about GPVs of words over three letter alphabet.

1. GPVs of all infinite words over $\Sigma = \{a, b, c\}$ lie on the plane $x + y + z = 1$ and conversely for each point on the plane $x + y + z = 1$, there exists a word in Σ^{ω} whose GPV is that point.
2. All words of length n over $\Sigma = \{a, b, c\}$ lie on the plane $x+y+z = \frac{2^n-1}{2^n}$.
3. The number of words of length n over $\Sigma = \{a, b, c\}$ lying on the plane $x+y+z = \frac{2^n-1}{2^n}$ is 3^n.
4. GPVs of all words over $\Sigma = \{a, b, c\}$ lie in the region bounded by the two planes $x+y+z = \frac{1}{2}$ and $x + y + z = 1$ and the coordinate planes.
5. GPVs of all words lie on and outside the boundary of the cube OABCDEF $x = 0$, $y = 0$, $z = 0$, $x = 1/2$, $y = 1/2$, $z = ½$ [see Fig.1].

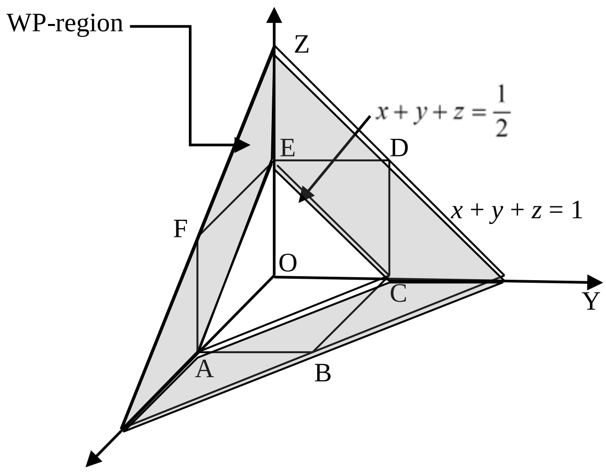

Fig.1. GPVs of words over three letter alphabet

Throughout this paper, we consider Σ to be a three letter alphabet. That is, $\Sigma = \{a, b, c\}$. The Generalized Parikh Vectors of words over Σ lie in space. This leads to the following definitions.

Definition 4.1: A language $L \subset \Sigma^{\infty}$ is defined as a Plane language if there exists a plane S in R^3 such that, $L = \{x \in \Sigma^{\infty}: p(x)$ lies on $S\}$. Then S is said to be the Language Plane of L.

Definition 4.2: A language $L \subset \Sigma^{\infty}$ is defined as a quasi-Plane language if $p(L)$ lie on a plane in R^3.

Definition 4.3: A Plane language L is said to be a finite Plane language if it contains only finite words.

Definition 4.4: Given a plane S, if there are words over $\Sigma = \{a, b, c\}$ such that their GPVs lie on S. we call S as a quasi-language plane.

Definition 4.5: A Plane language L is said to be a ω-plane language if it contains only ω-words (infinite words). The cardinality of the set L may be finite or infinite.

Definition 4.6: An infinitary Plane language contains both finite and infinite words.

Example 4.1: Let $L = \{a^2, b^2, c^2, ab, ba, ca, ac, bc, cb\}$. L is a Plane language and its Language plane is $x+y+z=\frac{2^2-1}{2^2}=\frac{3}{4}$. This is a finite Language plane of L.

Example 4.2: Σ^ω is a ω - plane language and the corresponding Language plane is $x + y + z = 1$.

Remark 4.1: Any plane of the form $x + y + z = c$ can be classified into two categories with respect to Plane languages.

1. If $c = 1$, then the plane $x + y + z = 1$ corresponds to the ω - plane language.
2. If $c=\frac{2^n-1}{2^n}$, then $x+y+z=\frac{2^n-1}{2^n}$ corresponds to the finite Plane language consisting of words of length n.

Theorem 4.1: A Language plane $x+y+z=\frac{2^n-1}{2^n}$ contains strictly square words only if $n = 2m$, $m = 1, 2, 3,\ldots,$.

Proof: For every word $w \in \Sigma^+$ of length m, the strictly square words $w^2 \in \Sigma^+$ are of length 2m and their GPVs lie on the Language plane $x+y+z=\frac{2^{2m}-1}{2^{2m}}$.

Example 4.3: Strictly square words of length 2 are $\{a^2, b^2\}$ and they lie on $x+y+z=\frac{3}{4}$.

Strictly square words of length 4 are $\{a^4, b^4, abab, baba, bcbc, cbcb, acac, caca\}$ and all these lie on the plane $x+y+z=\frac{2^4-1}{2^4}$.

Theorem 4.2: A Language33 plane $x+y+z=\frac{2^n-1}{2^n}$ contains strictly cube words only if $n = 3m$ for all $m \geq 1$.

Proof. For every word $w \in \Sigma^+$ of length m, the strictly cube word $w^3 \in \Sigma^+$ is of length 3m. Clearly, these words lie on the plane $x+y+z=\frac{2^{3m}-1}{2^{3m}}$. Thus any strictly cube word lies on a Language plane $x+y+z=\frac{2^n-1}{2^n}$ with $n = 3m$, $m = 1, 2, 3,\ldots$.

Remark 4.2: The Generalized Parikh Vectors of all words over Σ lie on the boundary and inside the region of the two planes $x+y+z=\frac{1}{2}$ and $x + y + z = 1$ and the coordinate planes. This region is said to be the WP-region.

Theorem 4.3: A plane $ax + by + cz = d$ is an infinitary language plane if (i) it intersects $x + y + z = 1$ in the WP-region (ii) it contains the GPVs of all words of a Plane language.

Theorem 4.4: The intersection of three Language planes π_1, π_2 and π_3 is a language point if they intersect.

Proof: If three planes intersect, they have a common point of intersection. This point P lies on π_1, π_2 and π_3. Since all three planes are Language planes that contain the GPVs of words over $\Sigma = \{a, b, \underline{c}\}$, their point of intersection will also correspond to the GPV of a word. Hence it is a language point.

Theorem 4.5: The intersection of two Language planes is a language line if they intersect.

Proof: Let π_1 and π_2 be two Language planes such that π_1 and π_2 are non-parallel. Then the intersection of π_1 and π_2 is the line ℓ. This line lies in the region $\frac{1}{2}\leq x+y+z\leq 1$ and contains the GPVs of all words common to the planes π_1 and π_2, thus satisfying the conditions of a language line.

Example 4.4: Let π_1 be $x+y+z=\frac{7}{8}$ and π_2 be $x - 2y - 4z = 0$. The intersection of π_1 and $\underline{\pi_2}$ is the line $\left(\frac{14}{24}-2t,\frac{7}{24}+5t,-3t\right)$. Since the planes are in three dimensions, the equation of the line of intersection is given in parametric form.

Definition 4.7: A language $L \subset \Sigma^\infty$ is said to be a surface language if there exists a surface S in $\mathbb{R}^3$ such that $L = \{x \in \Sigma^\infty : p(x)$ lies on the non-planar surface $S\}$ and S is said to be a Language surface.

Definition 4.8: A language $L \subset \Sigma^\infty$ is said to be a quasi-surface language if $p(L)$ lies on a surface in $\mathbb{R}^3$.

Definition 4.9: Given a surface Q, if there are words over $\Sigma = \{a, b, c\}$ such that their GPVs lie on Q, we call Q as a quasi- language surface.

Definition 4.10: A language $L \subset \Sigma^\infty$ is defined as a curve language if there exists a curve C containing non-collinear and non-planar points such that $L = \{x \in \Sigma^\infty : p(x)$ lies on $C\}$. The curve C is said to be the Language curve.

Example 4.5:

1. Consider the twisted cubic curve, $x = t$, $y = t^2$, $z = t^3$. The points $\left(\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3}\right)$, $\left(\frac{1}{2^2},\frac{1}{2^4},\frac{1}{2^6}\right)$, $\left(\frac{1}{2^3},\frac{1}{2^6},\frac{1}{2^9}\right)$, $\left(\frac{1}{2^4},\frac{1}{2^8},\frac{1}{2^{12}}\right)$ corresponds to the GPV of the words abc, $\lozenge a\lozenge b\lozenge c$, $\lozenge\lozenge a\lozenge\lozenge b\lozenge\lozenge c$ and $\lozenge\lozenge\lozenge a\lozenge\lozenge\lozenge b\lozenge\lozenge\lozenge c$.

   The above four points lie on the curve and are non-planar. Hence there is a surface S containing the curve, $x = t$, $y = t_2$, $z = t_3$, and is called the Language surface. The surface language corresponding to S is $\{\lozenge^n a\lozenge^n b\lozenge^n c \,/\, n \geq 0\}$.
2. The paraboloid $y = x^2$ is another example for Language surface. $S = \{(x, y, z)/y = x^2\}$ is a Language surface and its corresponding surface language is $\{\lozenge^{n-2}ca\lozenge^{n-1}b \,/\, n \geq 2\} \cup \{c^{n-1}ac^{n-1}bc^*: n \geq 1\}$.

   Let, $t_1=\frac{1}{2}$, $t_2=\frac{1}{4}$, $t_3=\frac{1}{8},\ldots,$ $t_n=\frac{1}{2^{n+1}}$. The points $\left(\frac{1}{4},\frac{1}{16},\frac{1}{2}\right)$, $\left(\frac{1}{8},\frac{1}{64},\frac{1}{4}\right)$, $\left(\frac{1}{16},\frac{1}{256},\frac{1}{8}\right)$ & $\left(\frac{1}{32},\frac{1}{1024},\frac{1}{16}\right)$ all lie on the paraboloid. i.e., The points

$\{(t_n, (t_n)^2, t_{n-1})\}$ lie on the paraboloid $y = x^2$ and is a Language surface, whose Surface language is $L = \{\Diamond^{n-2}ca\Diamond^{n-1}b \,/\, n \geq 2\} \cup \{c^{n-1}ac^{n-1}bc^*: n \geq 1\}$.

Remark 4.3:

1. Structures with linear expression are language points and language lines whereas; language planes, language surfaces and language curves are structures with non-linear expressions.
2. Surface languages are languages of partial words over $\Sigma = \{a, b, c\}$.

5. CLOSURE PROPERTIES

In this section we study some of the closure properties of Quasi-plane languages such as union, intersection, catenation, reversal and Kleene closure.

Theorem 5.1: Let L_1 and L_2 be two Quasi- Plane languages. The union of L_1 and L_2 is also a Quasi-Plane language if and only if $p(L_1)$ and $p(L_2)$ lie on the same plane.

Proof: L_1 and L_2 be two Quasi-Plane languages having the same Quasi- Language plane S. $p(L_1)$ and $p(L_2)$ lie on that same Quasi-Language plane S. Clearly $L_1 \cup L_2$ is a Quasi-Plane language.

Conversely, if $L_1 \cup L_2$ is a Quasi-Plane language then $p(L_1 \cup L_2)$ lies on S. clearly, $p(L_1)$ and $p(L_2)$ lie on that same Quasi-Language plane S.

Theorem 5.2: The intersection of two Quasi- Plane languages L_1 and L_2 is also a Quasi-Plane language if and only if $p(L_1)$ and $p(L_2)$ lie on the same plane.

Proof: Let L_1 and L_2 be two Quasi-Plane languages having their corresponding Quasi-language plane S. Then $p(L_1)$, $p(L_2)$ and $p(L_1 \cap L_2)$ all lie on S. Thus $L_1 \cap L_2$ is a Quasi-Plane language whose Quasi-Language plane is S.

Theorem 5.3: L_1 and L_2 be two Quasi-Plane languages. Then $L_1 \circ L_2$ is also a Quasi-Plane language if and only if their corresponding Quasi-Language planes are either,

(i) $x + y + z = \dfrac{2^k - 1}{2^k}$ and $x + y + z = \dfrac{2^m - 1}{2^m}$, $k, m \in N$

or

(ii) The same for both L_1 and L_2 and are of the form $x + y + z = \dfrac{2^n - 1}{2^n}$

Proof: Let L_1 and L_2 be two Quasi-Plane languages.

Case (i):

If L_1 has Quasi-Language plane $x + y + z = \dfrac{2^k - 1}{2^k}$ then all elements of L_1 are words of length k and $p(L_1)$ lie on the above plane.

If L_2 has Quasi-Language plane $x + y + z = \dfrac{2^m - 1}{2^m}$, then all elements of L_2 are words of length m and $p(L_2)$ lie on $x + y + z = \dfrac{2^m - 1}{2^m}$.

Then $p(L_1 \circ L_2) = \{p(x) + \dfrac{1}{2^{|x|}} p(L_2): x \in L_1\}$ and all are of length $k + m$.

Clearly $L_1 \circ L_2$ is again a Quasi-Plane language whose elements are words of length $k + m$, and their corresponding Quasi-Language plane is $x + y + z = \dfrac{2^{(k+m)} - 1}{2^{(k+m)}}$.

Case (ii):

If L_1 and L_2 are two Quasi-Plane languages having the Quasi-Language plane $x + y + z = \dfrac{2^n - 1}{2^n}$, then both L_1 and L_2 have elements which are words of length n. Clearly $L_1 \circ L_2$ is a Quasi-Plane language with the corresponding Quasi-Language plane $x + y + z = \dfrac{2^{2n} - 1}{2^{2n}}$.

Theorem 5.4: Let L_1 and L_2 be two Plane languages. Then, $L_1 \circ L_2$ is also a Plane language only if their corresponding Language planes are either of the form,

(i) $x + y + z = \dfrac{2^k - 1}{2^k}$ and $x + y + z = \dfrac{2^m - 1}{2^m}$, $k, m \in N$.

or

(ii) $x + y + z = \dfrac{2^n - 1}{2^n}$ and $x + y + z = 1$ for L_1 and L_2 respectively

Proof:

Case (i):

All words lying on $x + y + z = \dfrac{2^k - 1}{2^k}$ are of length k and those on $x + y + z = \dfrac{2^m - 1}{2^m}$ are of length m. Then, $p(L_1 \circ L_2) = \{p(x) + \dfrac{1}{2^{|x|}} p(L_2): x \in L_1\}$ and all are of length $k + m$.

Clearly, $L_1 \circ L_2$ is also a Plane language.

Case (ii):

$x + y + z = \dfrac{2^n - 1}{2^n}$ and $x + y + z = 1$ are Language planes for L_1 and L_2 respectively.

Then all words of L_1 are of length n and those of L_2 are of Σ^ω.

$L_1 \circ L_2 = \Sigma^\omega$ and is a Plane language.

Theorem 5.5: If L is a Plane language then L^R is also a Plane language only if its Language plane is either of the form $x + y + z = \dfrac{2^n - 1}{2^n}$ or the coordinate axes.

Proof: Let L be a Plane language with Language plane $x + y + z = \dfrac{2^n - 1}{2^n}$. Then all the elements of L are words of length n. The GPVs of their reversals lie on the same plane $x + y + z = \dfrac{2^n - 1}{2^n}$. All words lying on the coordinate axes $x = 0$, $y = 0$, $z = 0$ are respectively a^n, b^n and c^n: $n \geq 1$.

Theorem 5.6: If L is a Quasi-Plane language then L^R is also a Quasi-Plane language only if its Quasi-Language plane is of the form $x+y+z=\frac{2^n-1}{2^n}$ or the coordinate axes.

Proof: Let L be a Quasi- Plane language. $p(L)$ lies on $x+y+z=\frac{2^n-1}{2^n}$. Then $p(L^R)$ are all of length n and hence lie on $x+y+z=\frac{2^n-1}{2^n}$. If for the Quasi-Plane language, $p(L)$ lies on the coordinate axes, then $p(L^R)$ also lies on the coordinate axes.

Example 5.1: Let L = {*abc*, *aab*, *bab*, *cca*}. Then its Language plane is $x+y+z=\frac{7}{8}$. L^R = {*aba*, *baa*, *bab*, *acc*} also lies on the same Language plane.

Theorem 5.7: A Quasi- Plane language L is not closed under Kleene closure.

Proof: Let L be a Quasi- Plane language. The $L^* = \cup_{i\geq 0} L^i$, where $L^i = LL^{i-1}$ for $i \geq 1$ n and so L^* is a language that contains words over Σ of various lengths. There is no plane that contains $p(L^*)$.

Theorem 5.8: A Plane language L is not closed under the operation 'complement'.

Proof: Let L be a Plane language. Then its complement, namely L^c is a language having all words over Σ = {*a*, *b*, *c*} other than those in L. Hence it is not possible to find a Language plane consisting of all the words of L^c.

Theorem 5.9: A Quasi-Plane language L is not closed under the operation 'complement'.

Proof: Argument same as that of proof of Theorem 5.8.

6. CONCLUSION

In this paper we have introduced the concept of Plane languages, Language plane, Quasi-Plane languages, Quasi-language planes and have studied some geometrical properties of the same. Closure properties such as union, intersection, catenation, reversal, complement and Kleene star of Plane languages and Quasi- Plane languages have been discussed.

Closure properties such as union, intersection, catenation, reversal, complement and Kleene star of Plane languages and Quasi- Plane languages have been discussed.

All words over the three letter alphabet lie on various planes, all within the region between the two planes $x + y + z = ½$ and $x + y + z = 1$ and the coordinate planes. The study of plane languages is interesting with the discussion of their combinatorial properties. We plan to extend the study of combinatorial properties to Surface languages.

REFERENCES

[1] Rohit J. Parikh, "On Context-Free Languages", *Journal of the Association for Computing Machinery*, Vol. 13, No. 4, pp. 570-581, 1966.

[2] K. Sasikala, T. Kalyani, V.R. Dare and P.J. Abisha, "Line languages", *Electronic Notes in Discrete Mathematics*, Vol. 12, pp. 178-185, 2003.

[3] K. Sasikala, "A study on Partial Words, Partial Languages and Line Languages", Ph.D Dissertation, University of Madras, 2006.

[4] H. Sathyaseelan, V.R. Dare "On Generalized Parikh Vectors", *Proceedings of National Conference on Discrete Mathematics and its Applications*, pp. 160-165, 2007.

[5] Rani Siromoney and V. Rajkumar Dare, "A Generalization of Parikh Vectors for Finite and Infinite Words", *Lecture Notes in Computer Science*, Vol. 206, pp. 290-302, 1985.

[6] Rani Siromoney and V. Rajkumar Dare, "On Infinite Words Obtained by Selective Substitution Grammars", *Theoretical Computer Science*, Vol. 39, pp. 281-295, 1985.

[7] Grzegorz Rozenberg and Arto Salomaa, "*Handbook of Formal languages*", Vol. 3, Springer, 1997.