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Diffusion under time-dependent resetting

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We study a Brownian particle diffusing under a time-modulated stochastic resetting mechanism to a fixed position. The rate of resetting $r(t)$ is a function of the time t since the last reset event. We derive a sufficient condition on $r(t)$ for a steady-state probability distribution of the position of the particle to exist. We derive the form of the steady-state distributions under some particular choices of $r(t)$ and also consider the late time relaxation behavior of the probability distribution. We consider first passage time properties for the Brownian particle to reach the origin and derive a formula for the mean first passage time. Finally, we study optimal properties of the mean first passage time and show that a threshold function is at least locally optimal for the problem of minimizing the mean first passage time.

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I. INTRODUCTION

Over the last few decades there has been great interest in the study of search problems that appear in various contexts from animal foraging, protein binding on DNA, internet search algorithms to locate one's misplaced keys. In such different situations, one important issue is to consider optimal search strategies. For example, in protein binding on DNA, a protein molecule binds to a specific target binding site on DNA by an appropriate mixture of 3D diffusion and 1D sliding motion on DNA – called facilitated diffusion [1].

Quite often efficient search strategies involve a mixture of local steps and long-range moves. Such search strategies are called intermittent search process and are observed in foraging animals such as humming birds or bumblebees [2, 3]. The *E. Coli* bacteria alternatively uses ballistic moves, called “runs” with random “tumbles”, to change its direction in order to reach high food concentration regions [4].

Recently, an intermittent stochastic strategy containing such a mixture of local and long-range moves has been introduced [5–7] in which diffusion of a particle is interrupted by stochastically resetting it to a preferred position whereupon the diffusion process starts afresh. Examples of such stochastic resetting are found in a wide variety of situations. In daily life, while searching for some lost possession, after an unsuccessful search for some duration of time one often goes back to the starting place and recommences the search process again. In the ecological context, the movement of animals during foraging period often involves local diffusive search for food [3, 8]. Interestingly, such local diffusive movements are interrupted by long range moves to relocate themselves in other areas and after that local diffusive motion restarts from the relocation region [9]. Movement of free-ranging capuchin monkeys in the wild is described quite well by random walks with preferential relocations to places visited in the past [10]. There are other examples in nature where similar notions of stochastic resetting can also be found. For example, in the biological context several living organisms use stochastic switching between different phenotypic states to adapt in a fluctuating environment [11–15]. Stochastic restarts are often considered as a useful strategy to optimize computer search algorithms in hard combinatorial problems [16–18]. Also in the context of population growth, random catastrophic events may cause a sudden reduction in the population size and reset it to some lower value. [15].

The mechanism of stochastic resetting fundamentally affects the properties of diffusion process. Consider a particle, starting from x_0 , diffusing in one dimension. Let $x(t)$ be its position at time t . Along with diffusion, the particle is subject to a resetting mechanism in which its motion is interrupted stochastically such that the position of the particle is reset to some fixed position x_r at some rate r and after every such event the particle recommences its diffusive motion. In the absence of resetting, the Gaussian distribution of the position of the particle never reaches a steady state. The width of the distribution keeps on growing with time as $\sim \sqrt{t}$. On the other hand, in the presence of resetting the distribution of the position becomes a globally current-carrying non-equilibrium steady state with non-Gaussian fluctuations. Resetting also has another important consequence on the properties of diffusion process : the mean first passage time (MFPT) to a particular position of a diffusing particle is infinite whereas it becomes finite in presence of resetting. In fact there is an optimal resetting rate r for which this finite MFPT is minimum.

In the last decades, several theoretical works have been dedicated to the study of first-passage properties of intermittent search processes, and in particular to the question of the minimization of the mean first-passage time i.e. the mean time to locate some target [19–24]. These works use different analytical and numerical approaches to minimize the MFPT in a broad range of contexts such as search processes mixing slow diffusive movement and fast ballistic motion [20], persistent random walks [21], finding targets in a domain [22] and Brownian search in spatial heterogeneous media [24]. Also in [25], an intermittent search

process with Levy flights interrupted by random resettings has been studied where a first order phase transition associated to a discontinuous change in the optimal parameters has been observed.

Recently a number of generalizations of the simple diffusion with resetting have been made. The generalization from the one-dimensional case to higher dimensions has been considered in [26]. In the context of a target search process, the effect of partial absorption has been considered in [27]. Properties of non-equilibrium steady state for diffusion with resetting have been studied in the presence of a potential [28] or in a bounded domain [29]. Other generalizations include resetting to the current maximum of the Brownian particle [30], resetting in continuous-time random walks [31], in Lévy flights [25] etc. The effect of resetting in the dynamics of interacting multi-particle systems such as fluctuating interfaces [32], coagulation-diffusion process [33] and in general chemical reaction schemes [34] have been studied.

In this paper we consider a different generalization of the resetting process: we consider a resetting rate $r(t)$ that is *time-dependent*. This generalization is quite natural in the context of target search. While searching for a lost object, the searcher naturally would not like to reset in the beginning of the search, but as time progresses without success, the searcher would be more and more inclined to go back to the most likely location and start the search process anew. As a result it would be quite natural to consider a situation where the resetting rate $r(t)$ grows from zero as time increases. Specifically, we consider a situation where the resetting rate depends on the *time since the last reset*. Thus the rate as well as the searcher is reset at the resetting event.

In the case of time-independent resetting rates, the *renewal property* of Brownian motion has been exploited to compute the propagator (the probability density of the particle being at x at time t given that it began at x_0 at time 0) as well as the steady-state distribution of the position of the particle [6, 7, 26, 37]. More precisely, the propagator has been expressed as an integral over the time of the *last resetting events*. For time dependent resetting rates, we find that it is more convenient to express the propagator of the particle as an integral over the time of *first resetting events*. We refer to this formalism as the *first renewal picture* in contrast to the formalism used in [6, 7, 26, 37] which we refer to as the *last renewal picture*. We first recapitulate the constant resetting rate case in Sec. II A where we re-derive the known results using the *first renewal picture*. Then in Sec. II B we discuss the generalization of this approach to the time dependent case, where we compute the Laplace transform (w.r.t. time) of the propagator for a general resetting rate function $r(t)$.

Next we study the steady-state properties of the Brownian particle subject to such a resetting mechanism. In the absence of resetting, the Brownian particle keeps on diffusing freely over space with time; but the introduction of a resetting mechanism creates a current towards the reset position. This current may balance the outward current (flowing towards infinity) in the large time limit. As a result the system may reach a steady state eventually but *not* for all possible choices of the rate function $r(t)$. The question is then for what choices of $r(t)$ will a steady state be attained? We address this question in Sec. III, where we study the steady-state behavior of the Brownian particle in presence of a time-dependent resetting rate. We find a sufficient condition on the late time behavior of $r(t)$ for a steady state to be attained which is given by equation (21). Such a condition can be easily understood as follows : Let us consider the following two extreme cases, a) $r(t) \rightarrow \infty$ and b) $r(t) = 0$. In the former case, the particle is always instantaneously reset to the resetting position x_r and hence the probability distribution of the position is then given by $\delta(x - x_r)$ irrespective of the initial position x_0 . On the other hand, when $r(t) = 0, \forall t$, the particle diffuses freely and never attains a steady state distribution (on an infinite system). Hence if $r(t)$ decays fast enough to zero as time increases, then the particle may have finite probability of not being reset at all and as a result the distribution of the position of the particle may not reach a steady state. In fact we find that if $r(t)$ decays slower than $\sim 1/t$ for large t , then the particle reaches a steady state. The next question is: how does it relax to the steady state, if indeed one exists?

This question was first studied by Majumdar *et al* [38] in the context of constant resetting rate case. They found that for a given large time t , the distribution in an inner core region around the reset position has already relaxed to the (time-independent) steady state whereas the distribution in the outer region has not yet relaxed. The front dividing the relaxed from the non-relaxed region moves linearly with time through the system. In Sec. IV we observe the same phenomena for a specific class of time dependent resetting rate function $r(t) \sim t^\theta, \theta > -1$. However, depending on the value of θ , the motion of the front becomes linear, superlinear or sublinear with t .

One of the main motivations for studying Brownian motion with resetting is to improve our understanding of search paradigms, as discussed in the first and second paragraphs of the introduction. As a quantitative measure of the performance of a search process, one may consider the mean search time or the mean first passage time (MFPT) $T(x_0)$ to a static target for a given starting position x_0 and diffusion constant D . In the absence of resetting, the MFPT of a free Brownian particle to a static target at origin is infinite. On the other hand, in the presence of resetting to the initial position with constant rate $r(t) = r_c$, the MFPT becomes finite [6, 7]. Moreover, it has also been shown [6, 7] that there exists an optimal choice of the constant rate r_c^* for which MFPT $T(x_0)$ becomes minimum. In this work we ask the following questions : can this scenario be improved further if one considers a time-dependent resetting rate $r(t)$? What is the optimal rate function $r(t)$ for which the MFPT $T(x_0)$ becomes minimum ? We address these questions in Sec. V and Sec. VI respectively. We show that indeed there exists a few choices of $r(t)$ for which it is possible to achieve a MFPT lower than the minimum MFPT obtained using constant resetting rate r_c^* . In Sec. VI we study the optimal resetting rate function, where we conjecture that the optimal time-dependent rate function is given by a threshold function.

We note that several very recent works have also considered the problem of time-dependent resetting. Eule and Metzger [35] use a generalized (non-Markovian) master equation approach to consider the stationary distribution of the particle position and also some late time properties. In that work a Gamma-distribution of waiting times between successive reset events is considered

and mean first passage times are numerically determined. Nagar and Gupta [36] consider the particular case of a power law distribution of waiting times between resets. They discuss the steady states and some first passage time properties. In contrast, in our work we consider the general scenario of a time-dependent resetting rate $r(t)$ (with corresponding waiting time distribution $r(t)e^{-R(t)}$ given by (13)) for various different choices of $r(t)$. Furthermore we consider the late time relaxation of the probability distribution and we study analytically the problem of optimising the mean first passage time.

II. THE MODEL

First let us define diffusion with resetting with time-dependent resetting rate. We consider a single particle (or searcher) in one dimension with initial position x_0 at $t = 0$ and a resetting position x_r . The position $x(t)$ of the particle at time t is updated by the following stochastic rule: in a small time interval $t \rightarrow t + dt$ the position $x(t)$ becomes

$$x(t + dt) = x_0 \quad \text{with probability } r(t - \tau_l) dt \quad (1)$$

$$= x(t) + \eta(t)dt \quad \text{with probability } (1 - r(t - \tau_l) dt) \quad (2)$$

where τ_l is the time of the last resetting event. Thus the resetting rate $r(t - \tau_l)$ is a function of the *time elapsed since the last resetting event*. In (2) $\eta(t)$ is a Gaussian white noise with mean and two-time correlator given by

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t) \eta(t') \rangle = 2D\delta(t - t'), \quad (3)$$

where D is the diffusion constant. Here, angular brackets denote averaging over noise realizations. The initial condition is $x(0) = x_0$. To simplify matters, from now on we shall take the initial position to coincide with the resetting position

$$x(0) = x_0 = x_r, \quad (4)$$

unless otherwise specified. The dynamics thus consists of a stochastic mixture of resetting to the initial position with rate $r(t - \tau_l)$ (long range move) and ordinary diffusion (local move) with diffusion constant D .

We define $P_r(x, t|x_0, 0)$ as the probability of finding the particle at position x at time t , given that it was at x_0 at time $t = 0$ in the presence of *time-independent* stochastic resetting, $\mathbb{P}_r(x, t|x_0, 0)$ being the same quantity for the *time-dependent* stochastic resetting. In both cases the suffix ‘ r ’ indicates the presence of resetting.

A. Recap of constant resetting rate $r(t) = r_c$

For completeness we review here the formalism and the results for the case of constant resetting rate $r(t) = r_c$ [6, 26]. In this case, one can write down a Master equation for $P_r(x, t|x_0, 0)$ from the dynamical rules for the evolution of the particle given in the preceding section [6]

$$\frac{\partial P_r}{\partial t} = D \frac{\partial^2 P_r}{\partial x^2} - r_c P_r + r_c \delta(x - x_0), \quad (5)$$

with the initial condition $P_r(x, t = 0|x_0, 0) = \delta(x - x_0)$. For convenience, we have omitted the arguments of $P_r(x, t|x_0, 0)$ in the above equation. Here, the second and third terms on the right hand side (RHS) account for the resetting events, denoting, respectively, the negative probability flux $-r_c P_r$ from each point x and a corresponding positive probability flux into $x = x_r = x_0$. The steady-state solution for the time-independent case $P_r^{\text{st}}(x)$ satisfies

$$0 = D \frac{d^2 P_r^{\text{st}}}{dx^2} - r_c P_r^{\text{st}} + r_c \delta(x - x_0). \quad (6)$$

Alternatively, a renewal picture which we refer to as the *last renewal picture* may be used to write down an equation for $P_r(x, t|x_0, 0)$ in terms of the free propagator $G(x, t|x_0, 0)$ for a pure diffusive process (without resetting) as in [6, 26]

$$P_r(x, t|x_0, 0) = e^{-r_c t} G(x, t|x_0, 0) + r_c \int_0^t d\tau_l e^{-r_c(t-\tau_l)} G(x, t - \tau_l|x_0, 0). \quad (7)$$

Here, we have divided the process into two contributions. The first term in the RHS signifies that there has been no reset at all between time $(0, t)$ with the probability of no resets given by $e^{-r_c t}$. This probability is then simply multiplied by the free Brownian propagator

$$G(x, t|x_0, 0) = \frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{(x - x_0)^2}{4Dt}\right]. \quad (8)$$

The second term in the RHS accounts for the fact that there can be multiple resets: the integral sums over contributions in which the *last resetting* event takes place between time τ_l and $\tau_l + d\tau_l$. Subsequently, the particle propagates freely until the observation time scale t . It can be shown that (7) satisfies the master equation (5). The steady state can be obtained by taking the infinite time limit

$$P_r^{\text{st}}(x) = r_c \int_0^\infty d\tau_l e^{-r_c \tau_l} G(x, \tau_l | x_0, 0), \quad (9)$$

which satisfies the steady state equation given by (6). The integral (9) can be evaluated to yield [26]

$$P_r^{\text{st}}(x) = \frac{\alpha_0}{2} e^{-\alpha_0 |x-x_0|}, \quad \text{where, } \alpha_0 = \sqrt{\frac{r_c}{D}}. \quad (10)$$

The last resetting equation (7) also allows the long time relaxation to the steady state to be analyzed [38] as we shall review in Section IV.

In this work we introduce a *first renewal picture* where instead of the last resetting, we consider the *first resetting* between time τ_f and $\tau_f + d\tau_f$ having started from $t = 0$. Subsequently, the particle diffuses from τ_f until time t , in the presence of multiple resets. It is again straightforward to write down an equation for the probability

$$P_r(x, t | x_0, 0) = e^{-r_c t} G(x, t | x_0, 0) + r_c \int_0^t d\tau_f e^{-r_c(t-\tau_f)} P_r(x, t | x_0, \tau_f), \quad (11)$$

where the first term in the RHS corresponds to trajectories in which there are no resets at all. The integral in the second term sums over trajectories in which there has been a first reset between time τ_f and $\tau_f + d\tau_f$ and then there can be multiple resets which is taken care of by the reset propagator $P_r(x, t | x_0, 0)$ inside the integral. The equivalence between (7) and (11) is easy to show by taking Laplace transforms of both equations. We find that they result in identical expression in the Laplace space

$$\tilde{P}(x, s | x_0, 0) = \frac{r_c + s}{s} \tilde{G}(x, r_c + s | x_0, 0), \quad (12)$$

where $\tilde{P}(x, s | x_0, 0)$ is the Laplace transform of $P_r(x, t | x_0, 0)$ and $\tilde{G}(x, s | x_0, 0)$ is the Laplace transform of $G(x, t | x_0, 0)$. Taking the inverse Laplace transform with respect to s , one can easily obtain (7).

B. Time-dependent resetting rate

We now turn to the main subject of this paper, that of time-dependent resetting rates $r(t)$, as defined above. In this case one cannot simply write a Master equation for $\mathbb{P}_r(x, t | x_0, 0)$ in the presence of time-dependent resetting rate. This is because one must in addition keep track of the time since the last reset. The renewal pictures are more useful than the Master equation formalism to describe the time-dependent rate process. To this end, we define the following time-integrated quantity

$$R(\tau) = \int_0^\tau d\tau' r(\tau'). \quad (13)$$

Then the probability of no resets subsequent to an initial reset at $t = 0$ is given by $e^{-R(t)}$ and $r(t)e^{-R(t)}$ is the probability density for a first reset to occur in the interval $t \rightarrow t + dt$.

In the case of time-dependent resetting the *last renewal* equation (7) is modified to

$$\mathbb{P}_r(x, t | x_0, 0) = e^{-R(t)} G(x, t | x_0, 0) + \int_0^t d\tau_l \psi(\tau_l) e^{-R(t-\tau_l)} G(x, t - \tau_l | x_0, 0) \quad (14)$$

where $\psi(\tau_l)$ is the probability density for a reset (which turns out to be the last) to occur in $t \rightarrow t + dt$. We shall return to this picture in section IV where we consider the late time relaxation behavior.

However, to study the steady-state behavior it is most convenient to use the *first renewal* framework. In the presence of the time-dependent reset process the *first renewal* equation becomes

$$\mathbb{P}_r(x, t | x_0, 0) = e^{-R(t)} G(x, t | x_0, 0) + \int_0^t d\tau r(\tau) e^{-R(\tau)} \mathbb{P}_r(x, t - \tau | x_0, 0). \quad (15)$$

By taking the Laplace transform, we obtain

$$\tilde{\mathbb{P}}_r(x, s|x_0, 0) = \frac{\tilde{\mathbb{Q}}(x, s|x_0, 0)}{s\tilde{\mathbb{H}}_r(s)}, \quad (16)$$

where we have defined

$$\tilde{\mathbb{P}}_r(x, s|x_0, 0) = \int_0^\infty dt e^{-st} \mathbb{P}_r(x, t|x_0, 0), \quad (17)$$

$$\tilde{\mathbb{Q}}(x, s|x_0, 0) = \int_0^\infty dt e^{-st} e^{-R(t)} G(x, t|x_0, 0), \quad (18)$$

$$\tilde{\mathbb{H}}_r(s) = \int_0^\infty dt e^{-st} e^{-R(t)}. \quad (19)$$

Equation (16) is the main result of this section. In principle, the Laplace transform can be inverted although in practice this is difficult for arbitrary $r(t)$.

III. STEADY STATE BEHAVIOUR

In the previous section we have studied the propagator of a Brownian particle in presence of a time-dependent resetting mechanism using the *first renewal picture*. We have computed the Laplace transform (16) of the propagator $\mathbb{P}_r(x, t|x_0, 0)$ in terms of the Laplace transforms of the free propagator $G(x, t|x_0, 0)$ weighted by the probability $e^{-R(t)}$ of no reset in duration t . In this section we are interested in the large time behavior of $\mathbb{P}_r(x, t|x_0, 0)$. More precisely, we are interested in whether $\mathbb{P}_r(x, t|x_0, 0)$ takes a time independent form in $t \rightarrow \infty$ limit. If so then, in the Laplace transform language, this simply means that $\tilde{\mathbb{P}}_r(x, s|x_0, 0)$ should be expressible as $1/s$ multiplied by a quantity that does not depend on s in $s \rightarrow 0$ limit and the steady state distribution would be given by

$$\mathbb{P}_r^{\text{st}}(x) = \lim_{s \rightarrow 0} \frac{\tilde{\mathbb{Q}}(x, s|x_0, 0)}{\tilde{\mathbb{H}}_r(s)}, \quad (20)$$

provided the limit exists and is not zero. Now it is easy to check from the definitions (18) and (19) that if $\tilde{\mathbb{H}}_r(s \rightarrow 0) < \infty$ then also $\tilde{\mathbb{Q}}(x, s \rightarrow 0|x_0, 0) < \infty$. Thus a sufficient condition on the choice of $r(t)$ to achieve a steady state is that $\tilde{\mathbb{H}}_r(s \rightarrow 0) < \infty$ which implies that

$$\int_0^\infty e^{-R(t)} dt < \infty. \quad (21)$$

This condition implies that $R(t)$ must grow sufficiently fast for large t so that $e^{-R(t)} \rightarrow 0$ sufficiently quickly.

Recalling the definition $R(t) = \int_0^t d\tau r(\tau)$ (13), condition (21) will hold if $r(t)$ is an increasing function of time since then $R(t) \rightarrow \infty$. Thus for an increasing resetting rate, there always exists a steady state. Also if $r(t)$ tends to some constant value greater than zero for large t , then $R(t)$ grows linearly with time and again (21) is satisfied. On the other hand if $r(t)$ is a function decreasing to zero at large time since the last reset, to achieve a steady state, it must decrease sufficiently slowly that $R(t)$ diverges with time. Thus (21) implies that if the decreasing rate function $r(t)$ is bounded one fold i.e. if $r(t)$ decays *more slowly* than $\frac{1}{t}$, then there exists a unique steady state.

In the following, we explore few plausible choices of the time-dependent rates.

1. Case I: linear rate

As mentioned in the introduction, it is natural in the context of search processes to consider a rate which is an increasing function of time. The simplest such function is a linear function. For the linear resetting process $r(t) = b_0 t$, we find the following relations

$$\begin{aligned} R(t) &= \frac{b_0 t^2}{2}, \\ \tilde{\mathbb{H}}_r(s) &= \sqrt{\frac{\pi}{2b_0}} e^{s^2/2b_0} \operatorname{erfc}\left[\frac{s}{\sqrt{2b_0}}\right], \\ \tilde{\mathbb{Q}}(x, s|x_0, 0) &= \int_0^\infty dt e^{-st} e^{-b_0 t^2/2} \frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{(x-x_0)^2}{4Dt}\right]. \end{aligned} \quad (22)$$

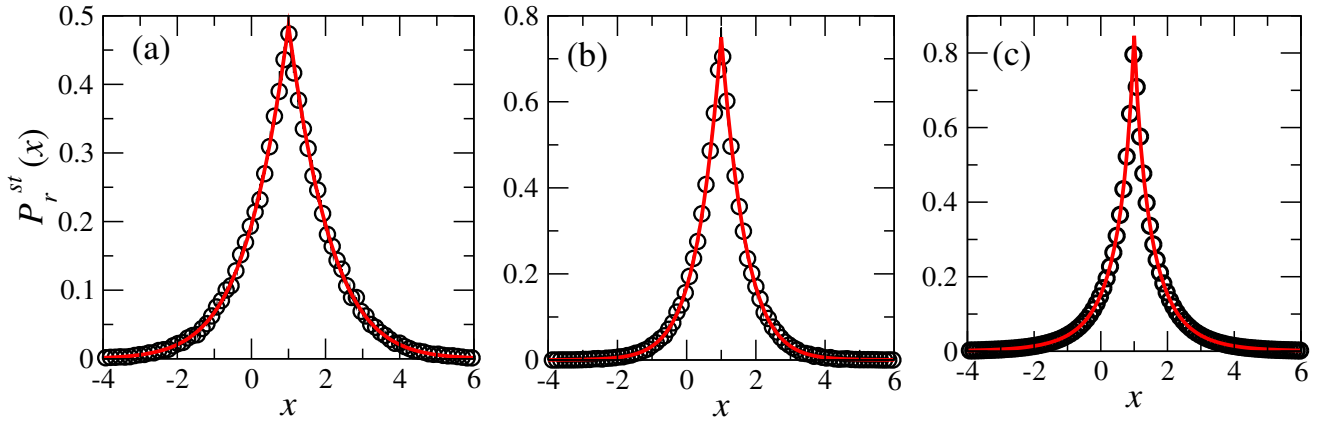


FIG. 1: (Color online) Stationary distributions $\mathbb{P}_r^{\text{st}}(x)$ of a single Brownian particle moving on a one-dimensional line in the presence of time-dependent resetting. In this plot we consider three choices of the resetting rate functions $r(t)$: (a) linearly increasing $r(t) = b_0 t$ and (b) increasing but bounded $r(t) = b_0(\frac{1}{\epsilon} - \frac{1}{t+\epsilon})$ and (c) decreasing rate $r(t) = \frac{b_0}{\sqrt{t}}$, where b_0 is some parameter in appropriate units. In all the figures the circles represent data obtained from direct numerical simulations whereas the solid lines correspond to the respective theoretical expressions (23), (29) and (32) in the main text. The intrinsic parameters of the system are : $D = 1$, $x_r = x_0 = 1$ and $b_0 = 1$ (in respective units) for all the three cases.

One can then write the steady state, in terms of a simple integral depending on one variable by inserting the relations (22) in (20)

$$\mathbb{P}_r^{\text{st}}(x) = \frac{\sqrt[4]{b_0}}{\sqrt{4\sqrt{2\pi D}}} \left(\frac{d}{d\ell} J(\ell) \right)_{\ell = \frac{|x-x_0|\sqrt[4]{b_0}}{\sqrt{4\sqrt{2\pi D}}}}, \quad (23)$$

$$\text{where, } J(\ell) = \int_0^\infty dz e^{-z^2} \text{erf}\left(\frac{\ell}{\sqrt{z}}\right). \quad (24)$$

This integral can be computed using *Mathematica* and its full expression is given in terms of hypergeometric functions in Appendix VIII. In Fig. 1a, we compare this result against the same obtained from direct numerical simulation of the dynamics and observe an excellent agreement.

2. Case II: increasing, but bounded rate

One unrealistic feature of an increasing rate, such as the linear rate considered in the previous paragraph, is that it may increase indefinitely. Here we consider a resetting processes in which the rate is an increasing function but bounded by an upper limit:

$$r(t) = b_0 \left(\frac{1}{\epsilon} - \frac{1}{t+\epsilon} \right). \quad (25)$$

Thus $r(0) = 0$ and $r(t) \rightarrow b_0/\epsilon$ as $t \rightarrow \infty$. In this case, the integrated rate $R(t)$ (13) is given by $R(t) = b_0(t/\epsilon) - b_0 \log[1 + t/\epsilon]$ implying $e^{-R(t)} \rightarrow 0$ for $t \rightarrow \infty$. Hence we expect to have a steady state which can be computed from (20). The function $\tilde{\mathbb{H}}_r(s)$ can be easily computed and its limit as $s \rightarrow 0$ is given by

$$\tilde{\mathbb{H}}_r(s)|_{s \rightarrow 0} = \epsilon e^{b_0 - (1+b_0)\ln b_0} \Gamma_g(1+b_0, b_0), \quad (26)$$

where $\Gamma_g(u, v) = \int_v^\infty dw w^{u-1} e^{-w}$ is an incomplete Gamma-function. To complete the evaluation of $\mathbb{P}_r^{\text{st}}(x)$, we now need to compute $\tilde{\mathbb{Q}}(x, s|x_0, 0)$ from (18) in the $s \rightarrow 0$ limit. After some straightforward manipulations one can show that $\tilde{\mathbb{Q}}(x, s|x_0, 0)|_{s \rightarrow 0}$ is given by

$$\tilde{\mathbb{Q}}(x, s|x_0, 0)|_{s \rightarrow 0} = \frac{\sqrt{\epsilon}}{4\sqrt{D}} \left(\frac{d}{d\ell} \mathcal{J}_{b_0}(\ell) \right)_{\frac{|x-x_0|}{\sqrt{4D\epsilon}}}, \quad \text{with} \quad (27)$$

$$\mathcal{J}_{b_0}(\ell) = \int_0^\infty dz e^{-b_0 z} (1+z)^{b_0} \text{erf}\left(\frac{b_0}{\sqrt{z}}\right), \quad (27)$$

$$= \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^k}{k!m!} (-2\ell)^m (b_0)_k \binom{m-2}{2}_k b_0^{m/2-k-1}. \quad (28)$$

where the symbol $(a)_k$ is given by $(a)_k = a(a-1)\dots(a-k+1)$. Hence, the steady state distribution $\mathbb{P}_r^{\text{st}}(x)$ is given by

$$\mathbb{P}_r^{\text{st}}(x) = \frac{e^{(1+b_0)\ln b_0 - b_0}}{\Gamma_g(1+b_0, b_0)} \frac{1}{4\sqrt{D\varepsilon}} \left(\frac{d}{d\ell} \mathcal{J}_{b_0}(\ell) \right) \Big|_{\frac{|x-x_0|}{\sqrt{4D\varepsilon}}}. \quad (29)$$

In Fig. 1b, we compare the analytical results against the same obtained from direct numerical simulation of the dynamics. The results are found to be in excellent agreement with the numerical results.

3. Case III: Rates decreasing as a power law $r(t) \sim b_0/t^\theta$

We now explore the case where the rate $r(t)$ decreases as a function of time. If the rate decreases too quickly then we expect from (21) that a steady state will not be attained. Therefore it is of natural interest to inquire more details of the condition for which the steady states exist and consequently their characteristic forms if they do.

We first consider the case $\theta < 1$ for which we may take

$$r(t) = b_0/t^\theta \quad \forall t \quad (30)$$

(it turns out that the singularity at $t = 0$ does not affect the steady state). Then we find

$$R(t) = b_0 t^{1-\theta}/(1-\theta) \quad \text{and} \quad R(t) \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty. \quad (31)$$

In this case, criterion (21) is satisfied and we have steady states given by

$$\mathbb{P}_r^{\text{st}}(x) = \frac{\tilde{\mathcal{Q}}(x, 0|x_0, 0)}{\tilde{\mathbb{H}}_r(0)} \quad (32)$$

where,

$$\begin{aligned} \tilde{\mathbb{H}}_r(0) &= b_0^{-1/(1-\theta)} (1-\theta)^{\theta/(1-\theta)} \Gamma\left(\frac{1}{1-\theta}\right), \\ \tilde{\mathcal{Q}}(x, 0|x_0, 0) &= \int_0^\infty dt e^{-b_0 t^{1-\theta}/(1-\theta)} \frac{\exp\left[-\frac{(x-x_0)^2}{4Dt}\right]}{\sqrt{4\pi Dt}}. \end{aligned} \quad (33)$$

We now consider the case for which $\theta > 1$. In order to ensure convergence of $R(t)$ we take

$$r(t) = b_0/(t+\varepsilon)^\theta. \quad (34)$$

The time-integrated rates can be found as

$$\begin{aligned} R(t) &\simeq \frac{b_0 t^{1-\theta}}{1-\theta} + \text{constant}, \\ e^{-R(t)} &\rightarrow \text{constant} > 0 \quad \text{as} \quad t \rightarrow \infty \end{aligned} \quad (35)$$

and therefore criterion (21) is not satisfied. Moreover, it can be checked that for small s , $\tilde{\mathbb{H}}_r(s) \sim 1/s$ whereas $\tilde{\mathcal{Q}}(x, s|x_0, 0) \sim s^{-1/2}$ so that (20) $\rightarrow 0$ as $s \rightarrow 0$ and there is no steady state.

Finally, we notice that, $\theta = 1$ is a marginal case. Taking $r(t)$ as in (34)

$$r(t) = b_0/(t+\varepsilon) \quad (36)$$

we obtain

$$e^{-R(t)} = \left[\frac{\varepsilon}{t+\varepsilon} \right]^{b_0}. \quad (37)$$

For $b_0 \leq 1$, $\tilde{\mathbb{H}}_r(s)$ diverges as $s \rightarrow 0$ whereas $\tilde{\mathcal{Q}}(x, s|x_0, 0)$ converges as $s \rightarrow 0$. Thus according to (20) there is no steady state. However, if $b_0 > 1$, $\tilde{\mathbb{H}}_r(s)$ converges as $s \rightarrow 0$, thus condition (21) is satisfied and there is a steady state given by (20) with

$$\begin{aligned} \tilde{\mathbb{H}}_r(0) &= \frac{\varepsilon}{b_0 - 1}, \\ \tilde{\mathcal{Q}}(x, 0|x_0, 0) &= \int_0^\infty dt \left[\frac{\varepsilon}{t+\varepsilon} \right]^{b_0} \frac{\exp\left[-\frac{(x-x_0)^2}{4Dt}\right]}{\sqrt{4\pi Dt}}. \end{aligned} \quad (38)$$

In Fig. 1c, we plot the steady state distribution corresponding to the power law choice of $r(t) = b_0/t^\theta$ with $\theta = 1/2$ and compare with the same obtained from numerical simulations. We again observe an excellent agreement.

IV. LATE TIME RELAXATION OF THE PROBABILITY DISTRIBUTION

For the case of time-independent constant resetting rate r_c , the late time relaxation of the probability distribution has been studied in [38]. It was shown that at large time t an inner region of the distribution $|x - x_0| < (4Dr_c)^{1/2}t$ has relaxed to the (time-independent) non-equilibrium steady state, whereas the outer region $|x - x_0| > (4Dr_c)^{1/2}t$ has not yet relaxed and the time-dependent probability is dominated by trajectories that have not yet undergone any resetting. Thus a front dividing the ‘equilibrated’ region from the ‘unequilibrated’ one moves linearly with time through the system with speed $v = (4Dr_c)^{1/2}$ [38].

Here we extend this analysis to the case of time-dependent resetting rates. We begin with the last renewal equation (14)

$$\begin{aligned} \mathbb{P}_r(x, t | x_0, 0) &= e^{-R(t)} G(x, t | x_0, 0) \\ &+ \int_0^t d\tau_l \psi(\tau_l) e^{-R(t-\tau_l)} G(x, t - \tau_l | x_0, 0), \end{aligned} \quad (39)$$

and assume that the second term in the right hand side (RHS) dominates at large t

$$\begin{aligned} \mathbb{P}_r(x, t | x_0, 0) &\simeq \int_0^t d\tau_l \psi(\tau_l) e^{-R(t-\tau_l)} G(x, t - \tau_l | x_0, 0) \\ &= t \int_0^1 dw \psi((1-w)t) e^{-R(wt)} G(x, wt | x_0, 0) \\ &= \left(\frac{t}{4\pi D}\right)^{1/2} \int_0^1 \frac{dw}{\sqrt{w}} \psi((1-w)t) e^{-R(wt) - \frac{(x-x_0)^2}{4Dwt}} \end{aligned} \quad (40)$$

where in the second line, a new integration variable w is defined through $\tau_l = (1-w)t$. We now wish to evaluate the integral by the saddle-point method for large t .

Consider a class of rate processes such as the following

$$r(t) = b_0 t^\theta \quad \text{where } \theta > -1. \quad (41)$$

This ensures us of both increasing and decreasing resetting rates along with the existence of a steady state in each case. Now, we have the integrated quantity

$$R(wt) = \frac{b_0 w^{1+\theta}}{1+\theta} t^{1+\theta}. \quad (42)$$

We observe in the scale where $|x - x_0| = ut^{1+\theta/2}$ i.e. we consider $|x - x_0| \rightarrow \infty$ and $t \rightarrow \infty$ with u held fixed. The integral (40) becomes

$$\mathbb{P}_r(x, t | x_0, 0) \simeq \left(\frac{t}{4\pi D}\right)^{1/2} \int_0^1 \frac{dw}{\sqrt{w}} \psi((1-w)t) e^{-t^{1+\theta} \left[\frac{b_0 w^{1+\theta}}{1+\theta} + \frac{u^2}{4Dw} \right]}. \quad (43)$$

Now we expect $\psi \rightarrow \text{constant} < \infty$ as $t \rightarrow \infty$ and therefore we may evaluate the integral in (43) by the saddle-point method. The saddle-point equation simply reads

$$\frac{\partial}{\partial w} \left[\frac{b_0 w^{1+\theta}}{1+\theta} + \frac{u^2}{4Dw} \right] = b_0 w^\theta - \frac{u^2}{4Dw^2} = 0, \quad (44)$$

which defines the value w^* of w that dominates the integral

$$w^* = \left(\frac{u^2}{4Db_0} \right)^{1/(2+\theta)}. \quad (45)$$

However this value can only dominate the integral if it is within the integration range $0 \leq w^* \leq 1$ which is the case if $u^2 < 4Db_0$ or equivalently

$$|x - x_0| < \sqrt{4Db_0} t^{1+\theta/2}. \quad (46)$$

On the other hand, if w^* is out of the integration range in which case (46) no longer holds and it turns out that the first term in the right hand side (RHS) of the last renewal equation (39) dominates. This contribution essentially represents those trajectories of the dynamics which have not participated in the resetting process yet.

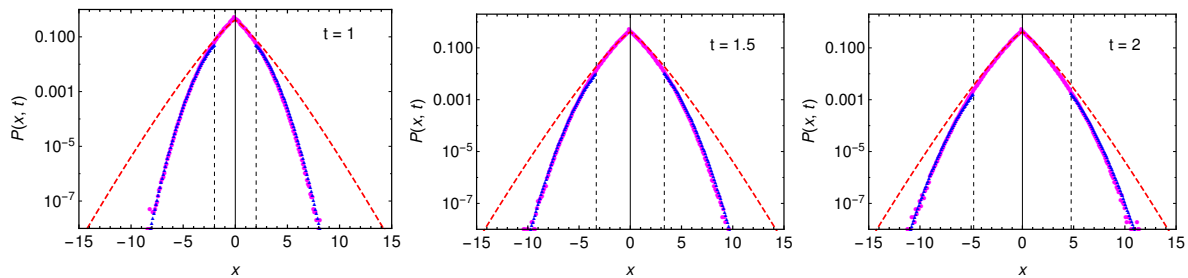


FIG. 2: (Color online) Dynamical relaxation of a single Brownian particle under the time-dependent stochastic resetting to the fixed point at the origin is shown. The resetting rate has the time dependence, according to (41), with $\theta = 0.5$ fixed for all the dynamical processes observed at different time scales $t = 1.0, 1.5, 2.0$ respectively. The intrinsic parameters for the system are : $D = 1, b_0 = 1$. The distribution is characterized by two regimes: inside one where the non-equilibrium steady state has already set in and the outside one where the system is still in a transient state. The front given by (46) separates these two regimes and have been depicted by the dashed vertical lines in the figures. The simulation results, indicated by magenta circles, agree well with the theoretical results obtained from (47), for both the steady state (dashed red curves) and the transient (the blue triangles) regimes.

The dominant asymptotic large time, and large $|x - x_0|$ behavior is therefore given by the following large deviation form:

$$\mathbb{P}_r(x, t | x_0, 0) \sim \exp \left[-t^{1+\theta} \mathbb{I}(u) \right], \quad \text{with } u = \frac{|x - x_0|}{t^{1+\theta/2}},$$

$$\text{and, } \mathbb{I}(u) = \begin{cases} b_0 \left(\frac{\theta+2}{\theta+1} \right) \left(\frac{u^2}{4Db_0} \right)^{\frac{1+\theta}{2+\theta}} & \text{for } u < u^* \\ \frac{b_0}{1+\theta} + \frac{u^2}{4D} & \text{for } u > u^* \end{cases} \quad (47)$$

where $u^* = \sqrt{4Db_0}$. The interpretation of this result is that the inner region $u < u^*$ or equivalently, $|x - x_0| < \sqrt{4Db_0} t^{1+\theta/2}$ has a time-independent behavior and therefore already has reached the non-equilibrium steady state form. This is the relaxed regime. On the other hand, the outer region $u > u^*$ or equivalently, $|x - x_0| > \sqrt{4Db_0} t^{1+\theta/2}$ is yet to equilibrate and still in a transient state. Thus an equilibration front moves through the system: the motion is superlinear if $\theta > 0$; linear if $\theta = 0$ with speed $u^* = \sqrt{4Db_0}$, and sublinear but superdiffusive if $-1 < \theta < 0$. The case $\theta = 0$ recovers the results of [38].

In Fig. 2, we observe this dynamical transition from numerical simulations, where we plot the distribution $\mathbb{P}(x, t | x_0, t)$ as a function of x for a given value of θ at three different observation time scales. For comparison we plot the theoretical large deviation form of the distribution given by (47) along with the simulation results. We find very good agreement between them in both the transient and the steady state regimes.

V. SURVIVAL PROBABILITY UNDER TIME-DEPENDENT RESETTING

We now look into the first passage probabilities of a diffusing particle under the influence of time-dependent stochastic resetting. First passage properties are generically of importance since they characterise the performance of search processes in various contexts. The most intuitive and important observable is the first passage time (FPT), for the particle to reach the origin say, which is itself a stochastic quantity and the PDF of FPT has its own rich characteristics. In particular, one of the main goals is to optimize the mean first passage time (MFPT) with respect to the system parameters. A review of the topic can be found in [39].

The problem of computing the survival probability in presence of a static target has been well studied for free Brownian motion (see e.g. [40] for a review). The MFPT was well investigated for diffusion under stochastic resetting at a constant rate in [6, 7]. It was found that the MFPT attains a minima with respect to the rate in contrast to the case of a single diffusive searcher in the absence of resetting. Thus there in fact exists an optimal resetting strategy which can be useful for all kinds of persistence problems. In this section, we address the question whether this strategy can be made even more efficient by introducing time-dependent resetting.

To this end, we compute the survival probability in presence of resetting at a time-dependent rate. To do so it is useful to introduce $q(x_0, t)$ that defines the survival probability of a free Brownian particle at 0 until time t , starting from x_0 . This result is well known (see e.g. [41]) and is given by

$$q(x_0, t) = \text{erf} \left[\frac{x_0}{\sqrt{4Dt}} \right]. \quad (48)$$

Let $Q_r(x_0, x_r, t)$ be the probability that the Brownian particle starting from x_0 survives until time t without being absorbed at 0 in the presence of stochastic resetting to x_r . The suffix ‘ r ’ indicates the presence of resetting, as before. Without any loss of generality, once again we will assume $x_r = x_0$.

Using the *first renewal* formalism, we now write the survival probability of the Brownian particle subject to time-dependent resetting

$$Q_r(x_0, x_0, t) = e^{-R(t)} q(x_0, t) + \int_0^t d\tau_f r(\tau_f) e^{-R(\tau_f)} q(x_0, \tau_f) Q_r(x_0, x_0, t - \tau_f). \quad (49)$$

The first term on the RHS represents trajectories in which there has been neither reset to x_0 nor absorption at the origin until time t . The integral in the second term represents a sum over first reset times τ_f and implies that there has been no reset and no absorption until time τ_f then a reset between τ_f and $\tau_f + d\tau_f$. The factor $r(\tau_f) e^{-R(\tau_f)} q(x_0, \tau_f)$ inside the integral gives the probability of no resetting and no absorption up to time τ_f and a resetting event between τ_f and $\tau_f + d\tau_f$. This is multiplied with the term $Q_r(x_0, x_0, t - \tau_f)$ which simply implies that no absorption occurs from τ_f to t .

Equation (49) belongs to a Wiener-Hopf class of integrals which can be solved in the Laplace space

$$\tilde{Q}_r(x_0, x_0, s) = \frac{\tilde{q}_r(x_0, s)}{s\tilde{q}_r(x_0, s) - \tilde{k}_r(x_0, s)}, \quad (50)$$

where we have defined the following quantities

$$\tilde{q}_r(x_0, s) = \int_0^\infty dt e^{-st} e^{-R(t)} q(x_0, t), \quad (51)$$

$$\tilde{k}_r(x_0, s) = \int_0^\infty dt e^{-st} e^{-R(t)} \frac{d}{dt} [q(x_0, t)]. \quad (52)$$

The mean first passage time to the origin starting from x_0 in the presence of stochastic resetting to $x_r = x_0$ is then given by

$$T(x_0) = - \int_0^\infty dt t \frac{\partial Q_r(x_0, x_0, t)}{\partial t} = \tilde{Q}_r(x_0, x_0, s \rightarrow 0), \quad (53)$$

which upon using (50) yields

$$T(x_0) = - \frac{\tilde{q}_r(x_0, 0)}{\tilde{k}_r(x_0, 0)}. \quad (54)$$

Inserting the expressions of $\tilde{q}_r(x_0, 0)$ and $\tilde{k}_r(x_0, 0)$ from eqs. (51) and (52) and performing some simple manipulations, one can write $T(x_0)$ in a compact form :

$$T(x_0) = -4 I(\beta) \left(\frac{d^2 I}{d\beta^2} \right)^{-1}, \quad \text{with } \beta = \frac{x_0}{\sqrt{4D}}. \quad (55)$$

The integral $I(\beta)$ is given by

$$I(\beta) = \int_0^\infty dt e^{-R(t)} \operatorname{erf} \left(\frac{\beta}{\sqrt{t}} \right). \quad (56)$$

For *time-independent* resetting rate $r(t) = r_c$, one can easily compute $I(\beta)$ to find $I(\beta) = \frac{1}{r_c} (1 - e^{-\alpha_0 x_0})$ where $\alpha_0 = \sqrt{\frac{r_c}{D}}$, is the inverse distance diffused between two resetting events. Now using this expression for $I(\beta)$ in (55) we recover the result

$$T(x_0) = \frac{1}{r_c} [e^{\alpha_0 x_0} - 1], \quad (57)$$

obtained in [6]. In [6], it was also shown that there is an optimal choice of the constant value of the rate $r_c^* = (2.53964\dots)(D/x_0^2)$ for given initial/resetting position x_0 and diffusion constant D , such that the value of MFPT becomes minimum : $T^*(x_0) = (1.54414\dots)(x_0^2/D)$.

The question now is whether one can make the search process more efficient by introducing time-dependent resetting rates. To answer this question, we compute the MFPT $T(x_0)$ for two choices of rate functions in the following. Then varying the respective parameters in the two rate functions we find that in both the cases one can get smaller $T(x_0)$ than the minimum $T^*(x_0)$ possible using a constant resetting rate r_c .

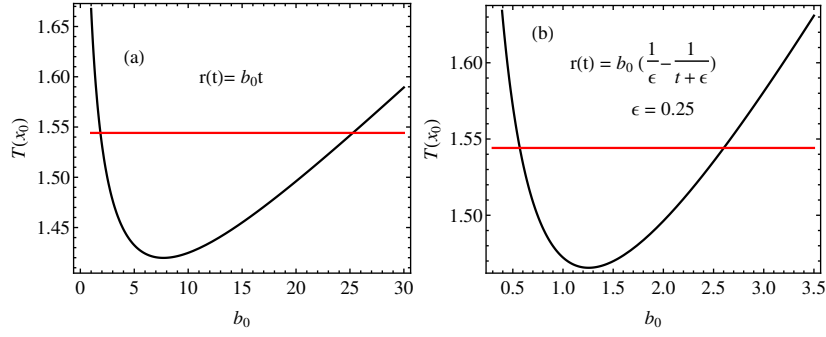


FIG. 3: (Color online) Plot of MFPT as function of the parameter b_0 for the following two reset rate functions : $r(t) = b_0 t$ and $r(t) = b_0 (1/\varepsilon - 1/(t + \varepsilon))$. The parameter b_0 in each cases of $r(t)$ is adopted in appropriate units. The diffusion constant $D = 1$ and the initial/resetting position has been fixed to $x_r = x_0 = 1$. We observe in both cases that there is a wide range of parameter for which the MFPT is much smaller than the minimum MFPT $T = 1.54414\dots$ (represented by the horizontal lines) possible using constant rate function $r(t) = r_c$ with the optimal value at r_c^* .

A. $r(t) = b_0 t$

For this case, $R(t) = b_0 t^2/2$ grows quadratically with time. Inserting this explicit form of $R(t)$ in (56) and performing some simple variable changes one finds that

$$I(\beta) = \sqrt{\frac{2}{b_0}} J\left(\beta \sqrt{\frac{b_0}{2}}\right), \quad (58)$$

where $J(\ell)$ is defined in (24) and given explicitly in Appendix VIII. Using this $I(\beta)$ in (55) one can find $T(x_0)$ for any given b_0 , x_0 and D . In Fig. 3a, we plot this MFPT as a function of b_0 for $x_0 = 1$ and $D = 1$. We observe a wide range of b_0 values for which $T(x_0)$ is much smaller than the minimum possible MFPT *i.e.* 1.54414... using a constant rate.

B. $r(t) = b_0(\frac{1}{\varepsilon} - \frac{1}{t+\varepsilon})$

In this case $R(t)$ is given by

$$R(t) = \int_0^t d\tau r(\tau) = b_0 \left[\frac{t}{\varepsilon} - \ln \left| 1 + \frac{t}{\varepsilon} \right| \right], \quad (59)$$

which, on inserting into (56) and performing some simple manipulations, yields $I(\beta) = \varepsilon \mathcal{J}_{b_0}(\beta/\sqrt{\varepsilon})$. The function $\mathcal{J}_{b_0}(\ell)$ is given in (27). Using Eq. (55), one thus obtains

$$T(x_0) = \frac{\sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{k+m}}{k!m!} \left(\frac{2x_0}{\sqrt{4D\varepsilon}} \right)^m (b_0)_k \left(\frac{m-2}{2} \right)_k b_0^{m/2-k-1}}{1 - \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{k+m}}{k!m!} \left(\frac{2x_0}{\sqrt{4D\varepsilon}} \right)^m (b_0)_k \left(\frac{m}{2} \right)_k b_0^{m/2-k}}, \quad (60)$$

where the symbol $(a)_k = a(a-1)(a-2)\dots(a-k+1)$. In Fig. 3b, we plot this MFPT as a function of b_0 for fixed ε , x_0 and $D = 1$. Here also we observe a wide range of b_0 values for which the $T(x_0)$ is much smaller than the minimum possible MFPT *i.e.* 1.54414... using constant rate.

VI. OPTIMAL RESETTING RATE FUNCTION

In this section we address the question of optimizing the search processes by minimizing the MFPT. We ask: What is the optimal form of the time-dependent resetting rate $r(t)$? Equation (54) is the primary equation for the MFPT and to seek the

optimized form let us take the functional derivative in (54) with respect to the function $r(t)$ at time t'

$$\frac{\delta T(x_0)}{\delta r(t')} = \frac{\int_{t'}^{\infty} dt e^{-R(t)} q(x_0, t)}{\tilde{k}_r(x_0, 0)} - \frac{\tilde{q}_r(x_0, 0)}{[\tilde{k}_r(x_0, 0)]^2} \int_{t'}^{\infty} dt e^{-R(t)} \frac{\partial q(x_0, t)}{\partial t}. \quad (61)$$

Setting the RHS to zero yields an equation which must be satisfied for all t' in order to have an extremal resetting function. However, it is clear that according to (61) one cannot find such a function: if the RHS of (61) is to be a constant value zero for all t' , then its derivative with respect to t' must vanish which yields

$$e^{-R(t')} \left[\tilde{k}_r(x_0, 0) q(x_0, t') - \tilde{q}_r(x_0, 0) \frac{\partial q(x_0, t)}{\partial t} \Big|_{t=t'} \right] = 0, \quad (62)$$

or, rearranging,

$$\frac{1}{q(x_0, t')} \frac{\partial q(x_0, t)}{\partial t} \Big|_{t=t'} = \frac{\tilde{k}_r(x_0, 0)}{\tilde{q}_r(x_0, 0)}, \quad (63)$$

and this equation cannot be satisfied for all t' since the RHS is constant but the LHS is a function of t' . Indeed the LHS does not depend on $r(t)$.

Since the optimal resetting function does not extremise the MFPT everywhere (i.e. it does not set the functional derivative to zero everywhere), it must instead involve values of rates at the boundary of the allowed rate space i.e. it must involve $r = 0$ (since a rate cannot be negative) and $r \rightarrow \infty$. This leads us to conjecture that a possible form for the optimal resetting function is

$$r(t) = \begin{cases} 0 & \text{for } t < t^* \\ \infty & \text{for } t > t^*. \end{cases} \quad (64)$$

That is, resetting does not occur up until time t^* then occurs instantaneously. Or in other words there is deterministic resetting with period t^* .

For $r(t)$ of the form (64) the MFPT is given by

$$T(x_0) = \frac{\int_0^{t^*} dt q(x_0, t)}{1 - q(x_0, t^*)}. \quad (65)$$

Extremising this expression with respect to t^* yields the condition

$$\frac{\int_0^{t^*} dt' q(x_0, t')}{1 - q(x_0, t^*)} = - \frac{q(x_0, t^*)}{\frac{\partial q(x_0, t)}{\partial t} \Big|_{t=t^*}}, \quad (66)$$

where, as usual, $q(x_0, t) = \text{erf}(x_0/\sqrt{4Dt})$. Note that the above equation can be expressed solely in terms of the rescaled variable $z = t/\beta^2$ as

$$\frac{\int_0^{z^*} dz' \text{erf}(1/\sqrt{z'})}{1 - \text{erf}(1/\sqrt{z^*})} = - \frac{\text{erf}(1/\sqrt{z^*})}{\frac{\partial \text{erf}(1/\sqrt{z^*})}{\partial z^*}}, \quad (67)$$

where $\beta = x_0/\sqrt{4D}$ as before. This implies that $t^* = z^* \frac{x_0^2}{4D}$ where z^* is the solution of the above equation. One can solve this equation numerically to find $z^* = 1.834011077\dots$. Using $t^* = z^* \frac{x_0^2}{4D}$ in (65) one finds that the optimum MFPT is given by $T(x_0) = 5.34354\dots(x_0^2/4D)$ which is much smaller than the minimum MFPT $6.17655\dots(x_0^2/4D)$ possible for any given x_0 and D using constant reset rate (see fig. 4). Secondly, the MFPT $T(x_0)$ and t^* constitute a linear relation $T(x_0) = 2.91358\dots t^*$, for any given x_0 and D (see fig. 4).

Furthermore we can check that the rate function $r(t)$ given by (64) with $t^* = z^* \frac{x_0^2}{4D}$, locally optimizes the MFPT with respect to variations of the rate function. First note that the functional derivative of the MFPT $\rightarrow 0$ for $t' > t^*$. For $t' < t^*$, after some simplification, we obtain

$$\frac{\delta T(x_0)}{\delta r(t')} = \frac{1}{[1 - q(x_0, t^*)]^2} K(x_0, t'), \quad (68)$$

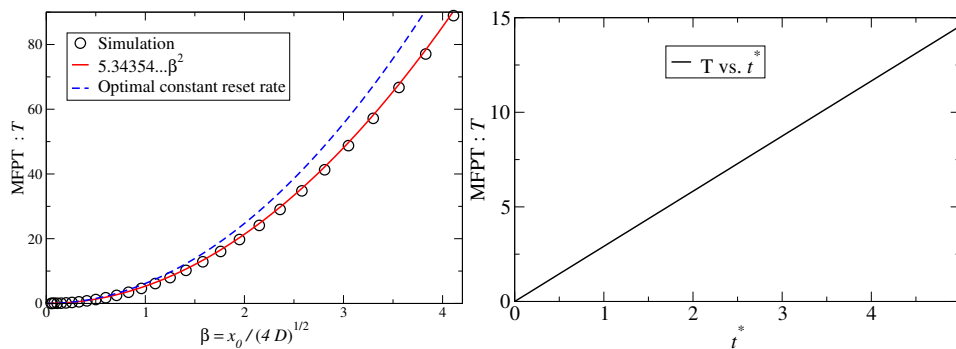


FIG. 4: (Color online) Left Panel: We plot MFPT $T(x_0)$ as a function of $\beta = x_0/\sqrt{4D}$ for the optimal resetting rate function (64). The circles are obtained from numerical simulation whereas the red solid line represents the theoretical expression $T = 5.34354\dots\beta^2$. The blue dashed line corresponds to the same for constant resetting rate with optimal value $r(t) = r_c^* = 2.53964\dots/\beta^2$. Right Panel: The linear relation between the MFPT $T(x_0)$ and t^* , as mentioned in the text, has been plotted.

where

$$K(x_0, t') = (1 - q(t^*)) \int_0^{t'} dt q(x_0, t) - (1 - q(t')) \int_0^{t^*} dt q(x_0, t). \quad (69)$$

Now consider the function $K(x_0, t')$: it takes value 0 and has positive derivative at $t' = 0$ while at $t' = t^*$ it takes value zero and has derivative equal to zero. In the domain $0 < t' < t^*$, the function $K(x_0, t')$ has a single turning point as can be checked numerically. Hence, in the domain $0 < t' < t^*$, the function $K(x_0, t') > 0$, is strictly positive as is the functional derivative $\frac{\delta T(x_0)}{\delta r(t')} > 0$. Thus $r(t)$ given by (64) is evidently a locally optimal rate function with respect to the MFPT.

VII. CONCLUSIONS

In this work we have considered a simple stochastic system of a Brownian particle subjected to a time-dependent resetting rate to a fixed position x_r . This is in contrast with earlier studies where a time independent rate was chosen. The rate of resetting $r(t)$ is a function of the time t since the last reset event. We have seen that for steady states to exist, the function $r(t)$ must respect certain features such as either increasing with time, tending to some finite constant as $t \rightarrow \infty$ or decay to zero sufficiently slow. A sufficient condition for the steady state is given by (21). In this context, various cases for the steady state distributions have been worked out in Section III.

We have also considered the late time relaxation behavior of the probability distribution for the specific choices of rate function $r(t) \propto t^\theta$ with $\theta > -1$. In analogy to recent work [38] we have shown that an inner core region around the resetting point settles into the steady state distribution and the boundaries of the inner core region is still in transient state and move outwards towards the tails of the distribution with time. Thus an ‘equilibration’ front moves through the system and its motion is superlinear if $\theta > 0$; linear if $\theta = 0$ (recovering the results of [38]) and sublinear but superdiffusive if $-1 < \theta < 0$.

Finally we have considered persistence properties of the Brownian motion in the presence of the time dependent resetting process. In particular, we have studied extensively the first passage time properties of the Brownian particle to reach the origin. We have derived a formula (55) for the mean first passage time. A primary motivation has been to optimize the mean first passage time with respect to the rate functions. To this end, we have studied thoroughly the optimal properties of the mean first passage time and have shown that a threshold function (64) with an optimised threshold t^* is at least locally optimal for the problem of minimizing the mean first passage time. It would be of great interest to see whether this is also the globally optimal function and if so, to provide a rigorous proof of this.

Other interesting generalizations would be to consider, for example, oscillatory reset functions or else random resetting rates chosen from a specified distribution. It would also be interesting to look at the properties of the entropy generation in the system subject to a resetting mechanism. Another propitious future direction might be to study the effects of time-dependent resetting on the dynamics of interacting multi-particle systems, which could prove useful in the context of cellular biology.

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VIII. APPENDIX

It has been shown in the main text that the steady state formula for the linear rate process can be written in terms of the function $J(\ell)$ as mentioned in (24). The integral $J(\ell)$ can be computed exactly using Mathematica and it is expressed in terms of hypergeometric functions as

$$J(\ell) = \frac{\ell \Gamma\left(\frac{1}{4}\right) {}_1F_3\left(\frac{1}{4}; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; -\frac{\ell^4}{4}\right)}{\sqrt{\pi}} + \frac{4\ell^3 \Gamma\left(\frac{3}{4}\right) {}_1F_3\left(\frac{3}{4}; \frac{5}{4}, \frac{3}{2}, \frac{7}{4}; -\frac{\ell^4}{4}\right)}{3\sqrt{\pi}} - 2\ell^2 {}_1F_3\left(\frac{1}{2}; \frac{3}{4}, \frac{5}{4}, \frac{3}{2}; -\frac{\ell^4}{4}\right). \quad (70)$$

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