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DIAGONALIZING MATRICES OVER AW*-ALGEBRAS

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Abstract. Every commuting set of normal matrices with entries in an AW*-algebra can be simultaneously diagonalized. To establish this, a dimension theory for properly infinite projections in AW*-algebras is developed. As a consequence, passing to matrix rings is a functor on the category of AW*-algebras.

1. Introduction

Diagonalization is a fundamental operation on matrices that can simplify reasoning about normal matrices. Every commuting set of normal $n \times n$ complex matrices can be simultaneously diagonalized. If A is a unital C*-algebra, it is well known that the ring $\mathbb{M}_n(A)$ of $n \times n$ matrices with entries in A is again a unital C*-algebra. The question naturally arises: over which C*-algebras can any commuting set of normal $n \times n$ matrices be diagonalized? To be precise, we say that A is *simultaneously n-diagonalizable* if, for any commuting set X of normal elements of $\mathbb{M}_n(A)$, there is a unitary u in $\mathbb{M}_n(A)$ making uxu^* diagonal for any $x \in X$. (Note that this property is stronger than the ability to diagonalize individual normal $n \times n$ matrices.) We prove that every AW*-algebra is simultaneously n-diagonalizable for any positive integer n.

This question has quite some history. Deckard and Pearcy first established in [6] that every individual normal matrix is diagonalizable in $\mathbb{M}_n(A)$ for a commutative AW*-algebra A in 1962. In 1977, Halpern showed that a single normal element of a properly infinite von Neumann algebra is diagonalizable (though this seems not to have been widely noticed [11, Lemma 3.2]). Since then, the problem of diagonalizing an individual matrix or operator has been studied in several contexts; for a brief survey and further references see [16, Chapter 6]. The question of whether an individual self-adjoint matrix over an AW*-algebras is diagonalizable was raised in [8]. Simultaneous diagonalization of matrices over noncommutative operator algebras was initiated by Kadison in 1982 ([12], see also [13, Volume IV, Exercises 6.9.18–6.9.35]). He proved that countably decomposable von Neumann algebras are simultaneously n-diagonalizable, relying on their decomposition into types (see also [14]). In 1984, Grove and Pedersen showed that for any $n \geq 2$, a commutative simultaneously n-diagonalizable C*-algebra is an AW*-algebra, and they asked whether Kadison's techniques extend to noncommutative AW*-algebras [10, 6.7]. We precisely accomplish this task.

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The bulk of the new results here concerns properly infinite AW*-algebras. In that case, our attack on the question requires a dimension theory, reducing equivalence of properly infinite projections to a problem about cardinal-valued dimensions. Kadison sidestepped such size issues by restricting to countably decomposable von Neumann algebras. By proving everything in full generality, our results are even new in the case of properly infinite von Neumann algebras. A dimension theory for AW*-algebras was given by Feldman already in 1956 [7]. Independently, Čilin studied a similar notion of dimension in 1980.[1] Tomiyama greatly extended Feldman's results in the case of von Neumann algebras in 1958 [17]; for a recent, and very general, account, see [9]. However, these studies into dimension theory do not interface seamlessly with Kadison's diagonalization results. Therefore, either the dimension theory or Kadison's methods have to be adapted; we chose the former. This requires a more intricate analysis of the dimension function. A crucial step here is a decomposition into so-called equidimensional projections. As a side note, we must mention that all these results depend heavily on the axiom of choice, and therefore are problematic in constructive settings.

Our original interest in diagonalization over AW*-algebras arose from the following problem. Let **Cstar** denote the category whose objects are unital C*-algebras and whose morphisms are unital $*$-homomorphisms. Let **AWstar** denote the subcategory of **Cstar** whose objects are the AW*-algebras and whose morphisms are those $*$-homomorphisms that preserve suprema of arbitrary sets of projections. Applying $*$-homomorphisms entrywise makes $\mathbb{M}_n$ into a functor $\mathbf{Cstar} \to \mathbf{Cstar}$. On objects, this functor sends AW*-algebras to AW*-algebras, by a combination of results due to Kaplansky and Berberian [2]. So it is natural to ask whether $\mathbb{M}_n$ restricts to a functor $\mathbf{AWstar} \to \mathbf{AWstar}$. As an application of the diagonalization theorem, we prove that this is indeed so.

As is clear from the historical introduction above, there is a fair amount of (routine) generalization from von Neumann algebras to AW*-algebras,[2] as well as piecing together fragmented results from the literature. To make the story reasonably self-contained, we include all such results in a uniform way with explicit proofs, relying upon [3] as our standard reference for the theory of AW*-algebras. The paper is structured as follows. After discussing preliminaries in Section 2, and the routine generalizations of Kadison's results to AW*-algebras of finite type in Section 3, the next few sections launch into the proof of simultaneous n-diagonalizability of AW*-algebras. Section 4 introduces the dimension theory, which is continued in Section 5, that concerns equidimensional projections. The dimension theory is then put to use in Section 6 to generalize Kadison's results to AW*-algebras of infinite type. Then Section 7 gathers all the ingredients to prove that AW*-algebras are simultaneously n-diagonalizable. Section 8 ends the paper with the functoriality of taking matrix rings of AW*-algebras. Finally, Appendix A contains additional technical results about dimensions that would disrupt the main development. Some open questions are mentioned at the end of Sections 6 and 7.

[1] Apparently it was published in [5], but we did not manage to locate that paper; instead we re-engineered, and generalized, the proofs of Theorems 4.6 and 5.3 below from Proposition 3.6.6 in Čilin's thesis. We thank S. Solovjovs for obtaining that thesis, and A. Akhvlediani for translating that proposition.

[2] See also Remark 8.3.

2. Preliminaries on AW*-algebras

An *AW*-algebra* is a C*-algebra in which the (right, and hence left) annihilator of any subset is generated by a single projection. This section recalls some general properties of these algebras, which were introduced by Kaplansky as a generalization of von Neumann algebras, preserving the purely algebraic content of their theory [15]. For example, the Gelfand spectrum of a commutative AW*-algebra is a Stonean space (*i.e.* a topological space in which the closure of an open set is again open, *i.e.* the Stone space of a complete Boolean algebra). To compare: the Gelfand spectrum of a commutative von Neumann algebra additionally satisfies a measure-theoretic property.

Maximal abelian subalgebras. We will use the abbreviated phrase *maximal abelian subalgebra* in place of "maximal abelian $*$-subalgebra" or "maximal abelian self-adjoint subalgebra". The notion of AW*-subalgebra is slightly subtle, but maximal abelian subalgebras are automatically AW*-subalgebras.

Projections. The main characteristic of AW*-algebras is that to a great extent they are algebraically determined by their projections. For example, any AW*-algebra A is the closed linear span of its projections $\mathrm{Proj}(A)$. Projections are partially ordered by $e \leq f$ if and only if $e = ef (= fe)$, and $\mathrm{Proj}(A)$ is a complete lattice. In the special case that $\{e_i\}$ is an orthogonal set of projections in A, we denote its supremum by $\sum e_i$. Projections $e, f \in \mathrm{Proj}(A)$ are equivalent when $e = vv^*$ and $f = v^*v$ for some $v \in A$. When the algebra in which they are equivalent must be emphasized, we write $e \sim_A f$, and similarly for the derived notions $e \precsim_A f$ (meaning $e \sim e' \leq f$ for some projection e') and $e \prec_A f$ (meaning $e \precsim f$ but $e \not\sim f$; we also allow $0 \prec 0$). Equivalence is additive: if $\{e_i\}$ and $\{f_i\}$ are orthogonal families of projections satisfying $e_i \sim f_i$, then $\sum e_i \sim \sum f_i$. Equivalence also satisfies Schröder–Bernstein: if $e \precsim f$ and $e \succsim f$, then $e \sim f$. It is a simple fact that if $z, e, f \in \mathrm{Proj}(A)$ are such that z is central and $e \sim f$, then $ze \sim zf$.

Comparison theorem. *Let e and f be projections in an AW*-algebra. There are orthogonal central projections x, y, z satisfying $x + y + z = 1$ and*

$$xe \prec xf, \qquad ye \sim yf, \qquad ze \succ zf.$$

Proof. Zorn's lemma produces a maximal orthogonal family $\{y_i\}$ of nonzero central projections satisfying $y_i e \sim y_i f$. Setting $y = \sum y_i$, then $ye \sim yf$. In fact, this y is the unique largest central projection with that property: if $we \sim wf$ for $w \in \mathrm{Proj}(Z(A))$ then $(1-y)we \sim (1-y)wf$, but $(1-y)w$ is orthogonal to y and must hence be zero by maximality of $\{y_i\}$.

There is a central projection w such that $we \precsim wf$ and $(1-w)e \succsim (1-w)f$ [3, Corollary 14.1]. Set $x = w(1-y)$ and $z = (1-w)(1-y)$. Then x, y and z are orthogonal and sum to 1. Clearly also $xe \precsim xf$, and because $xe \sim xf$ violates maximality of y as above, in fact $xe \prec xf$. Similarly $ze \succ zf$. $\square$

In fact, the x, y and z in the comparison theorem are unique, but we do not need this fact.

Passing to corner algebras. We will frequently use properties of corners of an AW*-algebra A, which we now list. For any $e \in \mathrm{Proj}(A)$, the corner algebra eAe and the centre $Z(A)$ are again AW*-algebras. Many relevant properties are preserved by passing to corners. For example, the following lemma shows that equivalence and maximality of abelian subalgebras are also well behaved when passing to corners.

Lemma 2.1. *Let e be a projection in an AW*-algebra A.*

(a) $\mathrm{Proj}(eAe) = \{p \in \mathrm{Proj}(A) \mid p \leq e\}$*;*
(b) For all $p, q \in \mathrm{Proj}(eAe)$ we have $p \sim_A q$ if and only if $p \sim_{eAe} q$.
(c) If C is a maximal abelian subalgebra of A, and $e \in \mathrm{Proj}(C)$, then eC is a maximal abelian subalgebra of eAe.

Proof. By definition $p \in eAe$ if and only if $p = eae$ for some $a \in A$. This is equivalent to $p = ep = pe$, that is, to $p \leq e$, establishing (a). For the non-trivial direction of (b), suppose $p \sim_A q$, say $v^*v = p$ and $vv^* = q$. Since we may assume that $v \in A$ is a partial isometry [3, Proposition 1.6], $v = vv^*vv^*v = qvp \in qAp \subseteq eAe$, so $p \sim_{eAe} q$.

For (c), observe that for any projection z in C one has $z(1-e) \leq 1-e$ in A, and so $ez(1-e) = 0$. Similarly $(1-e)ze = 0$. If $eae \in eAe$ commutes with eC, then

$$eaez = eaeze + eaez(1-e) = eaeze = ezeae + (1-e)zeae = zeae.$$

Hence eae commutes with C. So $eae \in C$ by maximality of C. Therefore $eae \in eC$, and eC is maximal. □

Central covers. We write $\mathrm{c}(e)$ for the least central projection above e, also called its central cover. If the AW*-algebra A must be emphasized, we write $\mathrm{c}_A(e)$ instead. Central covers and centres are also preserved by passing to corners.

Lemma 2.2. *If $f \leq e$ are projections in an AW*-algebra A, then $\mathrm{c}_{eAe}(f) = \mathrm{c}_A(f)e$. Hence $Z(eAe) = eZ(A)$.*

Proof. See Proposition 6.4 and Corollary 6.1 of [3]. □

We record two results of Kadison's on central covers, adapted to AW*-algebras.

Lemma 2.3. *If C is a maximal abelian subalgebra of an AW*-algebra A and $C \neq A$, then there are nonzero orthogonal projections e, f in C with $\mathrm{c}(e) = \mathrm{c}(f)$ and $e \precsim f$.*

Proof. If $p \in \mathrm{Proj}(C)$ satisfies $\mathrm{c}(p)\mathrm{c}(1-p) = 0$, then $p = \mathrm{c}(p)$, because

$$p \leq \mathrm{c}(p) \leq 1 - \mathrm{c}(1-p) \leq 1 - (1-p) = p.$$

So either each projection in C is central in A, or $q = \mathrm{c}(p)\mathrm{c}(1-p) > 0$ for some projection p in C. The former case is ruled out, because then $Z(A) = C$, and hence $C = A$ by maximality. Now qp and $q(1-p)$ are nonzero and $\mathrm{c}(qp) = \mathrm{c}(q(1-p))$. By the comparison theorem, there is a nonzero central projection $z \leq q$ with either $zp \precsim z(1-p)$ or $z(1-p) \precsim zp$. In any event, one of zp and $z(1-p)$ serves as e and the other as f, when A is not abelian. □

Lemma 2.4. *If A is an AW*-algebra without abelian central summands, then any maximal abelian subalgebra C contains a projection e with $\mathrm{c}(e) = 1 = \mathrm{c}(1-e)$ and $e \precsim 1-e$.*

Proof. Let $\{e_i\}$ be a family of nonzero projections in C maximal with respect to the properties that $\{\mathrm{c}(e_i)\}$ is orthogonal and $e_i \precsim 1-e_i$ for each i. From Lemma 2.3, C contains nonzero orthogonal projections $e_0 \precsim f_0$ $(\leq 1-e_0)$. Thus the family $\{e_i\}$ is not empty. Set $e = \sum_i e_i$. Then $\mathrm{c}(e) = \sum \mathrm{c}(e_i)$. If $z = \mathrm{c}(e) < 1$, then $(1-z)A$ is a nonabelian AW*-algebra (since A is assumed to have no central summands that are abelian) and $(1-z)C$ is a maximal abelian subalgebra. Again from Lemma 2.3, there is a nonzero projection e_1 in $(1-z)C$ with $e_1 \precsim (1-z)-e_1$. Adjoining e_1 to $\{e_i\}$ contradicts maximality of that family. Thus $z=1$. Since

$$e_i = \mathrm{c}(e_i)e_1 \precsim \mathrm{c}(e_i)(1-e_i) = \mathrm{c}(e_i) - e_i$$

for each i, we have $e = \sum_i e_i \precsim \sum_i(\mathrm{c}(e_i)-e_i) = 1-e$, and $\mathrm{c}(e) = z = 1$. □

Properly infinite projections. A projection e is *finite* when $e \sim f \leq e$ implies $e=f$; otherwise it is *infinite*. It follows from the comparison theorem that if $\mathrm{c}(e) \leq \mathrm{c}(f)$ for a finite projection e and an infinite projection f, then $ze \prec zf$ for nonzero central projection $z \leq \mathrm{c}(f)$. Following standard terminology, an AW*-algebra A is *properly infinite* if every nonzero central projection of A is infinite.

Lemma 2.5. *Let A be an AW*-algebra, and let e be a nonzero projection in A. The following are equivalent:*

(a) there exist projections $e_1 \sim e_2 \sim e$ in A such that $e = e_1+e_2$;
(b) there exists an infinite orthogonal set of projections $\{e_i\}$ in A such that $e_i \sim e$ for all i and $e = \sum e_i$;
(c) the AW-algebra eAe is properly infinite;*
(d) if $z \in A$ is a central projection, then ze is either zero or infinite.

Proof. That $(c) \Rightarrow (b) \Rightarrow (a)$ is essentially [3, Theorem 17.1]. Suppose (d) holds, and let x be a nonzero central projection in eAe. Lemma 2.2 provides $z \in \mathrm{Proj}(Z(A))$ such that $x = ze$, which is infinite in A by assumption. So $x \sim_A f < x$ for some f in A. But then $f \in eAe$ satisfies $x \sim_{eAe} f < x$ in eAe by Lemma 2.1. Hence x is infinite in eAe, establishing (c). Finally, we prove $(a) \Rightarrow (d)$. Let z be a central projection in A such that $ze > 0$. Then $ze_2 \sim ze$ is nonzero whence $ze \sim ze_1 < ze_1 + ze_2 = ze$. So ze is infinite. □

A nonzero projection e in an AW*-algebra is *properly infinite* if it satisfies the equivalent conditions of the previous lemma. Being properly infinite is preserved by equivalence of projections (*e.g.* by Lemma 2.5(c)). It also follows from the previous lemma that ze is properly infinite for any nonzero central projection $z \leq \mathrm{c}(e)$.

Lemma 2.6. *Let e be a projection in an AW*-algebra A.*

(a) A projection in eAe is properly infinite in A if and only if it is so in eAe.
(b) If e is infinite, there is a central projection $z \in A$ making ze finite and $(1-z)e$ properly infinite.

Proof. For (a), let $f \in eAe$ be a projection that is properly infinite in A. Then $f = a_1 + a_2$ and $a_1 \sim_A a_2 \sim_A f$ for some $a_1, a_2 \in \mathrm{Proj}(eAe)$ by Lemma 2.5. But since $a_i \leq f$, in fact $a_i \in \mathrm{Proj}(eAe)$ and $a_1 \sim_{eAe} a_2 \sim_{eAe} f$ by Lemma 2.1. So f is properly infinite in eAe. The converse is trivial.

For (b), let $\{z_i\}$ be a maximal orthogonal family of nonzero central projections such that $z_i e$ is finite for each i. Set $z = \sum z_i$. Then ze is finite [3, Proposition 15.8]. Moreover, if y is a central projection such that $y(1-z)e$ is finite, then $y(1-z)$ must be zero by maximality of $\{z_i\}$. So $(1-z)e$ is properly infinite by Lemma 2.5(d). □

Decomposition into types. Another property that survives passing to corners is the decomposition into types of an AW*-algebra. Recall that a projection e is abelian when eAe is abelian. An AW*-algebra is of type I if it has an abelian projection with central cover 1; it is of type II if it has a finite projection with central cover 1 but no nonzero abelian projections; and it is of type III if it has no nonzero finite projections. More specifically, type II_1 means type II and finite; type II_∞ means type II and properly infinite; type I_∞ means I and properly infinite. (Notice that the zero algebra is of all types.)

Lemma 2.7. *Let e be a nonzero projection in an AW*-algebra A.*

(a) eAe is finite if A is finite;
(b) eAe is of type I if A is of type I;
(c) eAe is of type I_∞ if A is of type I_∞ and e is properly infinite;
(d) eAe is of type II_1 if A is of type II_1;
(e) eAe is of type II_∞ if A is of type II_∞ and e is properly infinite;
(f) eAe is of type III if A is of III.

Proof. First, notice that a projection p in eAe is abelian in eAe if and only if $peAep = pAp$ is abelian, if and only if p is abelian in A.

For (a), suppose $f \in \mathrm{Proj}(eAe)$ is finite in A. That means that $p \leq f \sim_A p$ implies $p = f$ for all $p \in \mathrm{Proj}(A)$. As $f \leq e$, Lemma 2.1(b) makes this equivalent to: $p \leq f \sim_{eAe} p$ implies $p = f$ for $p \in \mathrm{Proj}(eAe)$. But this means that f is finite in eAe. Part (b) is [3, Exercise 18.2]. For (c): eAe is of type I by (a), and contains a properly infinite projection e by Lemma 2.6. Part (d) follows from (a) and the above observation about abelian projections. Part (f) follows from Lemma 2.6.

Finally, we turn to (e). If A is of type II_∞, it has a finite projection f with $\mathrm{c}(f) = 1$, and no nonzero abelian projections. So, by the above observation, also eAe has no nonzero abelian projections. Because e is properly infinite and f is finite, it follows from the comparison theorem that $\mathrm{c}(e)f \prec e$. Thus $\mathrm{c}(e)f \sim e_0 < e$ for some finite projection e_0 with $\mathrm{c}(e_0) = \mathrm{c}(e)$. It now follows from Lemmas 2.1(c) and 2.2 that e_0 is finite in eAe with $\mathrm{c}_{eAe}(e_0) = e$. Finally, e is properly infinite in eAe by Lemma 2.6(a), making eAe of type II_∞. $\square$

3. Relative comparison for AW*-algebras of finite type

We begin by quickly disposing of the relative comparison theory for AW*-algebras of finite type. This involves relatively straightforward generalizations of Kadison's results to AW*-algebras; the section is included in the interest of completeness. The results of this section and Section 6 will show that we can always find projections with various properties, not just in an AW*-algebra A, but in any maximal abelian subalgebra C of A. In this context, whenever we mention without specification concepts such as $\sim$, d, finite, infinite, abelian, or central cover, we mean the corresponding concepts in A (and not in C). We start by considering AW*-algebras of type II_1.

Proposition 3.1. *Let n be a positive integer, and let A be an AW*-algebra of type II_1. Let C be a maximal abelian subalgebra and $e \in \mathrm{Proj}(C)$.*

(a) There is a sequence $e_0, e_1, e_2, \ldots \in \mathrm{Proj}(C)$ with $e_0 = e$, $\mathrm{c}(e_i) = \mathrm{c}(e)$, $e_i \leq e_{i-1}$, and $e_i \precsim e_{i-1} - e_i$.
(b) If $f \in \mathrm{Proj}(A)$ satisfies $\mathrm{c}(e)\mathrm{c}(f) \neq 0$, then there is a nonzero $g \in \mathrm{Proj}(C)$ with $g \leq e$ and $g \precsim f$.
(c) If $f \in \mathrm{Proj}(A)$ satisfies $f \precsim e$, then $f \sim e_1 \leq e$ for some $e_1 \in \mathrm{Proj}(C)$.
(d) C contains n orthogonal equivalent projections with sum 1.

Proof. (a) If $e = 0$, choose $e_i = 0$ for each i. Suppose $e > 0$. Then eAe is of type II_1 by Lemma 2.7, and eCe is a maximal abelian subalgebra. In particular, eAe has no abelian central summands. Lemma 2.4 gives $e_1 \in \mathrm{Proj}(eCe)$ with $\mathrm{c}_{eAe}(e_1) = e$ and $e_1 \precsim e - e_1$. It follows from Lemma 2.2 that $\mathrm{c}_A(e_1) = \mathrm{c}_A(e)$. Induction now provides a sequence with the desired properties.

For (b): replacing A, C, e and f by zA, zC, ze and zf for $z = \mathrm{c}(e)\mathrm{c}(f)$, we may assume that $\mathrm{c}(e) = \mathrm{c}(f) = 1$. Now, if $ye_i \not\prec yf$ for each nonzero central projection y, then $f \precsim e_i$ by the comparison theorem. If $f \precsim e_i$ for each i, then $e_{i-1} - e_i$ has a subprojection equivalent to f for each i. In this case A contains an infinite orthogonal family of projections equivalent to f, which contradicts the assumption that A is finite. Thus, $ye_i \prec yf$ for some i and some nonzero central y. Now ye_i will serve as g.

For (c), let S be the set of pairs consisting of orthogonal families $\{e_i \in C \mid i \in \alpha\}$ and $\{f_i \in A \mid i \in \alpha\}$ of nonzero projections, where $e_i \sim f_i$ for all $i \in \alpha$, and $e_i \leq e$, and $f_i \leq f$. We can partially order S by

$$(\{e_i \mid i \in \alpha\}, \{f_i \mid i \in \alpha\}) \leq (\{e'_j \mid j \in \beta\}, \{f'_j \mid j \in \beta\})$$

when $\{e_i\} \subseteq \{e'_j\}$ and $\{f_i\} \subseteq \{f'_j\}$. Zorn's lemma provides a maximal element $(\{e_i\}, \{f_i\})$ in S. Set $e_1 = \sum_i e_i$ and $f_1 = \sum_i f_i$. Then $e_1 \sim f_1$ by additivity of equivalence. Now $e_1 \in C$, $e_1 \leq e$, and $f_1 \leq f$. Because A is finite and $f \precsim e$, we have $f - f_1 \precsim e - e_1$ [3, Proposition 17.5, Exercise 17.3]. From (b), there is a nonzero $e_0 \in \mathrm{Proj}(C)$ with $e_0 \leq e - e_1$ and $e_0 \sim f_0 \leq f - f_1$. But then $(\{e_0\} \cup \{e_i\}, \{f_0\} \cup \{f_i\})$ is an element of S properly larger than $(\{e_i\}, \{f_i\})$, contradicting maximality. It follows that $e_1 \sim f_1 = f$.

Finally, we turn to (d). By [3, Theorem 19.1] there are n orthogonal equivalent projections $f_1, \ldots, f_n$ in A with sum 1 since A has type II. Part (c) gives e_1 in $\mathrm{Proj}(C)$ with $e_1 \sim f_1$. From [3, Proposition 17.5], $1 - e_1 \sim 1 - f_1$ $(= f_2 + \cdots + f_n)$. Again from (c), there is $e_2 \leq 1 - e_1$ in C with $e_2 \sim f_2$. Continuing in this way, we find $e_1, \ldots, e_n \in \mathrm{Proj}(C)$ with $e_i \sim f_i$ and $e_1 + \cdots + e_n \sim f_1 + \cdots + f_n = 1$. Since A is finite, $e_1 + \cdots + e_n = 1$. □

Next, we turn to AW*-algebras of type I_n $(n = 1, 2, 3, \ldots)$: finite algebras of type I that have an orthogonal family $\{e_1, \ldots, e_n\}$ of equivalent abelian projections that sum to 1. Equivalently, such algebras are $*$-isomorphic to $\mathbb{M}_n(C)$ for a commutative AW*-algebra C.

Lemma 3.2. *Let A be an AW*-algebra of type I with no infinite central summand. For each positive integer n, let z_n be a central projection in A such that z_nA is of type I_n. Let C be a maximal abelian subalgebra of A.*

(a) Some nonzero subprojection of z_n in C is abelian in A.
(b) C contains an abelian projection with central cover 1.

Proof. Part (a) is proved by induction on n. If $n = 1$, then z_1 is a nonzero abelian projection in $Z(A) \subseteq C$. If $n > 1$, then z_nA is an AW*-algebra without abelian central summands, and z_nC is a maximal abelian subalgebra. From Lemma 2.4, z_nC contains a projection e_1 with $\mathrm{c}(e_1) = z_n$ and $e_1 \precsim z_n - e_1$. Now e_1Ae_1 is a type I AW*-algebra without infinite central summands by Lemma 2.7. Again, either e_1C has a nonzero abelian projection f, in which case $fAf = fe_1Ae_1f$ is abelian and f is an abelian projection in A, or there is a nonzero projection e_2 in e_1C with $e_2 \precsim e_1 - e_2$. Continuing in this way, we produce either a nonzero abelian projection in C or a set of n nonzero projection $e_1, \dots, e_n$ in z_nA with $e_{j+1} \precsim e_j - e_{j+1}$, $e_1 \precsim z_n - e_1$, and $e_{j+1} < e_j$. If $y = \mathrm{c}(e_n)$, then

$$e_n, y(e_{n-1} - e_n), y(e_{n-2} - e_{n-1}), \dots, y(e_1 - e_2), y(z_n - e_1)$$

are $n + 1$ orthogonal projections in yA with the same (nonzero) central cover, contradicting the fact that $yA = yz_nA$ is of type I_n [3, Proposition 18.2(2)]. Thus the process must end with a nonzero abelian subprojection of z_n in C before we construct e_n.

For (b), let $\{e_i\}$ be a family of nonzero projections in C abelian for A and maximal with respect to the property that $\{\mathrm{c}(e_i)\}$ is orthogonal. Set $p = \sum \mathrm{c}(e_i)$. If $p \neq 1$, then $(1-p)A$ is an AW*-algebra of type I with no infinite central summand. So [3, Theorem 18.3] implies that there is a nonzero central projection $z \leq 1-p$ and a positive integer n such that $z(1-p)A$ has type I_n. From part (a), the maximal abelian subalgebra zC of zA contains a nonzero abelian projection e_0. But then we may adjoin e_0 to $\{e_i\}$, contradicting maximality. Thus $p = 1$. Now $\sum e_i$ is abelian for A [3, Proposition 15.8], has central cover 1, and lies in C. □

Lemma 3.3. *Let e_1 be an abelian projection with $\mathrm{c}(e_1) = 1$ in an AW*-algebra A of type I_n for n finite. Then there is a set of n orthogonal equivalent projections with sum 1 in A containing e_1 (so that each is abelian in A), and $(1-e_1)A(1-e_1)$ is of type I_{n-1}.*

Proof. Fix orthogonal equivalent abelian projections $f_1, \dots, f_n$ with $\sum f_i = 1$. Then $e_1 \sim f_1$ ($\sim f_i$ for all i) by [3, Proposition 18.2(1)], and $1 - e_1 \sim 1 - f_1$ by [3, Proposition 17.5]. So $1 - e_1$ is the sum of $n - 1$ orthogonal equivalent abelian projections that are equivalent to $f_1 \sim e_1$ because the same is true for $1 - f_1 = f_2 + \cdots + f_n$. The claim now follows. □

Proposition 3.4. *Let A be an AW*-algebra of type I_n with n finite, and let C be a maximal abelian subalgebra.*

(a) There is an orthogonal set $\{e_1, \dots, e_n\}$ of equivalent abelian projections in C with sum 1 with central cover 1.

(b) C contains l orthogonal projections with sum 1 equivalent in A if $n = lm$ (with l and m positive integers).

Proof. Part (a) is proven by induction on n. If $n = 1$, then A is abelian, $C = A$, and 1 is a projection in C abelian in A with $\mathrm{c}(1) = 1$. Moreover, C is the centre of A. Suppose $n > 1$ and our assertion is established when A is of type I_k for $k < n$. Then A has no infinite central summands. Lemma 3.2(b) applies, giving an abelian projection $e_1 \in C$ with $\mathrm{c}(e_1) = 1$. It follows from Lemma 3.3 that $(1-e_1)A(1-e_1)$ is of type I_{n-1}, and $(1-e_1)C$ is a maximal abelian subalgebra. By the inductive hypothesis, $1 - e_1$ is the sum of $n - 1$ projections $e_2, \dots, e_n$ in $(1-e_1)C$ that are abelian in $(1-e_1)A(1-e_1)$ (and hence in A), and has central cover $1 - e_1$ in $(1-e_1)A(1-e_1)$. From Lemma 2.2 it follows that $1 = \mathrm{c}(e_j)$ for $j \geq 2$, and since $\mathrm{c}(e_1) = 1$ as well we must have $e_i \sim e_j$ for all j by [3, Proposition 18.1].

For (b), set $f_j = \sum_{k=0}^{m-1} e_{j+kl}$ for $j = 1, \dots, l$. Then $f_1, \dots, f_l$ are orthogonal projections in C with sum 1 equivalent in A. □

4. Dimension theory

Let e be a properly infinite projection in an AW*-algebra A. We are going to define a cardinal number $\mathrm{d}(e)$, that we think of as the "dimension" of e. The goal of this section is to prove that $e \precsim f$ and $\mathrm{c}(e) = \mathrm{c}(f)$ imply $\mathrm{d}(e) \leq \mathrm{d}(f)$. The next section will prove the converse in a special case of interest.

Let $\Gamma(e)$ denote the set of all orthogonal families $\{e_i\}$ of projections such that $e = \sum e_i$ and every $e_i \sim e$. Lemma 2.5 guarantees that $\Gamma(e)$ contains an infinite set. If Λ is a set of cardinals, we let $\sup^+ \Lambda$ denote the least cardinal that is strictly greater than every element of Λ. Evidently $\sup^+ \Lambda = \sup\{\alpha^+ \mid \alpha \in \Lambda\}$, where α^+ denotes the successor of a cardinal α.

Definition 4.1. For a properly infinite projection e in an AW*-algebra, define

$$\mathrm{d}(e) = \sup{}^+\{\operatorname{card} I \mid \{e_i\}_{i\in I} \in \Gamma(e)\},$$

$$\overline{\mathrm{d}}(e) = \sup\{\mathrm{d}(ze) \mid 0 < z \leq \mathrm{c}(e) \text{ is central}\}.$$

By convention, we agree that $\mathrm{d}(0) = \overline{\mathrm{d}}(0) = 0$. (When the algebra A in which $\mathrm{d}(e)$ and $\overline{\mathrm{d}}(e)$ are computed needs to be emphasized, we will write $\mathrm{d}_A(e)$ and $\overline{\mathrm{d}}_A(e)$.)

As a basic example, suppose that $e \in B(H)$ is a projection on a Hilbert space H whose range $e(H)$ is infinite dimensional. Then e is properly infinite and $\mathrm{d}(e)$ is the successor cardinal of the dimension of $e(H)$, that is, $\mathrm{d}(e) = (\dim e(H))^+$.

The definition of $\mathrm{d}(e)$ uses successors because it is not clear whether the supremum is achieved, *i.e.* whether there always exists a family in $\Gamma(e)$ with cardinality $\sup\{\operatorname{card} I \mid \{e_i\}_{i\in I} \in \Gamma(e)\}$. If this supremum is indeed achieved for all properly infinite projections in all AW*-algebras, then it would be more sensible to set $\mathrm{d}(e)$ equal to the supremum $\sup\{\operatorname{card}(I) \mid \{e_i\}_{i\in I} \in \Gamma(e)\}$; all results about $\mathrm{d}(e)$ proved below would still hold. The supremum is always achieved when the cardinal $\sup\{\operatorname{card} I \mid \{e_i\}_{i\in i} \in \Gamma(e)\}$ is not weakly inaccessible, and when A is a von Neumann algebra; see Appendix A. We leave the general question open, and move on to basic results about d.

Notice that if e is a properly infinite projection in an AW*-algebra A, then $\Gamma(e)$ and $\mathrm{d}(e)$ are the same whether "computed" in A or eAe. Thus d is invariant under passing to corners. Also, if $e \sim f$ in A then $\mathrm{d}(e) = \mathrm{d}(f)$.

Lemma 4.2. *Let e be a projection in an AW*-algebra A.*

(a) If $e = \sum_{i\in\alpha} e_i$ for an infinite cardinal α, with $\{e_i\}$ all nonzero and pairwise equivalent, then $e = \sum_{i\in\alpha} e'_i$ with $e'_i \sim e$. So e is properly infinite with $\alpha < \mathrm{d}(e)$.

(b) If e is properly infinite projection, and $\{e_i \mid i \in \alpha\}$ is an orthogonal set of projections with all $e_i \sim e$ for a cardinal $\alpha < \mathrm{d}(e)$, then $\sum e_i \sim e$.

Proof. For (a); since α is infinite, we have $\alpha^2 = \alpha$ (in cardinal arithmetic). So we can reindex $\{e_i \mid i \in \alpha\}$ as $\{e_{ij} \mid i,j \in \alpha\}$, and obtain $e = \sum_{i,j\in\alpha} e_{ij}$ with all e_{ij} equivalent and orthogonal. Set $e'_i = \sum_{j\in\alpha} e_{ij}$. Then each $e'_i \sim e$, and $\sum_{i\in\alpha} e'_i = e$.

We turn to (b). Because $\alpha < \mathrm{d}(e)$, there exists a set $\{f_i \mid i \in \alpha\} \in \Gamma(e)$. Then $f_i \sim e \sim e_i$ for all i, so additivity of equivalence gives $\sum e_i \sim \sum f_i = e$. □

Lemma 4.3. *If e is a properly infinite projection in an AW*-algebra A, then the set of cardinals $\{\operatorname{card} I \mid \{e_i\}_{i\in I} \in \Gamma(e)\}$ is downward-closed.*

Proof. Suppose that $\{e_i \mid i \in \beta\} \in \Gamma(e)$ for some cardinal β, and consider any cardinal $\alpha \leq \beta$. We will construct a set in $\Gamma(e)$ of cardinality α. Write $\beta = \bigsqcup_{j\in\alpha} \beta_j$ as a disjoint union of α-many subsets β_j which each have cardinality β (this is possible because $\alpha \cdot \beta = \beta$ in cardinal arithmetic). For each $j \in \alpha$, let $f_j = \sum_{i\in\beta_j} e_i$. By additivity of equivalence, $f_j \sim \sum_{i\in\beta} e_i = e$. Thus $\{f_j \mid j \in \alpha\} \in \Gamma(e)$. □

Lemma 4.4. *Let e be a projection in an AW*-algebra A.*

(a) If e is properly infinite, then $\mathrm{d}(e) \leq \mathrm{d}(ze)$ for central projections $0 < z \leq \mathrm{c}(e)$.
(b) If $e = \sum e_i$ for an orthogonal set $\{e_i\}$ of properly infinite projections, then e is properly infinite and $\mathrm{d}(e) \geq \min\{\mathrm{d}(e_i)\}$.
(c) If e is properly infinite and $\mathrm{c}(e) = \sum z_i$ for nonzero central projections z_i, then $\mathrm{d}(e) = \min\{\mathrm{d}(z_i e)\}$.
(d) If e is properly infinite, then $\overline{\mathrm{d}}(ze) \leq \overline{\mathrm{d}}(e)$ for any nonzero central projection z.

Proof. Part (a) follows from the observation that if $\{e_i\} \in \Gamma(e)$, then $\{ze_i\} \in \Gamma(ze)$ for any nonzero central projection $z \leq \mathrm{c}(e)$.

For (b), fix an infinite cardinal $\alpha < \min\{\mathrm{d}(e_i)\}$; then for each i there exists $\{e_{ij} \mid j \in \alpha\} \in \Gamma(e_i)$. For each $j \in \alpha$, define $e_j = \sum_i e_{ij}$. By additivity of equivalence, $e_j \sim \sum_i e_i = e$ for all j. Because $\sum e_j = e$, we find that e is properly infinite with $\{e_j \mid j \in \alpha\} \in \Gamma(e)$ and thus $\alpha < \mathrm{d}(e)$. This demonstrates that $\mathrm{d}(e) \geq \min\{\mathrm{d}(e_i)\}$.

Part (c) follows from (a) and (b). Part (d) follows by verifying the equations

$$\overline{\mathrm{d}}(e) = \sup\{\mathrm{d}(ye) \mid 0 < y \leq \mathrm{c}(e) \text{ is central}\},$$

$$\overline{\mathrm{d}}(ze) = \sup\{\mathrm{d}(ye) \mid 0 < y \leq \mathrm{c}(ze) = z\mathrm{c}(e) \text{ is central}\},$$

and noticing that the latter set over which the sup is quantified is a subset of the former. □

Theorem 4.6 below partly justifies the intuition that $\mathrm{d}(e)$ measures a "dimension" of e. If $e \leq f$ are properly infinite projections, then one might expect to have $\mathrm{d}(e) \leq \mathrm{d}(f)$; this is true under the additional hypothesis that e and f have the same central cover. The proof requires transfinite repetition of the following construction.

Lemma 4.5. *Let $p \leq q$ be projections in an AW*-algebra A, and suppose that p is properly infinite. There exist projections $p' \in A$ and central $z \leq \mathrm{c}(p)$ satisfying:*

- $zp \sim zq$;
- $(\mathrm{c}(p) - z)p \sim p' \leq (\mathrm{c}(p) - z)(q - p)$ *(in particular, $pp' = 0$ and $p' \leq q$);*
- $\mathrm{c}(p') = \mathrm{c}(p) - z$.

Proof. We may pass to the summand $\mathrm{c}(p)A$ and assume that $\mathrm{c}(p) = 1$. By generalized comparability, there exists a central projection z such that $z(q-p) \precsim zp$ and $(1-z)p \precsim (1-z)(q-p)$. Because p is properly infinite, we may write $p = p_1 + p_2$ for some projections $p_1 \sim p_2 \sim p$. Then $z(q-p) \precsim zp \sim zp_1$ and $zp \sim zp_2$. It follows that

$$zp \leq zq = z(q-p) + zp \precsim zp_1 + zp_2 = zp,$$

whence $zp \sim zq$. Because $(1-z)p \precsim (1-z)(q-p)$, there exists a projection $p' \leq (1-z)(q-p) \leq q - p$ such that $(1-z)p \sim p'$. Also, $p' \sim (1-z)p$ means that $\mathrm{c}(p') = \mathrm{c}((1-z)p) = (1-z)\mathrm{c}(p) = \mathrm{c}(p) - z$. □

The proof below will regard the cardinal $\mathrm{d}(e)$ as an initial ordinal: the smallest ordinal in its cardinality class.

Theorem 4.6. *Let e and f be properly infinite projections in an AW*-algebra A. If $\mathrm{c}(e) = \mathrm{c}(f)$ and $e \precsim f$, then $\mathrm{d}(e) \leq \mathrm{d}(f)$.*

Proof. Passing to the summand $\mathrm{c}(e)A$ and replacing e with an equivalent projection $e' \leq f$, we may assume that $\mathrm{c}(e) = 1 = \mathrm{c}(f)$ and $e \leq f$. We will build projections z_α and e_α for ordinals α with the following properties:

(a) $\{z_\alpha\}$ are central and orthogonal (and possibly zero);
(b) $\mathrm{c}(e_\alpha) = 1 - \sum_{\beta \leq \alpha} z_\beta$ (so if $e_\alpha = 0$ then $1 = \sum_{\beta \leq \alpha} z_\beta$);
(c) $\{e_\alpha\}$ are orthogonal projections below f;
(d) if $z_\alpha > 0$ then $\mathrm{d}(z_\alpha f) \geq \mathrm{d}(e)$;
(e) if $e_\alpha > 0$ then it is properly infinite and $\mathrm{d}(e_\alpha) \geq \mathrm{d}(e)$;

such that the process terminates exactly when $\sum z_\alpha = 1$. Notice that if $\sum e_\alpha = f$, then we must have $e_{\alpha+1} = 0$, so that the process terminates then by condition (b).

For $\alpha = 0$, set $z_0 = 0$ and $e_0 = e$. Now, suppose that z_β and e_β have already been constructed for all $\beta < \alpha$, and that $\sum_{\alpha<\beta} z_\beta < 1$. Notice by (b) that the $\mathrm{c}(e_\beta)$ form a decreasing chain and that

$$y := \bigwedge_{\beta<\alpha} \mathrm{c}(e_\beta) = 1 - \sum_{\beta<\alpha} z_\beta.$$

We are assuming that this central projection y is nonzero. For each β, condition (e) and $0 < y \leq \mathrm{c}(e_\beta)$ imply that ye_β is properly infinite with $\mathrm{d}(ye_\beta) \geq \mathrm{d}(e_\beta) \geq \mathrm{d}(e)$. Set

$$p = y \sum_{\beta<\alpha} e_\beta = \sum_{\beta<\alpha} ye_\beta.$$

By Lemma 4.4(b), p is properly infinite and $\mathrm{d}(p) \geq \min\{\mathrm{d}(ye_\beta) \mid \beta < \alpha\} \geq \mathrm{d}(e)$. Furthermore, $y \leq \mathrm{c}(e_\beta)$ for $\beta < \alpha$ by construction, so that $\mathrm{c}(p) = \bigvee \mathrm{c}(ye_\beta) = y$. Applying Lemma 4.5 to p and $q = f$ now gives projections $z_\alpha = z$ and $e_\alpha = p'$ with the following properties.

(a) By construction, z_α is central with $z_\alpha \leq \mathrm{c}(p) = 1 - \sum_{\beta<\alpha} z_\beta$. Therefore $z_\alpha \perp z_\beta$ for all $\beta < \alpha$, and $\{z_\beta \mid \beta \leq \alpha\}$ is orthogonal.
(b) We have $\mathrm{c}(e_\alpha) = \mathrm{c}(p) - z_\alpha = (1 - \sum_{\beta<\alpha} z_\beta) - z_\alpha = 1 - \sum_{\beta\leq\alpha} z_\beta$.
(c) Directly from Lemma 4.5 we have $e_\alpha \leq f - p$. So $e_\alpha \leq f$ and $e_\alpha \perp p$, which implies $e_\alpha \perp ye_\beta$ for all $\beta < \alpha$. Because $\mathrm{c}(e_\alpha) \leq \mathrm{c}(p) = y$, this means that $e_\alpha \perp e_\beta$ for all $\beta < \alpha$. Hence $\{e_\beta \mid \beta \leq \alpha\}$ is orthogonal.
(d) Next, z_α is chosen so that $z_\alpha p \sim z_\alpha f$. Combined with $z_\alpha \leq \mathrm{c}(p)$, we see that if $z_\alpha \neq 0$ then $z_\alpha f \sim z_\alpha p$ is properly infinite and $\mathrm{d}(z_\alpha f) = \mathrm{d}(z_\alpha p) \geq \mathrm{d}(p) \geq \mathrm{d}(e)$.
(e) Finally, assume $e_\alpha > 0$. The construction of e_α guarantees that $e_\alpha \sim \mathrm{c}(e_\alpha)p$. Since $0 < \mathrm{c}(e_\alpha) \leq \mathrm{c}(p)$, this means that e_α is properly infinite and we have $\mathrm{d}(e_\alpha) = \mathrm{d}(\mathrm{c}(e_\alpha)p) \geq \mathrm{d}(p) \geq \mathrm{d}(e)$.

Transfinite induction now gives us the desired projections $\{z_\alpha\}$, $\{e_\alpha\}$.

If there is a step α in the construction above for which $\sum_{\beta\leq\alpha} z_\beta = 1$, then by condition (d) we have that $f = \sum z_\beta f$ is a sum of properly infinite projections with $\mathrm{d}(z_\beta f) \geq \mathrm{d}(e)$ (ignoring those z_β which are zero). In this case, we conclude from Lemma 4.4(b) that $\mathrm{d}(f) \geq \mathrm{d}(e)$.

Finally, suppose that $\sum_{\beta\leq\alpha} z_\beta < 1$ at every step α. Then condition (b) guarantees that each e_α is nonzero. So the projections $\sum_{\beta\leq\alpha} e_\beta$ form a strictly increasing sequence below f. This chain cannot increase without bound (for instance, it is bounded by $\mathrm{card}(\mathrm{Proj}(fAf))^+$), so there exists α such that $\sum_{\beta\leq\alpha} e_\beta = f$. From condition (e) and Lemma 4.4(b) we once again conclude that $\mathrm{d}(f) \geq \mathrm{d}(e)$. $\square$

As an easy consequence, we see that $\overline{\mathrm{d}}$ behaves in the same way.

5. Equidimensional projections

The hypothesis in Theorem 4.6 that the projections e and f satisfy $\mathrm{c}(e) = \mathrm{c}(f)$ cannot be removed. For instance, let H and K be infinite dimensional Hilbert spaces with $\dim(H) < \dim(K)$ and consider the AW*-algebra $A = B(H) \oplus B(K)$. One can readily compute that $\mathrm{d}((1_H, 1_K)) = \dim(H)^+$ and $\mathrm{d}((0, 1_K)) = \dim(K)^+$. Thus $(0,1) < (1,1)$ but $\mathrm{d}((0,1)) > \mathrm{d}((1,1))$. The issue is that images of the projection $(1,1)$ in the two central summands $(1,0)A$ and $(0,1)A$ have different dimensions.

It will prove fruitful to focus on so-called equidimensional projections: those projections for which the above pathology does not occur. We will show that such projections are equivalent precisely when they have the same central cover and dimension. Moreover, we will prove that any properly infinite projection is a sum of equidimensional ones.

Definition 5.1. A properly infinite projection e in an AW*-algebra A will be called *equidimensional* if $\mathrm{d}(ze) = \mathrm{d}(e)$ for every nonzero central projection $z \leq \mathrm{c}(e)$. The AW*-algebra A is called *equidimensional* when 1_A is equidimensional. We say that e is *α-equidimensional* for a cardinal α if e is equidimensional with $\mathrm{d}(e) = \alpha$. By convention, we will also agree that $0 \in A$ is a 0-equidimensional projection.

It is straightforward to see that a properly infinite projection e in an AW*-algebra A is equidimensional if and only if there exists an infinite cardinal α such that, for every central projection $z \in A$, either $ze = 0$ or $\mathrm{d}(ze) = \alpha$.

Lemma 5.2. *Let e be a properly infinite projection in an AW*-algebra A.*
(a) eAe is equidimensional if and only if e is equidimensional.
(b) If e is equidimensional and $e \sim f$, then f is equidimensional.
Hence eAe is equidimensional when A is equidimensional and $e \sim 1$.

Proof. For (a), let e be equidimensional and let z central in eAe. Then $z = he$ for some central projection h of A by Lemma 2.2. Since e is equidimensional in A,

$$\mathrm{d}_{eAe}(z) = \mathrm{d}_{eAe}(he) = \mathrm{d}_A(he) = \mathrm{d}_A(e) = \mathrm{d}_{eAe}(e).$$

Conversely, assume that eAe is equidimensional, and let $h \in A$ be a central projection. Then he is central in eAe, so

$$\mathrm{d}_A(he) = \mathrm{d}_{eAe}(he) = \mathrm{d}_{eAe}(e) = \mathrm{d}_A(e).$$

For (b), notice that whenever $z \in A$ is a central projection, $ze \sim zf$ and thus $\mathrm{d}(ze) = \mathrm{d}(zf)$. The statement clearly follows. □

The following theorem establishes another desired property of a "dimension" measure. If the dimension of a projection e is strictly less than the dimension of a projection f, intuition developed in $B(H)$ might lead one to expect that $e \prec f$. The example $A = B(H) \oplus B(K)$ with $\dim(H) < \dim(K)$ infinite again shows that this cannot hold in full generality: fixing any orthogonal projection of K onto a subspace of dimension $\dim(H)$, we have $(1,p) < (1,1)$ and even $\mathrm{c}((1,p)) = \mathrm{c}((1,1))$, but $\mathrm{d}((1,p)) = \mathrm{d}((1,1))$. As mentioned above, the key assumption that both e and f be equidimensional makes the intuitive idea true. The proof below basically uses the same transfinite construction as the proof of Theorem 4.6, but with different termination conditions. For the sake of readability, we write it out in full.

Theorem 5.3. *Let e and f be properly infinite, equidimensional projections in an AW*-algebra. If $\mathrm{c}(e) = \mathrm{c}(f)$ and $e \prec f$, then $\mathrm{d}(e) < \mathrm{d}(f)$.*

Proof. Passing to the summand $\mathrm{c}(e)A$ and replacing e with an equivalent projection below f, we may assume that $\mathrm{c}(e) = 1 = \mathrm{c}(f)$, $e \leq f$, and $e \not\sim f$. We will build projections z_α and e_α for ordinals $\alpha < \mathrm{d}(e)$ (regarding $\mathrm{d}(e)$ as an initial ordinal) with the following properties:

(a) $\{z_\alpha\}$ are central and orthogonal (and possibly zero);
(b) $\mathrm{c}(e_\alpha) = 1 - \sum_{\beta \leq \alpha} z_\beta$;
(c) $\{e_\alpha\}$ are orthogonal projections below f;
(d) $e_\alpha \sim \mathrm{c}(e_\alpha)e$;
(e) $z_\alpha e \sim z_\alpha f$.

For $\alpha = 0$, set $z_0 = 0$ and $e_0 = e$. Now, suppose that z_β and e_β have already been constructed for all $\beta < \alpha$. Notice by (b) that the $\mathrm{c}(e_\beta)$ form a decreasing sequence and that

$$y := \bigwedge_{\beta<\alpha} \mathrm{c}(e_\beta) = 1 - \sum_{\beta<\alpha} z_\beta.$$

Condition (e) and $e \not\sim f$ guarantee that this central projection y is nonzero. For each β, condition (d) together with $y \leq \mathrm{c}(e_\beta)$ give $ye_\beta \sim ye$; notice $ye \neq 0$ because $\mathrm{c}(e) = 1$. Set

$$p = y \sum_{\beta<\alpha} e_\beta = \sum_{\beta<\alpha} ye_\beta.$$

Because $\operatorname{card} \alpha < \mathrm{d}(e)$ (as α is strictly below the initial ordinal $\mathrm{d}(e)$), Lemma 4.2(b) implies that $p \sim ye$. So p is properly infinite and $\mathrm{c}(p) = \mathrm{c}(ye) = y = 1 - \sum_{\beta<\alpha} z_\beta$. Applying Lemma 4.5 to p and $q = f$ now gives projections $z_\alpha = z$ and $e_\alpha = p'$ with the following properties.

(a)–(c) These follow just as conditions (a)–(c) in the proof of Theorem 4.6.
(d) Next, the construction of e_α along with $\mathrm{c}(e_\alpha) \leq \mathrm{c}(p) = y$ and $p \sim ye$ shows that $e_\alpha \sim \mathrm{c}(e_\alpha)p \sim \mathrm{c}(e_\alpha)e$.
(e) Finally, z_α is chosen so that $z_\alpha p \sim z_\alpha f$. Combined with $z_\alpha \leq y = \mathrm{c}(p)$ and $p \sim ye$, we have $z_\alpha f \sim z_\alpha p \sim z_\alpha e$.

Transfinite induction now gives us the desired projections z_α, e_α for $\alpha < \mathrm{d}(e)$.

Set $z = \sum_{\alpha<\mathrm{d}(e)} z_\alpha$, so that $zf \sim ze$ by (e) and $1 - z = \bigwedge_{\alpha<\mathrm{d}(e)} \mathrm{c}(e_\alpha)$ by (b). Since $e \not\sim f$, we must have $1 - z > 0$. Also (d) implies that $(1-z)e_\alpha \sim (1-z)e > 0$ for all $\alpha < \mathrm{d}(e)$. Furthermore, as each $\mathrm{c}((1-z)e_\alpha) = (1-z) = \mathrm{c}((1-z)f)$, we have $\mathrm{c}(\sum(1-z)e_\alpha) = 1 - z = \mathrm{c}((1-z)f)$. Therefore

$$\begin{aligned} \mathrm{d}(e) &< \mathrm{d}\left(\sum\nolimits_{\alpha<\mathrm{d}(e)} (1-z)e_\alpha\right) && \text{(by Lemma 4.2(a))} \\ &\leq \mathrm{d}((1-z)f) && \text{(by Theorem 4.6)} \\ &= \mathrm{d}(f), && (f \text{ is equidimensional}) \end{aligned}$$

as desired. □

Corollary 5.4. *Let e and f be properly infinite, equidimensional projections in an AW*-algebra. Then $e \precsim f$ if and only if $\mathrm{c}(e) \leq \mathrm{c}(f)$ and $\mathrm{d}(e) \leq \mathrm{d}(f)$. Therefore $e \sim f$ if and only if $\mathrm{c}(e) = \mathrm{c}(f)$ and $\mathrm{d}(e) = \mathrm{d}(f)$.*

Proof. If $\mathrm{c}(e) \leq \mathrm{c}(f)$ then $\mathrm{c}(e)f$ is equidimensional and $\mathrm{d}(\mathrm{c}(e)f) = \mathrm{d}(f)$. Replacing f by $\mathrm{c}(e)f$, we may assume $\mathrm{c}(e) = \mathrm{c}(f)$ and prove that $e \precsim f$ if and only if $\mathrm{d}(e) \leq \mathrm{d}(f)$.

One direction is just Theorem 4.6. For the other, suppose that $\mathrm{d}(e) \leq \mathrm{d}(f)$. The comparison theorem gives us a central projection z satisfying $ze \precsim zf$, and $(1-z)e \succ (1-z)f$, and $1-z \leq \mathrm{c}(e) = \mathrm{c}(f)$. If $z < 1$, then $(1-z)e$ and $(1-z)f$ are nonzero and properly infinite, so by equidimensionality and Theorem 5.3 we have $\mathrm{d}(e) = \mathrm{d}((1-z)e) > \mathrm{d}((1-z)f) = \mathrm{d}(f)$, which contradicts the assumption $\mathrm{d}(e) \leq \mathrm{d}(f)$. Thus $z = 1$ and $e \precsim f$. □

In order to make use of Corollary 5.4 in an arbitrary AW*-algebra, there must be a rich supply of equidimensional projections. This will be demonstrated in Theorem 5.6, after the following preparatory lemma.

Lemma 5.5. *Let e be a properly infinite projection in an AW*-algebra. Then:*

(a) e is equidimensional if $\overline{\mathrm{d}}(ze) = \overline{\mathrm{d}}(e)$ for all central projections $0 < z \leq \mathrm{c}(e)$;

(b) there exists a nonzero central projection $z \leq \mathrm{c}(e)$ making ze equidimensional.

Proof. To prove (a), suppose towards a contradiction that e is not equidimensional. Then $\mathrm{d}(e) < \mathrm{d}(z_0 e)$ for some nonzero central $z_0 \leq \mathrm{c}(e)$ by Lemma 4.4(a). Zorn's lemma allows us to extend $\{z_0\}$ to a maximal set $\{z_i\}$ of orthogonal nonzero projections such that $\mathrm{d}(e) < \mathrm{d}(z_i e)$. Because $\mathrm{d}(e) < \min\{\mathrm{d}(z_i e)\}$, it follows from Lemma 4.4(c) that $z = \mathrm{c}(e) - \sum z_i$ is nonzero. Applying that same lemma to the set of projections $\{z_i\} \cup \{z\}$ with sum $\mathrm{c}(e)$, we must have $\mathrm{d}(e) = \mathrm{d}(ze)$. Using the hypothesis,

$$\overline{\mathrm{d}}(ze) = \overline{\mathrm{d}}(e) = \overline{\mathrm{d}}(z_0 e).$$

Since $\mathrm{d}(e) < \mathrm{d}(z_0 e) \leq \overline{\mathrm{d}}(z_0 e) = \overline{\mathrm{d}}(ze)$, by definition of $\overline{\mathrm{d}}(ze)$ there is a nonzero central projection $y \leq \mathrm{c}(ze) = z$ such that $\mathrm{d}(ye) = \mathrm{d}(yze) > \mathrm{d}(e)$. But this contradicts the maximality of $\{z_i\}$. We conclude that e must be equidimensional.

As for (b): by well-ordering, there is a nonzero central projection $z \leq \mathrm{c}(e)$ minimizing $\overline{\mathrm{d}}(ze)$. Let $y \leq \mathrm{c}(ze) = z$ be a nonzero central projection. Then it follows from Lemma 4.4(d) that $\overline{\mathrm{d}}(ye) \leq \overline{\mathrm{d}}(ze)$. Therefore $\overline{\mathrm{d}}(ye) = \overline{\mathrm{d}}(ze)$ by minimality of $\overline{\mathrm{d}}(ze)$. Hence ze is equidimensional by (a). □

Theorem 5.6. *Let e be a properly infinite projection in an AW*-algebra A.*

(a) Each infinite cardinal $\alpha \leq \overline{\mathrm{d}}(e)$ allows a largest central projection $z_\alpha \leq \mathrm{c}(e)$ such that $z_\alpha e$ is α-equidimensional. These projections are orthogonal for distinct α.

(b) Letting α range as above, we have $\mathrm{c}(e) = \sum z_\alpha$.

Thus $e = \sum z_\alpha e$ is a sum of equidimensional projections.

Proof. Fix α as in part (a). Zorn's lemma produces a maximal orthogonal family $\{z_i\}$ of nonzero central projections $z_i \leq \mathrm{c}(e)$ where each $z_i e$ is α-equidimensional. Set $z_\alpha = \sum z_i$. It is straightforward to verify that $z_\alpha e = \sum z_i e$ is α-equidimensional using Lemma 4.4(c). Furthermore, if $z \leq \mathrm{c}(e)$ is central and ze is α-equidimensional, then the projection $z(\mathrm{c}(e) - z_\alpha)$ is central and orthogonal to all $\{z_i\}$. If it is nonzero then $z(\mathrm{c}(e) - z_\alpha)e$ is α-equidimensional. Maximality of $\{z_i\}$ thus requires $z(\mathrm{c}(e) - z_\alpha) = 0$, or $z \leq z_\alpha$. For cardinals α and β, if $z_\alpha z_\beta$ is nonzero then $\alpha = \mathrm{d}(z_\alpha z_\beta e) = \beta$. Thus $\alpha \neq \beta$ implies $z_\alpha z_\beta = 0$.

For (b), assume for contradiction that $y = \mathrm{c}(e) - \sum z_\alpha > 0$. Then $0 < ye \leq e$ with $\mathrm{c}(ye) = y$. Lemma 5.5(b) provides a nonzero projection $z \leq y$ such that ze is equidimensional, say with $\mathrm{d}(ze) = \beta$. But then $z \leq z_\beta \leq \sum z_\alpha$, contradicting that $0 < z \leq y = \mathrm{c}(e) - \sum z_\alpha$. So we must have $\mathrm{c}(e) = \sum z_\alpha$. □

We conclude this section by bringing our treatment more in line with the notions of dimension in the literature [7, 17, 9]. Such notions are traditionally defined in terms of $\mathrm{Spec}(Z(A))$, the Gelfand spectrum of the centre of an AW*-algebra A, rather than using central projections. We write φ for the canonical *-isomorphism from $Z(A)$ to the algebra of continuous complex-valued functions on $\mathrm{Spec}(Z(A))$. Given a properly infinite projection $e \in A$, we define a function D_e from $\mathrm{Spec}(Z(A))$ to the cardinals as follows. Let $\{z_\alpha\}$ be the family provided by the previous theorem, with the addition of $z_0 = 1 - \mathrm{c}(e)$; then $\mathrm{supp}(\varphi(z_\alpha))$ are disjoint clopens that cover $\mathrm{Spec}(Z(A))$ since $1 = \sum z_\alpha$. Therefore the function from $\bigsqcup_\alpha \mathrm{supp}(\varphi(z_\alpha))$ to the cardinals, mapping $\mathrm{supp}(\varphi(z_\alpha))$ to α, is continuous when we put the order topology on the cardinals [4, Section X.9]. Because $\sum z_\alpha = \mathrm{c}(e)$, this function is defined on a dense subset, and hence extends to a continuous function D_e from all of $\mathrm{Spec}(Z(A))$ to the cardinals [17, Lemma 5]. It follows from the previous theorem that

$$\mathrm{D}_e(t) = \mathrm{d}(ze) \qquad \text{for } t \in \mathrm{supp}(\varphi(z))$$

if ze is equidimensional. Write $\mathrm{D}_e \leq \mathrm{D}_f$ to mean that $\mathrm{D}_e(t) \leq \mathrm{D}_f(t)$ for all t. We show that properly infinite projections $e, f \in A$ are comparable if and only if the functions D_e and D_f are. Using the methods developed above, we reduce the problem to a test of the dimension of equidimensional summands.

Proposition 5.7. *For properly infinite projections e and f in an AW*-algebra A, $e \precsim f$ if and only if $\mathrm{D}_e \leq \mathrm{D}_f$.*

Proof. First, we claim that $\mathrm{D}_e \leq \mathrm{D}_f$ if and only if this holds on a dense subset. An inequality $\alpha \leq \beta$ of cardinals holds precisely when the equality $\beta = \max\{\alpha, \beta\} = \alpha\beta$ holds. Recall that a net $\{\beta_i\}$ converges to β in the order topology when there are a net $\{\alpha_i\}$ increasing to β and a net $\{\gamma_i\}$ decreasing to β such that $\alpha_i \leq \beta_i \leq \gamma_i$. It clearly makes cardinal multiplication continuous, and the claim follows.

Let $1 = \sum x_\alpha = \sum y_\beta$ be central decompositions as in the above discussion, so that $x_\alpha e$ is α-equidimensional and $y_\beta f$ is β-equidimensional. Since $1 = \sum x_\alpha y_\beta$ as well, $e \precsim f$ if and only if $x_\alpha y_\beta e \precsim x_\alpha y_\beta f$ for all α and β. Furthermore, the subsets $K_\alpha = \mathrm{supp}(\varphi(x_\alpha))$ and $L_\beta = \mathrm{supp}(\varphi(y_\beta))$ are (cl)open in $X = \mathrm{Spec}(Z(A))$, and $\bigcup K_\alpha$, $\bigcup L_\beta$ are open and dense in X, so $\bigcup(K_\alpha \cap L_\beta) = (\bigcup K_\alpha) \cap (\bigcup L_\beta)$ is again dense in X.

Thus it suffices to show that $x_\alpha y_\beta e \precsim x_\alpha y_\beta f$ if and only if $\mathrm{D}_e(t) \leq \mathrm{D}_f(t)$ for all $t \in K_\alpha \cap L_\beta$. Notice that $K_\alpha \cap L_\beta = \mathrm{supp}(\varphi(x_\alpha y_\beta))$. We may restrict to the case where $x_\alpha y_\beta > 0$, whence $K_\alpha \cap L_\beta \neq \emptyset$. In this case, $x_\alpha e$ is α-equidimensional and $y_\beta f$ is β-equidimensional. Furthermore, if $t \in K_\alpha \cap L_\beta$, then $\mathrm{D}_e(t) = \mathrm{d}(x_\alpha y_\beta e) = \alpha$ and $\mathrm{D}_f(t) = \mathrm{d}(x_\alpha y_\beta f) = \beta$. So by Corollary 5.4, $x_\alpha y_\beta e \precsim x_\alpha y_\beta f$ if and only if $\alpha \leq \beta$, if and only if $\mathrm{D}_e(t) \leq \mathrm{D}_f(t)$ for all $t \in K_\alpha \cap L_\beta$. $\square$

Using the known dimension theory of finite projections in AW*-algebras [3, Chapter 6] and Lemma 2.6(b), the definition of D_e can be extended to arbitrary projections e, still satisfying the property of the previous corollary, as in [17, 9].

6. Relative comparison for AW*-algebras of infinite type

Using the results about equidimensional projections, this section carries out the relative comparison theory for a maximal abelian subalgebra C of a properly infinite AW*-algebra A, as Section 3 did for finite algebras. Once again, whenever we mention without specification concepts such as $\sim$, d, finite, infinite, abelian, equidimensional, or central cover, we mean the corresponding concepts in A (and not in C). The results below are inspired by Kadison's [12], but are suitably adapted for algebras that need not be countably decomposable. Throughout this section, we will freely and repeatedly apply Lemmas 2.1(c) and 2.6.

We start by considering types II_∞ and III. The key application of the dimension theory developed in the previous two sections occurs in the proof of the following.

Proposition 6.1. *Let A be an AW*-algebra with a maximal abelian subalgebra C all of whose nonzero projections are properly infinite.*

(a) If A is nonzero, there are projections $0 < e \leq p$ in C satisfying $e \sim p \sim p - e$.
(b) There is a projection e in C with $e \sim 1 \sim 1 - e$.

Proof. For (a), choose a nonzero $p \in \mathrm{Proj}(C)$ such that $\overline{\mathrm{d}}_A(p) \leq \overline{\mathrm{d}}_A(f)$ for all $f \in \mathrm{Proj}(C)$; this can be done by well-ordering. Because d is invariant under passing to corners, $\overline{\mathrm{d}}_{pAp}(p)$ is also minimal, allowing us to drop the subscript. It follows from minimality of $\overline{\mathrm{d}}(p)$ and Lemma 4.4(d) that $\overline{\mathrm{d}}(p) = \overline{\mathrm{d}}(zp)$ for all nonzero central projections $z \leq \mathrm{c}(p)$. Hence Lemma 5.5(a) guarantees that p is equidimensional. Next, Lemma 2.4 provides a projection e in pC with $\mathrm{c}_{pAp}(e) = p = \mathrm{c}_{pAp}(p - e)$. In particular, e, p, and $p - e$ have the same central cover in pAp, and hence by Lemma 2.2 also in A. If $z \leq \mathrm{c}_{pAp}(e)$ is a nonzero projection in $Z(pAp) = pZ(A)$, then

$$\overline{\mathrm{d}}(ze) \leq \overline{\mathrm{d}}(e) \leq \overline{\mathrm{d}}(p) \leq \overline{\mathrm{d}}(ze)$$

by, respectively, Lemma 4.4(d), Theorem 4.6, and minimality of $\overline{\mathrm{d}}(p)$. The same inequalities with e replaced by $p - e$ hold, so $\overline{\mathrm{d}}(e) = \overline{\mathrm{d}}(p) = \overline{\mathrm{d}}(p - e)$, and e and $p - e$ are equidimensional. Thus $\mathrm{d}(e) = \mathrm{d}(p) = \mathrm{d}(p - e)$. Now e, p, and $p - e$ are equivalent by Corollary 5.4.

Proceeding to (b), Zorn's lemma produces a maximal set $\{p_i\}$ of orthogonal nonzero projections in C such that there exist projections $\{e_i\} \subseteq C$ with $e_i \leq p_i$ and $e_i \sim p_i \sim p_i - e_i$ for all i. Assume, towards a contradiction, that $\sum p_i \neq 1$; then $s = 1 - \sum p_i \in C$ is nonzero. By assumption, s is properly infinite. Projections in sCs are properly infinite in sAs by Lemma 2.6. So part (a) applies to sAs and its maximal abelian subalgebra sC, giving nonzero projections $e \leq p$ in sC with $e \sim p \sim p - e$. Thus we may enlarge $\{p_i\}$ with p, contradicting maximality.

Hence $\sum p_i = 1$. Define $e = \sum e_i$, so that $1 - e = \sum (p_i - e_i)$. Then $e \in C$, and additivity of equivalence provides $e \sim 1 - e \sim \sum p_i = 1$ as desired. □

Lemma 6.2. *If C is a maximal abelian subalgebra in an AW*-algebra A, and*

$$e = \bigvee \{f \in \mathrm{Proj}(C) \mid f \text{ is finite (in } A)\},$$

then projections in $(1 - e)C$ are properly infinite. If $\{f_i\}$ is a maximal orthogonal family of nonzero finite projections in C, then $e = \sum f_i$.

Proof. Let $p \in (1-e)C$. Then $p \in \mathrm{Proj}(C)$ with $p \perp e$. So for any central projection $z \in A$ such that $zp > 0$, also $C \ni zp \perp e$, making zp infinite by choice of e. Hence p is properly infinite by Lemma 2.5.

Let $\{f_i\}$ be a maximal orthogonal family of nonzero projections in C that are finite; such a family exists by Zorn's lemma. Clearly $\sum f_i \leq e$. If $f \in C$ is any finite projection, then $f(1 - \sum f_i)$ is both finite and orthogonal to each f_i. By maximality, this product is zero, so $f \leq \sum f_i$. By definition of e, this means $e \leq \sum f_i$. □

Lemma 6.3. *Let A be an AW*-algebra and $e, f \in \mathrm{Proj}(A)$.*

(a) If $p, q \in \mathrm{Proj}(A)$ satisfy $e \precsim p \perp q \succsim f$, then $e \vee f \precsim p + q$.
(b) If e is properly infinite and $f \precsim e$, then $e \sim e \vee f$.
(c) If e is properly infinite, f is finite, and $\mathrm{c}(f) \leq \mathrm{c}(e)$, then $e \vee f \sim e$.

Proof. By [3, Theorem 13.1], $(e \vee f) - f \sim e - e \wedge f \precsim p$. Since $((e \vee f) - f)f = 0$ and $f \precsim q$, we have $e \vee f = (e \vee f) - f + f \precsim p + q$, establishing (a).

We turn to (b). As e is properly infinite, there is a projection $g \in A$ with $g < e$ and $e \sim g \sim e - g$. Then $f \precsim e \sim e - g$. Part (a) implies that $e \vee f \precsim g + e - g = e$. Since $e \leq e \vee f$, we have $e \sim e \vee f$ from Schröder-Bernstein.

Toward (c), assume for contradiction that $f \not\precsim e$. The comparison theorem now gives a nonzero central projection $z \leq \mathrm{c}(f)$ with $ze \prec zf$. Because $z \leq \mathrm{c}(f) \leq \mathrm{c}(e)$, it follows from Lemma 2.5 that ze is (properly) infinite. This contradicts finiteness of zf, so $f \precsim e$. From part (b) we conclude that $e \vee f \sim e$. □

Proposition 6.4. *Let A be a properly infinite AW*-algebra without central summands of type I, and let C be a maximal abelian subalgebra of A. Then there exists a projection $e \in C$ such that $e \sim 1 \sim 1 - e$.*

Proof. First we will produce $e \in C$ such that $e \sim 1 - e$. Use Zorn's lemma to produce a maximal family $\{f_i\}$ of projections in C that are finite in A, and set $f = \sum f_i$. By Proposition 3.1, there exist projections $e_{i1} \sim e_{i2}$ for all i such that $f_i = e_{i1} + e_{i2}$. By Lemma 6.2, the projections of $(1 - f)C$ are properly infinite. As $(1 - f)C$ is a maximal abelian subalgebra of $(1 - f)A(1 - f)$, Proposition 6.1 provides projections $e'_1 \sim e'_2$ such that $(1 - f) = e'_1 + e'_2$. Thus for $j = 1, 2$, the projections $e_j = e'_j + \sum_i e_{ij}$ satisfy $e_1 \sim e_2$ and $1 = e_1 + e_2$. So we may take $e = e_1$.

It remains to show that $e_1 \sim 1 \sim e_2$. Let z be any central projection of A such that ze_1 is finite; then $ze_2 \sim ze_1$ is finite, so that $z = ze_1 + ze_2$ is finite. But A is properly infinite, so z must be zero. Thus e_1 is properly infinite by Lemma 2.5(d). Since $e_2 \sim e_1$, it follows from Lemma 6.3(b) that $e_1 \sim e_1 + e_2 = 1$, so that $e_2 \sim e_1 \sim 1$ as desired. □

Next, we turn to AW*-algebras of type I_∞.

Lemma 6.5. *Let A be a nonzero AW*-algebra of type I_∞, and let C be a maximal abelian subalgebra in which 1 is the supremum of projections in C finite in A.*

(a) C has a projection finite in A with central cover 1.
(b) C has a projection abelian in A with central cover 1.
(c) Some nonzero central projection $z \in A$ is the sum of infinitely many orthogonal equivalent projections in C.
(d) There is a projection $e \in C$ such that $e \sim 1 \sim 1 - e$.

Proof. For (a), let $\{f_j\}$ be a family of projections in C finite in A and maximal with respect to the property that $\{\mathrm{c}(f_j)\}$ is orthogonal. If $z = \sum \mathrm{c}(f_j)$ and $z < 1$, then $1 - z$ is a nonzero projection in C. If $1 - z$ is orthogonal to all finite projections of A in C, the supremum of these finite projections is not 1, contradicting the assumption. Thus there is a projection $f_0 \in C$ finite in A with $f_0(1 - z) > 0$. But then we may enlarge the family $\{f_j\}$ with $f_0(1-z)$, contradicting maximality. Then $f = \sum f_j$ is a projection in C finite with central cover 1 in A [3, Proposition 15.8].

Towards (b), fAf is an AW*-algebra of type I, and fC is a maximal abelian subalgebra. From Lemma 3.2(b), fC contains a projection e_0 abelian in fAf (and hence in A) with $c_{fAf}(e_0) = f$. Since $c_A(e_0) \geq c_{fAf}(e_0) = f$ and $c_A(f) = 1$, also $c_A(e_0) = 1$. Thus $e_0 \in C$ is a projection abelian with central cover 1 in A.

For (c), let $\{e_i\}$ be a maximal orthogonal family of projections in C that are abelian with central cover 1 in A, and set $e = \sum e_i$. By [3, Proposition 18.1], the e_i are pairwise equivalent. If $e < 1$, then $(1-e)A(1-e)$ is an AW*-algebra of type I in which $(1-e)C$ is a maximal abelian subalgebra. Moreover, $(1-e)$ is the supremum of projections in $(1-e)C$ finite in $(1-e)A(1-e)$. From part (b), $(1-e)C$ contains a projection e_1 abelian with central cover $1-e$ in $(1-e)A(1-e)$. It follows that e_1 is abelian with central cover $c(1-e)$ in A. If $c(1-e) = 1$, we can adjoin e_1 to $\{e_i\}$ contradicting maximality. Thus $z = 1 - c(1-e)$ is nonzero. Now $z(1-e) = 0$, so that $z = ze = \sum ze_i$ and $\{ze_i\}$ is a family of orthogonal equivalent projections in C with sum z. Because A is properly infinite and the ze_i are abelian, z is infinite and $\{ze_i\}$ cannot be a finite set; see [3, Theorem 17.3].

For (d), let $\{z_j \mid j \in \alpha\}$ be a maximal orthogonal family of central projections in A each with the property of z from (c). If $0 < 1 - \sum z_j =: z_0$, then $z_0 A$ is an AW*-algebra of type I_∞ and $z_0 C$ is a maximal abelian subalgebra with the property that z_0 is the supremum of projections in $z_0 C$ finite in $z_0 A$. Part (c) provides a nonzero central projection z_1 in $z_0 A$ that is the sum of infinitely many orthogonal equivalent projections in $z_0 C$. Adjoining z_1 to $\{z_j\}$ produces a family contradicting maximality of $\{z_j\}$. Hence $\sum z_j = 1$.

For each z_j fix an orthogonal set $\{e_{ij} \mid i \in \alpha_j\} \subseteq C$ of equivalent projections that sum to z_j for some infinite cardinal α_j. Partition the infinite set $\{e_{ij} \mid i \in \alpha_j\}$ into two subfamilies of the same cardinality, and let f_{kj} be the sum of the kth subfamily for $k = 1, 2$. Then $z_j = f_{1j} + f_{2j}$ and $f_{1j} \sim f_{2j} \sim z_j$ for all j. Set $e = \sum_j f_{1j}$ so that $1 - e = \sum_j f_{2j}$. Then $e \sim 1 - e \sim \sum z_j = 1$ as desired. □

Proposition 6.6. *Let A be an AW*-algebra of type I_∞. For any maximal abelian subalgebra C of A, there exists a projection $e \in C$ such that $e \sim 1 \sim 1 - e$.*

Proof. (We freely use Lemma 2.7 throughout this proof.) Let $g \in C$ be the supremum of the finite projections in C. By Lemma 6.2 the nonzero projections in $(1-g)C$ are properly infinite (in A and hence) in $(1-g)A(1-g)$, so $(1-g) = e_1 + e_2$ for orthogonal projections $e_i \in (1-g)C$ with $e_1 \sim e_2 \sim 1-g$ by Proposition 6.1. By Lemma 2.6 there exists a central projection $z \in A$ such that zg is finite and $(1-z)g$ is properly infinite or zero. Then $(1-z)g$ is a supremum of finite projections, so Lemma 6.5 applied to the maximal abelian subalgebra $(1-z)gC$ of $(1-z)gA(1-z)g$ shows that $(1-z)g = f_1 + f_2$ for orthogonal projections $f_i \in (1-z)gC$ with $f_1 \sim f_2 \sim (1-z)g$. In the sum of orthogonal projections

$$\begin{aligned} 1 &= zg + (1-g) + (1-z)g \\ &= zg + e_1 + f_1 + e_2 + f_2, \end{aligned}$$

set $e = zg + e_1 + f_1 \in C$. We will prove below that $zg + (1-g) \sim 1 - g$, from which it will follow that $(1-g) + (1-z)g \sim zg + (1-g) + (1-z)g = 1$ and thus $1 - e \sim e \sim 1$.

It remains to show that $zg + (1-g) \sim 1 - g$. We claim that $z \leq c(1-g)$. To see this, note that the central projection $y = z(1 - c(1-g))$ satisfies $y \leq z$ and $y(1-g) = 0$. Hence $y = yg \leq zg$ is finite. Because A is properly infinite, we conclude $z(1 - c(1-g)) = y = 0$, or $z \leq c(1-g)$. This gives the middle equality in

$$c(zg) \leq z = c(z(1-g)) \leq c(1-g).$$

If $1 - g = 0$ then $zg = 0$ and the claim is verified. If $1 - g > 0$ then it is properly infinite, and Lemma 6.3(c) implies that $zg + (1 - g) \sim 1 - g$. □

Finally, we combine the results for types I_∞, II_∞, and III to show that a maximal abelian subalgebra of any properly infinite algebra contains an infinite set of "diagonal matrix units".

Lemma 6.7. *If C is a maximal abelian subalgebra of a properly infinite AW*-algebra A, then there exists a projection $e \in C$ such that $e \sim 1 \sim 1 - e$.*

Proof. By [3, Theorem 15.3], $1 = z_1 + z_2$ is a sum of orthogonal central projections such that $z_1 A$ has type I and $z_2 A$ has no central summands of type I. Each $z_i C$ is a maximal abelian subalgebra of $z_i A$. By Propositions 6.4 and 6.6 there are projections $f_i \in z_i C$ such that $f_i \sim z_i \sim z_i - f_i$. Then $e = f_1 + f_2 \in C$ satisfies $e \sim 1 \sim 1 - e$. □

Theorem 6.8. *Let A be a properly infinite AW*-algebra, let C be a maximal abelian subalgebra of A, and let $1 \leq n \leq \aleph_0$ be a cardinal. Then there is a set $\{e_i\}$ of n orthogonal projections in C such that $1 = \sum e_i$ and every $e_i \sim 1$.*

Proof. By Lemma 4.3, it suffices to consider the case $n = \aleph_0$. For all positive integers k, we inductively decompose $1 = e_1 + \cdots + e_k + f_k$ as a sum of orthogonal projections in C where all $e_i \sim f_k \sim 1$. For $k = 1$ use Lemma 6.7, and for the inductive step suppose we have $e_1, \ldots, e_k, f_k \in C$ as above. Lemma 6.7 applied to the maximal abelian subalgebra $f_k C$ of the properly infinite algebra $f_k A f_k$ provides a projection $f \in f_k C$ such that $f \sim f_k \sim f_k - f$. Thus $e_{k+1} = f$ and $f_{k+1} = f_k - f$ satisfy $e_{k+1} \sim f_{k+1} \sim f_k \sim 1$ and $1 = e_1 + \cdots + e_{k+1} + f_{k+1}$ as desired. So $\{e_i\}_{i=1}^{\infty}$ is an orthogonal set of $\aleph_0$ projections in C that are equivalent to 1. In case $e = \sum e_i \in C$ is not equal to 1, we may replace e_1 with $e_1' = e_1 + (1 - e) \in C$; since $1 \sim e_1 \leq e_1' \leq 1$, Schröder–Bernstein implies that $e_1' \sim 1$. □

Theorem 6.8 naturally suggests the question of how large the cardinality n of a set of diagonal matrix units in C can become. In other words: given a maximal abelian subalgebra C of a properly infinite AW*-algebra A, for what (infinite) cardinals n does there exist an orthogonal set $\{e_i\} \subseteq \mathrm{Proj}(C)$ of cardinality n such that $\sum e_i = 1$ and each $e_i \sim 1$? It would be interesting to know to what extent the answer depends upon the particular subalgebra C.

7. Simultaneous diagonalization

We are now ready to prove simultaneous diagonalization over arbitrary AW*-algebras. Recall that if A is an AW*-algebra, then so is $\mathbb{M}_n(A)$ [2].

Lemma 7.1. *Let A be an AW*-algebra of type I_m for a positive integer m. If $1 = e_1 + \cdots + e_n$ for some equivalent projections $e_i \in A$, then m is divisible by n.*

Proof. We proceed by induction on m. The case $m = 1$ is evident. Let $f_1, \ldots, f_m$ be orthogonal equivalent abelian projections with sum 1. Since A is finite, so is each e_i; notice also that each $\mathrm{c}(e_i) = 1$. So for each $1 \leq i \leq n$ there is a projection g_i with $f_1 \sim g_i \leq e_i$ [3, Corollary 18.1]. Now the set $\{g_1, \ldots, g_n\}$ of orthogonal equivalent abelian projections has cardinality at most m [3, Proposition 2]. The projections $e_i - g_i$ remain equivalent [3, Exercise 17.3], and because $\sum g_i \sim (f_{m-n+1} + \cdots + f_m)$, similarly, the projection $h = \sum_{i=1}^{n} (e_i - g_i) = 1 - \sum g_i$ satisfies

$$h \sim 1 - (f_{m-n+1} + \cdots + f_m) = f_1 + \cdots + f_{m-n}.$$

On the one hand, hAh contains the n orthogonal equivalent projections $e_i - g_i$. But on the other hand, it has type I_{m-n} by the equation above. The inductive hypothesis implies that $m - n$ is divisible by n, whence m is divisible by n. □

Proposition 7.2. *Let A be an arbitrary AW*-algebra, and n a positive integer. If C is a maximal abelian subalgebra of $\mathbb{M}_n(A)$, then it contains n orthogonal projections with sum 1 equivalent in $\mathbb{M}_n(A)$.*

Proof. From [3, Theorem 15.3] and [3, Theorem 18.3], there are central projections $z_1, z_2, \ldots, z_c, z_\infty$ with sum 1 such that: $z_m\mathbb{M}_n(A)$ is of type I_m for every $m \in \{1, 2, \ldots\}$; $z_c\mathbb{M}_n(A)$ is of type II_1; and $z_\infty\mathbb{M}_n(A)$ is properly infinite. By Lemma 7.1, $z_m = 0$ when m is finite and not divisible by n. So $z_m > 0$ with m finite implies that $m = kn$ for some positive integer k, and z_mC contains n equivalent projections $e_{1m}, \ldots, e_{nm}$ with sum z_m by Proposition 3.4(c). Now, z_cC contains n equivalent projections $e_{1c}, \ldots, e_{nc}$ with sum z_c from Proposition 3.1, and $z_\infty C$ contains n equivalent projections $e_{1\infty}, \ldots, e_{n\infty}$ with sum z_∞ from Theorem 6.8. Set $e_j = e_c + e_{j\infty} + \sum_{m=1}^{\infty} e_{jm}$ for each $j \in \{1, \ldots, n\}$, where e_{jm} is defined to be 0 if $m < \infty$ does not divide n. Then $\{e_1 \ldots, e_n\}$ is a set of n equivalent projections in C with sum 1. □

Lemma 7.3. *Let A be an AW*-algebra and $\{e_1, \ldots, e_n\}$, $\{f_1, \ldots, f_n\}$ be two finite sets of projections in A, both summing to 1, with $e_1 \sim \cdots \sim e_n$, $f_1 \sim \cdots \sim f_n$. Then $e_j \sim f_j$ for each $j = 1, \ldots, n$.*

Proof. Lemma 2.6(b) gives a central projection $z \in A$ such that ze_1 is properly infinite or $z = 0$, and $(1 - z)e_1$ is finite. Then $(1 - z)e_2 \ldots, (1 - z)e_n$ are also finite [3, Proposition 15.3]. By [3, Theorem 17.3], $\sum_{j=1}^{n}(1 - z)e_j = 1 - z$ is finite. Hence $(1 - z)f_1, \ldots, (1 - z)f_n$ are finite. If $(1 - z)e_1$ is not equivalent to $(1 - z)f_1$, there is a central projection y in A such that either $y(1 - z)e_1 \prec y(1 - z)f_1$ or $y(1 - z)f_1 \prec y(1 - z)e_1$. In the former case,

$$y(1 - z)e_j \sim g_j < y(1 - z)f_j, \qquad \text{for } j = 1, \ldots, n.$$

Hence $y(1 - z) \sim \sum g_j < y(1 - z)$, contradicting the finiteness of $y(1 - z)$. Thus $(1 - z)e_1 \sim (1 - z)f_1$.

Suppose $z > 0$. Then ze_1 is properly infinite and by Lemma 6.3(b),

$$ze_1 \sim ze_1 + ze_2 \sim \cdots \sim \sum_{j=1}^{n} ze_j = z$$

since $ze_1 \sim \cdots \sim ze_n$. Now zf_1 is properly infinite, for if $z_0 \leq p$ is a nonzero central projection with z_0zf_1 is finite, then $\sum_{j=1}^{n} z_0f_j$ is finite. But z_0ze_1 and, hence, z_0 are infinite since ze_1 is properly infinite, contradicting finiteness of z_0. Since zf_1 is properly infinite, as before, $zf_1 \sim z$. Thus $ze_1 \sim zf_1$. It follows that $e_1 \sim f_1$. □

The previous lemma is known to hold for Rickart C*-algebras by a less elementary proof [1, Theorem 2.7] (see also [3, Proposition 13.2]). We arrive at our main theorem.

Theorem 7.4. *Let A be an AW*-algebra, and let $X \subseteq \mathbb{M}_n(A)$ be a commuting set of normal elements. There is a unitary $u \in \mathbb{M}_n(A)$ such that uau^{-1} is diagonal for each $a \in X$.*

Proof. Let C be a maximal abelian subalgebra of $\mathbb{M}_n(A)$ containing X. By Proposition 7.2, C contains n orthogonal equivalent projections $f_1, \ldots, f_n$ with sum 1. Let e_{jk} be the element of $\mathbb{M}_n(A)$ with 1 at the (j,k)-entry and 0 elsewhere. Then $\{e_{11}, \ldots, e_{nn}\}$ is an orthogonal family of equivalent projections in $\mathbb{M}_n(A)$ with sum 1. By Lemma 7.3, $e_{jj} \sim f_j$ for $j = 1, \ldots, n$.

Say $v_j^* v_j = f_j$ and $v_j v_j^* = e_{jj}$ with $v_j \in \mathbb{M}_n(A)$. Then $u = \sum_{j=1}^n v_j$ is a unitary element of $\mathbb{M}_n(A)$ and $u f_j u^{-1} = e_{jj}$. Since f_j commutes with every element in $C \supseteq X$, we see that e_{jj} commutes with uau^{-1} for all $a \in X$. Thus uau^{-1} is diagonal for all $a \in X$. □

It seems natural to ask whether the converse of Theorem 7.4 holds, in the following sense. To use a term defined in Section 1, the above theorem says that an AW*-algebra is simultaneously n-diagonalizable for all positive integers n. Conversely, if a C*-algebra A is simultaneously n-diagonalizable for all n, does it follow that A is an AW*-algebra? The question is especially tantalizing because Grove and Pedersen have answered this question affirmatively in the case where A is commutative, even under the weaker assumption that A is simultaneously n-diagonalizable for one fixed $n \geq 2$ (see Theorem 2.1 and the "Notes added in proof" of [10]). But it seems a subtle problem to decide the issue in a fully noncommutative setting.

8. Passing to matrix rings is functorial

Denote by **AWstar** the category of AW*-algebras and $*$-homomorphisms between them that preserve suprema of arbitrary sets of projections. As a consequence of our main result, Theorem 7.4, we can now show that passing to matrix rings is an endofunctor on this category.

Lemma 8.1. *A $*$-homomorphism $f\colon A \to B$ between AW*-algebras preserves suprema of arbitrary families of projections if and only if it preserves suprema of orthogonal families of projections.*

Proof. One direction is trivial. For the other, suppose that f preserves suprema of orthogonal sets of projections, and let $\{p_i\}$ be an arbitrary family of projections in A. Then $\ker(f) = zA$ for some central projection $z \in A$ [3, Exercise 23.8]. Hence $A = zA \oplus (1-z)A$. Since $p_i = zp_i + (1-z)p_i$ for each i, this yields $\bigvee p_i = \bigvee zp_i + \bigvee (1-z)p_i$, and so $f(\bigvee p_i) = f(\bigvee zp_i) + f(\bigvee (1-z)p_i) = f(\bigvee (1-z)p_i)$. Therefore we may pass to $(1-z)A$ and assume that $\ker(f) = \{0\}$. The proof in the case where f is injective is [3, Exercise 4.27]. □

Theorem 8.2. *There is a functor* **AWstar** $\to$ **AWstar** *extending the assignment $A \mapsto \mathbb{M}_n(A)$ on objects.*

Proof. It is well known that if A is an AW*-algebra, then $\mathbb{M}_n(A)$ is, too [2], and that if $f\colon A \to B$ is a $*$-homomorphism, then $\mathbb{M}_n(f)\colon \mathbb{M}_n(A) \to \mathbb{M}_n(B)$ is, too. The point is to show that $\mathbb{M}_n(f)$ preserves suprema of projections. By Lemma 8.1 it suffices to show that $\mathbb{M}_n(f)$ preserves suprema of orthogonal families of projections. Let $\{p_i\}$ be an orthogonal family of projections in $\mathbb{M}_n(A)$. Then $\{p_i\}$ is an abelian self-adjoint subset of $\mathbb{M}_n(A)$. Theorem 7.4 provides a unitary $u \in \mathbb{M}_n(A)$ making each up_iu^{-1} diagonal. Now

$$\begin{aligned}\mathbb{M}_n(f)\left(\sum p_i\right) &= \mathbb{M}_n(f)(u)^{-1}\cdot\mathbb{M}_n(f)\left(\sum up_iu^{-1}\right)\cdot\mathbb{M}_n(f)(u)\\ &= \mathbb{M}_n(f)(u)^{-1}\cdot\sum\mathbb{M}_n(f)(up_iu^{-1})\cdot\mathbb{M}_n(f)(u)\\ &= \sum\mathbb{M}_n(f)(p_i).\end{aligned}$$

The first and last equalities hold because $\mathbb{M}_n(f)$ is a $*$-homomorphism, and the middle equality holds because upu^{-1} is diagonal, f preserves suprema of projections, and the supremum of a set of diagonal projections is computed entrywise. $\square$

Remark 8.3. The previous theorem holds unabated if we replace $*$-homomorphisms by $*$-ring homomorphisms. In fact, due to the algebraic nature of our methods, the results in Sections 2–6 seem to hold for Baer $*$-rings with generalized comparability (GC) that satisfy the parallellogram law (P), in Berberian's terms [3]. For the results of Section 7 and the proof of Theorem 8.2 to carry through, one must further restrict to such $*$-rings A for which $\mathbb{M}_n(A)$ is again such a $*$-ring (for instance, properly infinite Baer $*$-rings A, where $\mathbb{M}_n(A)\cong A$ for all n). Lemma 8.1 additionally requires the "weak existence of projections" (WEP) axiom of [3].

Appendix A. Achieving the dimension

This appendix discusses two special cases in which the supremum in the definition of the dimension of properly infinite projections in an AW*-algebra A, Definition 4.1, is achieved with certainty. To be precise, fix a properly infinite projection $e\in A$, and define

$$\begin{aligned}\Delta(e) &= \{\operatorname{card} I \mid \{e_i\}_{i\in I}\in\Gamma(e)\} = \{\text{cardinals } \beta \mid \beta<\mathrm{d}(e)\},\\ \delta(e) &= \sup\Delta(e).\end{aligned}$$

The question is whether $\delta(e)\in\Delta(e)$, or equivalently, whether $\mathrm{d}(e)=\alpha^+$ for some $\alpha\in\Delta(e)$. The first special case in which we have a positive answer concerns properties of the cardinal $\delta(e)$ itself. Recall that a cardinal is *weakly inaccessible* if it is an uncountable regular limit cardinal.

Proposition A.1. *If the cardinal $\delta(e)$ is not weakly inaccessible, then $\delta(e)\in\Delta(e)$.*

Proof. Notice from Lemma 4.2 that $\aleph_0\in\Delta(e)$ necessarily. If $\delta(e)$ is either $\aleph_0$ or a successor cardinal, then from the definition $\delta(e)=\sup\Delta(e)$ it is clear that $\delta(e)\in\Delta(e)$. So we may assume that $\delta(e)$ is an uncountable limit cardinal that is not regular: it is strictly larger than the least cardinality of a cofinal set of cardinals below it. Let $\{\alpha_i \mid i\in\beta\}$ be a cofinal set of cardinals below $\delta(e)$, where $\beta<\delta$. Because $\beta<\delta(e)$, we can write $e=\sum_{i\in\beta}e_i$ with $e_i\sim e$. Next, we can also write $e_i=\sum_{j\in\alpha_i}e_{ij}$ for each $i\in\beta$ with $e_{ij}\sim e_i\sim e$. Then $\{e_{ij}\}\in\Gamma(e)$ has cardinality $\sup\{\alpha_i \mid i\in\beta\}$, which equals $\delta(e)$ by cofinality. $\square$

The answer is also positive when A is a von Neumann algebra. Recall that a projection is countably decomposable when any orthogonal family of nonzero subprojections is countable; A is countably decomposable when 1_A is.

Lemma A.2. *Every projection p in a von Neumann algebra A can be written as $p=\sum p_i$ for some orthogonal family $\{p_i\}$ of countably decomposable projections.*

Hence every central projection z can be written as $z = \sum z_i$ for an orthogonal family $\{z_i\}$ of central projections making each $z_i Z(A)$ countably decomposable.

Proof. Let A act faithfully on a Hilbert space H. Then $p = \sum p_i$ for an orthogonal set $\{p_i\}$ of projections cyclic in A for the action on H [13, Proposition 5.5.9]. But every cyclic projection in A is countably decomposable [13, Proposition 5.5.15]. $\square$

Lemma A.3. *Let A be a von Neumann algebra.*

(a) If $\{e_i \mid i \in \alpha\} \subseteq A$ is an orthogonal set of equivalent countably decomposable nonzero projections for some infinite cardinal α, then the projection $e = \sum_{i \in \alpha} e_i$ is properly infinite and satisfies $\alpha = \delta(e) \in \Delta(e)$.

(b) If $e \in A$ be a nonzero properly infinite projection, then there exists a nonzero central projection $z \leq \mathrm{c}(e)$ and an infinite set of projections $\{e_i \mid i \in \alpha\}$ as in (a) such that $ze = \sum_{i \in \alpha} e_i$.

Proof. For (a): by Lemma 4.2, e is properly infinite and $\alpha \in \Delta(e)$. On the other hand, if $\beta \in \Delta(e)$ there is a family $\{f_j\}_{j \in \beta} \in \Gamma(e)$. Because $\sum_{i \in \alpha} e_i = e = \sum_{j \in \beta} f_j$ and the e_i are countably decomposable, an easy adaptation of the proofs of [17, Lemma 1] and [13, Lemma 6.3.9] shows that $\beta \leq \alpha$. Because $\beta \in \Delta(e)$ was arbitrary, it follows that $\delta(e) \leq \alpha$. But $\alpha \in \Delta(e)$ further implies that $\delta(e) = \alpha \in \Delta(e)$.

For (b): by Lemma A.2, $\mathrm{c}(e)$ is a sum of central projections $\{z_i\}$ making each $z_i Z(A)$ countably decomposable. Passing to a direct summand, we may assume that $\mathrm{c}(e)Z(A)$ itself is countably decomposable. Then $\mathrm{c}(e) = \mathrm{c}(p)$ for some countably decomposable projection $p \in A$ [13, Propositions 5.5.16 and 5.5.15].

Write $e = \sum_{j=1}^{\infty} f_j$ with $f_j \sim e$. Notice that p is countably decomposable, $f_j \sim e$ are properly infinite, and $\mathrm{c}(p) = \mathrm{c}(e) = \mathrm{c}(f_j)$. It follows from [13, Theorem 6.3.4] that $p \precsim e \sim f_j$. So there exist orthogonal $g_j \leq f_j \leq e$ with $g_j \sim p$ for each j. Extend $\{g_j\}_{j=1}^{\infty}$ via Zorn's lemma to a maximal orthogonal set of projections $\{g_i \mid i \in \alpha\}$ such that $p \sim g_i \leq e$ for all $i \in \alpha$, where α is an infinite cardinal. Assume for contradiction that $e - \sum g_i$ is properly infinite with central cover $\mathrm{c}(e)$. Then $p \precsim e - \sum g_i$ by [13, Theorem 6.3.4]. This allows us to enlarge the set $\{g_i\}$, contradicting maximality. Therefore the situation reduces the following two cases.

Case 1: $e - \sum g_i$ is not properly infinite. Then there is a nonzero central projection $z \leq \mathrm{c}(e - \sum g_i) \leq \mathrm{c}(e)$ making $z(e - \sum g_i) > 0$ finite. Because $\mathrm{c}(g_i) = \mathrm{c}(e)$ for each i, it follows that $\mathrm{c}(\sum g_i) = \mathrm{c}(e) \geq \mathrm{c}(e - \sum g_i)$. Note that $\sum z g_i$ is properly infinite by Lemma 4.2(a). Furthermore, $z(e - \sum g_i)$ and $\sum z g_i$ have central cover z. It follows from Lemma 6.3(c) that

$$ze = z\left(e - \sum g_i\right) + \sum z g_i \sim \sum z g_i.$$

Thus $ze = \sum e_i$ for equivalent countably decomposable $e_i \sim z g_i$.

Case 2: $\mathrm{c}(e - \sum g_i)$ is strictly below $\mathrm{c}(e)$. Define $z = \mathrm{c}(e) - \mathrm{c}(e - \sum g_i)$. Then $0 < z \leq \mathrm{c}(e)$, and $z(e - \sum g_i) = 0$. Thus $ze = \sum z g_i$, where the $e_i = z g_i$ are pairwise equivalent and countably decomposable. $\square$

Proposition A.4. *If e is properly infinite projection in a von Neumann algebra A, then $\delta(e) \in \Delta(e)$.*

Proof. Applying Zorn's lemma to Lemma A.3(b) gives a maximal family $\{z_i\}$ of orthogonal nonzero central projections such that $z_i \leq \mathrm{c}(e)$, and $z_i e = \sum_j e_{ij}$ for some infinite orthogonal set $\{e_{ij} \mid j \in \alpha_i\}$ of equivalent countably decomposable projections. If $\sum z_i < \mathrm{c}(e)$, then $(\mathrm{c}(e) - \sum z_i)e$ is properly infinite, so the projection given by Lemma A.3(b) would violate maximality; therefore $\sum z_i = \mathrm{c}(e)$. Lemma A.3(a) also implies that $z_i e$ is properly infinite and $\delta(z_i e) = \alpha_i$. Then $\delta(e) = \min\{\alpha_i\} \in \Delta(e)$ by Lemma 4.4(c). $\square$

References

1. P. Ara, *Left and right projections are equivalent in Rickart C*-algebras*, Journal of Algebra **120** (1989), 433–448.
2. S. K. Berberian, *$N \times N$ matrices over an AW^*-algebra*, Amer. J. Math. **80** (1958), 37–44.
3. ———, *Baer $*$-rings*, Grundlehren der mathematischen Wissenschaften, vol. 195, Springer, 1972, Second printing 2011.
4. G. Birkhoff, *Lattice theory*, third ed., Amer. Math. Soc., 1948.
5. V. I. Čilin, *Equivalence of projectors in AW^*-factors of type* III, Tashkent. Gos. Univ. Sb. Nauchn. Trudov (1980), no. 623 Mat. Analiz i Geometriya, 78–83, 95, (Russian).
6. D. Deckard and C. Pearcy, *On matrices over the ring of continuous complex valued functions on a Stonian space*, Proc. Amer. Math. Soc. **14** (1963), 322–328.
7. J. Feldman, *Some connections between topological and algebraic properties in rings of operators*, Duke Math. J. **23** (1956), no. 2, 365–370.
8. M. Frank and V. M. Manuĭlov, *Diagonalizing "compact" operators on Hilbert W^*-modules*, Z. Anal. Anwendungen **14** (1995), no. 1, 33–41.
9. K. R. Goodearl and F. Wehrung, *The complete dimension theory of partially ordered systems with equivalence and orthogonality*, Memoirs, no. 831, Amer. Math. Soc., 2005.
10. K. Grove and G. K. Pedersen, *Diagonalizing matrices over $C(X)$*, J. Funct. Anal. **59** (1984), 65–89.
11. H. Halpern, *Essential central range and selfadjoint commutators in properly infinite von Neumann algebras*, Trans. Amer. Math. Soc. **288** (1977), 117–146.
12. R. V. Kadison, *Diagonalizing matrices*, Amer. J. Math. **106** (1984), no. 4, 1451–1468.
13. R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*, Academic Press, 1983.
14. V. Kaftal, *Type decomposition for von Neumann algebra embeddings*, J. Funct. Anal. **98** (1991), 169–193.
15. I. Kaplansky, *Projections in Banach algebras*, Ann. Math. **53** (1951), no. 2, 235–249.
16. V. M. Manuilov and E. V. Troitsky, *Hilbert C^*-modules*, Translations of Mathematical Monographs, vol. 226, Amer. Math. Soc., 2005, Translated from the 2001 Russian original by the authors.
17. J. Tomiyama, *Generalized dimension function for W^*-algebras of infinite type*, Tôhoku Math. J. (2) **10** (1958), 121–129.

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