The elephant in the ground: managing oil andsovereign wealth

THE ELEPHANT IN THE GROUND:

MANAGING OIL AND SOVEREIGN WEALTH

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Abstract

One of the most important developments in international finance and resource economics in the past twenty years is the rapid and widespread emergence of the $6 trillion sovereign wealth fund industry. Oil exporters typically ignore below-ground assets when allocating these funds, and ignore above-ground assets when extracting oil. We present a unified stylized framework for considering both. Subsoil oil should alter a fund's portfolio through additional leverage and hedging. First-best spending should be a share of total wealth, and any unhedgeable volatility must be managed by precautionary savings. If oil prices are pro-cyclical, oil should be extracted faster than the Hotelling rule to generate a risk premium on oil wealth. Finally, we discuss how our analysis could improve the management of Norway's fund in practice.

Keywords: oil revenue, portfolio allocation, sovereign wealth fund, leverage, hedging, optimal extraction, prudence, risk aversion

JEL codes: E21, F65, G11, G15, O13, Q32, Q33

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1. Introduction

Since 1994 the number of sovereign wealth funds has nearly quadrupled to 73 (SWF institute, 2013). These funds hold some of the largest portfolios in the world and globally account for over $6 trillion in assets (ibid.). Two thirds of the sovereign wealth fund industry (by size) has been funded by selling below-ground assets such as oil, natural gas, copper and diamonds ("oil" for short). These funds often comprise a large part of commodity exporters' wealth. Azerbaijan's US$ 34 billion fund accounts for almost half its GDP, Qatar's US$ 170 billion fund accounts for almost two thirds of GDP, Saudi Arabia's US$ 740 billion funds are approximately four-fifths of GDP, Norway's US$ 840 billion fund is nearly one and a half times GDP, and the United Arab Emirates' US$ 1 trillion funds are over two and a half times its GDP (SWF Institute, 2013; IMF, 2013).

The purpose of these funds is to smooth consumption of oil income: across generations because oil reserves are finite, and between periods because oil and asset prices are volatile. While such funds are professionally managed and often allocate their assets using modern portfolio theory, we argue that their investment strategies do not take due account of oil price volatility and subsoil reserves. Similarly, existing theories of optimal oil extraction do not take into account volatile financial markets. These are important issues for resource exporters, since commodity prices are notoriously volatile and below-ground assets can be worth much more than the above-ground fund.

Our aim is therefore to answer four questions about how below-ground resources should influence above-ground portfolios, and vice-versa. Firstly, how should one allocate above-ground assets given a volatile stock of below-ground assets? Secondly, how quickly should financial and oil wealth be consumed? Thirdly, how does this change if financial markets are incomplete, so that oil shocks cannot be completely hedged in the portfolio? Finally, how should the optimal extraction rate of below-ground assets be affected by risky above-ground assets?

We will show that policy-makers should adjust their above-ground portfolios to accommodate the volatility and erosion of below-ground oil stocks (hedging and

leverage effects respectively); consume a fixed share of total wealth; manage shocks that cannot be hedged with precautionary savings; and, if the marginal rent from extracting an additional barrel of oil, namely the oil price minus marginal extraction costs, co-varies positively with average equity market returns, then oil should be extracted faster.

Our analysis combines three large and previously unrelated strands of literature. Firstly, the allocation of financial assets is described by CAPM equations modified for subsoil oil wealth. This extends the continuous-time analysis of optimal consumption-saving and portfolio choice (Merton, 1990).[1] Secondly, consumption is described by a stochastic Euler equation,[2] extending the literature on prudence and precautionary savings to the case when both financial assets and oil extraction can be chosen.[3] Thirdly, the optimal rate of oil extraction is described by a stochastic Hotelling rule modified if the proceeds of extraction of below-ground wealth are invested in a risky above-ground financial portfolio.[4] Our intended contribution is to introduce a stylized framework that combines canonical insights from all three of these fields. These insights would be modified by including transaction costs and illiquidity premiums, which would help to explain why in practice fund managers do not adjust their portfolios too frequently by introducing some mean reversion into the portfolio decisions (Constantinides, 1986; Acharya and Pedersen, 2005; Garleanu and Pedersen, 2013; Jong and Driessen, 2015).

[1] This builds on classic portfolio theory (Tobin, 1958) and mean-variance theory (Markowitz, 1952; 1959). If investors have equal information and markets are complete, they hold the market portfolio as used in the CAPM (Sharpe, 1964). Our extension to allow for oil income is akin to those dealing with a non-tradable stream of income in the context of university endowments (Merton, 1993; Brown and Tiu, 2012), labor income including endogenous effort (Bodie et al., 1992; Wang et al., 2013), non-tradable and uninsurable income (Svensson and Werner, 1993; Koo, 1998) and non-financial stores of wealth such as housing (Flavin and Yamashita, 2002; Sinai and Souleles, 2005; Case et al., 2005).

[2] See Leland (1968), Sandmo (1970), Zeldes (1986), Kimball (1990), Carroll and Kimball (2008).

[3] This extends earlier work on precautionary saving in safe assets to cope with oil price volatility (Bems and de Carvalho Filho, 2011; van den Bremer and van der Ploeg, 2013).

[4] We require marginal extraction costs to be positive and increasing in the amount extracted but, unlike Pindyck (1980, 1981), we do not require them to be convex, which would create extractive prudence. Others treat extraction with stochastic oil prices, growth and capital, but abstract from above-ground financial assets (Gaudet and Khadr, 1991; Atewamba and Gaudet, 1992). Recent empirical evidence suggests that the Hotelling rule holds at the extensive margin of number of wells drilled, but not at the intensive margin (Anderson et al., 2014; Venables, 2014).

This paper is laid out as follows. Section 2 introduces our model for portfolio choice, saving and oil revenues. Section 3 shows how to allow for below-ground oil wealth with a predetermined path for oil production when the oil price is completely spanned by returns in asset markets. Section 4 deals with the case of investment restrictions which prevent the oil price being fully spanned. Section 5 derives the optimal path for oil extraction. Section 6 discusses the implications of our results and compares these with the policies adopted by the Norwegian fund. Finally, section 7 concludes and qualifies our results.

2. The model

Adopting Geometric Brownian motion processes for the oil price and asset returns, the problem is to choose the rate of public consumption C and portfolio asset weights w_i, $i = 1,.., n$, to maximize the expected present value of utility with discount rate $\rho > 0$:

$$J(F, P_O, t) = \max_{C, w_i} E_t \left[\int_t^\infty U\left(C(s)\right) e^{-\rho(s-t)} ds \right], \tag{1}$$

subject to the budget constraint:

$$dF = \sum_{i=1}^{m} w_i(\alpha_i - r)F dt + (rF + P_O O - C)dt + \sum_{i=1}^{m} w_i F \sigma_i dZ_i, \tag{2}$$

where the value function $J(F, P_O, t)$ depends on the size of the fund F, the oil price P_O and time t.[5] The rate of oil extracted at time t, $O(t)$, either declines exponentially at the rate κ with zero extraction costs (sections 3 and 4)[6] or is chosen optimally with convex costs (section 5). The fund has m risky assets, $i = 1,.., m$, with drift α_i and volatility σ_i and one safe asset, $i = m+1$, with return r and volatility $\sigma_{m+1} = 0$. There are thus $n \equiv m + 1$ assets. The fund holds N_i shares of assets, $i = 1,.., n$, each with price P_i, so $F = \sum_{i=1}^{n} P_i N_i$.

[5] This abstracts from all other public assets (e.g., future tax revenues) and liabilities (e.g., pensions).
[6] The results can readily be extended for the case of a constant windfall of finite duration.

The share of each asset in the fund is $w_i \equiv P_i N_i / F$, so $F = \sum_{i=1}^{n} w_i F$. The stochastic processes for the risky assets are:

$$dP_i = \alpha_i P_i dt + \sigma_i P_i dZ_i, \quad i = 1,..,m, \tag{3}$$

where dZ_i is a Wiener process with $\text{cov}(dZ_i\, dZ_j) = [\rho_{ij}]$ for $i = 1,.., m$. The returns of risky assets have covariance matrix $\Sigma = [\sigma_{ij}] = [\rho_{ij}\sigma_i\sigma_j]$. We abstract from mean reversion and stochastic volatility in asset prices, and ignore transaction costs (discussed in section 6). We thus assume that the coefficients in (3) are constant. The weight of the safe asset in the fund, $w_n = 1 - \sum_{i=1}^{m} w_i$, is positive or negative if the weight of the risky portfolio is smaller or larger than one, which corresponds to a long position ($w_n > 0$) or short position ($w_n < 0$) in the safe asset. Total holdings of risky assets is called the "portfolio", $(1 - w_n)F = \sum_{i=1}^{m} w_i F$, and its share in the fund is $w \equiv 1 - w_n$.

Preferences exhibit constant relative risk aversion, $U(C) = C^{1-1/\theta} / (1 - 1/\theta), \theta \neq 1$ and $U(C) = \ln(C), \theta = 1$, where θ is the coefficient of intertemporal substitution, $1/\theta$ the coefficient of relative risk aversion or the degree of intergenerational inequality aversion, and $1 + 1/\theta$ the coefficient of relative prudence. These are a member of the class of hyperbolic absolute risk aversion preferences and thus permit an analytical solution to the asset allocation problem (Merton, 1971). Section 3 also explores Epstein-Zin preferences, which allows one to disentangle risk aversion and intertemporal substitution (Epstein and Zin, 1989; Duffie and Epstein, 1992).[7]

The country is a small oil exporter that does not affect the oil price. The world oil price also follows a Geometric Brownian Motion process:

$$dP_O = \alpha_O P_O dt + \sigma_O P_O dZ_O, \tag{4}$$

[7] These have been used in empirical studies (e.g., Attanasio and Weber, 1989; Wang et al., 2013).

where the drift in the oil price is not too large, $\alpha_O < r$.[8] Again, we abstract from mean reversion, stochastic volatility and transaction costs. Risky assets are driven by a common set of shocks (e.g., to demand, supply, technology or the weather), $du \sim$ i.i.d. $N(0, dt)$. The correlation of each asset depends on how it is affected by these shocks, $dZ = \Lambda du$, where $\Lambda = \left[\lambda_{ij}\right]$ is an invertible $m \times m$ matrix and $dZ = \left[dZ_1, .., dZ_m\right]'$ is the vector of Wiener processes driving the returns on risky assets. The Wiener process driving oil returns is expressed as:

$$dZ_O = \lambda_{Oh} du_h + \Lambda_O du = \lambda_{Oh} du_h + M dZ, \tag{5}$$

where $M = \Lambda_O \Lambda^{-1}$. The vector $\Lambda_O = [\lambda_{O1}, .., \lambda_{Om}]$ determines how the oil price responds to the vector of underlying shocks, du, and $\text{cov}(dZ_O, dZ) = \Sigma M$.[9]

With complete markets, the fund has unrestricted access to all assets and the instantaneous return on oil can be perfectly replicated ("spanned") by a bundle of traded securities. Without loss of generality these securities represent equities and bonds rather than derivatives.[10] The unhedgeable component of oil prices is zero, $\lambda_{Oh} = 0$ (see sections 3 and 5). With incomplete markets, there is an unhedgeable component of the oil price with weight $\lambda_{Oh} = \sqrt{1 - \sum_{i=1}^{m-1} \lambda_{Oi}} \neq 0$ and $\Lambda_O = \left[\lambda_{O1}, .., \lambda_{Om-1}\right]$, where du_h is a residual oil-specific shock that is uncorrelated with the asset market shocks, du (see appendix A.1 and section 4).

3. Complete markets and a given path of oil extraction

With complete markets, oil wealth can be treated as tradable by replicating its properties with a synthetic bundle of traded financial assets. Accordingly, an arbitrage argument

[8] This is a sufficient condition for the present discounted value of a permanent oil windfall to be finite, and is consistent with empirical estimates (e.g., van den Bremer and van der Ploeg, 2013).
[9] Oil prices depend in general equilibrium on more fundamental shocks (Bodenstein et al., 2012).
[10] For large oil exporters, liquidity constraints make derivative hedging of oil prices impractical. Therefore we focus on long/short, equity/bond hedging strategies.

can be employed to derive the value of the stream of oil revenues (see appendix A.1. for a derivation):

$$V(P_O,t) = P_O(t)O(t)/\psi, \quad \psi \equiv r + \kappa - \alpha_O + \sum_{i=1}^{m} \beta_i(\alpha_i - r), \tag{6}$$

where $\beta_i = \frac{\sigma_O}{\sigma_i}[\Lambda_O \Lambda^{-1}]_i$ and $M_i \equiv [\Lambda_O \Lambda^{-1}]_i$. Total wealth, $W = F + V$, then satisfies:

$$dW = \sum_{i=1}^{m} \bar{w}_i W(\alpha_i - r)dt + (rW - C)dt + \sum_{i=1}^{m} \sigma_i \bar{w}_i W dZ_i, \tag{7}$$

where $\bar{w}_i \equiv (w_i F + \beta_i V)/(F+V), i = 1,..,m$.

The replicating bundle linearly combines exposures β_i to many financial assets, which depend on the correlation of each risky asset with the oil price and its uniqueness amongst other financial assets. This bundle matches the variance of oil revenues and the amount of the safe asset is chosen to match the drift. Oil wealth is current oil revenues divided by the effective discount rate ψ, where ψ is the safe return r plus the rate of decline of oil production κ minus the drift in the oil price α_O plus the adjustment to compensate risk-averse investors for bearing oil price risk.[11] Oil wealth reacts to the current oil price only, as (5) implies oil price shocks are permanent under our assumptions.

3.1. Asset allocation: leverage and hedging demands

If claims to oil can be securitized, the proceeds can be invested in a diversified portfolio and the problem reduces to that in Merton (1990). In practice, doing so may be difficult due to political and practical constraints[12]. Nevertheless, with the replicating bundle the problem reduces to choosing the net weight of each risky asset, $\bar{w}_i$ for $i = 1,.., m$, in total

[11] The value of an uncertain stream of income follows from discounting at the risk-free rate if the probability space is adjusted to a risk-neutral measure using a theorem due to Girsanov (1960).
[12] Politicians do not like the prospect of having sold oil for an ex-post low price, and risk-averse firms are unwilling to take on all price and production risk

above- and below-ground wealth, $W = F + V$. Evidently, the net weight of each risky asset in total wealth is constant:

$$\bar{w}_i = \delta_i \bar{w}, \quad i = 1 \ldots m, \quad \delta_i \equiv \frac{1}{v} \sum_{j=1}^{m} v_{ij} (\alpha_j - r), \tag{8}$$

and the net weight of all risky assets in total wealth is:

$$\bar{w} \equiv \sum_{i=1}^{m} \bar{w}_i = \theta v, \quad v \equiv \sum_{i=1}^{m} \sum_{j=1}^{m} v_{ij} (\alpha_j - r), \tag{9}$$

where $v_{ij} \equiv \left[\Sigma^{-1} \right]_{ij}$, and the share of safe assets in the total portfolio is $1 - \bar{w}$.

The weight of each risky asset in the above-ground fund is (see appendix A.2):

$$w_i = \bar{w}_i + \underbrace{\left(\bar{w}_i \frac{V}{F} \right)}_{\text{leverage demand}} + \underbrace{\left(-\beta_i \frac{V}{F} \right)}_{\text{hedging demand}}, \quad \beta_i = \frac{\sigma_O}{\sigma_i} M_i, \quad i = 1, .., m. \tag{10}$$

Sovereign wealth funds should thus be structured so that net exposure to each asset in total wealth is constant. The optimal portfolio of risky assets (8) is independent of preferences and the level of wealth, but depends as usual on the drift and covariance of asset returns. The optimal part of total wealth allocated to risky assets (9) is proportional to the overall risk-adjusted return of the portfolio v and the willingness to take risk θ (the inverse of the coefficient of relative risk aversion).[13]

To ensure that net exposure to each financial asset is a constant share of total wealth (8) , one requires offsetting leverage and hedging demands for each risky asset as a share of the above-ground fund (10). The allocation of the fund approaches its non-oil level, $\bar{w}_i$,

[13] If there is only one risky asset, (9) reduces to the Sharpe ratio, $\bar{w} = \theta(\alpha_1 - r) / \sigma_1^2$, so the portfolio is proportional to the excess return of the risky asset over the safe asset, and the willingness to take risk, and inversely proportional to the variance of the return on the risky asset. With multiple risky assets the overall risk-adjusted return is lower if assets are positively correlated, so there is less scope for fluctuations to offset each other and to hedge oil.

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as oil is depleted.[14] Leverage demand involves holding more of each risky asset in the above-ground fund. For example, if oil wealth matches the size of the fund and is uncorrelated with assets ($\beta_i = 0, \forall i$, $W = F + V = 2F$), the fund holds twice as much of each risky asset and can do so only by holding less or borrowing more of the risk-free asset. If there is only one risky asset, leverage demand is given by the Sharpe ratio, $\theta(\alpha_1 - r)\sigma_1^{-2}(V/F)$[15], clearly illustrating that, as oil is depleted, leverage demand vanishes by reallocating from risky to safe assets.

Furthermore, hedging demand offsets exposure to oil price risk. If oil is correlated with only one asset, $dZ_O = \rho_{Ok} dZ_k$, hedging demand is the oil-asset beta[16] multiplied by the leverage ratio, $-\rho_{Ok}\sigma_O / \sigma_k (V/F)$. If oil price risk is positively correlated with the financial asset ($\rho_{Ok} > 0$), hedging demand is negative. If the two are negatively correlated $(\rho_{Ok} < 0)$, the fund should hold more of the risky asset to hedge oil price risk. Again, as oil is extracted and the exposure to price risk falls, hedging demand vanishes. Equation (10) generalizes this insight to multiple risky financial assets. If all financial asset returns are independent (Λ is diagonal), oil should be hedged by investing more in assets that are negatively correlated (e.g., assets that use oil as an input such as manufacturing and consumer goods industries) and less in assets that are positively correlated (e.g., oil and gas stocks or substitutes like renewable energy), especially if oil reserves are large. One should then also leverage up all demands for risky assets that prevail in the absence of oil. If financial asset returns are correlated, hedging of oil must consider the covariance of each risky asset. It is then possible that the fund should invest less in assets that are negatively correlated with oil.[17] In practice one can implement this

[14] This assumes that withdrawals from the fund are not so rapacious (i.e., ρ is not too high, cf. (8)) that fund assets fall quicker than oil is extracted and V/F rises over time.
[15] Mean-variance analysis gives a similar expression (Gintschel and Scherer, 2008; Scherer, 2009).
[16] The slope coefficient of a regression of demeaned asset returns versus demeaned oil returns.
[17] For example, consider a shock du_G which affects oil and asset A but not others, $\lambda_{OG}, \lambda_{AG} > 0$ and $\lambda_{iG} = 0$, for all $i \neq A$. The other shocks du_j affect oil and asset A in opposite ways, $\lambda_{Oj} > 0$ and $\lambda_{Aj} < 0, j = 1,..,$

with a mix of the "market index", $\bar{w}_i$, and an "oil hedging index", β_i, constructed to replicate movements in the oil price. Over time the mix shifts from the second to the first index as oil is extracted from the ground (see (10)). Net demand may be negative for both risky assets (short positions) and riskless assets (leverage), which may not be practical for many SWFs. Section 4 addresses this by considering investment restrictions.

3.2. Consumption rules and precautionary saving

Oil wealth also affects precautionary saving and optimal consumption from the fund, as illustrated by the Euler equation governing the expected growth of consumption:

$$\frac{\frac{1}{dt}E_t[dC]}{C} = \theta(r-\rho) + \frac{1}{2}(1+1/\theta)\sigma_W{}^2\bar{w}^2, \quad \sigma_W \equiv \sqrt{\sum_{i=1}^m \sum_{j=1}^m \delta_i \delta_j \sigma_{ij}}. \qquad (11)$$

With complete markets, a closed-form solution for optimal consumption exists (Merton, 1990):

$$C = MPC \cdot W, \quad MPC \equiv r + \theta(\rho - r) + \frac{1}{2}\theta(1-\theta)\left(\frac{\alpha_W - r}{\sigma_W}\right)^2, \quad \alpha_W \equiv \sum_{i=1}^m \delta_i \alpha_i, \ (12)$$

where the drift and the volatility of total wealth are α_W and σ_W and total wealth also follows a Geometric Brownian Motion process:

$$dW = \alpha_W^* W dt + \sigma_W \bar{w} W dZ_W, \quad \alpha_W^* \equiv (\alpha_W - r)\bar{w} + r - MPC. \qquad (13)$$

The aggregate volatility of total wealth when portfolio weights are optimised is a weighted average of the volatility of each asset, $dZ_W = \frac{1}{\sigma_W}\sum_{i=1}^m \delta_i \sigma_i dZ_i$, and (13) has a solution $W(t) = W(0)\exp\left[\left(\alpha_W^* - \sigma_W^2 \bar{w}^2/2\right)t + \sigma_W \bar{w} Z_W(t)\right]$.

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m. It is then possible that oil and asset *A* are negatively correlated, $\sum_{j=1}^m \lambda_{Oj}\lambda_{Aj} < 0$, but the fund should nevertheless invest *less* in asset *A* to offset the exposure to shock *G*. The allocation of all other assets will have to adjust to hedge the effects of the remaining shocks, du_j for $j \neq g$.

Aggregate risk is managed by depressing consumption today to build a precautionary buffer of assets, as seen from the upward tilt of the expected consumption path in the final term of (11). The degree of tilt increases with the coefficient of relative prudence ($1 + 1/\theta$), the riskiness of the portfolio σ_W^2 , and the size of the risky portfolio in total wealth, $\bar{w}$. The buffer compensates future periods for bearing additional risk, but does not temporarily support consumption when asset prices are low, as here asset price shocks are random walks and thus persistent.

The optimal spending path can be achieved with a rule that consumes a fixed proportion of below and above-ground wealth, (12). The proportion is affected by a higher return on the safe asset through the intertemporal substitution effect (negative as future consumption has become cheaper) and the income effect (positive as lifetime wealth has gone up). The former dominates the latter if the elasticity of intertemporal substitution, θ, exceeds one. From (12) we see that the marginal propensity to consume, *MPC*, decreases with the return on the safe asset, *r*, and the average excess return on risky assets, $\alpha_W - r$; and increases with relative risk aversion, $1/\theta$, and fund volatility, σ_W . The proportion of total wealth consumed each period, *MPC*, should be less than its expected return $r_e = \bar{w}\alpha_W + (1-\bar{w})r$, so that both consumption and wealth rise over time.[18] The amount depends on prudence, as $MPC - r_e = -(1/2)(1+1/\theta)\bar{w}^2\sigma_W^2$, where $1+1/\theta$ is the coefficient of relative prudence and we have set $r = \rho$. This precautionary savings builds up a buffer of assets against future risk (Kimball, 1990) with absolute risk aversion, θ/C, falling as consumption rises.

[18] Optimal consumption is a fixed share of total wealth, but also incorporates precautionary saving. Oil is valued at a heavy discount rate but after extraction is replaced with less discounted financial assets, so the value of total wealth and consumption rises over time. Norway takes this to the limit, infinitely discounting future oil revenues and consuming only a fixed share of financial assets (see section 6).

With uncertain oil and asset prices and $r = \rho$, we observe from (13) how total above- and below-ground wealth evolves over time.[19] It rises due to the premium earned on risky assets, $\alpha_W \geq r$. It falls (rises) if the intertemporal substitution effect is dominated by the income effect in consumption[20] with the extent depending on the risk/return trade-off of total wealth, $-\theta(1-\theta)((\alpha_W - r)/\sigma_W)^2/2$.

3.3. Intergenerational equity and risk aversion: Epstein-Zin preferences

To capture intergenerational concerns relevant for the long investment horizons of sovereign wealth funds, it is important to separate the coefficient of relative risk aversion, *CRRA*, and the elasticity of intertemporal substitution, *EIS* or the coefficient of relative intergenerational inequality aversion, *IIA* = 1/*EIS* (Epstein and Zin, 1989). Restricting attention to one risky and one safe financial asset, we can show that the share of risky assets in total wealth and consumption are $\bar{w} = (\alpha - r)/(\sigma^2\ CRRA)$ and $C = \left(EIS \times \rho + (1-EIS)\left[r + (\alpha - r)^2/(2\ CRRA\ \sigma^2)\right]\right)W$ (see appendix A.4). These expressions extend (8) and (12) by departing from $EIS = 1/CRRA = 1/IIA = \theta$. If $EIS = 1$, the intertemporal substitution and income effects cancel out, so that the propensity to consume is independent of r, $C/F = \rho$. If $EIS > 1$ or $IIA < 1$, intertemporal substitution dominates and the risk-adjusted return in square brackets negatively impacts the propensity to consume. If $EIS < 1$, the income effect dominates and the risk-adjusted return increases the propensity to consume. The Euler equation becomes $(1/dt)E_t[dC] = [EIS \cdot (r - \rho) + EIS \cdot CRRA \cdot CRP \cdot w^2\sigma^2/2]C$, where the coefficient of relative prudence equals $CRP = 1 + 1/EIS = 1 + IIA$ for these preferences.

4. Investment restrictions and a given path of oil extraction

[19] Without oil or asset price uncertainty and $r = \rho$, any drop in below-ground wealth must be exactly compensated for by an increase in above-ground wealth to fully smooth consumption (Hartwick, 1977).
[20] That is, if the elasticity of intertemporal substitution θ is less (greater) than unity

4.1. Additional precautionary saving

Many funds restrict investment in certain asset classes for social and political reasons.[21] This is a form of incomplete markets which prevents the oil price being replicated by a bundle of traded financial securities. To illustrate this, assume that the fund cannot invest in a particular asset, so $\lambda_{Oh} \neq 0$ in (5) and the oil price is not fully spanned. In that case, there must be additional precautionary saving to cope with residual volatility.[22] With investment restrictions, the Euler equation can be approximated by (see appendix A.3):

$$\frac{\frac{1}{dt}E_t[dC]}{C} = \theta(r-\rho) + \frac{1}{2}(1+1/\theta)\left[\sigma_W^2\bar{w}^2 + \lambda_{Oh}^2\sigma_O^2\left(\frac{V}{W}\right)^2\right], \tag{14}$$

where $\bar{w}$ is given in (9) and σ_W in (11). Total wealth evolves according to:

$$\begin{aligned} dW = &\left(\sum_{i=1}^{m-1}\bar{w}_i W(\alpha_i - r) + \beta_h(\alpha_h - r) + rW - C\right)dt \\ &+\sigma_M\sum_{i=1}^{m-1}\bar{w}_i W dZ_i + \sigma_O\lambda_{Oh}V du_O. \end{aligned} \tag{15}$$

Hence, investment restrictions have both a precautionary and a wealth effect on consumption. The former arises as unspanned risk cannot be hedged optimally, whereas the latter because investment in a specific asset yielding high or low returns is not possible, as such an asset simply does not exist or investment in it is prohibited. Asset weights adjust to find the closest replicating bundle leaving only uncorrelated residual risk (see also appendix A.1.). The precautionary effect describes the additional savings needed because some oil price risk remains unhedged as in (14). The first term on the right-hand side is the usual slope of optimal consumption. The second term captures precautionary saving and is proportional to the coefficient of relative prudence, $CRP = (1 + 1/\theta)$. The term $\sigma_W^2\bar{w}^2$ inside the square brackets arises from the precautionary saving needed under complete markets when all oil price volatility is fully diversified.

[21] For example, Norway's fund does not invest in tobacco, military or coal assets amongst others.
[22] Earlier work ignored risky financial assets, an extreme case of incomplete markets (van den Bremer and van der Ploeg, 2013). Here we have risky assets too, but still allow for incomplete markets.

It is proportional to the variance of the portfolio of risky assets and the share of risky assets in the fund squared. The new term $\lambda_{Oh}^2\sigma_O^2\left(V/W\right)^2$ arises from the precautionary saving that is required if not all oil price volatility can be fully hedged. Less spanning of the oil price (a higher λ_{Oh}) implies that more precautionary saving is required, especially if oil wealth is volatile and comprises a large share of total wealth. Evidently, this effect diminishes as oil reserves are depleted and the ratio of *V* and *W* diminishes.

The wealth effect describes the change in the expected return on total wealth from not investing in a particular asset (see (15)). If an asset cannot be held by the fund (cf. asset *h* in (15)), there is still some exposure to it embodied in the oil price. With complete markets this exposure is offset inside the fund, so the net exposure is a constant share of total wealth. With incomplete markets this net exposure cannot be fully offset and will earn a rate of return, changing the expected return on total wealth. Its importance will diminish as oil reserves are depleted.

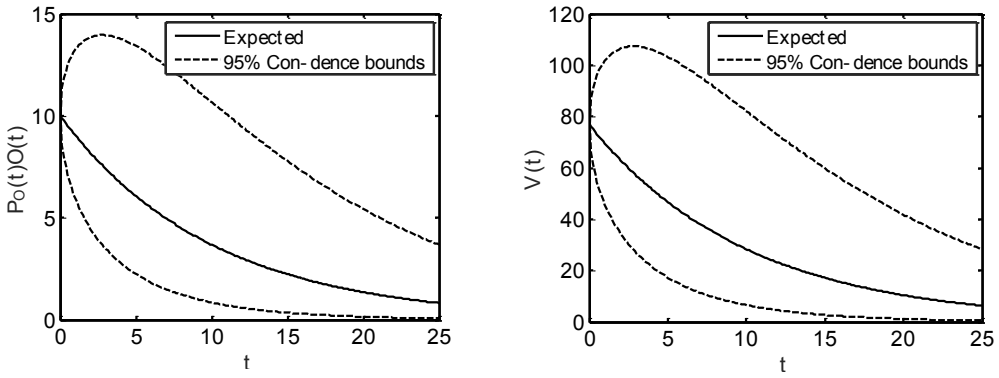
4.2. Stylized illustration of oil-CAPM model

We now illustrate how a sovereign wealth fund is affected by the presence of subsoil oil, depending on whether or not it has access to hedging assets. We suppose that there is a risk-free asset, *r*, and two risky assets: *1* uncorrelated with the oil price (the market asset) and *2* perfectly negatively correlated with the oil price (the hedging asset).[23] To ensure the latter asset is used for hedging only, we assume it has a zero excess return. This focuses our attention on the precautionary effect (and sets the wealth effect to zero). Figure 1 first gives the declining expected paths of oil revenues and oil wealth and their 95% confidence bounds.

Figure 1: Exogenous oil rents and the value of oil

(a) Oil revenues ***(b) Value of oil wealth***

[23] We set $F(0)=100$, $r=\rho=0.03$, $\theta = 0.5$ (or $\theta = 0.2$ when indicated) , $P_i(0) = 1$, $\sigma_i = 0.02$, $\rho_{ij} =0$ for $i,j=[A,B]$; $\alpha_1 = 0.07$, $\alpha_2 =0.03$, $S(0)=100$, $O(0) = 10$, $\kappa= 0.1$, $\alpha_O = 0$ and $\sigma_O = 0.25$.

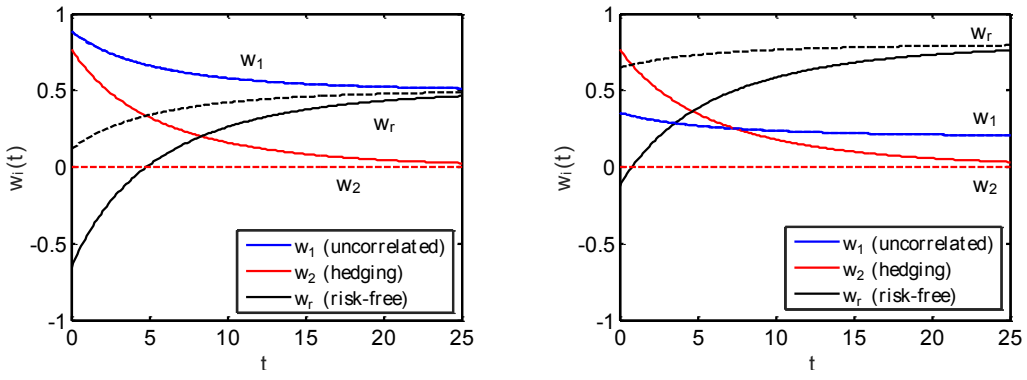


Complete markets

With complete markets there is leverage demand for both risky assets and a hedging demand for asset *2* that is negatively correlated with the oil price, as illustrated by the continuous lines in figure 2. These demands for each risky asset begin large but fall as oil reserves are depleted (cf. $\bar{w}_i V / F$) and exposure to oil prices diminishes.

Figure 2: Portfolio allocation without investment restrictions (solid) and with a ban on investing in the hedging asset *2* (dashed)

(a) Low prudence (CRP=3, CRRA=2) ***(b) High prudence (CRP=6, CRRA=5)***



To buy enough of risky asset *2* to fully hedge the oil price, the fund needs to borrow ("short") the risk-free asset. Without oil half of the fund is invested in the risky market asset and the other half in the risk-free asset: $\bar{w}_1 = 0.5,\ \bar{w}_2 = 0,\ \bar{w}_r = 0.5$. Increasing the coefficient of risk aversion will only reduce the demand for the market asset, leaving the

hedging demand unchanged, from equation (10) and as can be seen from comparing panels (a) and (b).

Figure 3: Optimal consumption and wealth with complete markets

(a) Consumption ***(b) Wealth***

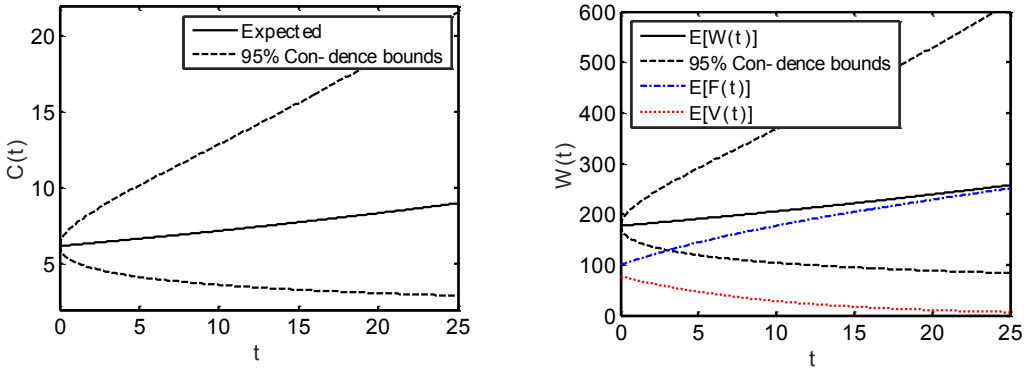


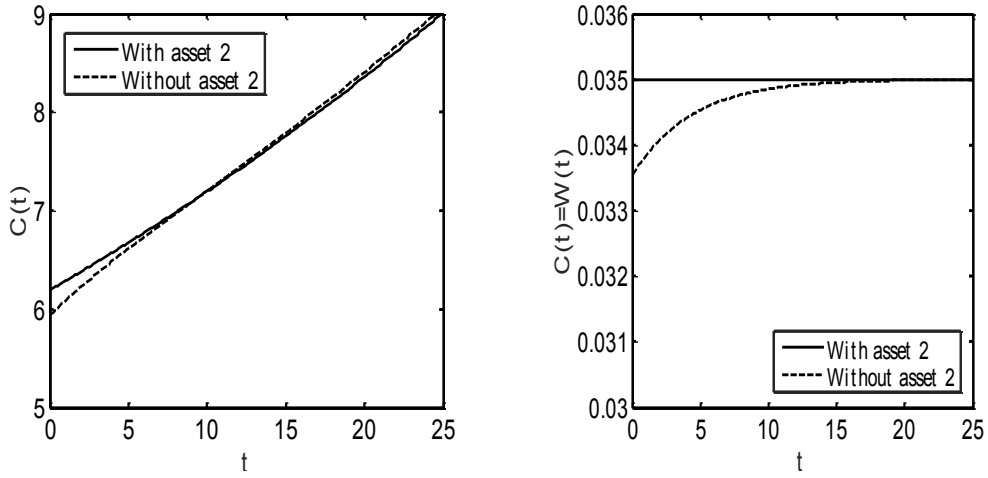
Figure 3 indicates that the consumption path is smoothed in face of declining and volatile oil revenues and grows in line with total above- and below-ground wealth to reflect precautionary saving. As oil wealth is run down (red dotted line in panel (b)), the fund is built up (blue dotted line) reflecting the basic insight that total wealth should grow at the same constant rate, if the oil price is completely spanned.

Investment restrictions: incomplete markets

Now consider the situation where the fund is prevented from investing in the risky hedging asset *2* or, equivalently, going short in an asset that correlates positively with the oil price. The dashed lines in figure 2 describe the case with investment restrictions, and indicate that the portfolio weight of the uncorrelated asset *1* is unaffected by restrictions on investing in the hedging asset (see (8)). The difference arises merely from the change in the drift of the fund *F* due to the precautionary effect discussed below. By restricting investment in the hedging asset (or, equivalently, preventing short positions in an asset that is positively correlated with the oil price), there is less need to borrow the safe asset (assuming pure hedging assets with zero excess return as in the numerical illustration, thus avoiding wealth effects). Residual volatility will then be managed by additional precautionary savings.

Figure 4: Optimal consumption without investment restrictions (solid), with a ban on investing in the hedging asset *2* (dashed)

(a) Consumption ***(b) Consumption share***



The effect of incomplete markets on consumption is illustrated by figure 4 for the case CRP =3 (CRRA = 2). Although not having access to the hedging asset (with zero excess return) does not have a direct effect on the expected evolution of total wealth, it leaves the consumer subject to additional now unhedgeable risk calling for additional precautionary savings. It is clear from panel (a) that initial consumption has to drop in favor of consumption at later times. This effect is larger for larger degrees of prudence, from equation (14). Panel (b) shows optimal consumption as a share of total wealth. If oil price risk cannot be hedged due to incomplete markets or investment prohibitions, the share of consumption in total wealth is no longer constant.

5. Portfolio allocation and spending with endogenous oil extraction

The optimal speed of extracting oil may be understood using the Hotelling rule. This states that the expected capital gains from keeping an additional barrel of oil in the ground must equal the return from extracting, selling and earning interest on it (Hotelling, 1931). We now extend this rule for volatile oil and financial asset prices.

5.1. Optimal rates of oil extraction

Since the data suggest that the oil price is positively correlated with financial assets, we proceed under this assumption.[24] Without loss of generality we assume that the oil price can be perfectly hedged with a single financial asset k, $dZ_O = dZ_k$. The policy maker chooses the consumption rate C, the rate of oil extraction O, and asset weights w_i, i =1, .., m to maximize expected welfare,

$$J(F,P_O,S,t) = \max_{C,w_i,O} E_t\left[\int_s^\infty U\left(C(s)\right)e^{-\rho(s-t)}ds\right], \tag{16}$$

subject to the budget constraint:

$$dF = \sum_{i=1}^{m} w_i(\alpha_i - r)Fdt + \left[rF + \Omega(P_O,O) - C\right]dt + \sum_{i=1}^{m} w_i F\sigma_i dZ_i, \tag{17}$$

the Geometric Brownian Motion processes for asset prices (3) and oil prices (4), and the reserve depletion equation

$$\frac{dS}{dt} = -O(t), \tag{18}$$

where oil rents are revenues minus extraction costs, $\Omega(P_O,O) \equiv P_O O - G(O)$, and total extraction costs are increasing in the extraction rate ($G'(O) > 0$) and convex to ensure a solution ($G''(O) > 0$) (cf., Pindyck, 1984) exists. Practically, the assumption of convexity corresponds to costs of extraction for a decision maker at a national level increasing more than proportionally when the rate of extraction is increased. [25] From the depletion equation (18), cumulative oil extraction cannot exceed initial reserves,

[24] Empirically the extent of this correlation varies over time, as is expected when the source of the oil price shock matters (Kilian, 2009). We abstract from this complication here.

[25] In practice, oil fields evolve stochastically as new fields are discovered and existing fields become economical (e.g., Pindyck, 1978). Extraction costs might be better captured by high upfront investment and small marginal costs. Reserves are also endogenous to exploration effort, but we abstract from these complications here.

$\int_0^\infty O(t)dt \le S_0$. It can be shown (see appendix A.5) that the optimal path for the expected rate of oil extraction satisfies the modified Hotelling rule:

$$\tfrac{1}{dt}E\left[d\Omega_O\right] = r\Omega_O + \left(-\frac{\tfrac{1}{dt}E\left[dJ_F d\Omega_O\right]}{J_F(F,P_O,S,t)}\right). \tag{19}$$

In the particular case of quadratic extraction costs, $G(O) = \gamma O^2/2, \gamma > 0,$ the stochastic path for oil extraction can be approximated by (for $\alpha_O = 0$):

$$dO \approx \left(-\tfrac{1}{\gamma}\left(r + \tfrac{\sigma_O}{\sigma_k}(\alpha_k - r)\right)P_O + \left(r + \tfrac{1}{2}\tfrac{\sigma_O}{\sigma_k}(\alpha_k - r)\right)O\right)dt + \tfrac{1}{2}O\sigma_O dZ_O. \tag{20}$$

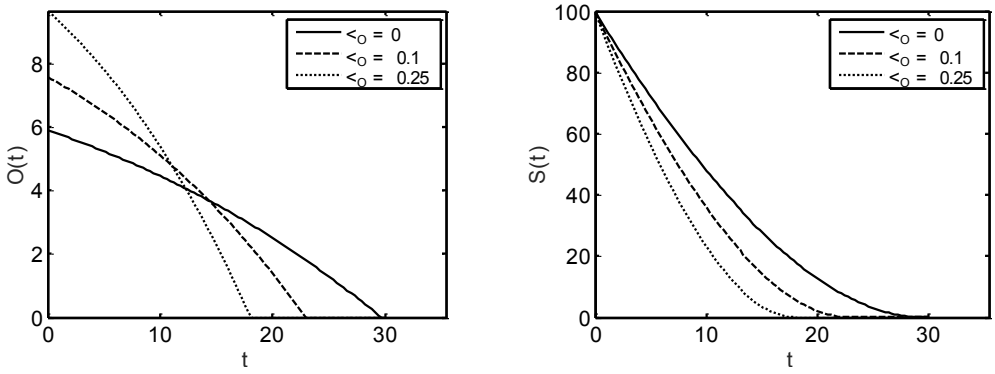
The stochastic Hotelling rule (19) states that the expected change in marginal oil rents must equal the return on safe assets plus a risk premium. Since we assume that oil and financial asset returns co-move positively, this premium is positive. High oil prices drive high marginal oil rents, which are associated with high fund values, *F*, and low marginal utility from an extra dollar in the fund ($\tfrac{1}{dt}E\left[dJ_F d\Omega_O\right] < 0$). The higher return compensates for the risk of holding oil in the ground (equal to $-\tfrac{1}{dt}E_t[dJ_F d\Omega_O]/(\mathrm{J}_F\,\Omega_O)$). If oil and asset markets are uncorrelated ($\tfrac{1}{dt}E\left[dJ_F d\Omega_O\right] = 0$), all oil price risk can be diversified and no risk premium is needed. The more correlated oil and asset markets are, the less oil price shocks can be diversified and the higher the risk premium. Figure 5 shows that oil price volatility implies that it is optimal to extract oil initially more quickly. As the rate of extraction drops, extraction costs fall non-linearly boosting the marginal return on oil extraction.

Equation (20) indicates that the optimal rate of oil extraction is positively correlated with the oil price, so that a sudden jump in the oil price requires a jump in the extraction rate to make the most of it. Oil price shocks affect the rate of extraction most when reserves (and in turn *O*) are highest, since this is when the majority of oil remains exposed to volatile prices. As the date of exhaustion approaches, the rate of oil extraction gets closer

to what it would be without volatile oil and asset prices. Note that the size of the fund does not matter for the optimal rate of oil extraction, only the properties of the assets in the background.

Figure 5: Endogenous oil extraction

a. Optimal rate of oil extraction ***b. Subsoil oil reserves***



Our finding that stochastic oil prices increase the oil extraction rate is consistent with earlier studies, but uses a different mechanism. Earlier work ignored financial assets and relied on "extractive prudence" driven by sufficiently convex marginal extraction costs, $G'''(O) > 0$ (Pindyck, 1981).[26] This means it is better to extract oil quickly because, once it is above ground and sold, it is no longer exposed to risk. By restricting our attention to quadratic extraction costs ($G'''(O) = 0$), we deliberately rule out this type of prudence. In our framework oil rents are still exposed to risk above the ground as they must be invested. Hence, oil should be treated as just another part of the total portfolio. The effect of risk on extraction is driven by "extractive risk aversion" ($G''(O)$) rather than by extractive prudence ($G'''(O)$) and so poses less onerous restrictions on extraction costs. Recent literature separates extraction and drilling decisions (e.g. Anderson et al., 2014). These models also display concavity in either or both choice

[26] Aggressive oil extraction also occurs with convex marginal utility arising from market power (van der Ploeg, 2010).

variable, so risk from above-ground financial markets will still speed up the optimal rate of extraction.[27]

5.2. Sovereign wealth funds with endogenous rates of oil extraction

With complete markets and without investment restrictions oil rents can be fully hedged by the fund, regardless of the path of oil extraction. This involves continuously adjusting the asset allocation so that the net exposure to risk remains a constant share of total above- and below-ground wealth. With complete markets oil wealth can be replicated with a bundle comprising the perfectly correlated asset k and the safe asset n, and the value of this bundle evolves according to (see appendix A.6 for a proof):

$$dV(t)+\Omega(t)dt=\left[rV(t)+(\alpha_k-r)\omega_k(O,t)V(t)\right]dt+\omega_k(O,t)V(t)\sigma_k dZ_k(t), \quad (21)$$

where $\omega_k(O,t)=N_kP_k/V$ is the continuously adjusted share of asset k in the replicating bundle. Total wealth evolves according to:

$$dW=\sum_{i=1}^{m}(\alpha_i-r)\bar{w}_iW+(rW-C)dt+\sum_{i=1}^{m}\sigma_i\bar{w}_iWdZ_i, \quad (22)$$

where

$$\bar{w}_i=w_i\left(F(t)/W(t)\right)\, i\neq k, \quad \bar{w}_k=w_k(t)\left(F(t)/W(t)\right)+\omega_k(O,t)\left(V(t)/W(t)\right). \quad (23)$$

Oil rents are no longer a Geometric Brownian Motion as in section 3, but driven by the drift $\mu_\Omega(P_O,S,t)dt$ and volatility $\sigma_\Omega(P_O,S,t)dZ_O$:

$$d\Omega=\mu_\Omega(P_O,S,t)dt+\sigma_\Omega(P_O,S,t)dZ_O. \quad (24)$$

[27] Other work estimates oil price volatility from options data and finds that it delays investment in Texas oil wells (Kellogg, 2014). However, this relies on a real options argument, whereas we focus on risk aversion and hedging.

The drift and volatility of oil rents are replicated by continuously reallocating the bundle of the perfectly correlated risky asset and the safe asset as P_O and S change. Holdings of asset k in the bundle are adjusted so that the change in oil rents, $\sigma_\Omega(P_O,S,t)dZ_O$, is matched by that in the bundle, $\omega_k(O,t)X(t)\sigma_k dZ_k$. The share of the safe asset is chosen so that the instantaneous drifts also match. As before, the fund is managed to ensure that net exposure to each financial asset is a constant share of total wealth: $\bar{w}_i = \delta_i w_i, i = 1,..,m$. Any exposure to asset k embodied in oil, $\omega_k(O,t)$, is offset by the asset's weight in the fund, $w_k(t)$, so that the net weight in total wealth is constant. By rearranging (23) holdings of each asset in the fund can, as before, be split up into a leveraged and a hedging component for the perfectly correlated asset k:

$$w_i = \bar{w}_i\left(\frac{F+V}{F}\right), i \neq k, \quad w_k(t) = \underbrace{\bar{w}_k + \bar{w}_k\left(\frac{V}{F}\right)}_{\text{leveraged demand}} + \underbrace{\left[-\omega_k(O,t)\left(\frac{V}{F}\right)\right]}_{\text{hedging demand}}. \tag{25}$$

As the asset allocation and consumption problems can be expressed in terms of total wealth (22), propositions 2 and 3 apply. Judicious management of the fund allows consumption to be smoothed in line with the permanent income hypothesis and to buffer consumption from oil price volatility by hedging it with traded financial assets.

6. Policy implications: Norway's Government Pension Fund Global

The policies of Norway's Government Pension Fund Global (GPFG) [28] closely follow standard CAPM recommendations ignoring oil wealth. Firstly, the GPFG uses the FTSE Global All Cap Index as the equity benchmark (with around 7,400 individual stocks, a

[28] At US$840 billion the GPFG is the largest single fund in existence, which was established in 1990 to smooth expenditure financed from oil after a period of fiscal volatility in the 1970s and 1980s. Evaluating governance, accountability and transparency, structure and behavior, the GPFG ranked first on the first two criteria and second overall, behind Alaska's US$45 billion permanent fund (Truman, 2008), and received the highest rating on the Linaburg-Maduell Transparency Index (SWF Institute, 2013). It has been called a "model" for sovereign wealth funds (Chambers, et al., 2012; Larsen, 2005).

close approximation of the market).[29] This is consistent with holding the optimal risky (or market) portfolio in (8) if $W = F$ instead of $W = F + V$. Secondly, the Ministry of Finance chooses the equity/bond mix, and in 2007 moved from 40/60% to 60/40%, as it was willing to accept more risk for a higher return. This is consistent with choosing the size of the risky portfolio based on preferences and the overall risk and return of the market, as in (9) with $W = F$. Thirdly, a fixed share of the fund (4% according to Norway's *handlingsregelen*) is consumed each year, as in (12) with $W=F$.

GPFG's management mandate does not mention oil wealth at all (NBIM, 2013), thus leaving Norway exposed to its large and volatile stock of oil wealth: the "elephant in the ground".[30] Norway, and other oil-rich countries with similar funds, would benefit by letting the asset allocation and the consumption rule in the GPFG vary over time.

Norway's asset allocation should vary over time to hedge as much of the volatility of remaining subsoil oil as possible.[31] In the first-best case described in section 3 this would involve taking large long positions in some industries, and large short positions in others (that may exceed the size of the fund), and reversing these positions as oil is extracted. Such highly leveraged positions expose the country to substantial risk if there are systematic shocks (Das and Uppal, 2004). They may also become illiquid, which invalidates the assumption of exogenous prices. Furthermore, the short positions assume that the covariance matrix is stable over time. In practice correlations between oil and each sector vary depending on the type of shock hitting the world economy (Kilian, 2009). As these correlations can only be estimated using past data and the size of the hedging positions are so large, there is the potential for large basis risk between oil and

[29] The benchmark is 60% equities, tracking the FTSE Global All Cap Index; up to 5% real estate, tracking the Investment Property Databank's Global Property Benchmark; and up to 40% bonds, of which 70% government and 30% corporate bonds, both tracking Barclays indices.

[30] Norway has proven reserves of nearly 9 billion barrels of oil and 73 trillion cubic feet of natural gas (BP, 2014). At 2013 prices these are worth US$ 945 billion and US$ 777 billion, respectively.

[31] Empirical simulations using the correlation of oil prices with financial assets indicate that Norway's exposure to aggregate oil price volatility is halved if oil wealth is hedged in the sovereign wealth fund (Gintschel and Scherer, 2008) and that the fund invests less aggressively in risky assets as it ages (Scherer, 2009; Balding and Yao, 2011). These studies focus on asset allocation but abstract from optimal consumption-saving decisions or oil extraction.

the hedging portfolio. Finally, as oil is extracted the highly leveraged positions must be reversed which will incur substantial transaction costs for a large fund.[32] Therefore the target index should not be rebalanced too frequently and portfolios should only be adjusted gradually.

A more pragmatic, second-best approach to asset allocation might be to only vary the equity/bonds mix.[33] This would be transparent and easy to explain to investors and the public. It would also notrequire short positions, have lower transaction costs, and would not rely on a large, time-varying correlation matrix covering all market assets. In this approach, the only risky asset is the overall equity market (e.g., the FTSE Global All Cap Index). If oil is sufficiently positively correlated with this market, the hedging demand to offset oil risk will exceed the leverage demand.[34] In this case, the GPFG should hedge the exposure of subsoil reserves to oil price risk by holding fewer equities and more safe assets while there is oil in the ground. Over time the oil reserves will be depleted and the exposure to equities embodied in subsoil oil will fall. This allows the above ground fund's equity exposure to rise, so that equities make up a greater share of the portfolio as oil is extracted.

The consumption rule should be a constant share of total assets, and thus should fall as a share of the fund as oil is extracted. If oil price risk is perfectly hedged as described in section 3, this rule should hold exactly. If hedging is imperfect, as would happen by only varying the equity/bond mix, slightly more precautionary savings would be needed. More precautionary saving is also needed if the fund faces a short-sales constraint. Recently, the fund has stopped investing in coal and oil stocks. If the aim is to hedge subsoil oil, it should go further by taking short positions in oil, gas and other stocks that

[32] See a recent report to the Norwegian *Storting* (Parliament) (Ministry of Finance, 2014a).
[33] Gintschel and Scherer (2008) impose short-sale constraints. This does not address the transactions costs that funds face by continuously rebalancing or potentially unstable correlations between assets.
[34] The correlation between the oil price and the overall equity market will also vary over time, though it will be more stable than a covariance matrix covering all 7,400 assets in the FTSE Global All Cap Index. Varying correlations will alter how quickly the equity share in the fund rises. Future work could account for this using regime-switching (cf. Ang and Bekaert, 2002).

are positively correlated with oil prices. If the aim is to protect the environment, spending should be curtailed to build up a buffer against less diversified risks. In general though, spending as a share of the fund should fall over time as above-ground assets account for an increasing share of total wealth.

These recommendations are relevant for the current debate in Norway. The fund excludes investments in certain assets for social and political reasons, such as tobacco and defense firms, and early 2015 also in assets affected by climate change and other environmental concerns such as coal, oil sands, cement and gold mining. In late 2014 Norway also established a government commission to assess its 4% spending rule due to concerns about excessive fiscal stimulus (Ministry of Finance 2014b). This follows declining spending as a share of GPFG assets, from nearly 6% in 2010 to below 3% in 2014, and there have been calls to limit spending to 3% in the future (Olsen, 2014).

7. Concluding remarks

Commodity exporters have two major types of national assets: natural resources below the ground and a sovereign wealth fund above it. Although some attempts to hedge commodity price volatility have been made, from long-term forward agreements in iron ore until 2010 to the purchase of oil options by Mexico in 2008, there is no evidence of systematic coordination of below- and above-ground assets. We have made the case for coordinating the management of these two types of asset by integrating the theories of portfolio allocation, precautionary saving, and optimal oil extraction under oil and asset price volatility.

Our main findings are as follows. Firstly, commodity exporters should change the allocation of their sovereign wealth fund by leveraging all risky assets and hedging subsoil oil risk. These effects are proportional to the ratio of oil and fund wealth, so unwind as resource reserves are depleted. Secondly, consumption should be a constant share of total oil and fund wealth. This stabilizes the mean and variance of spending as total wealth evolves steadily whilst oil reserves are replaced by financial assets, but

relies on the degree to which the oil price can be hedged by components of the above-ground portfolio. Thirdly, if oil wealth cannot be adequately hedged, less should be consumed initially in the interests of precautionary savings in the face of the additional unhedgeable risk that remains. Fourthly, the rate of oil extraction should be faster than predicted by the standard Hotelling rule if oil prices are volatile and positively correlated with financial markets. This generates a risk premium on subsoil oil, as convex extraction costs will fall faster than the rate of extraction. The size of the premium will depend on oil's correlation with the market, and disappears to zero if their returns are independent.

Our analysis attempts to offer a first step towards an integrated approach to managing sovereign wealth funds and natural resources under uncertainty. To do this we combine canonical models of asset allocation, precautionary savings and oil extraction. These models, while widely used and theoretically appealing, have received empirical criticism (Griffin, 1985; Jones, 1990; Fama and French, 2004; Anderson at al., 2014). Future work can address this along three dimensions. The first is to analyze the effect of financial assets on natural resources in more detail, allowing for the exploration and discovery of new reserves[35], and extraction decisions at the discrete well level (Kellogg, 2014; Anderson, et al., 2014; Venables, 2014). The second is to extend the analysis to include other non-financial assets such as domestic non-traded capital, human capital and pension liabilities, absorption constraints, general equilibrium effects of spending resource revenues,[36] and the benefits from structural reform to make the economy less vulnerable to commodity price volatility. Finally, there is scope for modelling oil and asset prices in more detail. In practice prices exhibit mean reversion (Wachter, 2002), stochastic volatility (Chacko and Viceira, 2005; Fouque et al., 2013), large jumps (Ngwira and Gerrard, 2007) and time-varying correlations (Bollerslev et al., 1988; Longin and Solnik, 1995). Although these extensions allow a better empirical testing of

[35] This would extend Pindyck (1978) to a setting with financial assets in order to understand how hedging oil price exposure affects exploration effort.
[36] Gaudet and Khadr (1991) and Atewemba and Gaudet (2012) allow for assets and capital scarcity.

our results, we conjecture that the qualitative nature of our policy insights will be unaffected.

References

Anderson, S., R. Kellogg and S. Salant (2014). Hotelling under pressure, Working Paper 20280, NBER, Cambridge, Mass

Ang, A. and G. Bekaert (2002). International asset allocation with regime shifts, *The Review of Financial Studies*, 15, 4, 1137-1187

Acharya, V. and L. Pedersen (2005). Asset pricing with liquidity risk, *Journal of Financial Economics*, 77, 375-410

Atewamba, C. and G. Gaudet (2012). Prices of durable nonrenewable natural resources under stochastic investment opportunities, Cahiers de Recherche 2012-01, CIREQ, University of Montreal.

Attanasio, O.P. and G. Weber (1989). Intertemporal substitution, risk aversion and the Euler equation for consumption, Economic Journal, 99, 395, 59-73.

Balding, C. and Y. Yao (2011). Portfolio Allocation for Sovereign Wealth Funds in the Shadow of Commodity-Based National Wealth. *International Finance Review*, 12, 293-312.

Bems, R. and I. de Carvalho Filho (2011). The Current Account and Precautionary Savings for Exporters of Exhaustible Resources, *Journal of International Economics*, 84, 1, 48–64.

Bodenstein, M., L. Guerrieri and L. Kilian (2012). Monetary policy responses to oil price fluctuations, *IMF Economic Review*, 60, 4, 470-504.

Bollerslev, T., Engle, R. F., and Wooldridge, J. M. (1988). A capital asset pricing model with time-varying covariances. *The Journal of Political Economy*, 116-131.

Bodie, Z., R.C. Merton and W.F. Samuelson (1992). Labor supply flexibility and portfolio choice in a life cycle model, *Journal of Economic Dynamics and Control*, 16, 3, 427-449.

BP (2014). Statistical Review of World Energy, BP, London, U.K.

Bremer, T.S. van den and F. van der Ploeg (2013). Managing and harnessing volatile oil windfalls, *IMF Economic Review*, 61, 1, 130-167.

Brown, K.C. and C.I. Tiu (2012). The interaction of spending policies, asset allocation strategies, and investment performance at university endowment funds, in How the Great Recession Affected Higher Education, *Chicago University Press*, Chicago.

Carroll, C.D. and M.S. Kimball (2008). Precautionary saving and precautionary wealth, *New Palgrave Dictionary of Economics*, 6, 579-584.

Case, K.E., J.M. Quigley and R.J. Shiller (2005). Comparing wealth effects: the stock market versus the housing market, *The B.E. Journal of Macroeconomics*, 5, 1, 1-34.

Chacko, G., and Viceira, L. M. (2005). Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets, *Review of Financial Studies*, 18(4), 1369-1402.

Chambers, D., E. Dimson and A. Ilmanen (2012). The Norway model, *Journal of Portfolio Management*, 38, 2, 67-81.

Constantinides, G.M. (1986). Capital market equilibrium with transaction costs, *Journal of Political Economy*, 94, 842-862.

Das, S. R., and R. Uppal (2004). Systemic risk and international portfolio choice. *Journal of Finance*, 59(6), 2809-2834.

Duffie, D. and L.G. Epstein (1992). Stochastic differential utility, *Econometrica*, 60(2), 353-394.

Epstein, L.G. and S.E. Zin (1989). Substitution, risk aversion, and the temporal behavior of consumption growth and assets returns: a theoretical framework, *Econometrica*, 57, 4, 937-969.

Fama, E. F., and French, K. R. (2004). The capital asset pricing model: theory and evidence. *Journal of Economic Perspectives*, 18, 25-46.

Flavin, M. and T. Yamashita (2002). Owner-occupied housing and the composition of the household portfolio, *American Economic Review*, 92, 1, 345-362.

Fouque, J. P., Sircar, R., & Zariphopoulou, T. (2013). Portfolio optimization & stochastic volatility asymptotics. Available at SSRN 2473289.

Gaudet, G. (2007). Natural resource economics under the rule of Hotelling, *Canadian Journal of Economics*, 40, 4, 1033-1059.

Gaudet, G. and A.M. Khadr (1991). The evolution of natural resource prices under stochastic investment opportunities: an intertemporal asset-pricing approach, *International Economic Review*, 441-455.

Garleanu, N. and L.H. Pedersen (2013). Dynamic trading with predictable returns and transaction costs, *Journal of Finance*, 68, 6, 2309-2340.

Gintschel, A. and B. Scherer (2008). Optimal asset allocation for sovereign wealth funds, *Journal of Asset Management*, 9, 3, 215-238.

Girsanov, I. V. (1960). On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. *Theory of Probability & Its Applications*, 5, 3, 285-301.

Griffin, J.M. (1985). OPEC behavior: a test of alternative hypotheses, *American Economic Review*, 75 (5), 954–63.

Hartwick, J.M. (1977). Intergenerational equity and the investing of rents from exhaustible resources, *American Economic Review*, 67, 5, 972-974.,

Hotelling, H. (1931). The economics of exhaustible resources, *Journal of Political Economy*, 39, 2, 137-175.

IMF (2013). World Economic Outlook Database, International Monetary Fund, Washington, D.C.

Jong, F. and J. Driessen (2015). Can large long-term investors capture illiquidity premiums, *Bankers, Market & Investors*, No. 134, January-February, 27-53

Jones, Clifton T. (1990). OPEC behaviour under falling prices: implications for cartel stability, *The Energy Journal*, 11 (3), 117–130.

Kellogg, R. (2014). The effect of uncertainty on investment: evidence from Texas oil drilling, *American Economic Review*, 104(6), 1698-1734.

Kimball, M.S. (1990). Precautionary saving in the small and in the large, *Econometrica*, 58, 1, 53-73.

Kilian, L. (2009). Not all oil price shocks are alike: Disentangling demand and supply shocks in the crude oil market. *American Economic Review*, 99, 3, 1053-1069.

Koo, H.K. (1998). Consumption and portfolio selection with labor income: a continuous time approach, *Mathematical Finance*, 8, 1, 49-65.

Larsen, E.R. (2005). Are rich countries immune to the resource curse? Evidence from Norway's management of its oil riches, *Resources Policy*, 30, 2, 75-86.

Leland, H.E. (1968). Saving and uncertainty: the precautionary demand for saving, *Quarterly Journal of Economics*, 82, 3, 465-473.,

Longin, F. and Solnik, B. (1995). Is the correlation in international equity returns constant: 1960–1990?, *Journal of International Money and Finance*, 14, 3-26.

Markowitz, H. (1952). Portfolio selection, *Journal of Finance*, 7, 1, 77-91.

Markowitz, H. (1959). Portfolio Selection: Efficient Diversification of Investments, *Yale University Press*, New Haven, Connecticut.

Merton, R.C. (1971). Optimal consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory*, 3, 4, 373-413.

Merton, R.C. (1990). Continuous-Time Finance, *Basil Blackwell*, New York and Oxford.

Merton, R. C. (1993). Optimal investment strategies for university endowment funds. In Studies of Supply and Demand in Higher Education (pp. 211-242). *University of Chicago Press*.

Ministry of Finance (2008). The management of the government pension fund in 2008, *Report to the Storting number 20*, Norwegian Ministry of Finance, Oslo.

Ministry of Finance (2014a). The management of the Government Pension Fund in 2013 (Section 2.4), *Report to the Storting number 19*, Norwegian Ministry of Finance, Oslo.

Ministry of Finance (2014b). *The National Budget 2014: A Summary*, Norwegian Ministry of Finance, Oslo.

NBIM (2013). Management Mandate, www.nbim.no, Norges Bank Investment Management, Oslo.

Ngwira, B. and Gerrard, R. (2007). Stochastic pension fund control in the presence of Poisson jumps, *Insurance: Mathematics and Economics*, 40(2), 283-292.

Olsen, O. (2014). Address by Governor Oystein Olsen to the Supervisory Council of Norges Bank and invited guests on Thursday 13 February 2014, Norges Bank, Oslo.

Pindyck, R.S. (1978). The optimal exploration and production of nonrenewable resources, *Journal of Political Economy*, 86, 5, 841-861.

Pindyck, R.S. (1980). Uncertainty and exhaustible resource markets, *Journal of Political Economy*, 88, 6, 1203-1225.

Pindyck, R.S. (1981). The optimal production of an exhaustible resource when price is exogenous and stochastic, *Scandinavian Journal of Economics*,83, 2, 277-288.

Pindyck, R.S. (1984). Uncertainty in the theory of renewable resource markets, *Review of Economic Studies*, 51, 2, 289-303.

Ploeg, F. van der (2010). Aggressive oil extraction and precautionary saving, *Journal of Public Economics*, 94, 5, 421-433.

Sandmo, A. (1970) The effect of uncertainty on saving decisions, *Review of Economic Studies*, 37, 3, 353-360.

Scherer, R. (2009). Portfolio choice for oil-based sovereign wealth funds, EDHEC-Risk Institute, EDHEC Business School, Nice, France.

Sharpe, W.F. (1964). Capital and asset prices: a theory of market equilibrium under conditions of risk, *Journal of Finance*, 19, 3, 425-442.,

Sinai, T. and N.S. Souleles (2005). Owner-occupied housing as a hedge against rent risk, *Quarterly Journal of Economics*, 120, 2, 763-789.

Solow, R.M. (1974). The economics of resources or the resources of economics, *American Economic Review*, 64, 2, 1-14.

Svensson, L.E. and I.M. Werner (1993). Nontraded assets in incomplete markets: pricing and portfolio choice, *European Economic Review*, 37, 5, 1149-1168.

SWF Institute (2013). Sovereign Wealth Fund Rankings, Webpage www.swfinstitute.org, Sovereign Wealth Institute.

Tobin, J. (1958). Liquidity preference as behavior towards risk, *Review of Economic Studies*, 25,2, 65-86.

Truman, E.M. (2008). A blueprint for sovereign wealth fund best practices, Policy Brief, Peterson Institute for International Economics, Washington, D.C.

Venables, A.J. (2014). Depletion and development: natural resource supply with endogenous field opening, *Journal of the Association of Environmental and Resource Economists*, 1, 3, 313-336.

Wachter, J. A. (2002). Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets, *Journal of Financial and Quantitative Analysis*, 37(01), 63-91.

Wang, C., N. Wang and J. Yang (2013). Optimal consumption and savings with stochastic income, *NBER Working Paper 19319*, Cambridge, Mass.

Zeldes, S.P. (1986). Optimal consumption with stochastic income: deviations from certainty equivalence, *Quarterly Journal of Economics*, 104, 2, 275-298.

Appendix A: Derivations

A.1. Valuing oil with exogenous oil extraction

Asset returns are assumed to be normally distributed and can be expressed as a linear combination of m independent shocks, $dZ = \Lambda^* du^*$ where du^* is an m x 1 vector. If the oil price is completely spanned by the market, it can be expressed as a linear combination of these m shocks: $dZ_O = \Lambda_O^* du^*$. Now, in order to study incomplete markets, remove one asset from the investment set. The returns on the remaining $m-1$ assets can now be expressed as a linear combination of $m-1$ (different) independent shocks, $dZ = \Lambda\, du$ where du is an $(m-1)$ x 1 vector. If oil returns are expressed in terms of these shocks, there is a residual part that is uncorrelated with the market, $dZ_O = \lambda_{Oh}\, du_h + \Lambda_O\, du$, as in (5). Although the asset that is removed from the investment set may be correlated with other assets, the unhedgeable component of the oil price is not. Here we are concerned with valuing subsoil oil and so ignore investment restrictions. Any asset that is outside

the investment set can still be observed and can be used to value oil wealth. The value thus derived is a market value. Taking equation (5) with $\lambda_{Oh} = 0$, we can express the oil price as:

$$P_O(t) = P_O(0)\exp(-\phi t)\prod_{i=1}^{m}\left[\frac{P_i(t)}{P_i(0)}\right]^{\beta_i}, \tag{A1}$$

with $\phi \equiv -\alpha_0 + \sum_{i=1}^{m}\beta_i\left(\alpha_i - \frac{1}{2}\sigma_i^2\right) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\beta_i\beta_j\sigma_{ij}$ and $\beta_i \equiv \sigma_O M_i / \sigma_i, M_i \equiv \left[\Lambda_O \Lambda^{-1}\right]_i$,

which can be readily verified using Ito's lemma and comparing coefficients with (4).

Lemma A1: With complete markets, the capitalized value of oil income is:

$$V(P_O, t) = P_O(t)O(t)/\psi, \quad \psi \equiv r + \kappa - \alpha_O + \sum_{i=1}^{m}\beta_i(\alpha_i - r). \tag{A2}$$

Derivation: Firstly, we construct a portfolio with value $V(P_1, .., P_m, t)$ which consists of assets $1, .., n$ that is identical to the capitalized value of oil and distributes an amount of cash equal to $P_O(t)O(t)$ per unit time. This value evolves according to:

$$dV = (\mu_V V - P_O O)dt + \sigma_V V dZ_V. \tag{A3}$$

With the aid of Ito's lemma the dynamics of the portfolio can be written as:

$$dV = \sum_{i=1}^{m}V_i dP_i + V_t dt + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}V_{ij}dP_i dP_j = \left[\sum_{i=1}^{m}\alpha_i V_i P_i + V_t + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sigma_{ij}V_{ij}P_i P_j\right]dt + \sum_{i=1}^{m}\sigma_i V_i P_i dZ_i, \tag{A4}$$

where $V_i = \partial V / \partial P_i$ and $\sigma_{ij} = \sigma_i\sigma_j\rho_{ij}$. Comparing coefficients with (A3) gives:

$$\mu_V V - P_O O = \sum_{i=1}^{m}\alpha_i P_i V_i + V_t + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sigma_{ij}V_{ij}P_i P_j, \quad \sigma_V V dZ_V = \sum_{i=1}^{m}\sigma_i P_i V_i dZ_i. \tag{A5}$$

Finally, let $dZ_V = \Lambda_V du$. This implies:

$$\sigma_V dZ_V = \sigma_V \Lambda_V du = \Gamma' dZ = \Gamma' \Lambda du, \quad \Gamma \equiv \left[V_1\sigma_1 P_1 / V, .., V_m\sigma_m P_m / V\right]. \tag{A6}$$

Secondly, we create another portfolio with value $X(t)$ that consists of oil wealth $V(t)$, the risky assets and the safe asset. This portfolio is dynamically constructed, so short positions offset long positions, there is no net risk, and the net value of the portfolio is

always zero. Hence, the weight of the safe asset in total wealth is $w_r = -w_V - \sum_{i=1}^{m} w_i$, where w_V is the weight of oil in total wealth. The return to this portfolio is:

$$\begin{aligned} dX &= w_V\left(\frac{dV + P_O O dt}{V}\right) + \sum_{i=1}^{m} w_i\left(\frac{P_i}{P}\right) + w_r r dt \\ &= \left[w_v(\mu_V - r) + \sum_{i=1}^{m} w_i(\alpha_i - r)\right]dt + w_V \sigma_V dZ_V + \sum_{i=1}^{m} w_i \sigma_i dZ_i \\ &= \left[w_v(\mu_V - r) + \sum_{i=1}^{m} w_i(\alpha_i - r)\right]dt + w_V \Gamma' \Lambda du + \Psi \Lambda du, \end{aligned} \tag{A7}$$

where the second equality follows from (A3), the third equality from (A6) and $\Psi \equiv [w_1\sigma_1, .., w_m\sigma_m]'$. Suppose that the weights in this new portfolio are dynamically constructed so that there is no risk: $w_V \Gamma' \Lambda du + \Psi \Lambda du = 0$ and the last two terms in the last equality of (A7) vanish. The weights that would achieve this are $w_i = -(V_i / V)P_i w_V, i = 1, .., m.$ Arbitrage dictates that such a constructed portfolio must have a zero expected excess return over the risk-free rate:

$$0 = w_V(\mu_V - r) + \sum_{i=1}^{m} w_i(\alpha_i - r), \qquad V(\mu_v - r) = \sum_{i=1}^{m} V_i P_i(\alpha_i - r). \tag{A8}$$

Combining (A8) with (A5) gives the following optimality condition:

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \sigma_{ij} P_i P_j V_{ij} + \sum_{i=1}^{m} r P_i V_i - rV + V_t + P_O O = 0. \tag{A9}$$

Thirdly, the proposed capitalized value of oil income and associated partials,

$$\begin{aligned} &V(P_O, t) = \frac{1}{\psi} P_O(0) \exp(-\phi t) \prod_{i=1}^{m}\left[\frac{P_i(t)}{P_i(0)}\right]^{\beta_i} O(0) \exp(-\kappa t), \quad V_i = \frac{\beta_i V}{P_i}, \\ &V_t = -\phi V, \quad V_{ii} = \frac{\beta_i(\beta_i - 1)V}{P_i^2}, \quad V_{ij} = \frac{\beta_i \beta_j V}{P_i P_j}, j = 1, .., m, \quad i = 1, .., m, \end{aligned} \tag{A10}$$

satisfy (A9) by substitution. Lemma A1 thus gives capitalized oil income.

Lemma A1 establishes (6). The instantaneous rate of change in the value of oil income is found by applying Ito's lemma to this equation to give:

$$P_O O dt + dV = \left[r + \sum_{i=1}^{m} \beta_i(\alpha_i - r)\right]V dt + \sigma_O V dZ_O. \tag{A11}$$

The result in (7) follows from substituting (A11), (2), and (5) with $\lambda_{O,h}=0$ into the expression for total wealth, $dW = dF + dV$. With an investment restriction on asset m, the derivation for the value of the windfall is analogous and (A1) still holds. Asset m is then replaced by the unspanned component of the oil price h and $\beta_h=\left(\sigma_O/\sigma_h\right)\lambda_{Oh}$.

A.2. Asset allocation with exogenous oil extraction

Here we derive the optimal portfolio weights in a sovereign wealth fund in the presence of oil, with and without investment restrictions based on Merton (1990). We begin by restricting investment in asset h, so $\lambda_{Oh}\neq 0$ and the fund holds m–1 securities. The unspanned component of the oil price is uncorrelated with other assets:

$$dP_h=\alpha_h P_h dt+\sigma_h P_h du_h, \tag{A12}$$

Note that m was a traded asset that was correlated with all other assets. Above-ground wealth is accumulated according to (2). We obtain:

$$\begin{aligned} dF &= \sum_{i=1}^{m-1} w_i F\left(\alpha_i - r\right)dt + \left(rF + P_O O - C\right)dt + \sum_{i=1}^{m-1} w_i F\sigma_i dZ_i \\ dV + P_O O dt &= \left(r + \sum_{i=1}^{m-1}\beta_i\left(\alpha_i - r\right) + \beta_h\left(\alpha_h - r\right)\right)V dt + \sigma_O V\left(M dZ + \lambda_{O,h} du_O\right). \end{aligned} \tag{A13}$$

The Hamilton-Jacobi-Bellman (HJB) equation is:

$$\max_{w_i, C}\left[U\left(C\right)e^{-\rho t} + \tfrac{1}{dt}E_t\left[dJ(F,V,t)\right]\right] = 0, \tag{A14}$$

$$\begin{aligned} \tfrac{1}{dt}E_t\left[dJ\right] = {} & J_t + J_F\left(\sum_{i=1}^{m-1} w_i F\left(\alpha_i - r\right) + rF + P_O O - C\right) + J_{VF} VF\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i\beta_j\sigma_{ij} \\ & + J_V\left(\left(r + \sum_{i=1}^{m-1}\beta_i\left(\alpha_i - r\right) + \beta_h\left(\alpha_h - r\right)\right)V - P_O O\right) \\ & + \frac{1}{2}J_{FF}F^2\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i w_j\sigma_{ij} + \frac{1}{2}J_{VV}V^2\left[\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\beta_i\beta_j\sigma_{ij} + \sigma_O^2\lambda_{O,0}^2\right]. \end{aligned} \tag{A15}$$

The first-order conditions with respect to C and w_i are:

$$U'\left(C\right)e^{-\rho t} - J_F = 0 \Rightarrow J_F = U'\left(C\right)e^{-\rho t} \tag{A16}$$

$$J_F F\left(\alpha_i - r\right) + J_{FF}F^2\sum_{j=1}^{m-1} w_j\sigma_{ij} + J_{FV}FV\sum_{j=1}^{m-1}\beta_j\sigma_{ij} = 0. \tag{A17}$$

Equation (A17) gives the optimal weights in the fund:

$$\begin{aligned} w_i &= -\frac{J_F}{FJ_{FF}}\sum_{j=1}^{m-1}\nu_{ij}\left(\alpha_j - r\right) - \frac{J_{FV}}{J_{FF}}\frac{V}{F}\beta_i \\ &= \frac{C/F}{\partial C/\partial F}\theta\sum_{j=1}^{m-1}\nu_{ij}\left(\alpha_j - r\right) - \frac{\partial C/\partial V}{\partial C/\partial F}\frac{V}{F}\beta_i. \end{aligned} \tag{A18}$$

If markets are complete, $\partial C/\partial F = \partial C/\partial V = \partial C/\partial W = MPC$ from (12). If markets are incomplete, instead of solving the arising partial differential equations numerically, we approximate these partials from the complete markets case or, alternatively, assume that consumption is a linear function of total wealth. With and without investment restrictions we then obtain:

$$w_i = \frac{W}{F}\theta\sum_{j=1}^{m-1}\nu_{ij}\left(\alpha_j - r\right) - \frac{V}{F}\beta_i. \tag{A19}$$

Defining $\bar{w}_i W \equiv w_i F + \beta_i V$, rearranging (A19) gives (8) and (10).

A.3. Optimal consumption with exogenous oil extraction

If markets are complete we can find a closed-form solution for the value function $J(F,V,t) = J(W \equiv F+V,t)$ from Merton (1990). Substituting the first-order conditions (A16) and (A17) into the HJB equation (A14) gives:

$$0 = \tfrac{1}{\theta-1}\exp(-\theta\rho t)\,\mathrm{J}_W^{1-\theta} + J_t + rWJ_W - \frac{J_W^2}{J_{WW}}\frac{(\alpha_W - r)^2}{2\sigma_W^2}. \tag{A20}$$

The closed-form solution to this stochastic partial differential equation is:

$$J(W,t) = \tfrac{\theta}{\theta-1}\exp(-\rho t)\left[\theta\rho - (\theta-1)\eta\right]^{-1/\theta} W^{(\theta-1)/\theta}, \quad \eta = r + \theta(\alpha_W - r)^2/2\sigma_W^2. \tag{A21}$$

(12) follows from substituting (A21) into (A16). Applying Ito's lemma to (A16):

$$\frac{\frac{1}{dt}E_t[dJ_F]}{J_F} = \frac{C''(C)}{U'(C)}\frac{\frac{1}{dt}E_t[dC]}{C} - \rho + \frac{1}{2}\frac{CU'''(C)}{U'(C)}\frac{\frac{1}{dt}E_t[dC^2]}{C}. \tag{A22}$$

Using Ito's lemma we obtain:

$$\begin{aligned} dJ_F(F,V,t) = {} & J_{FF}dF + J_{FV}dV + J_{Ft}dt + \tfrac{1}{2}J_{FFF}dF^2 \\ & + \tfrac{1}{2}J_{FVV}dV^2 + J_{FFV}dFdV. \end{aligned} \tag{A23}$$

In addition the derivative of (A14) with respect to F is:

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$$
\begin{aligned}
0 = {} & J_{tF} + J_{FF}\left(\sum_{i=1}^{m-1} w_i F\left(\alpha_i - r\right) + rF + P_O O - C\right) + J_F\left(\sum_{i=1}^{m-1} w_i\left(\alpha_i - r\right) + r\right) \\
& + J_{VF}\left(\left(r + \sum_{i=1}^{m-1}\beta_i\left(\alpha_i - r\right) + \beta_h\left(\alpha_h - r\right)\right)V - P_O O\right) \\
& + \frac{1}{2} J_{FFF} F^2 \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i w_j \sigma_{ij} + J_{FF} F \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i w_j \sigma_{ij} + J_{VF} V \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i \beta_j \sigma_{ij} \\
& + \frac{1}{2} J_{VVF} V^2 \left[\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\beta_i\beta_j\sigma_{ij} + \sigma_O^2\lambda_{Oh}^2\right] + J_{VFF} VF \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i \beta_j \sigma_{ij}.
\end{aligned}
\tag{A24}
$$

Substituting (A17) and (A23) into (A24) gives:

$$
0 = \tfrac{1}{dt} E_t\left[dJ_F\right] + J_F r. \tag{A25}
$$

We also have:

$$
\tfrac{1}{dt} E_t\left[\left(dC\right)^2\right] = C_F^2 \tfrac{1}{dt} E_t\left[\left(dF\right)^2\right] + C_V^2 \tfrac{1}{dt} E_t\left[\left(dV\right)^2\right] + 2C_V C_F \tfrac{1}{dt} E_t\left[dVdF\right], \tag{A26}
$$

$$
\begin{aligned}
& \tfrac{1}{dt} E_t\left[\left(dF\right)^2\right] = F^2 \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i w_j \sigma_{ij}, \quad \tfrac{1}{dt} E_t\left[dVdF\right] = VF \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i \beta_j \sigma_{ij}, \\
& \tfrac{1}{dt} E_t\left[\left(dV\right)^2\right] = V^2\left(\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\beta_i\beta_j\sigma_{ij} + \sigma_O^2\lambda_{Oh}^2\right).
\end{aligned}
\tag{A27}
$$

Combining (A26) and (A27), we obtain:

$$
\begin{aligned}
& \frac{1}{dt} E_t[dC^2] = C_W^2\left[\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}(w_i F + \beta_i V)(w_j F + \beta_j V)\sigma_{ij} + \lambda_{Oh}^2\sigma_O^2 V^2\right] \\
& = C_W^2\left(\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\bar{w}_i\bar{w}_j\sigma_{ij}W^2 + \lambda_{Oh}^2\sigma_O^2 V^2\right) = C_W^2\left(\bar{w}^2{\sigma_W}^2 W^2 + \lambda_{Oh}^2\sigma_O^2 V^2\right),
\end{aligned}
\tag{A28}
$$

where we use $C_W \approx C/W$, $C_W \approx C_F \approx C_V$. The stochastic Euler equations (11) and (14) follow from substituting (A25) and (A28) into (A22). Equation (11) assumes complete markets, so $\lambda_{Oh} = 0$ and $C_W = C/W$ from (12). The result is exact. Equation (14) assumes incomplete markets, so $\lambda_{Oh} \neq 0$ and we use $C_W \approx C/W$ in order to obtain an approximate solution in the absence of a closed-form one.

A.4. Complete markets and exogenous oil paths: Epstein-Zin preferences

The results in section 3.3 follow from solving the HJB equation in the undiscounted value function *J*(*F*) modified for Epstein-Zin preferences (Duffie and Epstein, 1992):

$$0 = \max_{w,C}\left\{\frac{\rho}{1-1/EIS}\frac{C^{1-1/EIS}-\left((1-CRRA)J(F)\right)^{\frac{1-1/EIS}{1-CRRA}}}{\left[(1-CRRA)J(F)\right]^{\frac{CRRA-1/EIS}{1-CRRA}}}+\ldots\right.$$
$$\left.+J'(F)\left[rF-C+w(\alpha-r)F\right]+\frac{1}{2}J''(F)w^2\sigma^2F^2\right\}. \quad \text{(A29)}$$

It can be verified that (A29) has the solution:

$$J(F)=\frac{(\Theta F)^{1-CRRA}}{1-CRRA},\Theta=\left(\rho\cdot EIS+\left[r+(\alpha-r)^2/\left(2\sigma^2\cdot CRRA\right)\right](1-EIS)\right)^{\frac{1}{1-EIS}}\rho^{\frac{EIS}{EIS-1}}. \quad \text{(A30)}$$

A.5. Endogenous oil extraction

The HJB equation for the problem in (16), (17), (3), (4) and (18) is:

$$0=\max_{C,w_i,O}\left\{U(C)e^{-\rho t}+\tfrac{1}{dt}E_t\left[dJ(F,P_O,S,t)\right]\right\},$$
$$\tfrac{1}{dt}E_t\left[dJ(F,P_O,S,t)\right]=J_F\left[\sum_{i=1}^{m}w_i(\alpha_i-r)F+rF-C+P_OO-G(O)\right]+J_P\alpha_OP_O \quad \text{(A31)}$$
$$-J_SO+J_t+\frac{1}{2}J_{FF}F^2\sum_{i=1}^{m}\sum_{j=1}^{m}w_iw_j\sigma_{ij}+\frac{1}{2}J_{PP}\sigma_O^2P_O^2+J_{FP}\sigma_OP_OF\sum_{i=1}^{m}w_i\sigma_i\rho_{iO}.$$

The first-order conditions are:

$$U'(C)e^{-\rho t}=J_F, \quad \text{(A32)}$$

$$J_FF(\alpha_i-r)+J_{FF}F^2\sum_{j=1}^{m}w_j\sigma_{ij}+J_{FP}F\sigma_i\sigma_O\rho_{iO}P_O=0 \quad \forall i, \quad \text{(A33)}$$

$$J_F\left[P_O-G'(O)\right]-J_S=0. \quad \text{(A34)}$$

Differentiating (A31) with respect to the states invoking the envelope theorem

$$\tfrac{1}{dt}E\left[dJ_F\right]+J_F\left[r+\sum_{i=1}^{m}w_i(\alpha_i-r)\right]+J_{FF}F\sum_{i=1}^{m}\sum_{j=1}^{m}w_iw_j\sigma_{ij}+J_{FP}\sigma_OP_O\sum_{i=1}^{m}w_i\sigma_i\rho_{iO}=0, \quad \text{(A35)}$$

$$\tfrac{1}{dt}E\left[dJ_S\right]=0, \quad \text{(A36)}$$

$$\tfrac{1}{dt}E\left[dJ_P\right]+J_FO+J_P\alpha_O+J_{PP}\sigma_O^2P_O+J_{FP}F\sum_{i=1}^{m}w_i\sigma_i\sigma_O\rho_{iO}=0. \quad \text{(A37)}$$

Upon substitution of (A33) into (A35), we obtain:

$$\tfrac{1}{dt}E\left[dJ_F\right]=-rJ_F. \quad \text{(A38)}$$

Applying Ito's lemma to (A32) and combining the result with (A38) gives:

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$$\frac{\frac{1}{dt}E_t[dC]}{C}=\left[\frac{-U'(C)}{CU''(C)}\right](r-\rho)-\left[\frac{CU'''(C)}{U''(C)}\right]\frac{\frac{1}{dt}E_t\left[dC^2\right]}{C^2}. \tag{A39}$$

Applying Ito's lemma to (A34) gives:

$$\frac{dJ_S}{J_S}=\frac{dJ_F}{J_F}+\frac{d\Omega_O}{\Omega_O}+\frac{dJ_F d\Omega_O}{J_F\Omega_O},\quad \Omega_O=P_O-G'(O). \tag{A40}$$

Combining (A36), (A38) and (A40) yields (19). Ito's lemma yields:

$$dJ_F=-rJ_F dt+J_{FF}F\sum_{i=1}^{m}w_i\sigma_i dZ_i+J_{FP}P_O\sigma_O dZ_O, \tag{A41}$$

$$dO=\mu_O(F,P_O,S,t)dt+O_F F\sum_{i=1}^{m}w_i\sigma_i dZ_i+O_P P_O\sigma_O dZ_O, \tag{A42}$$

where use (A38) and $\mu_O(F,P_O,S,t)=(1/dt)E_t[dO]$ is the to be determined expected rate of growth of the rate of oil extraction. Applying Ito's lemma to $\Omega_O=P_O-G'(O)=P_O-\gamma O$ gives:

$$\begin{aligned}d\Omega_O&=dP_O-G''(O)dO-\frac{1}{2}G'''(O)dO^2\\&=\left[\alpha_O P_O-\gamma\mu_O(F,P_O,S,t)\right]dt-\gamma O_F F\sum_{i=1}^{m}w_i\sigma_i dZ_i+(1-\gamma O_P)P_O\sigma_O dZ_O,\end{aligned} \tag{A43}$$

with $G'''(O)=0$ for quadratic costs. Multiplying (A41) and (A43) gives:

$$\begin{aligned}\frac{dJ_F d\Omega_O}{J_F\Omega_O}=&-\frac{\gamma O_F F}{J_F\Omega_O}\left\{\sum_{i=1}^{m}w_i\sigma_i\left[J_{FP}\sigma_O P_O\rho_{iO}+J_{FF}F\sum_{j=1}^{m}w_j\sigma_j\rho_{ij}\right]\right\}dt\\&+\frac{(1-\gamma O_P)P_O\sigma_O}{J_F\Omega_O}\left(J_{FP}P_O\sigma_O+J_{FF}F\sum_{i=1}^{m}w_i\sigma_i\rho_{iO}\right)dt.\end{aligned} \tag{A44}$$

Substituting (A33) for all assets (the first term on the right-hand side) and for the perfectly correlated asset k ($\rho_{kO}=1$, the second term) gives:

$$\frac{dJ_F d\Omega_O}{J_F\Omega_O}=\frac{1}{\Omega_O}\left[\gamma O_F F\sum_{i=1}^{m}w_i(\alpha_i-r)-(1-\gamma O_P)\left(\frac{\alpha_k-r}{\sigma_k}\right)P_O\sigma_O\right]dt. \tag{A45}$$

Substituting (A43) and (A45) into (19) gives:

$$d\Omega_O = r\Omega_O dt - \gamma O_F \sum_{i=1}^{m}\left[(\alpha_i - r)dt + \sigma_i dZ_i\right] w_i F + (1-\gamma O_P)\left[(\alpha_k - r)dt + \sigma_k dZ_O\right] P_O \frac{\sigma_O}{\sigma_k}. \tag{A46}$$

Result: If all prices are deterministic, $\dot{\Omega}_O = \ldots = 0$. If the oil price is also without drift, $\alpha_O = 0$, the date of exhaustion is $T = -\frac{1}{r}\ln(\gamma O(0)/P_O(0))$ and the optimal rate of oil extraction () is to leading-order approximation:

$$O(t) = \sqrt{2\tfrac{r}{\gamma} S(t) P_O(t)}. \tag{A47}$$

Derivation: Using $\Omega_O = P_O - \gamma O$ in the deterministic Hotelling rule, we get $\dot{O} = \ldots = \gamma(\alpha_O - r)P_O(0)e^{\alpha_O t}$ which can be solved to give:

$$O(t) = O(0)e^{rt} + \tfrac{1}{\gamma} P_O(0)\left(e^{\alpha_O t} - e^{rt}\right). \tag{A48}$$

We exclude $\alpha_O \geq r$ as price growth would delay extraction indefinitely. Provided $\Omega_O(0) > 0$ and, $\alpha_O < r$, the extraction rate remains finite. The optimal initial extraction rate satisfies, $S(t) = \int_t^T O(\tau)d\tau$, and the date of exhaustion *T* must satisfy *O(T)* = 0. The date of exhaustion only has an explicit solution if $\alpha_O = 0$:

$$T = -\tfrac{1}{r}\ln(1-R), \tag{A49}$$

$$S(0) = -\tfrac{1}{r\gamma} P_O(0)\left(\ln(1-R) + R\right), \tag{A50}$$

where $0 < R = \gamma O(0)/P_O(0) < 1$ is small. As (A50) only defines the initial rate of extraction, $O(0) = f(S(0), P_O(0))$, we use asymptotic methods to find a series-solution and get the leading-order effect. Since $-\ln(1-R) = \sum_{n=1}^{\infty} \frac{1}{n} R^n$ we obtain

$$\frac{r\gamma S(0)}{P_O(0)} = \sum_{n=2}^{\infty} \frac{R^n}{n}. \tag{A51}$$

This can be inverted to give:

$$O(t) = S(t)\left[\sqrt{2}\xi(t)^{-1} - \tfrac{2}{3} + \tfrac{1}{9\sqrt{2}}\xi(t) + \tfrac{2}{135}\xi(t)^2 + \tfrac{1}{540\sqrt{2}}\xi(t)^3 + o(\xi(t)^4)\right], \tag{A52}$$

where $\xi(t) = \sqrt{r\gamma S(t)/P_O(t)}$ and the coefficients stem from the series inversion and are independent of parameters. The leading order yields (A47). Lemma A2 gives:

$$\partial O/\partial F = 0, \qquad \partial O/\partial P_O = O/(2P_O) = \sqrt{rS/(2\gamma P_O)}. \tag{A53}$$

Assuming the effect of uncertainty is modest, we use these partial derivatives in the analogous problem (analogous to taking the leading-order terms in a perturbation expansion in the volatility of the oil price). Substituting these partials into (A46) gives:

$$d\Omega_O = r\Omega_O dt + (P_O - \gamma \tfrac{1}{2} O)\frac{\sigma_O}{\sigma_k}\left[(\alpha_k - r)dt + \sigma_k dZ_O\right]. \tag{A54}$$

Combining (A43) and (A54) and setting $\alpha_O = 0$ as in (A46) gives:

$$\tfrac{1}{dt} E[dO] = \mu_O(F, P_O, S, t) = -\tfrac{1}{\gamma}\left(r + \tfrac{\sigma_O}{\sigma_k}(\alpha_k - r)\right)P_O + \left(r + \tfrac{1}{2}\tfrac{\sigma_O}{\sigma_k}(\alpha_k - r)\right)O. \tag{A55}$$

Equation (20) is found by substituting (A55) into (A42) and solving the initial value problem subject to the exhaustion condition $O(t=T) = S(t=T) = 0$. To obtain the results in figure 5, we use the full series solution in (A51), which becomes exact in the limit of an infinite number of terms.

A.6. Asset allocation with endogenous oil extraction

Endogenous oil rents can be replicated with a bundle of N_k shares of asset k and N_r shares of the safe asset, $X \equiv N_k P_k + N_r P_r$. This replicating bundle can be constructed as follows. To finance the dividend, the price must increase or shares must be sold:

$$-\Omega dt = \sum\nolimits_{i=k,r} dN_i dP_i + dN_i P_i \tag{A56}$$

Equation (21) combines this expression for the dividends with the path for the replicating bundle. By Ito's lemma the replicating bundle must satisfy:

$$\begin{aligned} dX + \Omega dt &= \sum_{i=k,r}\left(N_i dP_i + dN_i dP_i + dN_i P_i\right) + \Omega dt \\ &= \sum_{i=k,r}\left(N_i dP_i\right) = \omega_k X(\alpha_k - r)dt + rXdt + \omega_k X\sigma_k dZ_k. \end{aligned} \tag{A57}$$

where $\omega_k(t) \equiv N_k(t)P_k(t)/X(t)$. The weights $\omega_k(t)$ are updated continuously to match the stochastic path of oil rents (A46). As oil wealth and the replicating bundle have the same properties they must also have the same value, $X = V$, giving (21). We have focused on $dV(t) + \Omega(t)dt$. $V(t)$ is found using contingent claims (Merton, 1990) if oil rents follow the Ito process $d\Omega(t) = a(.)\Omega dt + s(.)\Omega dZ_O$ and $a(.)$ and $s(.)$ are not constants. The value of oil rents must be that of the replicating bundle, $V(t) = X(t)$. Equation (22) states that the problem can be summarized in terms of $W(t) = F(t) + V(t)$. Combining (17) and (A57) gives (22). The weight of asset k in the fund adjusts continuously so that the net

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weight of oil in total wealth is constant. The weight of all other assets in the fund remains constant, as in (23).