Error Exponents for Multi-keyhole MIMO Channels

ISSN 0032-9460, Problems of Information Transmission, 2015, Vol. 51, No. 1, pp. 1–19. © Pleiades Publishing, Inc., 2015.
Original Russian Text © J. Xue, T. Ratnarajah, C. Zhong, 2015, published in Problemy Peredachi Informatsii, 2015, Vol. 51, No. 1, pp. 3–22.

INFORMATION THEORY

Error Exponents for Multi-Keyhole MIMO Channels

J. Xue[a,1], **T. Ratnarajah**[a,1], **and C. Zhong**[b,2]

[a] *Institute for Digital Communications, School of Engineering, The University of Edinburgh, Edinburgh, United Kingdom*
e-mail: J.Xue@ed.ac.uk, T.Ratnarajah@ieee.org
[b] *Institute of Information and Communication Engineering, Zhejiang University, Zhejiang, China*
National Mobile Communications Research Laboratory, Southeast University, Nanjing, China
e-mail: caijunzhong@zju.edu.cn

Received January 27, 2014; in final form, November 29, 2014

Abstract—Along with the channel capacity, the error exponent is one of the most important information-theoretic measures of reliability, because it sets ultimate bounds on the performance of communication systems employing codes of finite complexity. In this paper, we derive closed-form expressions for the Gallager random coding and expurgated error exponents for multi-keyhole multiple-input multiple-output (MIMO) channels, which provide insights into a fundamental tradeoff between the communication reliability and information rate. We investigate the effect of keyholes on the error exponents and cutoff rate. Moreover, without an extensive Monte-Carlo simulation we can easily compute the codeword length necessary to achieve a predefined error probability at a given rate, which quantifies the effects of the number of antennas, channel coherence time, and the number of keyholes. In addition, we derive exact closed-form expressions for the ergodic capacity and cutoff rate based on the easily computable Meijer G-function. Finally, we extend our study to Rayleigh-product MIMO channels and keyhole MIMO channels.

DOI: 10.1134/S0032946015010019

1. INTRODUCTION

One of the critical information-theoretic measures of wireless communication systems is the channel capacity, which, however, only provides the knowledge of the maximum error-free transmission rate achieved with an infinitely long code. However, in practice, we are more interested in scenarios where finite length codes are used; hence, a natural question arises of how to understand the fundamental relationship between the reliability and information rate. In fact, such a relationship can be characterized by the error exponent, which is a function of the code length L and information rate R. In general, it is very difficult to obtain the exact error exponent of a particular channel. Nevertheless, a tight lower bound of the error exponent, also refereed to as the random coding error exponent (RCEE), was proposed by Gallager [1].

Being a function of both the transmission rate and the code length, the RCEE provides an alternative measure to study the fundamental tradeoff between the communication reliability and information rate of communication systems. Therefore, it has gained enormous attentions from

[1] Supported in part by the UK Engineering and Physical Sciences Research Council (EPSRC), grant no. EP/I037156/2, funded by the UK government.
[2] Supported in part by the National High-Tech. R&D Program of China under grant 2014AA01A705, the National Natural Science Foundation of China (61201229), the Zhejiang Science and Technology Department Public Project (2014C31051), the Fundamental Research Funds for Central Universities (2014QNA5019), and the open research fund of National Mobile Communications Research Laboratory, Southeast University (no. 2013D06).

the research community, and a number of works have investigated the RCEE in various practical channels. Specifically, the work [2] studied the RCEE with average and peak transmit power constraints for single-input single-output (SISO) channels, while the authors in [3] extended the analysis to single-input multiple-output (SIMO) channels. More interesting multiple-input single-output (MISO) channels were later addressed in [4]. For the most general multiple-input multiple-output (MIMO) channels, exact closed-form expressions for the RCEE and cutoff rate have been derived in [5–7]. In the low signal-to-noise ratio (SNR) regime, [8] investigated the impact of the number of transmit and receive antennas on the error exponent of MIMO channels.

While the aforementioned works have significantly improved our understandings on the behavior of the RCEE in MIMO channels, a common limitation of these works is that they all assume an ideal rich scattering propagation environment, and hence flat fading MIMO Rayleigh channels. In practice, MIMO channels may exhibit reduced-rank behavior due to the lack of scatterers around the transmitter and receiver terminals. For this reason, a more general channel model accounting for the rank deficiency behavior as well as the antenna correlation effect of wireless channels was proposed in [9], and it is generally referred to as double-scattering MIMO channels. The authors extended the single-keyhole channel model by introducing multi-keyhole channels and derived asymptotic results in [10]. Because of its generality and practical significance, a good deal of works have investigated its performance in different settings [11–17]. However, to the best of the authors' knowledge, the behavior of the error exponent in double-scattering MIMO channels has not been reported in the literature.

Motivated by this, in this paper we make an attempt to study the error exponents for multi-keyhole MIMO channels.[3] The main contributions of the paper include new closed-form expressions for negative moments of the determinant of the random matrices of interest. Based on this, exact analytical expressions for the RCEE of multi-keyhole MIMO channels are derived. In addition, general expressions for the ergodic capacity and cutoff rate are also obtained from the error exponent analysis. Based on this, the impact of the number of scatters and correlation on the performance of a system is investigated.

The rest of the paper is organized as follows: The system model and some mathematical preliminaries are presented in Section 2. Section 3 gives a comprehensive account of the RCEE, expurgated error exponent, cutoff rate, and ergodic capacity of multi-keyhole MIMO channels. Section 4 analyzes the required codeword length. Finally, Section 5 concludes the paper.

Throughout the paper, we adopt the following notation. Matrices and vectors are denoted by bold uppercase and bold lowercase letters, respectively. By $\boldsymbol{I}_n$ we denote the $n \times n$ identity matrix; $(\cdot)^\dagger$ denotes the conjugate transpose of a matrix or vector, $\mathbb{C}^{n\times m}$ denotes the set of $n \times m$ complex matrices, and $\mathcal{CN}(\mu, \sigma^2)$ is the circularly symmetric complex Gaussian distribution with mean μ and variance σ^2. By $\mathrm{tr}(\cdot)$ and $\det(\cdot)$ we denote the trace and determinant of a matrix; $\{\boldsymbol{A}\}_{i,j}$ is the (i,j)th element of a matrix $\boldsymbol{A}$; $|\cdot|$ denotes the absolute value; $\mathbf{E}[\cdot]$ and $\ln(\cdot)$ are the expectation operation and natural logarithm, respectively.

2. SYSTEM MODEL AND PRELIMINARIES

We consider a MIMO communication system with n_t transmit antennas and n_r receive antennas. We assume that there are n_k keyholes between the transmitter and receiver for multi-keyhole MIMO channels. The channel remains constant for T_c symbol periods and changes independently to a new value for each coherence time. We consider the channel coding over N_b independent coherence intervals. Therefore, for an observation interval of N_bT_c symbol periods, received signals are given

[3] Mathematically, multi-keyhole MIMO channels is a special case of the general double-scattering MIMO channels with only correlation among the scatters.

by

$$\boldsymbol{Y}_i = \sqrt{\frac{P}{n_t}}\boldsymbol{H}_i\boldsymbol{X}_i + \boldsymbol{W}_i, \quad i = 1, 2, \ldots, N_b. \tag{1}$$

where $\boldsymbol{Y}_i \in \mathbb{C}^{n_r\times T_c}$ and $\boldsymbol{X}_i \in \mathbb{C}^{n_t\times T_c}$ are the received and transmitted signal matrices, respectively. We assume that the total available transmit power for every T_c symbol period is P, i.e., $\mathbf{E}[\mathrm{tr}(\boldsymbol{X}_i^\dagger\boldsymbol{X}_i)] \le P$; $\boldsymbol{W}_i \sim \mathcal{CN}_{n_r,T_c}(0, N_0\boldsymbol{I}_{n_r}\otimes\boldsymbol{I}_{T_c})$ are the additive white Gaussian noise matrices, and $\boldsymbol{H}_i \in \mathbb{C}^{n_r\times n_t}$ denote the channel matrices. Since the channel is memoryless with identical channel statistics for each coherence time interval, we drop the index i in the following.

For multi-keyhole MIMO channels, the channel matrix is given by $\boldsymbol{H} = \boldsymbol{H}_r\boldsymbol{A}\boldsymbol{H}_t^\dagger$, where $\boldsymbol{H}_r \in \mathbb{C}^{n_r\times n_k}$ and $\boldsymbol{H}_t \in \mathbb{C}^{n_t\times n_k}$ are mutually independent matrices with elements following the distribution $\mathcal{CN}(\mathbf{0},\mathbf{1})$. Let $\boldsymbol{A} = \mathrm{diag}(a_1,\ldots,a_{n_k}) \in \mathbb{C}^{n_k\times n_k}$, where a_k is the complex gain for the kth keyhole. Without loss of generality, we assume that the diagonal elements of matrix $\boldsymbol{A}$ are ordered according to their magnitude, i.e., $|a_1| \le |a_2| \le \ldots \le |a_{n_k}|$. We define $\boldsymbol{F} \triangleq \boldsymbol{H}^\dagger\boldsymbol{H} = \boldsymbol{H}_t\boldsymbol{A}^\dagger\boldsymbol{H}_r^\dagger\boldsymbol{H}_r\boldsymbol{A}\boldsymbol{H}_t^\dagger$, $\boldsymbol{Q} \triangleq \boldsymbol{A}^\dagger\boldsymbol{H}_r^\dagger\boldsymbol{H}_r\boldsymbol{A}$, $\boldsymbol{B} \triangleq \boldsymbol{A}\boldsymbol{A}^\dagger$, and assume that the channels are normalized, which means that $\mathbf{E}_{\boldsymbol{H}_r,\boldsymbol{H}_t}[\mathrm{tr}(\boldsymbol{F})] = n_tn_r$ and $\mathrm{tr}(\boldsymbol{B}) = 1$.

When there is no correlation among the keyholes, multi-keyhole MIMO channels reduce to the so-called Rayleigh-product MIMO channels, i.e., $\boldsymbol{H} = \dfrac{1}{\sqrt{n_k}}\boldsymbol{H}_r\boldsymbol{H}_t^\dagger$.

2.1. Preliminaries

In this subsection, we present some key mathematical results which will be invoked in the subsequent analysis of information-theoretic measures such as the RCEE, expurgated error exponent, ergodic capacity, and cutoff rate.

Theorem 1. *The expectation of the determinant of the matrix* $\left(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{F}\right)^{-\tau}$ *is given by*

$$\mathbf{E}_{\boldsymbol{H}_r,\boldsymbol{H}_t}\left[\det\left(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{F}\right)^{-\tau}\right] = \frac{\det(\boldsymbol{\Delta})}{\Gamma(\tau)\prod\limits_{i=1}^{n_k}\Gamma(n_t-i+1)\Gamma(n_r-i+1)\prod\limits_{i<j}^{n_k}(b_j-b_i)}, \tag{2}$$

where $\boldsymbol{\Delta}$ *is an* $n_k\times n_k$ *matrix with entries*

$$[\boldsymbol{\Delta}]_{i,j} = \begin{cases} b_i^{j-1}, & j \le n_k - p;\\ b_i^{j-1}G_{3,1}^{1,3}\left[\dfrac{\gamma b_i}{\varpi}\,\middle|\,\begin{matrix} n_k-n_r-j+1, 1-n_t+n_k-j, 1-\tau\\ 0\end{matrix}\right], & j > n_k - p,\end{cases}$$

where $G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right]$ *is the Meijer G-function* [18]; γ, ϖ, *and* τ *are constants*; *and* $b_1 \le b_2 \le \ldots \le b_{n_k}$ *are the diagonal elements of* $\boldsymbol{B}$.

Proof. Let us define $\alpha \triangleq \max(n_r,n_t)$, $\varrho \triangleq \min(n_r,n_t)$, $q \triangleq \max(\varrho,n_k)$, $p \triangleq \min(\varrho,n_k)$, $d \triangleq \max(\alpha,n_k)$, $s \triangleq \min(\alpha,n_k)$, and $\vartheta \triangleq \min(n_r,n_k)$. We have

$$\mathbf{E}_{\boldsymbol{H}_r,\boldsymbol{H}_t}\left[\det\left(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{H}_t\boldsymbol{A}^\dagger\boldsymbol{H}_r^\dagger\boldsymbol{H}_r\boldsymbol{A}\boldsymbol{H}_t^\dagger\right)^{-\tau}\right] = \mathbf{E}_{\boldsymbol{Q}}\left[\mathbf{E}_{\boldsymbol{H}_t|\boldsymbol{Q}}\left[\det\left(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{H}_t\boldsymbol{Q}\boldsymbol{H}_t^\dagger\right)^{-\tau}\right]\right]. \tag{3}$$

Define $\boldsymbol{\Psi} = \boldsymbol{H}_t\boldsymbol{Q}\boldsymbol{H}_t^\dagger$. Then the conditional p.d.f. of joint eigenvalues of $\boldsymbol{\Psi}$ is given by [19, equations (86) and (87)]

$$f(\lambda_1,\ldots,\lambda_{n_k}\,|\,\boldsymbol{Q}) = \frac{\det(\boldsymbol{\Delta}_1)\prod\limits_{i<j}^{n_k}(\lambda_j-\lambda_i)}{n_k\prod\limits_{i=1}^{n_k}\Gamma(n_t-i+1)\prod\limits_{i<j}^{n_k}(q_j-q_i)}, \tag{4}$$

where $q_1, \ldots, q_\vartheta$ are the ϑ nonzero eigenvalues of $\boldsymbol{Q}$, $\lambda_1, \ldots, \lambda_{n_k}$ are the ordered eigenvalues of $\boldsymbol{\Psi}$, and $\boldsymbol{\Delta}_1$ is $\vartheta \times \vartheta$ matrix with entries

$$\boldsymbol{\Delta}_1 = \begin{pmatrix} 1 & q_1 & \ldots & q_1^{\vartheta-n_t-1} & q_1^{\vartheta-n_t-1} e^{-\frac{\lambda_1}{q_1}} & \ldots & q_1^{\vartheta-n_t-1} e^{-\frac{\lambda_{n_k}}{q_1}} \\ 1 & q_2 & \ldots & q_2^{\vartheta-n_t-1} & q_2^{\vartheta-n_t-1} e^{-\frac{\lambda_1}{q_2}} & \ldots & q_2^{\vartheta-n_t-1} e^{-\frac{\lambda_{n_k}}{q_2}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & q_\vartheta & \ldots & q_\vartheta^{\vartheta-n_t-1} & q_\vartheta^{\vartheta-n_t-1} e^{-\frac{\lambda_1}{q_\vartheta}} & \ldots & q_\vartheta^{\vartheta-n_t-1} e^{-\frac{\lambda_{n_k}}{q_\vartheta}} \end{pmatrix}. \tag{5}$$

Therefore, using the probability density function (p.d.f.) of (4) and assuming that $n_k \ge p$, we have

$$\mathbf{E}\Big[\det\Big(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{H}_t\boldsymbol{Q}\boldsymbol{H}_t^\dagger\Big)^{-\tau} \,|\, \boldsymbol{Q}\Big] = \frac{\int\limits_{D_{\text{ord}}} \det(e^{-x_j/q_i}) \prod\limits_{i=1}^{n_k} \Big(1 + \frac{\gamma}{\varpi}x_i\Big)^{-\tau} x_i^{n_t-n_k} \det(x_i^{j-1})\, dx_1 \ldots dx_{n_k}}{\prod\limits_{i=1}^{n_k} q_i^{1+n_t-n_k} \Gamma(n_t - i + 1) \prod\limits_{i<j}^{n_k} (q_j - q_i)}.$$

Defining the integration region as $D_{\text{ord}} = \{\infty \ge x_1 \ge \ldots \ge x_{n_k} \ge 0\}$ and using the method of [20] and [18, equation (7.813.1)], we have

$$\mathbf{E}\Big[\det\Big(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{H}_t\boldsymbol{Q}\boldsymbol{H}_t^\dagger\Big)^{-\tau} \,|\, \boldsymbol{Q}\Big] = \frac{\det(\boldsymbol{\Delta}_2)}{\Gamma(\tau) \prod\limits_{i=1}^{n_k} \Gamma(n_t - i + 1) \prod\limits_{i<j}^{n_k} (q_j - q_i)},$$

where $\boldsymbol{\Delta}_2$ is a $\vartheta \times \vartheta$ matrix with entries

$$\{\boldsymbol{\Delta}_2\}_{i,j} = q_i^{j-1} G_{2,1}^{1,2}\left[\frac{\gamma q_i}{\varpi} \,\middle|\, \begin{matrix} 1 - n_t + n_k - j, 1 - \tau \\ 0 \end{matrix}\right].$$

When $n_r \ge n_k$, the joint p.d.f. of the n_k ordered eigenvalues of $\boldsymbol{Q}$ is given in [21, equation (42)]. Therefore, from [18, equation (7.813.1)], the expectation over $\boldsymbol{Q}$ is given by

$$\mathbf{E}_{\boldsymbol{H}_r,\boldsymbol{H}_t}\Big[\det\Big(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{H}_t\boldsymbol{A}^\dagger\boldsymbol{H}_r^\dagger\boldsymbol{H}_r\boldsymbol{A}\boldsymbol{H}_t^\dagger\Big)^{-\tau}\Big] = C_1 \det(\boldsymbol{\Delta}_3),$$

where

$$C_1 = \frac{1}{\Gamma(\tau) \prod\limits_{i=1}^{n_k} \Gamma(n_t - i + 1) \prod\limits_{i=1}^{n_k} \Gamma(n_r - i + 1) \det(\boldsymbol{B})^{n_r-n_k+1} \prod\limits_{i<j}^{n_k} (b_j - b_i)}$$

and

$$\{\boldsymbol{\Delta}_3\}_{i,j} = b_i^{n_r-n_k+j} G_{3,1}^{1,3}\left[\frac{\gamma b_i}{\varpi} \,\middle|\, \begin{matrix} n_k - n_r - j + 1, 1 - n_t + n_k - j, 1 - \tau \\ 0 \end{matrix}\right].$$

Following similar mathematical manipulations as in [22, Appendix D] and according to the definition of $\det(\boldsymbol{B})^{n_r-n_k+1}$, we have

$$\mathbf{E}_{\boldsymbol{H}_r,\boldsymbol{H}_t}\Big[\det\Big(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{H}_t\boldsymbol{A}^\dagger\boldsymbol{H}_r^\dagger\boldsymbol{H}_r\boldsymbol{A}\boldsymbol{H}_t^\dagger\Big)^{-\tau}\Big] = C_2 \det(\boldsymbol{\Delta}_4), \tag{6}$$

where $C_2 = \dfrac{1}{\Gamma(\tau) \prod\limits_{i=1}^{n_k} \Gamma(n_t - i + 1)\Gamma(n_r - i + 1) \prod\limits_{i<j}^{n_k} (b_j - b_i)}$ and

$$\{\boldsymbol{\Delta}_4\}_{i,j} = b_i^{j-1} G_{3,1}^{1,3}\left[\frac{\gamma b_i}{\varpi} \,\middle|\, \begin{matrix} n_k - n_r - j + 1, 1 - n_t + n_k - j, 1 - \tau \\ 0 \end{matrix}\right].$$

When $n_r < n_k$, the joint p.d.f. of the n_r ordered eigenvalues of $\boldsymbol{Q}$ is given in [23]. Therefore, following the similar procedure as was described in the case of $n_r \geq n_k$ and using the identity [18, equation (7.813.1)], we derive Theorem 1 as is shown above. △

Corollary. *The expectation of the determinant of the matrix* $\left(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{F}\right)^{-\tau}$ *for independent and identically distributed single keyhole Rayleigh fading MIMO channels is given by*

$$\mathbf{E}_{\boldsymbol{H}_r,\boldsymbol{H}_t}\left[\det\left(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{F}\right)^{-\tau}\right] = \frac{1}{\Gamma(\tau)\Gamma(n_t)\Gamma(n_r)} G_{3,1}^{1,3}\left[\frac{\gamma}{\varpi} \,\middle|\, \begin{matrix} 1-n_r, 1-n_t, 1-\tau \\ 0 \end{matrix}\right]. \tag{7}$$

Proof. This result can be derived straightforwardly from Theorem 1 when $n_k = 1$ and $b = 1$. △

It is worth noting that a similar result has already been presented in [24], which studied the error exponent of i.i.d. single keyhole Nakagami-m fading MIMO channels.

Theorem 1 essentially presents a closed-form expression for the Laplacian transformation of the mutual information for multi-keyhole MIMO channels, which is valid for arbitrary keyhole structures. Hence, the corresponding expression for the special case where there is no correlation among the keyholes, i.e., $b_i = \frac{1}{n_k}$, could be extracted from the general result presented in Theorem 1. Indeed, this can be achieved by introducing a small perturbation into b_i: let $b_j - b_i \xrightarrow{\lim} 0$ for all $i, j \in (1, \ldots, n_k)$, and then successive application of the L'Hôpital law leads to the final result. Although such an approach is feasible, it is nevertheless computationally expensive. Hence, in the following, we directly derive the corresponding analytical expression for Rayleigh-product MIMO channels using some results of random matrix theory.

Theorem 2. *The expectation of the determinant of the matrix* $\left(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{H}\boldsymbol{H}^\dagger\right)^{-\tau}$ *for Rayleigh-product MIMO channels is given by*

$$\mathbf{E}_{\boldsymbol{H}}\left[\det\left(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{H}\boldsymbol{H}^\dagger\right)^{-\tau}\right] = M_1 \det(\boldsymbol{\Psi}), \tag{8}$$

where $u = \min(n_r, n_k)$, $v = \max(n_r, n_k)$, $m = \min(u, n_t)$, $n = \max(u, n_t)$,

$$M_1 = \frac{(-1)^{(u-n)(u+m-1)/2} n_k^{uv}}{\prod\limits_{i=1}^{u} (u-i)!\,(v-i)! \prod\limits_{j=1}^{m} (n_t - j)!},$$

and $\boldsymbol{\Psi}$ *is a* $u \times u$ *matrix with entries*

$$\psi_{i,j} = \begin{cases} b_{i,j}, & i = 1, \ldots, u,\ j = 1, \ldots, m, \\ c_{i,j}, & i = 1, \ldots, u,\ j = m+1, \ldots, u, \end{cases} \tag{9}$$

where

$$b_{i,j} = \frac{1}{\Gamma(\tau)} n_k^{-v+n_t-i-j-n+u+1} \times G_{3,1}^{1,3}\left[\frac{\gamma}{\varpi n_k} \,\middle|\, \begin{matrix} -v+n_t-i-j-n+u+2, -j-n+u+1, 1-\tau \\ 0 \end{matrix}\right]$$

and

$$c_{i,j} = (-1)^{u-j}(v-m-n+i+j-2)!\, n_k^{-(v-m-n+i+j-1)}.$$

Proof. For the Rayleigh-product MIMO channels, the p.d.f. of the ordered eigenvalues of $\boldsymbol{H}\boldsymbol{H}^\dagger$ is given by [25]

$$p_{\boldsymbol{\lambda}}(\boldsymbol{x}) = M_1 |\Phi(\boldsymbol{x})| |\Xi(\boldsymbol{x})| \prod_{i=1}^{m} x_i^{n-u}, \tag{10}$$

where $|\boldsymbol{x}| = \det(\boldsymbol{x})$, $\infty > x_1 \geq \ldots \geq x_m \geq 0$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$, $\boldsymbol{x} = (x_1, \ldots, x_m)$.

The matrices $\Phi(\boldsymbol{x})$ and $\Xi(\boldsymbol{x})$ are given by

$$\{\Phi(\boldsymbol{x})\}_{i=1,\ldots,u,j} = \begin{cases} 2\Big(\dfrac{x_j}{n_k}\Big)^{(v-n_t+i-1)/2} K_{v-n_t+i-1}(2\sqrt{n_k x_j}) & \text{if } j = 1,\ldots,m, \\ (-1)^{u-j}(v-m-n+i+j-2)!\, n_k^{-(v-m-n+i+j-1)} & \text{if } j = m+1,\ldots,u, \end{cases}$$

and $\{\Xi(\boldsymbol{x})\}_{i,j} = x_i^{j-1}$, $i = 1,\ldots,m$, $j = 1,\ldots,m$, where $K_v(\cdot)$ is the modified Bessel function of the second kind [18].

Then we have

$$\mathbf{E}_{\boldsymbol{H}}\Big[\det\Big(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{H}\boldsymbol{H}^\dagger\Big)^{-\tau}\Big] = \int\limits_{D_1} \prod_{i=1}^{m} \Big(1 + \frac{\gamma\lambda_i}{\varpi}\Big)^{-\tau} p_{\lambda_i}(\lambda_i)\, d\lambda_1 \ldots d\lambda_m \tag{11}$$

with the integration region $D_1 = \{0 < \lambda_1 < \ldots < \lambda_m < \infty\}$, where $(\lambda_1,\ldots,\lambda_m)$ are the ordered eigenvalues of $\boldsymbol{H}\boldsymbol{H}^\dagger$. Using the identity [26]

$$(1-x)^{-\alpha} = \frac{1}{\Gamma(\alpha)} G_{1,1}^{1,1}\Big[-x \,\Big|\, \begin{matrix} 1-\alpha \\ 0 \end{matrix}\Big], \tag{12}$$

and the p.d.f. of (10), we have

$$\mathbf{E}_{\boldsymbol{H}}\Big[\det\Big(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{H}\boldsymbol{H}^\dagger\Big)^{-\tau}\Big] = \int\limits_{D_1} \prod_{i=1}^{m} \frac{M_1|\Phi(\boldsymbol{\lambda})||\Xi(\boldsymbol{\lambda})|}{\Gamma(\tau)} G_{1,1}^{1,1}\Big[\frac{\gamma\lambda_i}{\varpi} \,\Big|\, \begin{matrix} 1-\tau \\ 0 \end{matrix}\Big] \prod_{i=1}^{m} \lambda_i^{n-u}\, d\lambda_1 \ldots d\lambda_m. \tag{13}$$

Using the property of matrix determinant and applying the technique proposed in [20, 27], the multiple integral can be solved as

$$\mathbf{E}_{\boldsymbol{H}}\Big[\det\Big(\boldsymbol{I}_{n_r} + \frac{\gamma}{\varpi}\boldsymbol{H}\boldsymbol{H}^\dagger\Big)^{-\tau}\Big] = M_1 \det(\widetilde{\boldsymbol{\Psi}}),$$

where $\widetilde{\boldsymbol{\Psi}}$ is the matrix with entries

$$\widetilde{\psi}_{i,j} = \begin{cases} \widetilde{b}_{i,j}, & i = 1,\ldots,u,\ j = 1,\ldots,m, \\ c_{i,j}, & i = 1,\ldots,u,\ j = m+1,\ldots,u, \end{cases}$$

where

$$\widetilde{b}_{i,j} = \frac{1}{\Gamma(\tau)} \int\limits_0^\infty \{\Phi(x)\}_{i,j}\{\Xi(x)\}_{i,j} G_{1,1}^{1,1}\Big[\frac{\gamma x}{\varpi} \,\Big|\, \begin{matrix} 1-\tau \\ 0 \end{matrix}\Big] x^{n-u}\, dx,$$

$$c_{i,j} = (-1)^{u-j}(v-m-n+i+j-2)!\, n_k^{-(v-m-n+i+j-1)}.$$

Noticing that the integral of $\widetilde{b}_{i,j}$ can be solved with the help of the identity [18, equation (7.821.3)] and that ordered eigenvalues are used in this paper, Theorem 2 can be obtained with some basic algebraic manipulations. △

3. ERROR EXPONENT ANALYSIS

In this section, we study the error exponents of multi-keyhole MIMO channels and derive closed-form expressions for the RCEE, expurgated error exponent, cutoff rate, and ergodic capacity.

3.1. RCEE

For a communication system, the error exponent is defined by [28]

$$E(R) := \limsup_{L\to\infty} \frac{-\ln P_e^{\rm opt}(R,L)}{L},$$

where $P_e^{opt}(R,L)$ is the average error probability of a communication system employing an optimal code with length L and rate R [5–7]. Due to the presence of the supremum function, it is in general quite difficult to find the exact error exponent. However, a tight lower bound, which can easily be computed, can be expressed as [1]

$$P_e^r \le \left(\frac{2e^{r\delta}}{\xi}\right)^2 e^{-N_b T_c E_r(p_{\boldsymbol X}(\boldsymbol X),R,T_c)},$$

where $r \ge 0$, $\delta \ge 0$, $\xi \approx \dfrac{\delta}{\sqrt{2\pi N_b \sigma_\xi^2}}$, and $\sigma_\xi^2 = \int\limits_{\boldsymbol X} [\mathrm{tr}(\boldsymbol X\boldsymbol X^\dagger) - T_c P]^2 p_{\boldsymbol X}(\boldsymbol X)\,d\boldsymbol X$. The RCEE, denoted by $E_r(p_{\boldsymbol X}(\boldsymbol X),R,T_c)$, is given by

$$E_r(p_{\boldsymbol X}(\boldsymbol X),R,T_c) = \max_{0\le\rho\le1}\Big\{\max_{r\ge0} E_0(p_{\boldsymbol X}(\boldsymbol X),\rho,r,T_c) - \rho R\Big\}, \tag{14}$$

where

$$\begin{aligned} &E_0(p_{\boldsymbol X}(\boldsymbol X),\rho,r,T_c)\\ &= -\frac{1}{T_c}\ln\Bigg\{\int\limits_{\boldsymbol H} p_{\boldsymbol H}(\boldsymbol H)\int\limits_{\boldsymbol Y}\Bigg(\int\limits_{\boldsymbol X} p_{\boldsymbol X}(\boldsymbol X) e^{r[\mathrm{tr}(\boldsymbol X\boldsymbol X^\dagger)-T_cP]} p(\boldsymbol Y|\boldsymbol X,\boldsymbol H)^{\frac{1}{(1+\rho)}}\,d\boldsymbol X\Bigg)^{1+\rho} d\boldsymbol Y\,d\boldsymbol H\Bigg\}, \end{aligned} \tag{15}$$

where $P_{\boldsymbol X}(\boldsymbol X)$, $p_{\boldsymbol H}(\boldsymbol H)$, and $p(\boldsymbol Y|\boldsymbol X,\boldsymbol H)$ denote the distributions of the input signal, channel, and received signal [5,6], respectively. Since the Gaussian signaling is capacity achieving, we have

$$p_{\boldsymbol X}(\boldsymbol X) = \pi^{-n_tT_c}\det(\boldsymbol Q)^{-T_c}\,\mathrm{etr}(-\boldsymbol Q^{-1}\boldsymbol X\boldsymbol X^\dagger) \tag{16}$$

with power constraint $\mathrm{tr}(\boldsymbol Q) \le P$. Assuming no channel state information at the transmitter, the equal power allocation scheme is adopted, i.e., $\boldsymbol Q = \dfrac{P}{n_t}\boldsymbol I_{n_t}$.

In this case, $E_0(p_{\boldsymbol X}(\boldsymbol X),\rho,r,T_c)$ and $E_r(p_{\boldsymbol X}(\boldsymbol X),R,T_c)$ are denoted by $\widetilde E_0(\rho,\beta,T_c)$ and $E_r(R,T_c)$, respectively. Therefore, for Rayleigh fading MIMO channels, $\widetilde E_0(\rho,\beta,T_c)$ is given by

$$\begin{aligned} \widetilde E_0(\rho,\beta,T_c) &\triangleq E_0\Big(\frac{P}{n_t}\boldsymbol I_{n_t},\rho,r,T_c\Big)\Big|_{\beta=n_t-rP}\\ &= \mathcal K(\rho,\beta) - \frac{1}{T_c}\ln\bigg\{\mathbf E\bigg[\det\bigg(\boldsymbol I_{n_r} + \frac{\gamma\boldsymbol H\boldsymbol H^\dagger}{\beta(1+\rho)}\bigg)^{-T_c\rho}\bigg]\bigg\}, \end{aligned} \tag{17}$$

where γ denotes the SNR, $\beta = n_t - rP$, and $\mathcal K(\rho,\beta) = (1+\rho)(n_t-\beta) + n_t(1+\rho)\ln\Big(\dfrac{\beta}{n_t}\Big)$.

Proposition 1. *A closed-form expression for the RCEE of multi-keyhole MIMO channels is given by*

$$E_r(R,T_c)^{\text{multi-keyhole}} = \max_{0\le\rho\le1}\Big\{\max_{0\le\beta\le n_t}\Big\{\mathcal K(\rho,\beta) - \frac{1}{T_c}\ln\{C_3\det(\boldsymbol\Delta_5)\} - \rho R\Big\}\Big\}, \tag{18}$$

where $C_3 = \dfrac{1}{\Gamma(T_c\rho)\prod\limits_{i=1}^{n_k}\Gamma(n_t-i+1)\Gamma(n_r-i+1)\prod\limits_{i<j}^{n_k}(b_j-b_i)}$ *and*

$$\{\boldsymbol{\Delta}_5\}_{i,j} = \begin{cases} b_i^{j-1}, & j \le n_k - p,\\ b_i^{j-1} G_{3,1}^{1,3}\left[\dfrac{\gamma b_i}{\beta(1+\rho)} \,\middle|\, \begin{matrix} n_k - n_r - j + 1, 1 - n_t + n_k - j, 1 - T_c\rho \\ 0 \end{matrix}\right], & j > n_k - p. \end{cases} \tag{19}$$

Proof. From (17), $\widetilde{E}_0(\rho,\beta,T_c)$ of Rayleigh fading multi-keyhole MIMO channels is given by

$$\widetilde{E}_0(\rho,\beta,T_c) = \mathcal{K}(\rho,\beta) - \frac{1}{T_c}\ln\left\{\mathbf{E}\left[\det\left(\boldsymbol{I}_{n_r} + \frac{\gamma\boldsymbol{F}}{\beta(1+\rho)}\right)^{-T_c\rho}\right]\right\}. \tag{20}$$

Substituting (2) into (20) and utilizing the result presented in Theorem 1, we obtain the desired result. △

Similarly, for Rayleigh-product MIMO channels, we have the following key results.

Proposition 2. *The RCEE for Rayleigh-product MIMO channels*, $E_r(R,T_c)$, *is given by*

$$E_r(R,T_c) = \max_{0\le\rho\le1}\left\{\max_{0\le\beta\le n_t}\left\{\mathcal{K}(\rho,\beta) - \frac{1}{T_c}\ln\{M_1\det(\boldsymbol{\Psi}^r)\} - \rho R\right\}\right\}, \tag{21}$$

where $\boldsymbol{\Psi}^r$ *is the* $u\times u$ *matrix with entries*

$$\psi_{i,j}^r = \begin{cases} b_{i,j}^r, & i = 1,\dots,u,\ j = 1,\dots,m,\\ c_{i,j}, & i = 1,\dots,u,\ j = m+1,\dots,u, \end{cases} \tag{22}$$

where

$$b_{i,j}^r = \frac{1}{\Gamma(\rho T_c)} n_k^{-v+n_t-i-j-n+u+1} \times G_{3,1}^{1,3}\left[\frac{\gamma}{\beta(1+\rho)n_k} \,\middle|\, \begin{matrix} -v+n_t-i-j-n+u+2, -j-n+u+1, 1-\rho T_c \\ 0 \end{matrix}\right].$$

Proof. The desired result can easily be obtained by invoking Theorem 2 with parameters $\tau = T_c\rho$ and $\varpi = \beta(1+\rho)$. △

An important case of the general multi-keyhole MIMO channels is the keyhole channel, which represents the extreme case where only a single path exists between the multiple antennas transmitter and receiver. For this particular channel, the RCEE can be deduced as

$$\begin{aligned} E_r(R,T_c)_{\text{keyhole}} &= \max_{0\le\rho\le1}\left\{\max_{0\le\beta\le n_t}\widetilde{E}_0^{\text{keyhole}}(\rho,\beta,T_c) - \rho R\right\} \\ &= \max_{0\le\rho\le1}\left\{\max_{0\le\beta\le n_t}\left\{\mathcal{K}(\rho,\beta) - \frac{1}{T_c}\ln\left\{\frac{\psi_{1,1}}{(n_r-1)!\,(n_t-1)!}\right\} - \rho R\right\}\right\}, \end{aligned} \tag{23}$$

where

$$\psi_{1,1} = \frac{1}{\Gamma(\rho T_c)} G_{3,1}^{1,3}\left[\frac{\gamma}{\beta(1+\rho)} \,\middle|\, \begin{matrix} -n_r+1, -n_t+1, 1-\rho T_c \\ 0 \end{matrix}\right].$$

3.2. Expurgated Error Exponent

In this section, we look at an enhanced version of the RCEE, which is in general referred to as the expurgated error exponent. The key idea behind the expurgated error exponent is to distinguish good and bad codewords, and expurgate the bad codewords from the code ensemble. The improved error probability bound can be expressed as [1]

$$P_e^{\mathrm{ex}} \le e^{-N_b T_c E_{\mathrm{ex}}(p_{\boldsymbol{X}}(\boldsymbol{X}),R,T_c)+o(1)},$$

where $o(1) \to 0$ when $N_b \to \infty$. The expurgated error exponent $E_{ex}(p_{\boldsymbol{X}}(\boldsymbol{X}), R, T_c)$ is defined as

$$E_{\mathrm{ex}}(p_{\boldsymbol{X}}(\boldsymbol{X}), R, T_c) = \max_{\rho\ge 1}\Big\{\max_{r\ge 0} E_x(p_{\boldsymbol{X}}(\boldsymbol{X}), \rho, r, T_c) - \rho R\Big\}, \tag{24}$$

with

$$E_x(p_{\boldsymbol{X}}(\boldsymbol{X}), \rho, r, T_c) = -\frac{1}{T_c}\ln\Bigg\{\int\limits_{\boldsymbol{H}} p_{\boldsymbol{H}}(\boldsymbol{H})\Bigg\{\int\limits_{\boldsymbol{X}'}\int\limits_{\boldsymbol{X}} p_{\boldsymbol{X}}(\boldsymbol{X})p_{\boldsymbol{X}}(\boldsymbol{X}')e^{r[\mathrm{tr}(\boldsymbol{X}\boldsymbol{X}')\,\mathrm{tr}(\boldsymbol{X}'\boldsymbol{X}'^{\dagger})-2T_cP]}$$
$$\times\Bigg\{\int\limits_{\boldsymbol{Y}} \sqrt{p(\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{H})}\sqrt{p(\boldsymbol{Y}|\boldsymbol{X}',\boldsymbol{H})}\,d\boldsymbol{Y}\Bigg\}^{\frac{1}{\rho}} d\boldsymbol{X}\,d\boldsymbol{X}'\Bigg\}^{\rho} d\boldsymbol{H}\Bigg\}, \tag{25}$$

where $\boldsymbol{X}'$ denotes the input signal of the good codewords and has the same distribution as $\boldsymbol{X}$.

With the above definition, we now present the following key result.

Proposition 3. *The expurgated error exponent of multi-keyhole MIMO channels is given by*

$$E_{\mathrm{ex}}(R, T_c) = \max_{\rho\ge 1}\Big\{\max_{0\le\beta\le n_t} \mathcal{K}'(\rho,\beta) - \frac{1}{T_c}\ln\{C_3\det(\boldsymbol{\Delta}_6)\} - \rho R\Big\}, \tag{26}$$

where

$$\{\boldsymbol{\Delta}_6\}_{i,j} = \begin{cases} b_i^{j-1}, & j \le n_k - p,\\ b_i^{j-1} G_{3,1}^{1,3}\left[\dfrac{\gamma b_i}{2\beta\rho}\,\middle|\, \begin{matrix} n_k - n_r - j + 1, 1 - n_t + n_k - j, 1 - T_c\rho \\ 0 \end{matrix}\right], & j > n_k - p. \end{cases} \tag{27}$$

Proof. Similarly to (17), we have

$$\widetilde{E}_x(\rho,\beta,T_c) \triangleq E_x\Big(\frac{P}{n_t}\boldsymbol{I}_{n_t}, \rho, r, T_c\Big)\Big|_{\beta=n_t-rP}$$
$$= \mathcal{K}'(\rho,\beta) - \frac{1}{T_c}\ln\Big\{\mathbf{E}\Big[\det\Big(\boldsymbol{I}_{n_r} + \frac{\gamma\boldsymbol{H}\boldsymbol{H}^{\dagger}}{2\rho\beta}\Big)^{-\rho T_c}\Big]\Big\}, \tag{28}$$

where $\mathcal{K}'(\rho,\beta) = 2\rho(n_t-\beta) + 2\rho n_t\ln\Big(\dfrac{\beta}{n_t}\Big)$.

From (28), $\widetilde{E}_x(\rho,\beta,T_c)$ for multi-keyhole MIMO channels is given by

$$\widetilde{E}_x(\rho,\beta,T_c) = \mathcal{K}'(\rho,\beta) - \frac{1}{T_c}\ln\Big\{\mathbf{E}\Big[\det\Big(\boldsymbol{I}_{n_r} + \frac{\gamma\boldsymbol{F}}{2\rho\beta}\Big)^{-\rho T_c}\Big]\Big\}. \tag{29}$$

Substituting (2) into (29) yields the desired result. △

Similarly, a closed-form expression for the expurgated error exponent of Rayleigh-product MIMO channels can be obtained as follows.

Proposition 4. *The expurgated error exponent $E_{ex}(R, T_c)$ for Rayleigh-product MIMO channels is given by*

$$E_{\text{ex}}(R, T_c) = \max_{\rho \geq 1} \Big\{ \max_{0 \leq \beta \leq n_t} \Big\{ \mathcal{K}'(\rho, \beta) - \frac{1}{T_c} \ln\{M_1 \det(\mathbf{\Psi}')\} \Big\} - \rho R \Big\}, \tag{30}$$

where $\mathbf{\Psi}'$ is the $u \times u$ matrix with entries

$$\psi'_{i,j} = \begin{cases} b'_{i,j}, & i = 1, \ldots, u, \ j = 1, \ldots, m, \\ c_{i,j}, & i = 1, \ldots, u, \ j = m+1, \ldots, u, \end{cases} \tag{31}$$

where

$$b'_{i,j} = \frac{1}{\Gamma(\rho T_c)} n_k^{-v+n_t-i-j-n+u+1} \times G^{1,3}_{3,1}\Big[\frac{\gamma}{2\beta\rho n_k} \ \Big| \ \begin{matrix} -v + n_t - i - j - n + u + 2, -j - n + u + 1, 1 - \rho T_c \\ 0 \end{matrix} \Big].$$

Proof. From (24), the expurgated error exponent $E_{ex}(R, T_c)$ for Rayleigh-product MIMO channels is given by

$$E_{\text{ex}}(R, T_c) = \max_{\rho \geq 1} \Big\{ \max_{0 \leq \beta \leq n_t} \widetilde{E}_x(\rho, \beta, T_c) - \rho R \Big\}, \tag{32}$$

where $\widetilde{E}_x(\rho, \beta, T_c)$ is given by

$$\widetilde{E}_x(\rho, \beta, T_c) = \mathcal{K}'(\rho, \beta) - \frac{1}{T_c} \ln\Big\{ \mathbf{E}\Big[\det\Big(\boldsymbol{I}_{n_r} + \frac{\gamma \boldsymbol{H}\boldsymbol{H}^\dagger}{2\beta\rho} \Big)^{-T_c\rho} \Big] \Big\}. \tag{33}$$

Then the desired result follows by invoking Proposition 2 with $\tau = T_c\rho$ and $\varpi = 2\beta\rho$. △

With $n_k = 1$, the expurgated error exponent of keyhole MIMO channels is given by

$$\begin{aligned} E_{\text{ex}}(R, T_c)_{\text{keyhole}} &= \max_{\rho \geq 1} \Big\{ \max_{0 \leq \beta \leq n_t} \widetilde{E}_x^{\text{keyhole}}(\rho, \beta, T_c) - \rho R \Big\} \\ &= \max_{\rho \geq 1} \Big\{ \max_{0 \leq \beta \leq n_t} \Big\{ \mathcal{K}'(\rho, \beta) - \frac{1}{T_c} \ln\Big\{ \frac{\psi'_{1,1}}{(n_r - 1)!\,(n_t - 1)!} \Big\} - \rho R \Big\} \Big\}, \end{aligned} \tag{34}$$

where $\psi'_{1,1} = \frac{1}{\Gamma(\rho T_c)} G^{1,3}_{3,1}\Big[\frac{\gamma}{2\beta\rho} \ \Big| \ \begin{matrix} -n_r + 1, -n_t + 1, 1 - \rho T_c \\ 0 \end{matrix} \Big]$.

Figure 1 shows[4] the effect of the correlation between keyholes on the RCEE, $E_r(R, T_c)$, and expurgated error exponent, $E_{ex}(R, T_c)$, for multi-keyhole MIMO channels as a function of rate R. We see that for a fixed antenna configuration, both the error exponents $E_r(R, T_c)$ and $E_{ex}(R, T_c)$ decrease with R, as one expects. Also, the error exponents increase when the correlation between the keyholes is reduced, i.e., $b_j - b_i$ goes to 0, $\forall i \neq j$. The values are bounded by the two special cases which are product MIMO channels (upper bound) and keyhole MIMO channels (lower bound). Moreover, with $n_k \to \infty$ and no correlation between the keyholes, the error exponents of multi-keyhole MIMO channels respectively denote the error exponent of the general MIMO channels. Therefore, with a particular antenna configuration at the transmitter and receiver, the error exponent of MIMO channels (with $n_k = 500$) is greater than that of keyhole MIMO channels. On the other hand, Fig. 2 shows the effect of the number of keyholes on the $E_r(R, T_c)$ and $E_{ex}(R, T_c)$

[4] Note that in all simulations, unless otherwise specified, the number of keyholes and the corresponding power distributions are as follows: for $n_k = 2$, $\{b_1, b_2\} = \{0.4, 0.6\}$ and for $n_k = 4$, $\{b_1, b_2, b_3, b_4\} = \{0.1, 0.2, 0.3, 0.4\}$.

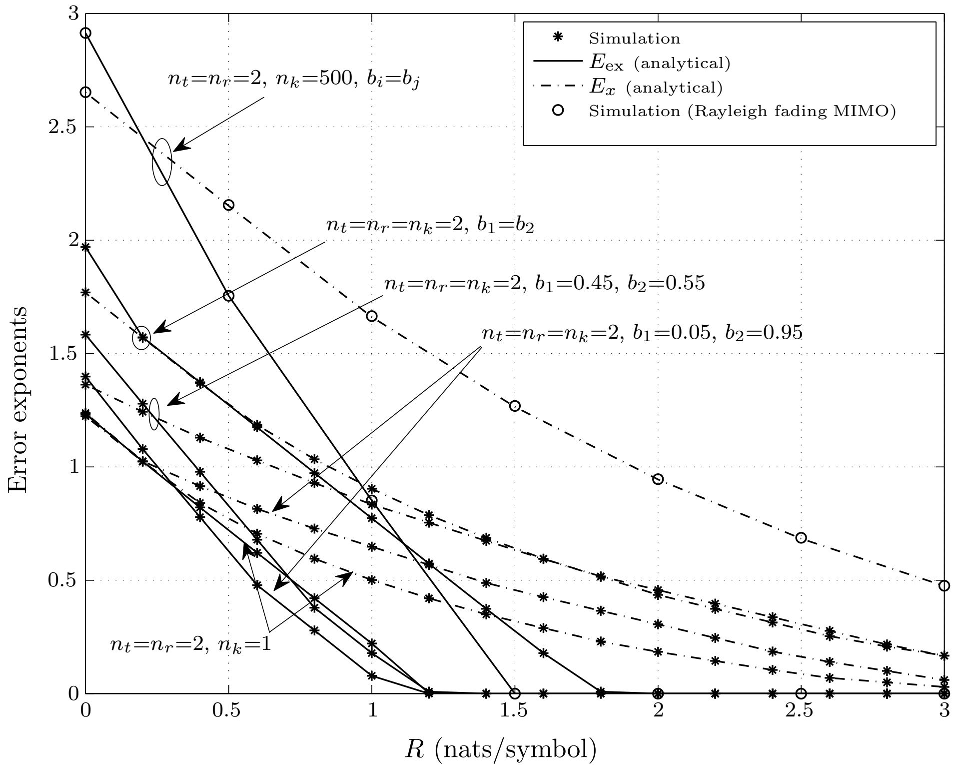

Fig. 1. RCEE and the expurgated error exponent for multi-keyhole MIMO channels with $T_c = 5$ and $\gamma = 15\,\text{dB}$.

for multi-keyhole MIMO channels. It is observed that both the $E_r(R, T_c)$ and $E_{ex}(R, T_c)$ increase with the number of keyholes.

Figure 3 shows the upper bounds of the error probability [1]

$$P_e(E_r) = \frac{8\pi e^2 (n_t - \beta)^2 N_b}{n_t T_c} e^{-N_b T_c\, E_r(R,T_c)}, \tag{35}$$

$$P_e(E_{\rm ex}) = \frac{8\pi e^2 (n_t - \beta)^2 N_b}{n_t T_c} e^{-N_b T_c\, E_{\rm ex}(R,T_c)} \tag{36}$$

for Rayleigh-product MIMO channels as a function of R. It shows $P_e(E_r)$ and $P_e(E_{ex})$ for selected values of T_c. We see that the error probability increases with T_c (in Fig. 3), which in turn causes a reduction and improvement in the error exponents.

3.3. Cutoff Rate

The cutoff rate R_o can be seen as a lower bound to the channel capacity. This important parameter determines the zero-rate error exponent magnitude and the rate regime in which an arbitrarily high level of reliability can be reached by increasing the codeword length. The value of $\widetilde{E}_o(\rho, \beta, T_c)$ at $\rho = 1$ and $\beta = n_t$ is the cutoff rate [5] of the channel. Therefore, the cutoff rate can be expressed as

$$R_0 = \widetilde{E}_0(\rho, \beta, T_c)\big|_{\rho=1,\, \beta=n_t}. \tag{37}$$

From (37), the cutoff rate for multi-keyhole MIMO channels is given by

$$R_0 = -\frac{1}{T_c} \ln \mathbf{E}\left[\det\left(\boldsymbol{I}_{n_r} + \frac{\gamma \boldsymbol{F}}{2n_t}\right)^{-T_c}\right]. \tag{38}$$

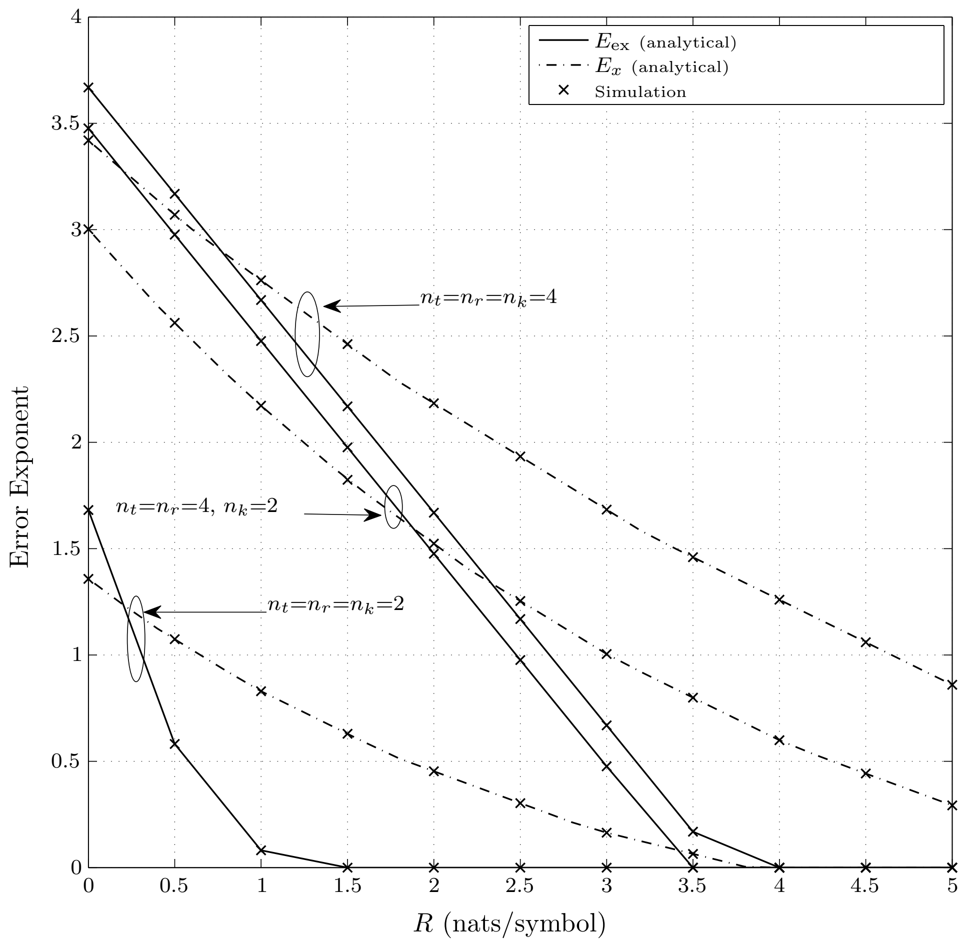

Fig. 2. Random coding and expurgated error exponents for multi-keyhole MIMO channels with $T_c = 5$ and $\gamma = 15\,\mathrm{dB}$ for selected values of n_k, n_t, and n_r.

When $\tau = T_c$ and $\varpi = 2n_t$, a closed-form expression for the cutoff rate of Rayleigh fading multi-keyhole MIMO channels is given by

$$R_0^{\text{multi-keyhole}} = -\frac{1}{T_c}\ln\left(\frac{\det(\boldsymbol{\Delta}_7)}{\Gamma(T_c)\prod\limits_{i=1}^{n_k}\Gamma(n_t-i+1)\Gamma(n_r-i+1)\prod\limits_{i<j}^{n_k}(b_j-b_i)}\right), \tag{39}$$

where

$$\{\boldsymbol{\Delta}_7\}_{i,j} = \begin{cases} b_i^{j-1}, & j \le n_k - p, \\ b_i^{j-1} G_{3,1}^{1,3}\left[\dfrac{\gamma b_i}{2n_t}\,\middle|\, \begin{matrix} n_k - n_r - j + 1, 1 - n_t + n_k - j, 1 - T_c \\ 0 \end{matrix}\right], & j > n_k - p. \end{cases} \tag{40}$$

For Rayleigh-product MIMO channels, the cutoff rate is given by

$$R_0 = -\frac{1}{T_c}\ln \mathbf{E}\left[\det\left(\boldsymbol{I}_{n_r} + \frac{\gamma \boldsymbol{H}\boldsymbol{H}^\dagger}{2n_t}\right)^{-T_c}\right]. \tag{41}$$

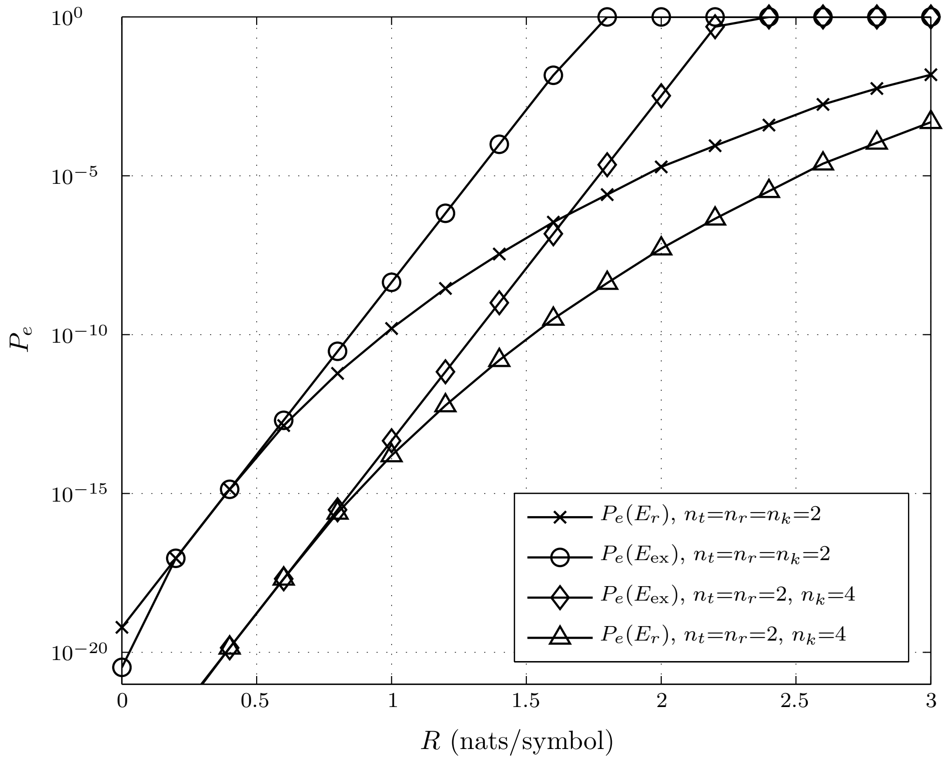

Fig. 3. Upper bound of the error probabilities $P_e(E_r)$ and $P_e(E_{ex})$ for Rayleigh-product MIMO channels with $n_r = n_t = 2$, $N_b = 5$, and $\gamma = 15\,\mathrm{dB}$ for selected values of T_c.

Based on the p.d.f. of (10), we have

$$R_0 = -\frac{1}{T_c}\ln\int\limits_{\boldsymbol\lambda}\prod_{\boldsymbol\lambda}\Big(1+\frac{\gamma\boldsymbol\lambda}{2n_t}\Big)^{-T_c}p_{\boldsymbol\lambda}(\boldsymbol\lambda)\,d\boldsymbol\lambda$$
$$= -\frac{1}{T_c}\ln\bigg\{\int\limits_{\lambda_1}\dots\int\limits_{\lambda_m}\prod_{i=1}^{m}\frac{1}{\Gamma(T_c)}G_{1,1}^{1,1}\bigg[\frac{\gamma\lambda_i}{2n_t}\,\bigg|\,\begin{matrix}1-T_c\\0\end{matrix}\bigg]M_1|\Phi(\boldsymbol\lambda)||\Xi(\boldsymbol\lambda)|\prod_{i=1}^{m}\lambda_i^{n-u}\,d\lambda_1\dots d\lambda_m\bigg\}. \quad (42)$$

Following the process in the proof of Theorem 2, a closed-form expression for the cutoff rate of Rayleigh-product MIMO channels is given by

$$R_0 = -\frac{1}{T_c}\ln\bigg\{M_1\det\begin{pmatrix}b_{i,j}^{R_0}, & i=1,\dots,u,\ j=1,\dots,m\\ c_{i,j}, & i=1,\dots,u,\ j=m+1,\dots,u\end{pmatrix}\bigg\}, \quad (43)$$

where

$$b_{i,j}^{R_0} = \frac{1}{\Gamma(T_c)}n_k^{-v+n_t-i-j-n+u+1} \times G_{3,1}^{1,3}\bigg[\frac{\gamma}{2n_tn_k}\,\bigg|\,\begin{matrix}-v+n_t-i-j-n+u+2, -j-n+u+1, 1-T_c\\0\end{matrix}\bigg]. \quad (44)$$

Substituting $n_k = 1$ into (43), the cutoff rate for keyhole MIMO channels is given by

$$R_0^{\rm keyhole} = -\frac{1}{T_c}\ln\bigg\{\frac{1}{(n_r-1)!\,(n_t-1)!\,\Gamma(T_c)}G_{3,1}^{1,3}\bigg[\frac{\gamma}{2n_t}\,\bigg|\,\begin{matrix}-n_r+1, -n_t+1, 1-T_c\\0\end{matrix}\bigg]\bigg\}. \quad (45)$$

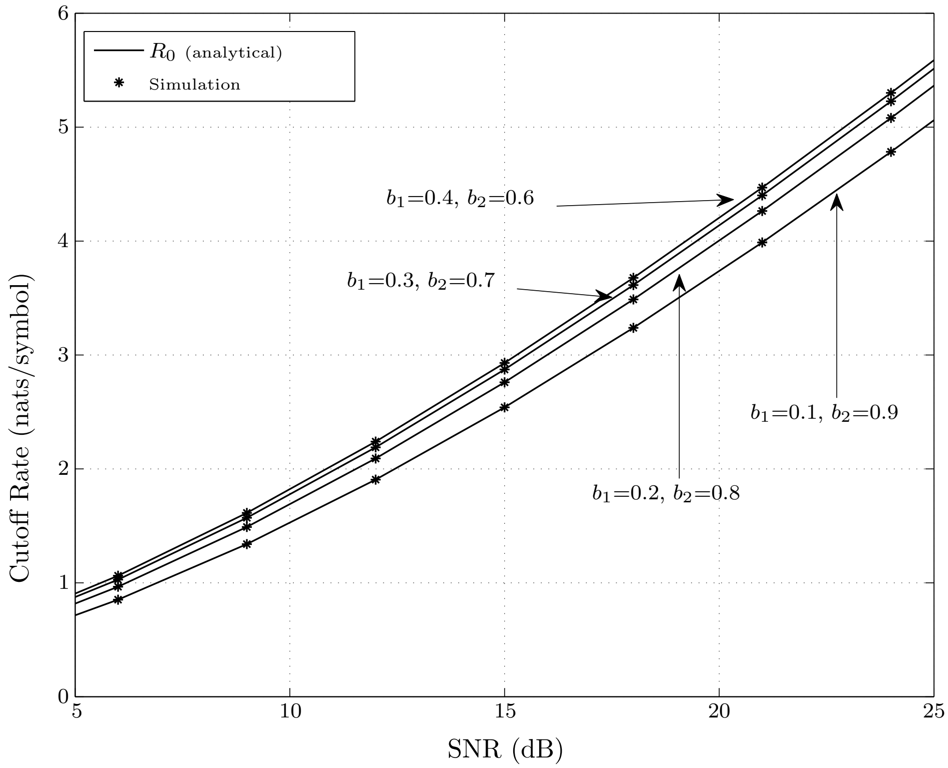

Fig. 4. Cutoff rate R_0 for multi-keyhole MIMO channels with $T_c = 5$.

Figures 4 and 5 show the cutoff rates for multi-keyhole MIMO channels as a function of the SNR (dB). In Fig. 4, we show the cutoff rate with different correlations between the keyholes. As can readily be observed, the cutoff rate improves when the keyholes are less correlated. We can observe that the cutoff rate increases when either n_t, n_r, or n_k increases.

3.4. Ergodic Capacity

We can reach the maximum value of the exponent in (14) by defining the information rate R as

$$R \triangleq \left[\frac{\partial \widetilde{E}_0(\rho, \beta, T_c)}{\partial \rho}\right]\Big|_{\beta=\beta^*(\rho)}, \tag{46}$$

where $\beta^*(\rho)$ is the solution of $\left[\frac{\partial \widetilde{E}_0(\rho, \beta, N_c)}{\partial \beta} = 0\right]$ for all $0 \le \rho \le 1$ and is always in the range $0 < \beta \le n_t$. We should mention that the value of R is the critical rate when $\rho = 1$. The value of R at $\rho = 0$ (i.e., $\beta^*(0) = n_t$) is the ergodic capacity of the channel and is given by

$$\langle C \rangle \triangleq \left[\frac{\partial \widetilde{E}_0(\rho, \beta, T_c)}{\partial \rho}\right]\Big|_{\rho=0,\, \beta=n_t}. \tag{47}$$

From (17), the ergodic capacity of Rayleigh fading MIMO channels is given by

$$\langle C \rangle = \left[\frac{\partial \widetilde{E}_0(\rho, \beta, N_c, m)}{\partial \rho}\right]\Big|_{\rho=0,\, \beta=n_T} = \mathbf{E}\Big[\ln\det\Big(\boldsymbol{I}_{n_r} + \frac{\gamma}{n_t}\boldsymbol{F}\Big)\Big] = m\,\mathbf{E}_\lambda\Big[\ln\Big(1 + \frac{\gamma}{n_t}\lambda\Big)\Big]. \tag{48}$$

Note that we derive expression (48) from the error exponent, and it is independent of T_c.

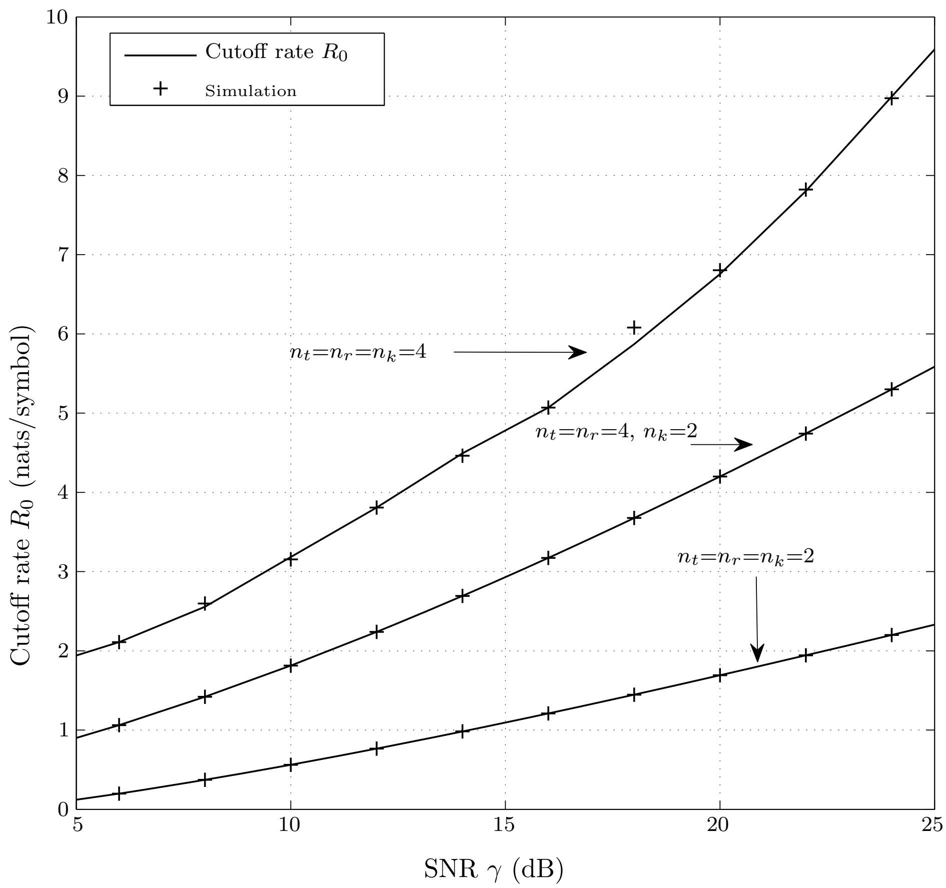

Fig. 5. Cutoff rate R_0 for multi-keyhole MIMO channels with $T_c = 5$ for selected values of n_k, n_t, and n_r.

From (48), the ergodic capacity can be evaluated by

$$\langle C\rangle = m\int\limits_0^\infty \ln\Big(1+\frac{\gamma}{n_t}\lambda\Big)g_\lambda(\lambda)\,d\lambda. \tag{49}$$

The marginal p.d.f. of an unordered eigenvalue of $\boldsymbol{F}$, denoted by $g_\lambda(\lambda)$, is given in [22]. Therefore, substituting the value of $g_\lambda(\lambda)$ into (49) and using the identity of (54) and [18, equation (7.821.3)], a closed-form expression for the ergodic capacity of multi-keyhole MIMO channels with equal-power allocation is given by

$$\langle C\rangle^{\text{multi-keyhole}} = \frac{1}{\prod\limits_{i<j}^{n_k}(b_j-b_i)}\sum_{i=1}^{n_k}\sum_{j=n_k-p+1}^{n_k}\frac{G_{4,2}^{1,4}\Big[\dfrac{\gamma b_i}{n_t}\ \Big|\ \begin{matrix} n_k+j-1-\alpha, n_k+j-1-\varrho, 1, 1\\ 1, 0\end{matrix}\Big]}{b_i^{1-j}D_{i,j}^{-1}\Gamma(\varrho-n_k+j)\Gamma(\alpha-n_k+j)}, \tag{50}$$

where $D_{i,j}$ denotes the (i,j)th cofactor of the matrix $\boldsymbol{\Lambda}$, which is defined as

$$[\boldsymbol{\Lambda}]_{i,j} = b_i^{j-1}, \quad 1\le i,j\le n_k. \tag{51}$$

We should note that our expression of the ergodic capacity is also in agreement with the previous result in [22], where a similar result was derived with a small constant parameter difference.

For Rayleigh-product MIMO channels, the marginal p.d.f. of an unordered eigenvalue λ of $\boldsymbol{F}$ is given by [19]

$$f_\lambda(\lambda) = \frac{2M_3}{m}\sum_{i=1}^{u}\sum_{j=u-m+1}^{u}\frac{\lambda^{(n_t+2j+v+i-2u-3)/2}}{\Gamma(n_t-u+j)}K_{v-n_t+i-1}(2\sqrt{\lambda})D_{i,j}, \tag{52}$$

where $M_3 = \left(\prod\limits_{l=1}^{u}\Gamma(u-l+1)\Gamma(v-l+1)\right)^{-1}$ and $D_{i,j}$ is the (i,j)th cofactor of the $u\times u$ matrix $\boldsymbol{\Theta}$ with entries $\{\boldsymbol{\Theta}\}_{m,n} = \Gamma(v-u+m+n-1)$. Using the p.d.f. of (52), we have

$$\begin{aligned}\langle C\rangle = m\int\limits_{\lambda}\ln\Bigl(1+\frac{\gamma}{n_t}\lambda\Bigr)f_\lambda(\lambda)d\lambda = 2M_3\sum_{i=1}^{u}\sum_{j=u-m+1}^{u}\frac{D_{i,j}}{\Gamma(n_t-u+j)}\\ \times\int\limits_{\lambda}\ln\Bigl(1+\frac{\gamma}{n_t}\lambda\Bigr)\lambda^{(n_t+2j+v+i-2u-3)/2}K_{v-n_t+i-1}(2\sqrt{\lambda})\,d\lambda.\end{aligned} \tag{53}$$

Using the identity of [29, equation (8.4.6.5)]

$$\ln(1+ax) = G_{2,2}^{1,2}\left[ax \,\middle|\, \begin{matrix}1,1\\1,0\end{matrix}\right] \tag{54}$$

and [18, equation (7.821.3)], we obtain the ergodic capacity for Rayleigh-product MIMO channels as

$$\langle C\rangle = M_3\sum_{i=1}^{u}\sum_{j=u-m+1}^{u}\frac{D_{i,j}}{\Gamma(n_t-u+j)}G_{4,2}^{1,4}\left[\frac{\gamma}{n_t}\,\middle|\, \begin{matrix}-v-i-j+u+2,-j-n_t+u+1,1,1\\1,0\end{matrix}\right]. \tag{55}$$

Substituting $n_k = 1$ into (55), a closed-form expression for the ergodic capacity of keyhole MIMO channels is given by

$$\langle C\rangle_{\text{keyhole}} = \frac{1}{\Gamma(n_t)}G_{4,2}^{1,4}\left[\frac{\gamma}{n_t}\,\middle|\, \begin{matrix}1-n_r,1-n_t,1,1\\1,0\end{matrix}\right]. \tag{56}$$

4. CODEWORD LENGTH ANALYSIS

In order to investigate the effect of the channel coherence time, SNR, antenna diversity, and the number of keyholes on the codeword length required to achieve a certain upper bound of the decoding error probability, we estimate the required code length.

The decoding error probability is defined as [30]

$$P_e(E_r) = \left(\frac{2e^{r\delta}}{\xi}\right)^2 e^{-N_bT_c\,E_r(R,T_c)}. \tag{57}$$

A computable equation (35) can be derived by using $\xi \approx \dfrac{\delta}{\sqrt{2\pi N_b\sigma_\xi^2}}$, $\beta = n_T - rP$, and minimizing the factor $\left(\frac{2e^{r\delta}}{\xi}\right)^2$ over δ for large N_b.

Let $L = \lceil T_cN_b\rceil$ denote the estimated codeword lengths for N_b calculated from (35), where $\lceil\cdot\rceil$ denotes the smallest integer larger than or equal to an enclosed quantity. The effect of the channel coherence time on L_r is described in Table 1, to achieve the upper bound of the decoding error probability $P_e(E_r) = 10^{-6}$ at the transmission rate 2.0 bits/symbol with $n_r = n_t = 2$ and $\gamma = 15\,\text{dB}$. Since error probability increases with T_c, in order to maintain a fixed SNR with a certain power allocation and noise variance, codeword length should be increased with T_c as is illustrated in the table. Moreover, we see that when T_c increases, there is a significant increase in

Table 1. Effect of channel coherence time T_c on the codeword lengths L_r required to achieve the upper bound of the decoding error probability $P_e^r \leq 10^{-6}$ at a rate 2.0 bits/symbol with $n_r = n_t = 2$ and $\gamma = 15\,\mathrm{dB}$

Product MIMO ($n_k = 2$)				MIMO ($n_k = 500$)				Keyhole MIMO ($n_k = 1$)				Multi-keyhole MIMO ($n_k = 2$)			
T_c	L_r	γ (dB)	L_r	T_c	L_r	γ (dB)	L_r	T_c	L_r	γ (dB)	L_r	T_c	L_r	γ (dB)	L_r
1	14	11	94	1	9	11	39	1	33	11	336	1	16	11	100
2	21	12	78	2	11	12	30	2	48	12	218	2	20	12	71
3	26	13	59	3	13	13	24	3	65	13	142	3	25	13	55
4	32	14	46	4	15	14	20	4	69	14	104	4	31	14	44
5	38	15	38	5	18	15	18	5	82	15	82	5	36	15	36
6	43	16	31	6	19	16	16	6	92	16	67	6	42	16	31
7	50	17	26	7	21	17	14	7	108	17	54	7	43	17	27
8	55	18	23	8	24	18	12	8	129	18	44	8	45	18	23
9	56	19	20	9	26	19	11	9	137	19	37	9	50	19	20
10	61	20	18	10	28	20	10	10	144	20	32	10	56	20	18

Table 2. Effect of the number of keyholes on the codeword lengths L_r required to achieve the upper bound of the decoding error probability $P_e^r \leq 10^{-6}$ at a rate 2.0 bits/symbol with $n_r = n_t = 4$

	Multi-keyhole			Multi-keyhole	
T_c	n_k	L_r	γ (dB)	n_k	L_r
1	2	6	12	2	16
	4	5		4	11
3	2	9	14	2	12
	4	6		4	9
5	2	13	16	2	9
	4	9		4	7

coding complexity relative to the single-symbol coherence time. When T_c goes from 2 to 10, in the case of product MIMO channels (with $n_k = 2$), the increase in L_r ranges from 50% to 336%, while for MIMO channels (with $n_k = 500$), the increase in L_r ranges from 22% to 211%. In the case of keyhole MIMO channels (with $n_k = 1$), the increase in L_r ranges from 45% to 245%, while for multi-keyhole MIMO channels (with $n_k = 2$), the increase in L_r ranges from 25% to 250%. As is seen from the table, the required codeword length decreases with n_k; therefore, for the same coherence time, keyhole MIMO channels require longer codeword length than product MIMO, multi-keyhole MIMO, and MIMO channels to achieve a certain level of reliability. Table 1 also shows the effect of the SNR on L_r with $T_c = 5$. We see that the required code length decreases with the SNR, but the decreasing rate is not constant. For example, in the case of product MIMO channels, increasing the SNR from 11 dB to 12 dB reduces 17% of the codeword length, but the codeword length reduces only 10% when the SNR increases from 19 dB to 20 dB.

Table 2 shows the effect of keyholes on L_r to achieve the decoding error probability $P_e(E_r) = 10^{-6}$ at the transmission rate of 2.0 bits/symbol with $T_c = 5$ and $\gamma = 15\,\mathrm{dB}$. We see that for fixed T_c and γ, the required codeword length decreases with the number of keyholes. However, the rate of reduction in the codeword length increases with T_c up to a certain value and decreases with γ. For example, (i) when $T_c = 1$, increasing n_k from 2 to 4 reduces L_r by 17%; (ii) when $T_c = 3$, increasing n_k from 2 to 4 reduces L_r by 33%; and (iii) when $T_c = 5$, increasing n_k from 2 to 4 reduces L_r by 31%. On the other hand, (i) when $\gamma = 12\,\mathrm{dB}$, increasing n_k from 2 to 4 reduces L_r by 31.6%; (ii) when $\gamma = 14\,\mathrm{dB}$, increasing n_k from 2 to 4 reduces L_r by 28.6%; and (ii) when $\gamma = 16\,\mathrm{dB}$, increasing n_k from 2 to 4 reduces L_r by 27.3%.

5. CONCLUSION

In this paper, we have studied the error exponents of multi-keyhole MIMO channels. Closed-form expressions were presented for the key performance measures, which provide clear insights into impact of key system parameters on the performance of a system. The effect of the channel coherence time, SNR, and diversity on the codeword length required to achieve a certain upper bound of the decoding error probability were also investigated. Our findings suggested that, for a fixed symbol coherence time, the coding complexity decreases with the SNR, but the decreasing rate is slow in the high SNR regime. In addition, the required codeword length for multi-keyhole MIMO channels decreases with the number of keyholes.

REFERENCES

1. Gallager, R.G., *Information Theory and Reliable Communication*, New York: Wiley, 1968.
2. Ahmed, W.K.M., Information Theoretic Reliability Function for Flat Fading Channels, *PhD Thesis*, Queen's Univ., Kingston, Ontario, Canada, 1997.
3. Kaplan, G. and Shamai (Shitz), S., Error Probabilities for the Block-Fading Gaussian Channel, *Arch. Electronik Übertragungstech. (AEÜ)*, 1995, vol. 49, no. 4, pp. 192–205.
4. Grant, A., Rayleigh Fading Multi-Antenna Channels, *EURASIP J. Adv. Signal Process.*, 2002, no. 3, pp. 316–329.
5. Sarkar, M.Z.I., Shin, H., and Win, M.Z., Random Coding Exponent for MIMO Channels, in *Proc. 2008 IEEE 67th Vehicular Technology Conf. (VTC'2008 Spring), Marina Bay, Singapore, May 11–14, 2008*, pp. 559–563.
6. Shin, H. and Win, M.Z., Gallager's Exponent for MIMO Channels: A Reliability-Rate Tradeoff, *IEEE Trans. Commun.*, 2009, vol. 57, no. 4, pp. 972–985.
7. Telatar, I.E., Capacity of Multi-Antenna Gaussian Channels, *Eur. Trans. Telecommun.*, 1999, vol. 10, no. 6, pp. 585–595.
8. Wu, X. and Srikant, R., MIMO Channels in the Low-SNR Regime: Communication Rate, Error Exponent, and Signal Peakiness, *IEEE Trans. Inform. Theory*, 2007, vol. 53, no. 4, pp. 1290–1309.
9. Gesbert, D., Bolcskei, H., Gore, D.A., and Paulraj, A.J., Outdoor MIMO Wireless Channels: Models and Performance Prediction, *IEEE Trans. Commun.*, 2002, vol. 50, no. 12, pp. 1926–1934.
10. Levin, G. and Loyka, S., Multi-Keyhole MIMO Channels: Asymptotic Analysis of Outage Capacity, in *Proc. 2006 IEEE Int. Sympos. on Information Theory (ISIT'2006), Seattle, WA, July 9–14, 2006*, pp. 1305–1309.
11. Shin, H. and Lee, J.H., Capacity of Multiple-Antenna Fading Channels: Spatial Fading Correlation, Double Scattering, and Keyhole, *IEEE Trans. Inform. Theory*, 2003, vol. 49, no. 10, pp. 2636–2647.
12. Jin, S., McKay, M.R., Wong, K.-K., and Gao, X., Transmit Beamforming in Rayleigh Product MIMO Channels: Capacity and Performance Analysis, *IEEE Trans. Signal Process.*, 2008, vol. 56, no. 10, Part 2, pp. 5204–5221.
13. Levin, G. and Loyka, S., On the Outage Capacity Distribution of Correlated Keyhole MIMO Channels, *IEEE Trans. Inform. Theory*, 2008, vol. 54, no. 7, pp. 3232–3245.
14. Levin, G. and Loyka, S., From Multi-Keyholes to Measure of Correlation and Power Imbalance in MIMO Channels: Outage Capacity Analysis, *IEEE Trans. Inform. Theory*, 2011, vol. 57, no. 6, pp. 3515–3529.
15. Shin, H. and Win, M.Z., MIMO Diversity in the Presence of Double Scattering, *IEEE Trans. Inform. Theory*, 2008, vol. 54, no. 7, pp. 2976–2996.
16. Zhong, C., Jin, S., and Wong, K.-K., MIMO Rayleigh-Product Channels with Co-channel Interference, *IEEE Trans. Commun.*, 2009, vol. 57, no. 6, pp. 1824–1835.

17. Zhong, C., Jin, S., Wong, K.-K., and McKay, M.R., Outage Analysis for Optimal Beamforming MIMO Systems in Multikeyhole Channels, *IEEE Trans. Signal Process.*, 2010, vol. 58, no. 3, Part 1, pp. 1451–1463.

18. Gradshteyn, I.S. and Ryzhik, I.M., *Table of Integrals, Series, and Products*, New York: Academic, 2000, 7th ed.

19. Jin, S., McKay, M.R., Zhong, C., and Wong, K.-K., Ergodic Capacity Analysis of Amplify-and-Forward MIMO Dual-Hop Systems, *IEEE Trans. Inform. Theory*, 2010, vol. 56, no. 5, pp. 2204–2224.

20. Shin, H., Moe, F., Win, Z., Lee, J.H., and Chiani, M., On the Capacity of Doubly Correlated MIMO Channels, *IEEE Trans. Wireless Commun.*, 2006, vol. 5, no. 7, pp. 2253–2265.

21. Kang, M. and Alouini, M.-S., Capacity of Correlated MIMO Rayleigh Channels, *IEEE Trans. Wireless Commun.*, 2006, vol. 5, no. 1, pp. 143–155.

22. Zhong, C., Jin, S., Wong, K.-K., and McKay, M.R., Ergodic Mutual Information Analysis for Multi-Keyhole MIMO Channels, *IEEE Trans. Wireless Commun.*, 2011, vol. 10, no. 6, pp. 1754–1763.

23. Smith, P.J., Garth, L.M., and Loyka, S., Exact Capacity Distributions for MIMO Systems with Small Numbers of Antennas, *IEEE Commun. Lett.*, 2003, vol. 7, no. 10, pp. 481–483.

24. Xue, J., Sarkar, M.Z.I., and Ratnarajah, T., Error Exponents for Nakagami-m Fading Channels, *Probl. Peredachi Inf.*, 2014, vol. 50, no. 2, pp. 31–59 [*Probl. Inf. Trans.* (Engl. Transl.), 2014, vol. 50, no. 2, pp. 144–170].

25. Zhang, H., Jin, S., Zhang, X., and Yang, D., On Marginal Distributions of the Ordered Eigenvalues of Certain Random Matrices, *EURASIP J. Adv. Signal Process.*, 2010, Article ID 957243, 12 pp.

26. `http://functions.wolfram.com/HypergeometricFunctions/MeijerG/03/01/03/11` (Meijer G-Function. The Wolfram Functions Site).

27. Chiani, M., Win, M.Z., and Zanella, A., On the Capacity of Spatially Correlated MIMO Rayleigh-Fading Channels, *IEEE Trans. Inform. Theory*, 2003, vol. 49, no. 10, pp. 2363–2371.

28. Shannon, C.E., Probability of Error for Optimal Codes in a Gaussian Channel, *Bell Syst. Tech. J.*, 1959, vol. 38, no. 3, pp. 611–656.

29. Prudnikov, A.P., Brychkov, Yu.A., and Marichev, O.I., *Integraly i ryady*, vol. 3: *Spetsial'nye funktsii. Dopolnitel'nye glavy* (Special Functions. Advanced Chapters), Moscow: Nauka, 1986. Translated under the title *Integrals and Series*, New York: Gordon & Breach, 1986.

30. Ahmed, W.K.M. and McLane, P.J., Random Coding Error Exponents for Two-Dimensional Flat Fading Channels with Complete Channel State Information, *IEEE Trans. Inform. Theory*, 1999, vol. 45, no. 4, pp. 1338–1346.