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Nick Polydorides[1], Taweechai Ouypornkochagorn[1], Hugh McCann[1]

[1]School of Engineering, University of Edinburgh, UK. n.polydorides@ed.ac.uk

Abstract: We propose a methodology for speeding up the computation of a nonlinear inverse solution, by approximating efficiently the Jacobian of the forward model. This has practical advantages not only in linearising the inverse problem but also in computing the regularised solution.

1 Introduction

We consider the inverse problem of estimating a conductivity function $\sigma \in X$ from a finite set of electrical potential measurements $y \in \Re^m$ from the nonlinear model

$$y = F(\sigma) + n, \tag{1}$$

where n is additive random noise, $F : X \to \Re^m$ is an analytic forward mapping, with continuous first and second weak derivatives and X is the space of bounded positive functions spanning over a closed domain Ω. For a choice of regularisation term $R : X \to \Re$, the inverse problem is to estimate

$$\hat{\sigma} = \arg\min_{\sigma \in X} \Big\{ \|F(\sigma) - y\|^2 + R(\sigma) \Big\}. \tag{2}$$

Using a gradient-based algorithm, such as the Gauss-Newton, a local minimum can be traced iteratively for $n = 0, 1, \ldots$

$$\sigma_{n+1} = \sigma_n + h_n \tag{3}$$

$$h_n = \arg\min\{\|DF_{\sigma_n}(h) - \delta y_n\|^2 + R(h)\}, \tag{4}$$

where DF_{σ_n} is the Fréchet derivative of F at a feasible $\sigma_n \in X$, and $\delta y_n = y_n - F(\sigma_n)$. The need for fast image reconstruction in EIT has been addressed in the past, and methods like single-step inexact Newton-based regularised reconstructions [1], [2] have been popular in this context, and to a lesser extend Newton-Kantorovich algorithms Broyden methods [3].

2 Approximate gradient method

Let $|\Omega|$ denote the volume of the domain to be imaged, then consider the weighted mean conductivity value

$$\bar{\sigma} = \frac{1}{|\Omega|} \int_\Omega \mathrm{d}x \sigma(x) w(x), \quad \text{where} \quad \int_\Omega \mathrm{d}x\, w(x) = 1. \tag{5}$$

As any $\sigma \in X$ admits a decomposition $\sigma = \bar{\sigma} + p$, then we claim that

$$DF_\sigma(h) \approx DF_{\bar{\sigma}}(h) = \frac{1}{\bar{\sigma}^2} DF_{\mathbf{1}}(h), \tag{6}$$

where $\mathbf{1}(x)$ is the unit conductivity function in Ω. Options for the weights may include

$$w(x) \propto \mathbf{1}(x) \quad \text{or} \quad w(x) \propto \sum_{i=1}^{d} \big| \sigma(x) \nabla u_i(x) \big|, \tag{7}$$

with $u_i(x) = F(\sigma)$ denoting the forward solution for the ith boundary current. By the analyticity of F we have

$$DF_\sigma = DF_{\bar{\sigma}} + \frac{1}{2} D^2 F_{\bar{\sigma}}\, p + o(p^2), \tag{8}$$

and thus to first-order accuracy

$$\|DF_\sigma - DF_{\bar{\sigma}}\| \leq \frac{1}{2} \|D^2 F_{\bar{\sigma}}\| \|p\|, \tag{9}$$

and we can replace DF_{σ_n} in the Gauss-Newton iteration with $DF_{\bar{\sigma}_n}$, such that at each iteration we obtain the approximate Jacobian by scaling accordingly a precomputed Jacobian $DF_{\mathbf{1}}$. To see the computational advantage of this scheme let $\mathbf{J}_{\mathbf{1}} = \sum_{i=1}^m u_i s_i v_i'$ the Jacobian matrix at discrete model with $k > m$ elements and unit conductivity, and $R(h) = \lambda \|h\|^2$, then the iterative solution is given by

$$\sigma_{n+1} = \sigma_n + \sum_{i=1}^{m} \gamma_i^n v_i, \text{ for } \gamma_i^n = \langle u_i, \delta y_n \rangle \Big(\frac{s_i \bar{\sigma}_n^2}{s_i^2 + \lambda \bar{\sigma}_n^4} \Big) \tag{10}$$

3 Simulations

We implement the approximate gradient scheme using a regularised Gauss-Newton algorithm and compare convergence and timings against the respective implementation involving the exact Jacobian. These tests were based on simulations performed with EIDORS 3D [3] using a model with 8640 elements and 208 measurements, with a targeted oscillatory conductivity function $0.01 \leq \sigma(x) \leq 110$. The timings for 6 iterations on a macbook pro running Matlab were 114 s and 8 s for the exact and approximate schemes respectively, while the errors in data and image are those appearing in fig. 1.

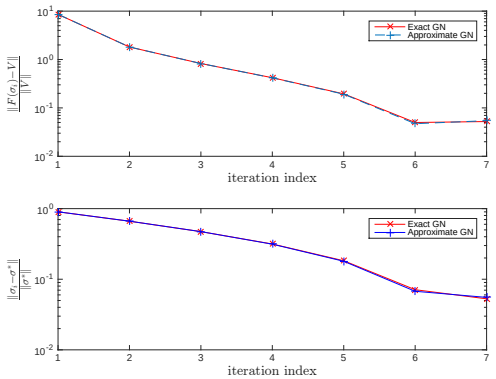

Figure 1: Numerical comparison of the Gauss-Newton method with exact and approximate gradients using $w(x) \propto \mathbf{1}(x)$.

4 Conclusions

The numerical results suggest that the compromise in the convergence of the nonlinear solver due to the approximate Jacobian is very small compared to the profound computational savings.

References

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