THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Revisiting James Watt's linkage with implicit functions and modern techniques

Citation for published version:
Sangwin, CJ 2008, 'Revisiting James Watt's linkage with implicit functions and modern techniques' Mathematics Magazine, vol. 81, no. 2, pp. 116-126.

Revisiting James Watt's linkage with implicit functions and modern techniques

Christopher J Sangwin
Maths, Stats and OR Network,
School of Mathematics,
University of Birmingham,
Birmingham, B15 2TT,
United Kingdom
C.J.Sangwin@bham.ac.uk

The engineer James Watt (1736–1819) was a pioneer of steam power in the United Kingdom. His practical work revolutionised the rather inefficient atmospheric engines of his predecessors such as Newcomen. He vastly improved these engines in a variety of ways so that steam power became the "prime mover" of his age. In doing so he accelerated the Industrial Revolution and helped to usher in the modern industrial era.

In this article I want to re-examine a mechanism he invented to constrain the piston of a steam engine to move in a straight line. It consists of the simple linkage system illustrated in Figure 1. This may appear rather trivial to us now but with the rise of the importance of mechanical engineering during the Victorian period this, and related mechanisms, had many important applications. Such linkages still find many contemporary uses and modern research in robotics and flexible structures rely on the geometry which we are going to examine here.

Watt published his linkage in a patent dated August 24th 1784 and it is important to remember that he did not claim it produced a true straight line. He understood its importance and in his old age he wrote to his colleague Matthew Boulton

> Although I am not over anxious after fame, yet I am more proud of the parallel motion than of any other invention I have ever made.

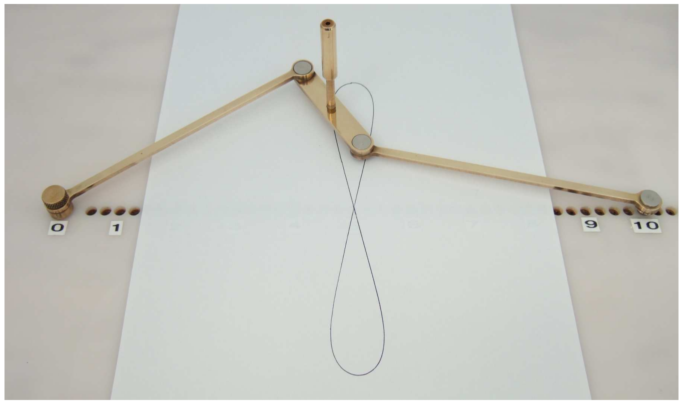

Figure 1: Watt's linkage

The *parallel motion* is a simple development of the linkage shown in Figure 1 and this is the phrase used in historical engineering books.

Mathematical functions

The mathematical description of Watt's linkage will be in terms of *implicit functions* and so we shall consider these first. The usual modern definition of a function f is a rule which takes each element of the *domain* X and assigns a unique element in the *codomain* Y. Sometimes we think of the rule as a mapping or a procedure. Sometimes we write the function $f : X \to Y$ as $y = f(x)$, where $x \in X$ and $y \in Y$.

Some examples, where X and Y are both the set $\mathbb{R}$ of real numbers, are $f(x) = x^3$ and $f(x) = e^x$. It is surprising to learn that this particular definition of function is a relatively recent innovation. I'd like in this article to point out an older notion of function which, because of some rather exciting new techniques for solving systems of polynomial equations, is likely to become more important again. These techniques rely on the pure mathematics of rings and groups, but already have important applications in mechanical engineering and robotics design.

Just over one hundred years ago the English mathematician G. H. Hardy, in his famous book [5], made the following remarks about functions.

> We must point out that the simple examples of functions mentioned above possess three characteristics which are by no means involved in the general idea of a function, viz:
>
> 1. y is determined *for every value of* x;
> 2. to each value of x for which y is given corresponds *one and only one value of* y;
> 3. the relation between x and y is expressed by means of *an analytical formula*.
>
> [...] All that is essential is that there should be some relation between x and y such that to some values of x at any rate correspond values of y.

Hardy then goes on to give a number of further examples to illustrate these ideas which can be broadly separated into two groups. Firstly are those which involve a formula, equation or *algebraic* expression in x and y. This might include an infinite sum such as a series. The second are when the relationship between x and y follow from some *geometrical* construction. In this article we shall also look at both these constructions. In particular we shall find an equation which describes the geometric curve shown in Figure 1 by algebraic means with the help of a computer algebra system.

To begin, for a clearer separation between algebraic and the more geometric notions of function, we shall go back even further and look at the work of Leonhard Euler. Euler wrote rather a lot of mathematics. For us, the separation between algebraic and geometric notions of function are clearly explained by him in the two volumes [2] and [3].

> §4. *A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.* Hence every analytic expression, in which all component quantities except the variable z are

> constants, will be a function of that z; thus $a + 3z$; $az - 4z^2$; $az + \sqrt{a^2 - z^2}$; c^z; etc. are functions of z. [2]

Here Euler proposes that a *function* is that which can be expressed using an analytic expression. This notion is not that of an input-output machine, in which the domain and codomain are distinguished. "§*16. If y is any kind of function of z, then likewise, z will be a function of y.*" It is important to realise in connection with this statement that the *same algebraic expression* represents both functions. For example, if we think about $y = x^3$ again, for Euler, this is as much a function of y as it is a function of x. It all depends on how you are thinking about it at any moment.

A result of this is that functions can be multiple valued: "§*10. Finally we make a distinction between single-valued and multiple-valued functions.*" In particular, Euler gives $\sqrt{2z + z^2}$ as an example of a two-valued function.

> Whatever value is assigned to z, the expression $\sqrt{2z + z^2}$ has a twofold significance, either positive or negative.

As another example, the usual way of expressing a circle, for example

$$x^2 + y^2 = 1, \tag{1}$$

is to Euler a "function". It fails to be a function in the modern sense, even if restricted to the domain $(-1, 1)$, since to each value of x in this range there are two choices for y. Likewise, for each value of $-1 < y < 1$ there are two possible values of x. The modern definition requires only one value for each x in the domain. This is not just nit-picking, but an important restriction on what can be a function.

We compare this with the opening of the next volume [3], in which quite a different notion of function is examined. This is of a single-valued function of a real variable, which can be represented by a *graph*.

> Thus any function of x is translated into geometry and determines a line, either straight or curved, whose nature is dependent on the nature of the function. [3, §6]

Conversely, explains Euler, *"a curve can define a function"*. It is this notion of function as *curve in space* to which [3] is devoted. In particular, the topic of curves generated by an algebraic equation relating x and y is developed in detail in [3].

What both these definitions have in common is the notion we would describe as an *implicit function*. Implicit functions do not, I think, have the popularity they deserve. In this article I want to show some situations in which using them leads to tidier mathematical results, and then to explain some applications in which they arise naturally. In particular, the curve shown in Figure 1 will be described by an implicit function.

The straight line

The straight line is usually described by the equation $y = mx + c$. The first point to note is that the value of y is given as an *explicit* algebraic expression in x, namely $mx + c$. So, it is clear that to each x is assigned a unique value of y. As a result of this we can draw a graph, and we find that m is the slope, and c the intersection of the line with the y-axis.

Conversely, if we have a line in the plane, then unless the line is vertical, we can write the equation relative to a pair of axes. But such a description cannot capture the case in which the line is parallel to the y-axis. Here, we have only one value of x for which there exists values of y, and indeed every value of y is identified with this value of x. We would need to write this line as $x = a$, say.

However, if we expand our notion of function to include "expressions composed howsoever from the quantities", we may include equations such as the following.

$$ax + by = p. \tag{2}$$

In this, we can recover $y = mx + c$ by division, provided $b \neq 0$. If $b = 0$ then, $ax = p$, which expresses a vertical line. Initially this equation appears more complex, having three unknowns a, b and p instead of the usual two. But it is more general. I shall now explain why I prefer this form with some more substantial observations.

Let us assume that we have two different points (x_a, y_a) and (x_b, y_b). The task is to find a straight line between them. It turns out that the expression representing a straight line through these two points is given by the standard slope-intercept form by a rather complicated equation

$$y = \frac{(y_a - y_b)x + x_a y_b - x_b y_a}{x_a - x_b}. \tag{3}$$

If $x_a = x_b$ then we would divide by zero, which is forbidden. This corresponds to a vertical line, and as before, we cannot express this in the form $y = mx+c$.

If we re-write (3) in the form (2), then we have

$$(y_b - y_a)x + (x_a - x_b)y = x_a y_b - x_b y_a.$$

This appears to be complex, but there is a symmetry between the x_a, x_b, y_a and y_b which is arguably easier to see, and hence remember, than in (3). Putting the point a on the y-axis as $(0, a)$, and the point b on the x-axis as $(b, 0)$ this reduces to the form

$$ax + by = ab.$$

So an easy way to remember the formula is to look at the two axis intercepts: there is no need to calculate the slope, just to find the equation of the line. If we define $p := \frac{ab}{\sqrt{a^2+b^2}}$ then this can again be re-written in the form

$$\sin(t)x + \cos(t)y = p,$$

where t is the angle of the line to the x-axis and p now represents the perpendicular distance of the line from the origin. In this form we recover an equation in only two unknowns, t and p. In all these forms the symmetry between x and y, and the two interpolated points, is arguably more natural than in the traditional form of the equation for a straight line.

Circles are almost always expressed in an implicit way, as is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If we use the form (2) for the equation of a line, then it an exercise for you to show that the tangent to this ellipse, through the point (x_0, y_0) is given by

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1.$$

Again, this seems to be simpler and more general than the form $y = mx + c$ would permit.

Solving the cubic equation

In this section I want to illustrate how the straight lines (2) can help us solve a cubic equation

$$w^3 + aw^2 + bw + c = 0,$$

by graphical means. Our first observation is that by defining $z = w - \frac{a}{3}$ we have

$$z^3 + pz + q = 0. \tag{4}$$

Notice the z^2 term is missing. The equation (4) is known as the *reduced cubic* and it is the first step in the method of finding the general formula for the roots of the cubic, which [4] develops in full detail.

We shall divide by z^3 and then define $p = x$ and $q = y$. This gives us the equation

$$\frac{x}{z^2} + \frac{y}{z^3} = -1.$$

For each value of z this gives us a straight line. Furthermore, for each point on this straight line the equation (4) holds. If we think of the plane as the (p, q) space of all cubic equations (4), then points on these straight lines are solutions of the equation (4). Hence, to solve a particular equation (4) we look to see which line(s), if any, pass through the point (p, q).

These lines have been plotted in Figure 2. Let us examine the line labeled $z = -2$. This certainly passes through the point $p = 100$, $q = 208$. Hence, we know that the reduced cubic $z^3 + 100z + 208 = 0$ has a real root $z = -2$. Furthermore, from Figure 2 it appears that every point in the (p, q) plane

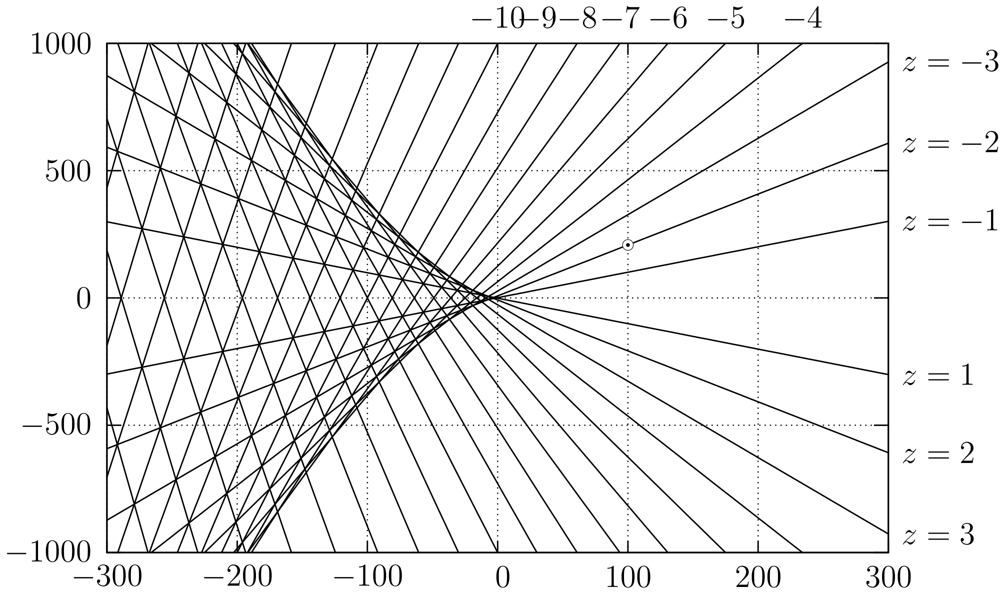

Figure 2: Graphical solution to the cubic equation: real solutions

has at least one line through it, so every reduced cubic has at least one real root. To the left of the figure it appears that we have a region in which three lines always pass through a given point (p, q). Hence, here we always have three real roots for (p, q) in this region.

The boundary of the region in which the cubic has three real roots is given by the equation

$$4p^3 = -27q^2,$$

which is another example of an implicit function.

Visualizing implicit functions

One clear advantage of the contemporary function-as-function-machine approach is the ease with which such functions can be visualised. You can simply plot the graph, or have a machine approximate this for you. Functions defined implicitly by equations are hard to vizualise, at least initially.

Figure 3: Other variations of the four-bar linkage

The following is a key observation when trying to sketch the graph of an expression such as $p(x, y) = 0$. We begin by factoring $p(x, y)$ over the real numbers. The expression is satisfied if any of the factors equal zero. Hence, we plot the graphs of each of the factors separately, and then combine them by superposition. The graph of $p(x, y) = 0$ is the superposition of the graphs of its factors. For example, the graph of the expression $x^2 = y^2$, or rather $(x - y)(x + y) = 0$, is the superposition of the two lines $y = x$ and $y = -x$.

Irreducible expressions, such as (1), have their own particular *forms*, something which [3] examines at length. We know this, from familiarity, to be a circle of radius 1, centered at the origin. Familiarity with the other second order curves, that is to say conic sections, is something which comes with regular use. Your might like to discover why adding x^2y^2 to the right hand side of (1) might be described as *"squaring the circle"*.

Computer algebra systems, or other technology, can help here, although you should beware that many CAS's fail to plot simple implicit functions by failing to factor the expressions and act on this simple observation. For example, $(x - 2)^2 = 0$ fails to change sign for any x and y, and as a result an embarrassing number of mainstream CAS's fail to plot this convincingly as the line $x = 2$.

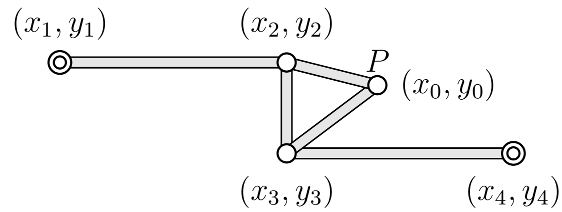

Figure 4: A schematic of general four-bar linkages

Linkages

To illustrate somewhat more substantially the contemporary importance of implicit functions, we shall examine Watt's linkage, and four-bar linkages in general. Watt's linkage is shown in Figure 1. This consists of three movable bars, fixed to a base which constitutes a fourth bar. We are interested in the path of the pen, fixed to the middle bar, over all physically realistic positions of the linkage. You can see this path in the figure.

The original purpose was to constrain the movement of a piston in the cylinder of a steam engines to move in an approximately straight line. Other four-bar linkages which were proposed for this purpose are shown in Figure 3. In the linkage to the left, the pen is on an extension of the middle link. In the linkage to the right, the pen is connected to the middle linkage, but is offset from the linkage itself, and this represents the most general situation. For more details of the history of the applications of this problem see [1].

From our point of view, all these will be treated in an identical way, as shown in Figure 4. From this sketch, we are most interested in the locus of P, for all positions of the linkage. It is very worthwhile making physical models of these linkages, either from commercially available model kits or from materials you have on hand. Alternatively you might like to implement these on a dynamic geometry package, such as GeoGebra (`http://www.geogebra.at/`).

We shall take a more algebraic approach. We first notice that the distance between (x_1, y_1) and (x_2, y_2) is fixed, at r_1 say. Hence by the Pythagorean Theorem we have

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = r_1^2.$$

Indeed, to describe the whole linkage three more applications of the Pythagorean Theorem provide us with the following equations.

$$(x_2 - x_3)^2 + (y_2 - y_3)^2 = r_2^2,$$

$$(x_3 - x_4)^2 + (y_3 - y_4)^2 = r_3^2,$$

$$(x_4 - x_1)^2 + (y_4 - y_1)^2 = r_4^2.$$

Now, to describe the position of $P = (x_0, y_0)$, relative to (x_2, y_2) and (x_3, y_3) we need two further applications

$$(x_2 - x_0)^2 + (y_2 - y_0)^2 = r_5^2,$$

$$(x_3 - x_0)^2 + (y_3 - y_0)^2 = r_6^2.$$

Since we fix both ends of the linkage, we specify (x_1, y_1) and (x_4, y_4), hence defining r_4 and making one equation redundant. The task is to solve the system consisting of the remaining five equations. By "solve", we mean to specify the lengths of the links $r_1, \ldots, r_6$, and then to eliminate (x_2, y_2) and (x_3, y_3) to leave a single equation in only (x_0, y_0) as the solution.

This looks hopeless, but in fact it can be done with computer algebra in a straightforward way using a concept known as Gröbner bases. If you have a computer algebra system, such as Maple, Mathematica, Maxima or some other CAS, you will probably already have the software necessary to do this.

For reference, if you have Maple 9.5, then the commands look something like this.

```
> restart:with(Groebner):with(Ore_algebra):
> P1:=(x1-x2)^2+(y1-y2)^2-r1^2;
> P2:=(x2-x3)^2+(y2-y3)^2-r2^2;
> P3:=(x3-x4)^2+(y3-y4)^2-r3^2;
> P4:=(x4-x1)^2+(y4-y1)^2-r4^2;
> P5:=(x0-x2)^2+(y0-y2)^2-r5^2;
> P6:=(x0-x3)^2+(y0-y3)^2-r6^2;
```

Notice that we are using an expression $(x_1 - x_2)^2 + (y_1 - y_2)^2 - r_1^2$ instead of an equation. This is really only an input syntax issue, and from this point onwards it is implied that such an expression represents an equation with right hand side zero.

Next to examine in more detail a specific example we assign some lengths to these expressions.

```
> y1:=0; y4:=0; x1:=-5; x4:=5;
> r1:=5; r2:=2; r3:=5;
> r5:=1; r6:=1;
> S := [P1,P2,P3,P5,P6];
```

Notice that we have not included the redundant equation `P4` in the list `S`, which is the resulting system of expressions representing our equations. Next the CAS computes the "Gröbner basis" for this system, and then we remove any expressions which have any of the variables x_2, y_2, x_3, or y_3.

```
> LinkageGB:=gbasis(S,lexdeg([x2,y2,x3,y3],[x0,y0])):
> Linkage:=op(remove(has,LinkageGB,{x2,y2,x3,y3})):
> factor(Linkage);
```

The result of this calculation, which took approximately six minutes to complete, is the expression

$$\left({y_0}^6 + 2\,{y_0}^4 - 99\,{y_0}^2 + 3\,{x_0}^2{y_0}^4 - 96\,{x_0}^2{y_0}^2 + 2401\,{x_0}^2 + 3\,{x_0}^4{y_0}^2 - 98\,{x_0}^4 + {x_0}^6\right)^2. \tag{5}$$

In terms of the solution to the original system, this reduces to

$${y_0}^6+2\,{y_0}^4-99\,{y_0}^2+3\,{x_0}^2{y_0}^4-96\,{x_0}^2{y_0}^2+2401\,{x_0}^2+3\,{x_0}^4{y_0}^2-98\,{x_0}^4+{x_0}^6 = 0. \tag{6}$$

While we have not been able to find y_0 in terms of an explicit expression in x_0, even finding this implicit function is quite an achievement. In fact, it is hopeless to suppose that the figure of eight curve shown in Figure 1 could result in a single-valued $y_0 = f(x_0)$.

At this point you may be feeling some disquiet that I am not going to explain exactly what a Gröbner basis *is* or exactly what you have *done* with it. That is your task to investigate using the many available references. A good place to start is the "help" files on your computer algebra system. For example, in Maple type `help(Grobner);`. What I do hope to convince you of is that these (relatively) new computational techniques are particularly useful by applying them to a classical problem which does not appear to be solvable by traditional means, such as those of [6]. Indeed, so useful are they for solving apparently hopeless systems, such as that above, that I confidently predict that the implicit function itself will become much more important and widely used.

In fact, we can do rather a lot more with these techniques. In particular, rather than specifying the end points and linkage lengths at the outset, we shall keep these variables in the system of equations. Now we shall solve the same system, and eliminate (x_2, y_2) and (x_3, y_3), finding a single implicit equation for (x_0, y_0), in terms of the end points and linkage lengths.

This appears to be an even more hopeless a task, since we have five nonlinear equations with four variables to eliminate and a further nine parameters which will be left. And yet it can be done. Specifying only that $y_1 = 0$, so that one end of the linkage is effectively anchored on the x-axis, Maple is (eventually) able to find the required equation. The restriction $y_1 = 0$ is not necessary, but it does make the computations finish in a sensible amount of time. Unfortunately even then, this is rather too long to print here, having some 27255 terms in the equation.

Having obtained this equation we can use it to investigate the general behavior of four-bar linkages in which one end is anchored to the x-axis. A first experiment is suggested by Figure 1. Notice that the end points of the linkage can be moved to various positions along the x-axis. While these have been labeled from 0 upwards on the diagram, it makes sense for us to have

$$(x_1, y_1) = (-r, 0), \quad (x_4, y_4) = (r, 0)$$

to obtain symmetry, and then as before take

$$r_1 = r_3 = 5, \quad r_2 = 2, \quad r_5 = r_6 = 1.$$

Doing this we obtain, as a polynomial in r, the following.

$$r^4 \left(y_0^2 + x_0^2\right) + 2r^2 \left(y_0^4 - 26y_0^2 - x_0^4 + 24x_0^2\right) + \left(y_0^2 + x_0^2 - 24\right)^2 \left(y_0^2 + x_0^2\right) = 0.$$

Not surprisingly, by substituting $r = 5$ into this equation we recover (6).

Figure 5 illustrates the locus of the center point in the middle arm of Watt linkages, for various values of r. When $r = 0$ both long arms are fixed at the origin, and we have the circle $y_0^2 + x_0^2 = 24$. The figure of eight shape shown in the model of Figure 1 is clearly reproduced for $r = 5$ in Figure 5, and the algebraic expression for this curve is given in (6).

Recall that Watt's original intention was to draw an approximate straight line. For $r = 2$, Figure 5 appears to show a much longer, approximately straight section in the curve. Perhaps moving the fixed points to $r = 2$, rather than $r = 5$, gives a better straight line? Indeed it does, and this configuration was actually proposed for this purpose by the Russian mathematician Pafnuty Chebyshev (1821–1894), who was fascinated by linkages. We shall refer to this as "Chebyshev's approximate straight line" and you are encouraged to actually make this for yourself, or at the very least sketch the linkage. If you do this you will see how the two disconnected parts of the curve correspond to physical configurations of the linkage.

Notice also that the curve shown to the left of Figure 3 looks very similar indeed to that for Watt's linkage with $r = 2$. And yet Watt's linkage has quite a different form than this model, in particular the pen on Watt's linkage lies mid-way along the center link. This suggests another line of inquiry: can we find more than one linkage which generates a particular curve?

To do this let us start with our general expression for the four-bar linkage. Into this we substitute the values for Chebyshev's approximate straight line to obtain the equation

$$\left({y_0}^6 - 40\,{y_0}^4 + 384\,{y_0}^2 + 3\,{x_0}^2{y_0}^4 - 96\,{x_0}^2{y_0}^2 + 784\,{x_0}^2 + 3\,{x_0}^4{y_0}^2 - 56\,{x_0}^4 + {x_0}^6\right)^2 = 0. \tag{7}$$

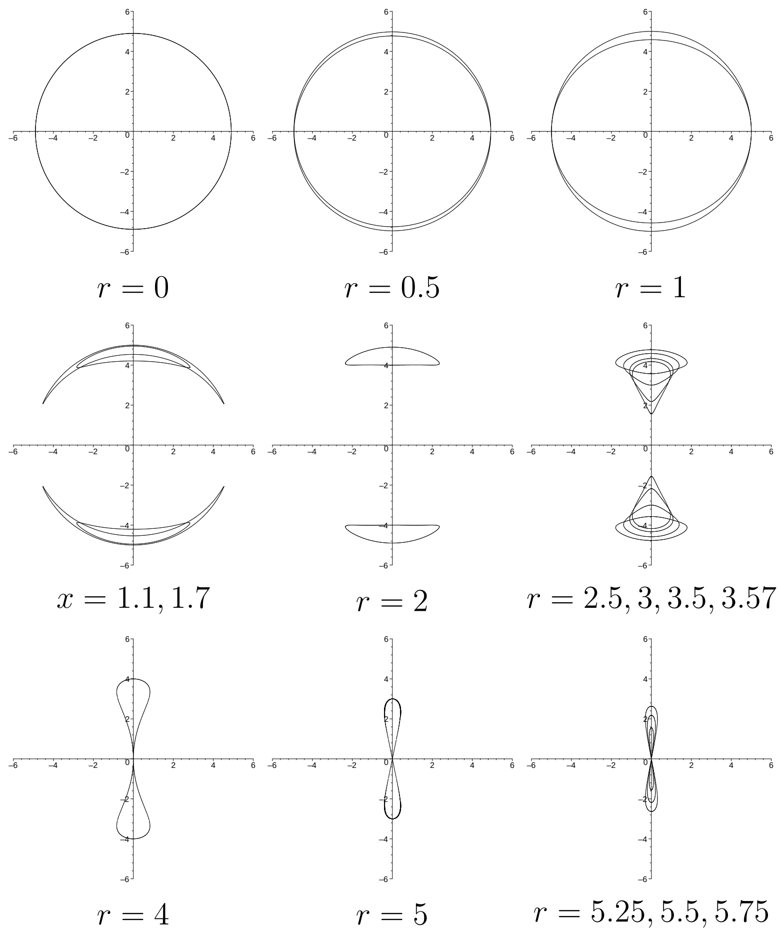

Figure 5: The locus of Watt's linkage for various separations r.

This is plotted in Figure 5, and labeled $r = 2$. Now we compare the coefficients of (7) with those of the 27255 terms in the general equation. Each comparison provides an equation in the unknown positions of the end points, and also the lengths of the links. Again we have a large number of non-linear equations in nine unknowns. How can we possibly hope to solve these, and hence find alternative four-bar linkages which produce Chebyshev's approximate straight line? This is simple; we apply the Gröbner basis technique again to solve this system of equations, just as we did before.

One result is

$$(x_1, y_1) = (0, 0), \quad (x_4, y_4) = (2, 0)$$

and

$$r_1 = \frac{5}{2}, \quad r_2 = \frac{5}{2}, \quad r_3 = 1, \quad r_5 = \frac{5}{2}, \quad r_6 = 5.$$

A model of this linkage is shown to the left of Figure 3. Having found this result by algebraic techniques, it is relatively straightforward to find a simple and purely geometrical proof that the curves generated are identical by drawing the two alternative linkages on the same diagram. If you use dynamical geometry then the proof "jumps out" as the linkages move in unison.

What about the most general case? If we take a linkage, such as that shown to the right of Figure 3, then it is always possible to find exactly *two* others which generate the same curve. This is the famous *triple generation theorem*, and a simple purely geometric proof is given in, for example, [7].

What these algebraic expressions, and their associated graphs, lack is the *movement* obtained by the linkages. For example, if the point (x_2, y_2) is rotated at a constant speed, then how does the velocity of P change? These linkages have a satisfying aesthetic quality to them, which can only be experienced by making the linkages. This can be either with a physical model of your own, or virtually in dynamic geometry. For a particularly intriguing example, try $r_1 = r_3 = 5$, $r_2 = 6$, the point P the midpoint of the bar, that is to say $r_5 = r_6 = 3$, and with $(x_1, y_1) = (-2, 0)$, and $(x_1, y_1) = (2, 0)$. Many other configurations are possible.

The four-bar linkage is perhaps the simplest of mechanisms: a three bar linkage forms a triangle and hence is rigid, and any more bars give the potential for greater degrees of freedom. The techniques I have sketched above are becoming widely used for the design of mechanisms in general, and the design of robots in particular. They allow the user to accurately model the movement of complex joints, both in the plane and in three dimensions. They allow a designer to search for alternative configurations of links with the same, or similar, paths. Furthermore, there are many other situations in mathematics which generate systems of polynomial equations. Where these need to be manipulated, the Gröbner basis technique is invaluable. All that can be hoped for as an outcome in general is an implicit function. As a result of this, I predict that implicit functions will become much more important in the near future.

References

[1] J. Bryant and C. J. Sangwin. *How Round is your Circle?* Princeton University Press, 2007.

[2] L. Euler. *Introduction to Analysis of the Infinite*, volume I. Springer, 1988. Translated by Blanton, J. from the Latin *Introductio in Analysin Infinitorum, 1748.*

[3] L. Euler. *Introduction to Analysis of the Infinite*, volume II. Springer, 1990. Translated by Blanton, J. from the Latin *Introductio in Analysin Infinitorum, 1748.*

[4] L. Euler. *Elements of Algebra.* Tarquin Publications, 2006. ISBN 978-1-89961-873-6.

[5] G. H. Hardy. *A course of Pure Mathematics.* Cambridge University Press, 1908.

[6] J. A. Hrones and G. L. Nelson. *Analysis of the Four Bar Linkage*. Press of the Massachusetts Institute of Technology, and Wiley, New York, 1951.

[7] R. C. Yates. *Geometrical Tools: a mathematical sketch and model book*. Educational Publishers, Incorporated Saint Louis, 1949.