Endogenous Coalition Formation in Contests

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Abstract

This paper analyzes coalition formation in a model of contests with linear costs. Agents first form groups and then compete by investing resources. Coalitions fight for prizes that are assumed to be subject to rivalry, so their value is non-increasing in the size of the group. This formulation encompasses as particular cases some models proposed in the rent-seeking literature. We show that the formation of groups generates positive spillovers and analyze two classes of games of coalition formation. A contest among individual agents is the only stable outcome when individual defections leave the rest of the group intact. More concentrated coalition structures, including the grand coalition, are stable when groups collapse after a defection, provided that rivalry is not too strong. Results in a sequential game of coalition formation suggest that there exists a non-monotonic relationship between the level of underlying rivalry and the level of social conflict.

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"If men were supplied with every thing in the same abundance, justice and injustice would be equally unknown among mankind."

David Hume (1740), *A Treatise of Human Nature.*

1 Introduction

Conflict and confrontation have been constant phenomena in human societies. Nations engage in war over territories and resources. Social classes struggle to obtain political power. Firms lobby for government support. Individuals compete for social status. In all these contexts, agents often spend staggering amounts of resources in order to prevail. Why they do not settle peacefully instead?

Scarcity is the sire of conflict. States, social groups, firms and individuals resort to confrontation because under universal peace territories and political power must be shared, resources are overexploited, property rights dilute. In other words, agents engage in costly clashes because some degree of *rivalry* underlies their interactions. Conflict constitutes then the missing link between scarcity and injustice in Hume's argument: It grants exclusive rights to the winners and allows them to avoid congestion, at the expense of the losers.

But we also observe that groups form in many conflict situations. Members agree not to fight each other, at least temporarily, so facing fewer rivals is a reason to seek for allies. In addition, coalitions may enjoy a higher chance of success than individual agents on their own because members pool their efforts. On the other hand, a conflict of interests persists within the winning group when the spoils of victory must be divided. Moreover, free-riding may undermine the effectiveness of coalitions. How these opposite forces shape the incentives to form groups?

The main objective of the present paper is to study the interplay between rivalry and the incentives to break up peace and form coalitions in contests. We study group formation in a model where players compete for a prize that is subject to congestion. In a first stage, agents join in coalitions, and then invest resources in a contest that determines the winning group. Because the formation of the grand coalition is equivalent to universal peace, our model can help us to predict both under which circumstances conflict will erupt and what group structures we should expect to arise in that case.

We impose two mild conditions on the prizes these groups contest for: They must depend upon the size of the coalition and not on the identity of its members (*Anonymity*); and they must be non-increasing in that size (*Rivalry*). These prizes can thus be considered as the reduced form of the interaction that would take place within the group in case of victory. This allows us to encompass as particular cases some models already proposed in the literature and, most importantly, to analyze how the stability of coalition structures depends on the level of intra-group rivalry.

We first characterize the unique equilibrium of the contest game given a coalition structure. As it is well known in the rent-seeking literature, when prizes are purely private and the cost of investments is linear, agents who value victory less tend to remain inactive (Ueda (2002) refers to this phenomenon as *oligopolization*). Therefore, bigger groups drop out of the contest because their members fight for a smaller prize due to rivalry. Equilibrium uniqueness allows us to derive the *valuation* (Bloch, 1997), a mapping which associates to each coalition structure a vector of individual payoffs. The valuation depends upon the entire coalition structure because the number and sizes of rivals have a direct impact on the outcome of the contest. In the second step of our analysis, we study the properties of the valuation. We show that coalition formation generates positive spillovers, that is, non-members are better off whenever two groups merge. We also show that the valuation satisfies some other properties proposed in the literature of endogenous coalition formation.

Spillovers complicate considerably the analysis of group formation. Here we will consider two families of models of coalition formation. First, we study two simultaneous games of exclusive membership proposed by Hart and Kurz (1983): In the δ game, players stick together if one member withdraws; in the γ game, the coalition breaks apart completely. We obtain that the singleton structure, that is, a contest among all individual players, is the only Nash equilibrium of the former game for any degree of rivalry. This is because individual agents can free ride on the existence of bigger and less aggressive coalitions when they leave their own group. More concentrated coalition structures, including the grand coalition, can be supported as equilibria of the γ game when rivalry is not too strong. The reason is that a defection in this case triggers a fierce contest among ex-members. However, when intra-group rivalry is very intense, individual defections become again too attractive for any coalition to be stable.

We then analyze a sequential game of coalition formation proposed by Bloch (1996) in which players' reaction to deviations are fully endogenous. In this game, results depend again on the level of rivalry and they seem to display a bang-bang pattern: For low levels of rivalry, an asymmetric two-group contest can be supported. For high levels, either the grand coalition or the singleton structure are stable, and under very strong rivalry only the latter is stable. We have not been able to obtain results for the intermediate range, but examples indicate that only the grand coalition is stable in those cases. All this suggests that rivalry has a non-monotonic effect on the level of social conflict. The grand coalition is not formed when rivalry is either weak or very strong; whereas obvious in the latter case, it happens in the former because when the first player to move forms a singleton, it still pays to the rest of players to form the complementary coalition rather than to trigger a contest among singletons that would in turn leave the initial defector worse off. Hence, peace breaks up.

The first analysis of the role of groups in contests dates back to the pioneering work by Olson (1965). He established what is often referred to as the group-size paradox of collective action: Small groups are more effective than larger ones in pursuing their targets. Olson argued that there exist two reasons for this: First, the perceived effect of an individual defection decreases with the size of the group, so free-riding intensifies. The second force is precisely the concept of rivalry we employ here: When the prize is private and must be divided up, individual prizes get smaller as groups become larger.[1]

However, the literature has remained relatively silent on the issue of coalition formation in contests. Baik and Shogren (1995), Baik and Lee (1997) and Baik and Lee (2001) explored models that proceeded in three stages: First, players form coalitions, then choose the sharing rule to be employed within them and they finally compete. The resulting stable group structures tend to contain one coalition comprising half of the players. These games differ from ours in two important stands: Firstly, they use open membership games where players cannot exclude others from joining their group. We see this feature as a caveat in the context of confrontations and contests. We employ instead simultaneous and sequential games of exclusive membership where the formation of a coalition requires a certain degree of agreement among members. Secondly, their sharing rule depends directly upon the contest investments by members. In contrast, we do not model sharing rules explicitly and take individual benefits from victory as given when groups compete at the second stage of the game (see footnote 2).

The closest contributions to ours are Garfinkel (2004a) and (2004b) and Bloch et al. (2006). The first two papers apply the concept of *farsighted stability* (Chwe, 1994) to a model of contests in which members engage in a new contest after victory. Because deviations under this solution concept must be stable themselves, relatively more group structures end up being stable compared to our framework. Both models obtain that symmetric and nearly symmetric coalition structures are farsighted stable, but not the grand coalition when rivalry is strong. On the other hand, Bloch et al. (2006) study coalition formation in a general model of contests with convex costs. Because in their model there is a minimal degree of rivalry within groups, the grand coalition is always the efficient group structure, and consequently constitutes the main focus of their analysis. We obtain parallel results for that case, but we are able to extend our stability analysis to other coalition structures and under different degrees of intra-group rivalry.

The remainder of the paper is organized as follows. Section 2 lays down the basic elements of the model. Section 3 characterizes the equilibrium

[1] Olson's argument has been explored in depth later onby Nitzan (1991), Lee (1995), Esteban and Ray (2001) and Ueda (2002).

of the contest stage given a coalition structure, derives the valuation and studies its properties. In Section 4, group formation is studied under two simultaneous games of exclusive membership. Section 5 analyzes the sequential game of coalition formation. Section 6 concludes.

2 The model

Let us suppose that interactions across individuals occur in two stages, as in the games of coalition formation developed by Hart and Kurz (1983), Bloch (1996) and Ray and Vohra (1997). In the first stage, agents form groups. In the second stage, they engage in a contest that will determine the winning coalition; victory grants benefits to members of the group. A third stage determining how the spoils from victory are distributed could eventually be added but we will not model it here explicitly.

Formally, consider a set N composed by n players who are *ex-ante* identical and have the same strategy set. A *coalition structure* $C = \{C_1, C_2, ..., C_K\}$ is a partition of N into a collection of disjoint coalitions indexed by k. Let us denote by $|C_k|$ the cardinality of C_k. Without loss of generality, we will order groups within any coalition structure C in ascending order of sizes so $|C_k| \leq |C_{k+1}|$. A coalition structure is called *symmetric* when all coalitions in it are of the same size.

Once a coalition has formed, agents in C_k invest resources in order to make their group win the contest. The formation of a group implies thus that members pool their efforts and commit to not fighting each other in the second stage of the game. Let us denote by r_i the resources spent by agent i and by $\mathbf{R}(C) = (R_1, R_2, ..., R_K)$, where $R_k = \sum_{i \in C_k} r_i$, the vector of coalitional efforts. The result of this contest is driven by the *contest success function* that maps $\mathbf{R}(C)$ into a vector $\mathbf{p} = \{p_k\}_{k=1}^{K}$ of coalitional winning probabilities (with probability p_k the coalition C_k attains the control of the resource and so on). We will adopt a simple functional form initially proposed by Tullock (1967), and axiomatized by Skaperdas (1996): Coalition C_k wins the contest with probability

$$p_k(\mathbf{R}) = \frac{R_k}{\sum\limits_{k=1}^{K} R_k} = \frac{R_k}{R}. \tag{1}$$

Observe that coalitions care only about the supply of effort made by other groups and not about the exact composition of C. We will assume that, as in the case of all-pay-auctions, the cost of investments is linear and independent of the outcome of the contest.

In the present setup, agents form coalitions because by doing so they can absorb potential rivals and pool their efforts. Hence, coalitions have no other specific objective, like supporting an ideology or an ethnic group, rather than victory. For this reason, we assume that all members of group C_k receive the same payoff, denoted by π_k in case of winning the contest, and that they receive nothing if their coalition is defeated. Therefore, the payoff of an agent i belonging to group C_k is given by

$$u_k^i(C_k, \mathbf{R}(C)) = p_k \pi_k - r_i = \frac{R_k}{R}\pi_k - r_i. \tag{2}$$

Throughout the paper we will impose two mild conditions on the individual payoff π_k.

Assumption 1 (Anonymity) *For any two distinct coalitions C_k and C_j such that $|C_k| = |C_j|$, it holds that $\pi_k = \pi_j$.*

Assumption 2 (Rivalry) *The aggregate coalitional payoff $\Pi_k = |C_k| \cdot \pi_k$ is non-increasing in $|C_k|$.*

These assumptions have very simple motivations. Anonymity stems naturally from the assumption of ex-ante identical players. Hence, individual prizes should not depend on the exact identity of the members of a group.

On the other hand, observe that Rivalry implies that the individual payoff π_k must be strictly decreasing in the size of the group $|C_k|$. It (that closely relates to *S-convexity*, employed by Bloch et al., 2006) holds when the prize is peacefully shared among members or when some internal struggle determines the allocation of the rent. This property is also naturally satisfied by several models of contests proposed in the literature, as shown below.

Example 1.1: Continuing contest. This example is based on Garfinkel and Skaperdas (2006). After the contest among groups takes place, rivalry may remain strong enough within the winning group to prompt members to engage in a new contest. Still, there may be some degree of cooperation that (partially) binds them to not fight each other. Individual payoffs for the members of the coalition will then depend on how much they invest in this new contest, denoted by s_i, and the degree of cooperation within the group, denoted by μ

$$\pi_k^i = (\frac{\mu}{|C_k|} + (1-\mu)\frac{s_i}{\sum_{j \in C_k} s_j})V - s_i. \tag{3}$$

This new contest admits a unique symmetric Nash equilibrium yielding the expected payoff

$$\pi_k = \frac{V}{|C_k|}\frac{\mu(|C_k|-1)+1}{|C_k|},$$

that satisfies Assumptions 1-2. This formulation encompasses several models proposed in the rent-seeking literature[2]: The case when $\mu = 1$, as in Bloch et. al. (2006), corresponds to a scenario of no conflict within the winning group; the prize is thus equally shared or, alternatively, distributed through a fair lottery among members. When $\mu = 0$, rivalry is resolved by means of a full-fledged contest. Examples of the latter type of interaction can be found in Katz and Tokatlidu (1996), Wärneryd (1998), Stein and Rapoport (2004) and Garfinkel (2004b).[3]

Example 1.2: Exclusion from an open-access resource. This example is based on Sánchez-Pagés (2006). It assumes that coalitions fight for the right to exploit an open access resource. Rivalry stems from the 'Tragedy of the commons' outcome due to the non-cooperative exploitation of the resource. Suppose that production is carried through the iso-elastic function $F(L_k) = (L_k)^\alpha$, where $L_k = \sum_{i \in C_k} l_i$ is the sum of individual labor inputs and $\alpha \in [0, 1)$. Following the classical exposition of the commons' problem by Cornes and Sandler (1983) the individual payoff is given by

$$\pi_k^i = \frac{l_i}{L_k}(L_k)^\alpha - l_i. \tag{4}$$

Again, this game admits a unique Nash equilibrium yielding the equilibrium payoff

$$\pi_k = \frac{1-\alpha}{|C_k|^2}\left(\frac{|C_k| - 1 + \alpha}{|C_k|}\right)^{\frac{\alpha}{1-\alpha}}, \tag{5}$$

which also satisfies Assumptions 1-2 for any value of α. Overexploitation becomes more severe as the size of the group that gains access to the resource increases.[4]

3 The contest stage

In this Section we study the game agents play once a particular coalition structure C has formed, derive the valuation and analyze its properties.

The individual payoff for an individual $i \in C_k$ in the contest stage is given by expression (2). Players choose simultaneously their investments in

[2]Nitzan (1991), Baik and Shogren (1995), Lee (1995), Baik and Lee (1997), Baik and Lee (2001) and Ueda (2002) have considered sharing rules that are the weighted average between equal sharing and sharing proportional to effort contributions in the group contest. In contrast, under (3) the second and third stage decisions are not directly linked.

[3]All these contributions assume that no group can be formed in this new contest. This issue by Skaperdas (1998), Tan and Wang (2000) and Esteban and Sakovics (2003), but their models differ substantially from those considered here.

[4]In Garfinkel (2004a) agents in the winning group also engage in production at this third stage and in addition spend resources to secure shares of the output jointly produced. The resulting equilibrium payoffs satisfy both Anonimity and Rivalry as well.

the contest aiming to maximize (2) taking as given the investments made by their fellow group members and the outsiders.

With linear costs of effort, it is well-known that in contests with heterogeneous agents some players may remain *inactive* and make no investment. These players are those who value victory less and thus prefer to drop out. In the present set-up, agents within bigger coalitions receive lower payoffs in case of victory so big groups will tend to remain inactive. The next Lemma, that follows Hillman and Riley (1989), characterizes the number of coalitions that will be active in the contest.

Lemma 1 *The number of active coalitions in C is the largest integer $\kappa \geq 2$ such that*

$$\pi_\kappa > \frac{\kappa-2}{\sum_{j=1}^{\kappa-1}\frac{1}{\pi_j}} = \frac{\kappa-2}{\kappa-1}H_{\kappa-1}, \tag{6}$$

where $H_{\kappa-1}$ is then the harmonic mean of the individual payoff π_k for coalitions $k = 1, ..., \kappa-1$.

Proof. It is easy to see that (2) is a strictly concave function so the individual decision problem always admits an interior solution. We will show now that condition (6) characterizes the set of active coalitions. The first order condition for the member of one of those groups is

$$\frac{\partial u_k^i}{\partial r_i} = \frac{R-R_k}{R^2}\pi_k - 1 = 0. \tag{7}$$

Solving for R_k and summing up across coalitions yields,

$$R^* = \frac{\kappa-1}{\sum_{j=1}^{\kappa}\frac{1}{\pi_j}} = \frac{\kappa-1}{\kappa}H_\kappa,$$

and after plugging it back to (7), the equilibrium individual level of effort is

$$r_i^* = \frac{\kappa-1}{\kappa}\frac{H_\kappa}{|C_k|}(1-\frac{H_\kappa}{\pi_k}\frac{\kappa-1}{\kappa}).$$

This implies that coalition k exerts positive effort if and only if

$$\pi_k > H_\kappa\frac{\kappa-1}{\kappa}.$$

that can be rewritten as the condition stated in the text. Now it remains to check that it does not pay for agents in coalitions $k = \kappa+1, ..., K$, for who condition (6) does not hold, to exert any effort. This hinges of the sign of the derivative

$$\begin{aligned}\frac{\partial u_{\kappa+1}^i}{\partial r_i} &= \frac{R^*-R_{\kappa+1}}{(R^*)^2}\pi_{\kappa+1} - 1 \\ &= (1-p_{\kappa+1})\frac{\pi_{\kappa+1}}{\frac{\kappa-1}{\kappa}H_\kappa} - 1 < 0,\end{aligned}$$

so members of any coalition $k \geq \kappa + 1$ exert no effort in the contest. ■

Therefore, only the κ smallest coalitions in C will be active in the contest. In fact, singletons will always be active. Moreover, all groups are active if C is a symmetric coalition structure. The next Lemma shows that this generalizes to other coalition structures if groups sizes are not too unequal.

Lemma 2 *The biggest active coalition in C is strictly smaller than the sum of the two smallest coalitions.*

Proof. Take coalition C_3 and suppose contrary to our statement that $|C_3| \geq |C_1| + |C_2|$. By Rivalry,

$$\frac{1}{\pi_1} + \frac{1}{\pi_2} \leq \frac{|C_1| + |C_2|}{|C_3|} \frac{1}{\pi_3} \leq \frac{1}{\pi_3},$$

whereas for C_3 to be active it must hold that

$$\pi_3 > \frac{1}{\frac{1}{\pi_1} + \frac{1}{\pi_2}} \Rightarrow \frac{1}{\pi_1} + \frac{1}{\pi_2} > \frac{1}{\pi_3},$$

so we have reached a contradiction. By the same token, $\frac{1}{\pi_1} + \frac{1}{\pi_2} \leq \frac{1}{\pi_k}$ for any coalition $k > 3$, but at the same time it must hold that

$$\pi_k \geq \frac{k-2}{\frac{1}{\pi_1} + \frac{1}{\pi_2} + \sum_{j=3}^{k} \frac{1}{\pi j}} > \frac{k-2}{\frac{1}{\pi_1} + \frac{1}{\pi_2} + (k-3)(\frac{1}{\pi_1} + \frac{1}{\pi_2})} = \frac{1}{\frac{1}{\pi_1} + \frac{1}{\pi_2}}.$$

■

This Lemma implies that singleton coalitions will be active if C contains at least two of them.

We are now in the position of characterizing the existence of an equilibrium of the subgame induced by any coalition structure.

Proposition 1 *The contest induced by coalition structure C with κ active coalitions has a unique Nash equilibrium where individual investments in the contest are given by*

$$r_k^* = \begin{cases} \frac{\kappa-1}{\kappa} \frac{H_\kappa}{|C_k|}(1 - \frac{H_\kappa}{\pi_k} \frac{\kappa-1}{\kappa}) & \text{if } k \leq \kappa \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof follows directly from the proof of Lemma 1. ■

Simple inspection of the equilibrium profile shows that members of big groups will spend less effort in the contest. This is due to two reasons: As groups become large, free-riding intensifies and the value of the prize they are fighting for decreases. In the limit, when coalitions are sufficiently big relative to the rest of groups in the coalition structure, they remain inactive.

We can finally derive the closed form solution for the equilibrium payoff for a member of coalition C_k.

$$u(|C_k|, C) = \begin{cases} \frac{\pi_k}{|C_k|}(1 - \frac{\kappa-1}{\kappa}\frac{H_\kappa}{\pi_k})(|C_k| - \frac{\kappa-1}{\kappa}\frac{H_\kappa}{\pi_k}) & \text{if } k \leq \kappa \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Notice that this payoff depends only on the size of the coalition agents belong to and on the size of the other coalitions in C via π_k.

Strictly speaking, the payoff function in (8) is called a *valuation* because it allows agents to evaluate the payoff they get from each possible coalition structure. Next, we will show that our valuation satisfies some properties proposed in the literature of coalition formation with spillovers.

A useful classification of valuations is based on whether coalition formation creates positive or negative externalities to nonmembers.

Definition *A valuation $u(|C_k|, C)$ exhibits positive externalities if $u(|C_k|, C) < u(|C_k|, C')$,where and $C_k \in C, C'$ and C' is obtained by merging two coalitions in $C \backslash \{C_k\}$.*

Well-known models of coalition formation, as cartels in oligopoly (Bloch, 1995), research joint ventures (Yi and Shin, 2000) or public good provision (Ray and Vohra, 2001) generate positive externalities.

Note that the merger of two groups can potentially create very strong spillovers in our set-up, because such a merger may substantially alter the constellation of active coalitions. Next we show that the sign of these spillovers will indeed be positive in the range of relevant cases.

Proposition 2 *The valuation (8) exhibits positive externalities whenever the merging coalition C' remains active.*

Proof. This result is trivial for previously inactive coalitions. Lemma A below (see the appendix) shows that no active coalition can become inactive after the merging if C' is active. Hence, we need to consider the rest of cases: Suppose first that none of the inactive coalitions become active. Simple inspection of the valuation (8), after being rewritten as

$$u(|C_k|, C) = \frac{\pi_k}{|C_k|}(1 - (\kappa - 1)\frac{\frac{1}{\pi_k}}{\sum_{j=1}^{\kappa}\frac{1}{\pi_j}})(|C_k| - (\kappa - 1)\frac{\frac{1}{\pi_k}}{\sum_{j=1}^{\kappa}\frac{1}{\pi_j}}),$$

shows that our claim is true since the number of active coalitions has been reduced by one and the ratio $\frac{1}{\pi_k}/\sum_{j=1}^{\kappa}\frac{1}{\pi_j}$ has increased given that $\frac{1}{\pi_l}+\frac{1}{\pi_m} \leq \frac{1}{\pi_{l \cup m}}$. Consider now that some previously inactive coalitions become active after the merger. Denote by $\kappa' \geq \kappa$ the index under C of the biggest group among the newly active coalitions. In order to show the existence of positive spillovers in this case we need to check simply that

$$\frac{\kappa - 1}{\sum_{j=1}^{\kappa} \frac{1}{\pi_j}} \geq \frac{\kappa' - 2}{\sum_{j=1}^{\kappa'} \frac{1}{\pi_j} - \frac{1}{\pi_l} - \frac{1}{\pi_m} + \frac{1}{\pi_{l \cup m}}}.$$

Note that the left hand side above is not smaller than $\pi_{\kappa+1}$ since $C_{\kappa+1}$ was previously inactive. Now, first suppose that $|C_{l \cup m}| \leq C_{\kappa'}$. Then the right hand side is smaller or equal than $\pi_{\kappa'}$ since $C_{\kappa'}$ is now active. So the expression holds true. If $|C_{l \cup m}| > C_{\kappa'}$ then

$$\frac{\kappa' - 2}{\sum_{j=1}^{\kappa'} \frac{1}{\pi_j} - \frac{1}{\pi_l} - \frac{1}{\pi_m} + \frac{1}{\pi_{l \cup m}}} < \frac{\kappa' - 3}{\sum_{j=1}^{\kappa'} - \frac{1}{\pi_l} - \frac{1}{\pi_m}} \leq \pi_{\kappa'},$$

where the first inequality holds from $\frac{1}{\pi_{l \cup m}} > \frac{1}{\pi_{\kappa'}}$, and the second again from the fact that $\pi_{\kappa'}$ is active. This finally proves the existence of positive spillovers. ■

Notice that the Proposition 2 applies to the range of relevant cases because an active coalition will clearly prefer to not merge if the resulting group becomes inactive.

In contests then, outsiders benefit from a reduction in the number of contenders due to a merger. This is so because the new group is less aggressive than the original coalitions given that the private benefit of winning the contest for their members has gone down. This result rests on the fact that the merger can only make inactive groups active, but not in the other way around. The interest of this observation is not merely technical. We often see that groups previously silent in some social dispute suddenly mobilize and become active after the configuration of other social groups changes. In our case, the merger decreases the relative cost of effort for non-members because the new group is less aggressive than the old ones, so previously inactive coalitions have more incentives to enter the contest.

Our valuation also satisfies other conditions proposed in the literature (see Bloch, 1997). They will prove useful later on when analyzing the stability of different coalition structures.

Proposition 3 *Under Assumptions 1-2 the valuation (8) satisfies the following properties:*

(*i*) ***Negative association:*** *Given C, the members of smaller coalitions get higher payoffs, i.e. $u(|C_k|, C) > u(|C_l|, C)$ for $|C_k| < |C_l|$.*

(*ii*) ***Inverse monotonicity:*** *If a member of coalition C_k leaves it to join a larger coalition C_l, then the members in C_l become worse off, i.e. $u(|C_l|, C) < u(|C_l| + 1, C \backslash \{C_k, C_l\} \cup \{C_k \backslash i, C_l \cup i\})$ when $|C_k| < |C_l|$.*

Proof. The first property entails no change in the number of active coalitions. It only requires to show that (8) is decreasing in size for a fixed C. It can be conveniently rewritten as

$$u(|C_k|, C) = \pi_k(1 - R(|C_k|))(1 - \frac{R(|C_k|)}{|C_k|}), \tag{9}$$

where $R(|C_k|) = (\kappa - 1)\frac{\frac{1}{\pi_k}}{\sum_{j=1}^{\kappa}\frac{1}{\pi_j}}$. It is clear that the two first factors are strictly decreasing in $|C_k|$ by Assumption 2. Hence it is enough to show that

$$\frac{R(|C_k|)}{|C_k|} \leq \frac{R(|C_l|)}{|C_l|},$$

that reduces to the inequality $\frac{1}{\Pi_k} \leq \frac{1}{\Pi_l}$, that is also satisfied because of Assumption 2.

For Inverse monotonicity, notice that the addition of a player to coalition C_l can make it inactive. Then, the property immediately follows. If $C_l \cup i$ remains active, we have to distinguish several cases.

Suppose first that the number of active coalitions does not decrease. Suppose that the index under C of the last active coalition after the change was $\kappa' \geq \kappa$. To show that the valuation decreases for members in C_k it is enough to show that

$$\frac{R(|C_l|)}{|C_l|} \leq \frac{R(|C_l| + 1)}{|C_l| + 1}, \tag{10}$$

but because of Rivalry it is enough to show that

$$\frac{\kappa - 1}{\sum_{j=1}^{\kappa}\frac{1}{\pi_j}} \geq \frac{\kappa' - 1}{\sum_{j=1, j\neq l,k}^{\kappa}\frac{1}{\pi_j} + \frac{1}{\pi_{k\setminus i}} + \frac{1}{\pi_{l\cup i}} + \sum_{j=\kappa+1}^{\kappa'}\frac{1}{\pi_j}}. \tag{11}$$

By employing Rivalry repeatedly one can obtain that

$$\frac{1}{\pi_{l\cup i}} - \frac{1}{\pi_l} \geq \frac{1}{|C_l|}\frac{1}{\pi_l} \geq \frac{1}{|C_k|}\frac{1}{\pi_k} = \frac{1}{\pi_k} - \frac{|C_k| - 1}{|C_k|}\frac{1}{\pi_k} > \frac{1}{\pi_k} - \frac{1}{\pi_{k\setminus i}}, \tag{12}$$

so the above inequality holds if

$$\begin{aligned}(\kappa - 1)(\sum_{j=1}^{\kappa}\frac{1}{\pi_j} + \sum_{j=\kappa+1}^{\kappa'}\frac{1}{\pi_j}) &\geq (\kappa' - 1)\sum_{j=1}^{\kappa}\frac{1}{\pi_j} \Leftrightarrow \\ \frac{\kappa - 1}{\sum_{j=1}^{\kappa}\frac{1}{\pi_j}} &\geq \frac{\kappa' - \kappa}{\sum_{j=\kappa+1}^{\kappa'}\frac{1}{\pi_j}}.\end{aligned}$$

In the right hand side of the last inequality, we have the harmonic mean of the individual payoffs corresponding to the newly active coalitions. But because the smaller of them, $C_{\kappa+1}$, was inactive before we know that

$$\frac{\kappa' - \kappa}{\sum_{j=\kappa+1}^{\kappa'}\frac{1}{\pi_j}} \leq \pi_{\kappa+1} < \frac{\kappa - 1}{\sum_{j=1}^{\kappa}\frac{1}{\pi_j}},$$

and this proves that the valuation has decreased.

Let us now show that the alternative, that is, that coalitions previously active become inactive after the change, cannot occur. To show that, it is enough to show that C_κ remains active, or formally that

$$\pi_\kappa > \frac{\kappa' - 1}{\sum_{j=1,j\neq l,k}^{\kappa} \frac{1}{\pi_j} + \frac{1}{\pi_{k-1}} + \frac{1}{\pi_{l+1}} + \sum_{j=\kappa+1}^{\kappa'} \frac{1}{\pi_j}}.$$

Note that in the previous discussion, we have just shown that the inequality (11) holds, so given that $\pi_\kappa > (\kappa - 1)/\sum_{j=1}^{\kappa} \frac{1}{\pi_j}$, our statement is true. This concludes the proof. ■

The intuitive reason why our valuation satisfies Negative association and Inverse monotonicity[5] is due both to Rivalry and the strong free-riding that exists within big groups. Small coalitions not only enjoy higher benefits from winning the contest (and are thus more aggressive) but also face rivals whose members both value victory less and have more incentives to free-ride.

4 Exclusive membership games of group formation

As is evident from the discussion above, the complexity of the analysis of group formation in our setup stems from the presence of spillovers. Because of this difficulty, there is not a unique theoretical approach to tackle the problem. In the next two sections, we will explore two different classes of games that seem suitable to the study of coalition formation in contests.

The first class of rules of group formation that we will consider here belong to the more general family of simultaneous games in which agents announce the coalition they would like to belong to. Contrary to the open membership games considered by Baik and Shogren (1995), Baik and Lee (1997) and Baik and Lee (2001), in the games of exclusive membership, players are not free to join an existing group. This requires some degree of agreement among members.

We will analyze two exclusive membership games first proposed by Hart and Kurz (1983). In these games, the strategy space of the players is the set of all coalitions they can belong to, i.e. $C^i = \{C_k \subseteq N \ / \ i \in C_k\}$. Players simultaneously announce a group $c^i \in C^i$ and coalitions form according to one of the following two rules of group formation:

(i) *The γ game*, in which a coalition forms if and only if all its members announced that coalition; unanimity is required, i.e. a group C_k forms only if all members i of C_k have chosen $c^i = C_k$. In that case, if a member of C_k deviates, the coalition breaks apart.

[5] These properties are called (P.2) and (P.3) respectively in Yi (1997).

(*ii*) *The δ game*, in which a group forms among those players who announced the same coalition even though some of its prospective members announced something else. Formally, for any possible coalition C_k, let $C(k) = \{i \ / \ c^i = C_k\}$ denote the group formed by all players who announced C_k. Hence, if a member of $C(k)$ deviates all other members remain together.

These games are called of exclusive membership because players announce a list of fellow members. The basic solution concept to be employed is the Nash equilibrium (NE henceforth). This set of equilibria is often very large so refinements like Strong Nash (Aumann, 1959) and Coalition-proof equilibrium (Bernheim et al., 1987) have been also considered in the literature.

A very useful concept when characterizing the NE coalition structures under these two games is *stand-alone stability.*

Definition *A coalition structure $C = \{C_1, ..., C_K\}$ is said to be stand-alone stable if and only if for any player $i \in C_k$, $u(|C_k|, C) \geq u(1, C')$, for $C' = C \backslash \{C_k\} \cup \{C_k \backslash i, i\}$.*

A coalition structure is stand-alone stable if no player is better off by leaving her coalition to become a singleton, holding the rest of the coalition structure fixed.

The importance of this property is clear: It is straightforward to show that a coalition structure is a NE of the δ game if and only if it is stand-alone stable (Yi and Shin, 2000). The next Proposition fully characterizes the set of NE of the δ game.

Proposition 4 *The unique NE (stand-alone stable) coalition structure of the δ game is the singleton structure $\{1, ..., 1\}$.*

Proof. First of all, notice any group structure in which at least one coalition is inactive cannot be stand alone stable since any member of an inactive coalition can secure a positive payoff by deviating. On one extreme, the singleton structure is stand-alone stable by definition. On the other, the grand coalition is stand alone stable if and only if

$$\pi_N \geq \pi_i \left(\frac{\frac{1}{\pi_{N \backslash i}}}{\frac{1}{\pi_i} + \frac{1}{\pi_{N \backslash i}}} \right)^2,$$

where $\pi_{N \backslash i}$ denotes the individual payoff for a member of the complementary coalition. By Rivalry, $\frac{1}{\pi_{N \backslash i}} \geq \frac{n-1}{\pi_i}$ so

$$\pi_i \left(\frac{\frac{1}{\pi_{N \backslash i}}}{\frac{1}{\pi_i} + \frac{1}{\pi_{N \backslash i}}} \right)^2 \geq \pi_i \left(\frac{n-1}{n} \right)^2 \geq \frac{\pi_N}{n} (n-1)^2 > \pi_N,$$

and we reach a contradiction. Finally, we show that for any other coalition structure, a member of the largest group has an incentive to defect and become a singleton.

- Case (i): Suppose C is a symmetric coalition structure. Then we want to show that

$$\pi_i(\frac{\frac{1}{\pi_{k\backslash i}}}{\frac{1}{\pi_i}+\frac{1}{\pi_{k\backslash i}}})^2 \geq \frac{\pi_k}{|C_k|}\frac{1}{\kappa}(|C_k|-\frac{\kappa-1}{\kappa}) = \frac{\pi_k}{n}(|C_k|-1+\frac{|C_k|}{n}),$$

since after i's defection all coalitions except i and $C_k\backslash\{i\}$ become inactive by Lemma 2. By applying Rivalry twice, one time at each side of above inequality, it is enough to show that

$$\pi_i(\frac{|C_k|-1}{|C_k|})^2 \geq \frac{\pi_i}{n}(\frac{|C_k|-1}{|C_k|}+\frac{1}{n}) \Leftrightarrow \frac{|C_k|-1}{|C_k|} \geq \frac{\sqrt{5}+1}{2}\frac{1}{n},$$

and since $n \geq 4$ and $|C_k| \geq 2$, the latter inequality holds true. Hence, no symmetric coalition structure apart from the singleton structure can be stand-alone stable.

- Case (ii): Suppose that there exists a coalition k such that $|C_j| = |C_1|+1$ for $j = k,...,\kappa$ and $|C_j| = |C_1|$ otherwise. In that case, our aim is to show that

$$\frac{\pi_\kappa}{|C_\kappa|}(1-(\kappa-1)\frac{\frac{1}{\pi_\kappa}}{\sum_{j=1}^{\kappa}\frac{1}{\pi_j}})(|C_\kappa|-(\kappa-1)\frac{\frac{1}{\pi_\kappa}}{\sum_{j=1}^{\kappa}\frac{1}{\pi_j}}) < \pi_i(1-k\frac{\frac{1}{\pi_i}}{\frac{1}{\pi_i}+k\frac{1}{\pi_1}})^2,$$

since after i's defection all coalitions C_j such that $k \leq j < \kappa$ will become inactive. Note that

$$\begin{aligned}\frac{\pi_\kappa}{|C_\kappa|}(1-(\kappa-1)\frac{\frac{1}{\pi_\kappa}}{\sum_{j=1}^{\kappa}\frac{1}{\pi_j}})(|C_\kappa|-(\kappa-1)\frac{\frac{1}{\pi_\kappa}}{\sum_{j=1}^{\kappa}\frac{1}{\pi_j}}) &\leq \frac{\pi_i}{|C_\kappa|^2}\frac{1}{\kappa}(|C_\kappa|-1+\frac{1}{\kappa}) \\ &= \frac{\pi_i}{(|C_1|+1)^2}\frac{1}{\kappa}(|C_1|+\frac{1}{\kappa}) \\ &\leq \frac{\pi_i}{(|C_1|+1)^2}\frac{1}{k}(|C_1|+\frac{1}{k}),\end{aligned}$$

since $\frac{1}{\pi_\kappa}/\sum_{j=1}^{\kappa}\frac{1}{\pi_j} \geq \frac{1}{\kappa}$ and $\kappa \geq k$. At the same time, notice that by Rivalry $\frac{1}{\pi_1} \geq \frac{|C_1|}{\pi_i}$ so

$$\pi_i(1-k\frac{\frac{1}{\pi_i}}{\frac{1}{\pi_i}+k\frac{1}{\pi_1}})^2 \geq \pi_i(\frac{|C_1|k-k+1}{|C_1|k+1})^2.$$

Hence for our purposes it is enough to show that

$$\frac{\pi_i}{(|C_1|+1)^2}\frac{1}{k}(|C_1|+\frac{1}{k}) < \pi_i(\frac{|C_1|\,k-k+1}{|C_1|\,k+1})^2.$$

Note that the left hand side is decreasing in $|C_1|$ whereas the right hand side is increasing. Suppose that $|C_1| \geq 2$. Then it is enough to show that

$$\frac{\pi_i}{9}\frac{1}{k}(2+\frac{1}{k}) < \pi_i(\frac{k+1}{2k+1})^2 \Leftrightarrow 0 < 9k^4+10k^3-3k^2-6k-1,$$

which holds true for any $k \geq 2$. If instead $|C_1| = 1$ we can perform direct computations since then it must be the case that $\kappa = 2$ (otherwise some coalition would be inactive to start with). Hence, we have to check that

$$\frac{\pi_\kappa}{2}(1-(\kappa-1)\frac{\frac{1}{\pi_\kappa}}{(\kappa-1)\frac{1}{\pi_\kappa}+\frac{1}{\pi_i}})(2-(\kappa-1)\frac{\frac{1}{\pi_\kappa}}{(\kappa-1)\frac{1}{\pi_\kappa}+\frac{1}{\pi_i}}) \leq \frac{\pi_i}{9},$$

but again, we can use the fact that $\frac{1}{\pi_\kappa} \geq \frac{2}{\pi_i}$, so the above holds if

$$\frac{\pi_i}{2}\frac{\kappa}{(2\kappa-1)^2} \leq \frac{\pi_i}{9} \Leftrightarrow 0 \leq 8\kappa^2-17\kappa+2,$$

which is true for any $\kappa \geq 2$. Hence, this coalition structure is not stand-alone stable either.

- Case (iii): Suppose that $|C_\kappa| \geq |C_1|+2$ and that there exists a coalition $k \leq \kappa$ such that $|C_k| \geq |C_1|+1$ and $|C_j| = |C_1|$ for $j = 1, ..., k-1$. In that case, all coalitions bigger than $|C_1|$ become inactive after i's defection. By applying the same procedure as above, it is enough to show that

$$\frac{\pi_i}{(|C_1|+2)^2}\frac{1}{k}(|C_1|+1+\frac{1}{k}) \leq \pi_i(\frac{|C_1|\,(k-1)-(k-2)}{|C_1|\,(k-1)+1})^2.$$

Suppose again that $|C_1| \geq 2$ then the above holds if

$$\frac{\pi_i}{16}\frac{1}{k}(3+\frac{1}{k}) \leq \pi_i(\frac{k}{2k-1})^2 \Leftrightarrow 0 \leq 16k^4-12k^3+8k^2+k-1,$$

which holds true for any k. Finally, if $|C_1| = 1$ notice that again it must be that $\kappa = 2$, otherwise all coalitions bigger than $|C_1|$ would be inactive. In that case, the above comparison reduces to $\frac{5}{36}\pi_i \leq \frac{1}{4}\pi_i$, which holds true. This coalition structure is not stand alone stable either. This exhausts all possible cases and finishes the proof. ■

This Proposition shows that players have very strong incentives to break up the group they belong to if defections leave unchanged the rest of the coalition structure. That is natural since by becoming a singleton the agent obtains the maximum prize in case of victory and faces bigger and thus less aggressive groups. However, as we will see below, it may be unreasonable to expect that no further deviations will take place. Observe also that the set of Nash, Strong Nash, and Coalition-proof equilibrium coalition structures of the δ game coincide.[6]

On the other hand, the γ game assumes that individual defections trigger the collapse of the group into singletons. We know from Bloch (1997) that if, as in our case, the valuation displays positive spillovers and satisfies Negative association and Inverse monotonicity, the γ game tends to support more concentrated structures than the δ game. This is because although members of big coalitions receive lower payoffs before the break up, they also face a higher number of rivals afterwards, so the incentives to defect are weaker[7].

Unfortunately, less clear-cut results can be obtained in this case. For this reason, in the remainder of the paper, we will in occasions adopt a specific functional form for the individual payoff given by

$$\pi_k = \frac{V}{|C_k|^\rho}, \tag{13}$$

where $\rho \geq 1$. In line with the examples in Section 2, the parameter ρ measures the degree of intra-group rivalry: The cases $\rho = 1$ and $\rho = 2$ coincide with $\mu = 1$ and $\mu = 0$ respectively in Example 1.1; this range also matches the degree of rivalry in Example 1.2. Hence, we will refer to the case $\rho \in (1, 2)$ as an intermediate level of rivalry, whereas we will consider that the case $\rho > 2$ corresponds to an extremely intense conflict of interests within groups.

The following Proposition partially characterizes the set of Nash equilibria of the γ game.

Proposition 5 *In the γ game of coalition formation,*

(i) The grand coalition $\{N\}$ is a NE if and only if $\pi_N n^2 > \pi_i$.

(ii) Any coalition structure with at least one singleton cannot be supported in equilibrium.

[6] By Lemma 2, only the coalition composed by $n-1$ players would be active if it deviates collectively. But, as Proposition 6 below shows, members of such group are indifferent between $\{1, ...1\}$ and $\{n-1, 1\}$ when Π_k is constant in size and strictly prefer the former partition otherwise. Hence, the singleton structure constitutes the unique Strong Nash equilibrium too. Finally, since the set of Coalition-proof equilibria contains the set of Strong Nash and it is a subset of the set of Nash equilibria, the three sets coincide.

[7] Sadly, this result cannot help us to sharpen our characterization of the equilibria; given that only the singleton structure $\{1, ..., 1\}$ was δ-stable, any equilibrium structure under the γ game will be necessarily more concentrated.

(*iii*) *Assume that π_k is given by (13). Then the set of Nash equilibria coalition structures consists of any structure satisfying*

$$|C_k|\,(n-|C_k|\,(K-1))(n-K+1)>n^2, \qquad \textit{for } k=1,...,K, \quad (14)$$

when $\rho=1$, *and the singleton structure* $\{1,...,1\}$ *when* $\rho\geq 2$.

Proof. The valuation under the grand coalition is simply π_N. An individual deviation would imply the collapse of N into the singleton structure, and the valuation is then $\frac{\pi_i}{n^2}$. The comparison between these two yields the stated condition on the γ stability of the grand coalition. On the other hand, suppose that some group in C is already a singleton. Then, the rest of coalitions in C must be of the same size in order to be γ stable. Otherwise, at least one would be inactive. Then, it is straightforward to show that the valuation for all the coalitions is smaller or equal than

$$\frac{\pi_i}{|C_k|}\frac{(|C_k|-1)(K-1)+1}{(|C_k|\,(K-1)+1)^2}\leq\frac{\pi_i}{(|C_k|+1)^2},$$

where the last term coincides with the valuation after i's deviation. Hence, a coalition structure with at least one singleton cannot constitute a NE of the γ game.

Finally, when individual payoffs are such that $\pi_k=\frac{V}{|C_k|^\rho}$, the valuation can be rewritten as

$$\frac{V}{|C_k|^\rho}(1-(K-1)\frac{|C_k|^\rho}{\sum_{j=1}^K|C_j|^\rho})(1-(K-1)\frac{|C_k|^{\rho-1}}{\sum_{j=1}^K|C_j|^\rho}).$$

After i's deviation, and assuming that no coalition in C is a singleton, her payoff is simply $\frac{V}{|C_k|^2}$. Then checking whether C constitutes a NE of the γ game reduces to show that

$$\frac{1}{|C_k|^{\rho-2}}(1-(K-1)\frac{|C_k|^\rho}{\sum_{j=1}^K|C_j|^\rho})(1-(K-1)\frac{|C_k|^{\rho-1}}{\sum_{j=1}^K|C_j|^\rho})>1 \qquad \text{for } k=1,...,K.$$

When $\rho=1$, the condition becomes the one stated in the text of the Proposition. When $\rho\geq 2$ it is straightforward to see that the left hand side of the above expression attains its minimum value for coalition C_K. Knowing that,

$$\begin{aligned}\frac{1}{|C_K|^{\rho-2}}(1-(K-1)\frac{|C_K|^\rho}{\sum_{j=1}^K|C_j|^\rho})(1-(K-1)\frac{|C_K|^{\rho-1}}{\sum_{j=1}^K|C_j|^\rho}) &\leq \frac{|C_K|\,K-K+1}{|C_K|^{\rho-1}K^2}\\ &\leq \frac{|C_K|\,K-K+1}{|C_K|\,K^2}\\ &\leq \frac{2\,|C_K|-1}{4\,|C_K|}<1,\end{aligned}$$

so C cannot be an equilibrium of the γ game. ■

This result shows that the set of NE coalition structures of the γ game is rather limited. Again notice that no partition in which at least one group is inactive can be supported as an equilibrium of this game. Hence, equilibrium structures will tend to be formed by groups of similar size. But this Proposition also illustrates the importance of intra-group rivalry: The grand coalition will fall apart if rivalry is strong enough, i.e. $\rho > 2$, and so will do any other coalition structure apart from the singleton one. Only when rivalry is not very intense, i.e. $\rho < 2$, other coalition structures can be supported as Nash equilibria of the γ game.[8]

5 Sequential coalition formation

In the previous Section, the behavior of players following a defection was exogenously imposed. After a deviation, the rest of members may not necessarily stick together nor break apart. This shortsightedness in reactions can be avoided by employing sequential games in which the extensive form fully describes the process of group formation. In this case, players can rationally predict the coalition structure that outsiders will form after any move they make. The drawback of this approach is common to all games in extensive form, namely that many alternative protocols can be considered. Here, we will follow Bloch (1996) game of coalition formation in which groups form *à la* Rubinstein. The game proceeds as follows: The first player in a pre-determined protocol makes a proposal for a coalition she belongs to. The players in this proposed coalition decide sequentially whether to accept or not this proposal. The process stops when all members accept or one rejects. In the former case, the proposed coalition forms and the next available player in the protocol is called to move. In the latter case, the rejector must make a counter-offer and propose the formation of another coalition she belongs to. Bloch (1996) shows that when, as in our case, valuations are *symmetric,* that is, payoffs depend on the size of the coalition and not on the particular identity of its members, this game yields the same stationary subgame perfect equilibrium coalition structure as the much simpler "Size Announcement game": The first player in the protocol proposes a coalition of size $|C_1|$ that immediately forms. Then player $|C_1|+1$ in the protocol proposes a coalition $|C_2|$ and so on until the player set is exhausted. This game

[8]Bloch et al. (2006) obtain parallel results for the grand coalition when rivalry is minimal, i.e. $\rho = 1$. On the other hand, Garfinkel (2004b) employs the concept of *farsighted stability*, that requires deviations to be δ-stable themselves. The author shows that any symmetric and nearly symmetric coalition structure, except the grand coalition, is inmune to such deviations. This is because, given that only the singleton structure is δ-stable, any Pareto superior partition is farsighted stable.

is solved by backward induction and generically admits a unique subgame perfect equilibrium (SPE henceforth).

This sequential game of coalition formation is very well suited to our particular application since it can uncover the dynamics of coalition formation in contests. If for instance, we focus our attention on the stability of the grand coalition, this game can help us to disentangle the incentives of any player to break up peace, since the game fully endogenizes how events will unravel afterwards.

We can use Proposition 5 to state the following Lemma that partially characterizes player's optimal strategies in this game:

Lemma 3 *Assume the aggregate coalitional payoffs* Π_k *are strictly decreasing in size. Then in a SPE of Bloch's game of sequential coalition formation if any player forms a singleton all subsequent players choose to form singletons too.*

Proof. Suppose that $S \leq n$ players remain in the game. By subgame perfection, players know that after they move all coalitions subsequently formed will be active. Then by Lemma 2, if player $n-S+1$ in the protocol announces a singleton all coalitions formed after that must be of the same size. Note that if before that player is called to move a singleton already formed, the Lemma holds straightaway.

Suppose $M \geq 1$ coalitions already formed before player $n-S+1$ was called to move, and denote by $|C_2| \leq S-1$ the size of the group that player $n-S+2$ announces. Again, if $|C_2| = 1$ the Lemma trivially holds. We know that $|C_2|$ will not be in any case bigger than the smallest among the M groups, otherwise it would be inactive. So either all those M groups become inactive or at most $m \leq M$ of them remain active, but they must be of exactly size $|C_2|$. Now we can establish that this coalition structure cannot constitute a SPE of the Bloch game for $|C_2| \geq 2$ by using Proposition 5 to show that the last player that is called to move prefers instead to form a singleton. To see this notice that we are exactly in case (*ii*) of that Proposition, since by forming a singleton two of them have formed and it is optimal for all players after $n-S+2$ to form singletons too. Hence the proposed coalition structure is a SPE of the Bloch game only if it is γ-stable. And we know it cannot be the case when aggregate coalitional payoffs are strictly decreasing in size (if they are constant, player $n+S-1$ is indifferent between announcing a singleton and announcing $S-1$ when $m=0$). Notice finally that this argument also holds if $M=0$. ■

This Lemma shows that cooperation is very fragile in our setup. A player contemplating the formation of a coalition of size two or more knows that her group will become inactive if one of the subsequent players in the protocol were to form a singleton. At the same time, she can ensure to herself a positive payoff by forming one. This indicates the existence of the bang-bang outcomes that we will find below.

Beyond this point, a full characterization of the subgame perfect equilibria of the game requires a relatively simple closed form solution for the valuation. The main difficulty we face here is that as coalition structures change, the constellation of active coalitions also changes. This makes players' optimal decisions hard to characterize. One possible way to proceed is to employ the functional form for individual payoffs introduced in Proposition 5, and state the results in terms of the parameter ρ measuring the degree of intra-group rivalry.

Proposition 6 *Assume that π_k is given by (13). Then the SPE coalition structure of the Bloch game of coalition formation*

(i) is $\{1, n-1\}$ when $\rho = 1$.

(ii) are the grand coalition and the singleton structure $\{1, ..., 1\}$ when $\rho = 2$.

(iii) is the singleton structure $\{1, ..., 1\}$ when $\rho > 2$.

Proof. Assume $\rho = 1$. The closed form solution for the valuation is then

$$u(|C_k|, C) = \frac{V}{|C_k|} \frac{n' - (\kappa - 1)|C_k|}{n'} \frac{n' - \kappa + 1}{n'},$$

where n' is the number of players belonging to the active coalitions in C. Simple inspection shows that the best case scenario for the first player in the protocol is to form a singleton and that $\kappa = 2$, $n' = n$. Next we show that this can be supported as a SPE of the Bloch game of coalition formation: The second player in the protocol knows that for any announcement $|C_2|$ that she makes, any other coalition that will form after that will be of size no bigger than that and moreover, by subgame perfection, that all these coalitions will end up being active. If C_2 becomes inactive after that, the result trivially holds. If not, then it must be that $n' = n$ and $\kappa \geq \frac{n-1}{|C_2|} + 1$. But for $|C_2| \geq 2$, Proposition 5 again implies that the last coalition formed will prefer to break up into singletons, so C_2 will be inactive in that case too. Then, player 2 has only two options, either to announce $n - 1$ or to form a singleton, and then trigger $\{1, ..., 1\}$. Direct computations show that she is indifferent between the two, so $\{1, n-1\}$ constitutes a SPE.

Parts *(ii)* and *(iii)* can be proved by noting first that when $\rho \geq 2$, for any $2 \leq |C_k| < n$, the valuation $u(|C_k|, C)$ is strictly smaller than $u(N) = V/|C_k|^2$, the payoff under a contest among $|C_k|$ singletons. Then, in any SPE coalition structure, the last group formed cannot be of such size, since the player who announced it will always prefer to form a singleton. But even if the last group formed is a singleton, Proposition 5 implies that the second to last coalition would like to break up too unless it is a singleton itself. So the only two candidates left for a SPE structure are the grand coalition and the singleton structure. In the special case when $\rho = 2$ both payoffs are equal to $\frac{V}{n^2}$. Hence, both can constitute a SPE. For $\rho > 2$ however, the payoff under $\{N\}$ becomes lower, so the singleton structure is the unique SPE of the game. ■

This Proposition shows the complex effect of rivalry on coalition formation. For relatively low levels, an asymmetric two-sided contest emerges, with the first player forming a singleton against all the rest. Since rivalry is not very strong, it pays for the second player in the protocol to avoid further conflict by forming a grand coalition among the remaining players. Notice that this is not efficient from the social point of view, since when $\rho = 1$ it is the grand coalition the structure that maximizes the sum of payoffs. On the other hand, when rivalry is relatively strong, the rest of players always prefer to form singletons after a coalition has been formed. Then, the first player in the protocol can either form the grand coalition or trigger a contest among singletons. Only when $\rho = 2$ he is indifferent between the two. When $\rho > 2$ however, intra-group rivalry outweights the benefits of avoiding conflict by absorbing rivals, so conflict ensues.

The following example illustrates the outcome of the game in the intermediate case, i.e. $\rho \in (1, 2)$.

Example 2: Suppose that $N = \{a, b, c, d, e\}$, $\rho = 1.5$ and $V = 100$. The following table describes the valuation for all possible coalition structures.

	$u_a(C)$	$u_b(C)$	$u_c(C)$	$u_d(C)$	$u_e(C)$
$\{N\}$	9	9	9	9	9
$\{abcd\}, \{e\}$	1.1	1.1	1.1	1.1	79
$\{abc\}, \{de\}$	5.3	5.3	5.3	18	18
$\{abc\}, \{d\}\{e\}$	0	0	0	25	25
$\{ab\}, \{cd\}, \{e\}$	3	3	3	3	49
$\{ab\}, \{c\}, \{d\}, \{e\}$	0	0	11.1	11.1	11.1
$\{a\}, \{b\}, \{c\}, \{d\}, \{e\}$	4	4	4	4	4

Table 1: Valuation when $\rho = 1.5$ and $n = 5$.

Take player a, the first in the protocol. We know from Lemma 3 that forming a singleton triggers the singleton contest, so she prefers to announce $\{N\}$. Announcing sizes 3 or 4 are also dominated by the grand coalition since even in the best case scenario (the complementary coalition forms) they yield payoffs below 9. Finally, if she announces a coalition of size 2, player c can either form the grand coalition among the remaining players, form a group of size 2 or trigger a contest among the three last players. She prefers the last option, so player a would become inactive if she announces 2. Hence, the grand coalition is the SPE coalition structure of this sequential game of coalition formation. Notice that the efficient coalition structure in this case is one singleton against the rest of players.

This example suggests the existence of a non-monotonic relationship between the degree of underlying rivalry in society and the level of conflict. When rivalry is not too strong the formation of a small coalition may trigger a fierce conflict among singletons. On the other hand, big coalitions give too low payoffs since they are heavily disadvantaged in the contest. Hence, the first player may prefer to form the grand coalition and avoid conflict. As rivalry becomes stronger, the payoff from preserving peace decreases, whereas the payoff under the singleton contest remains constant. In the limit, when $\rho = 2$, they coincide and the game admits two SPE as we have seen above; this is the tipping point between these two partitions. Hence for $\rho > 1$, the sequential game of coalition formation presents bang-bang outcomes, either universal peace for intermediate levels of rivalry, or a "war of all against all" when rivalry is strong.

This parallels somehow the results obtained by Bloch et. al. (2006). There, the dynamics of coalition formation also displayed this choice faced by the first player between the grand coalition and the singleton structure. However, the former was always the efficient partition in their setup, so when choosing between these two symmetric partitions, the first player always preferred to form the grand coalition.

In other models of coalition formation with positive spillovers, like cartel formation under Cournot competition (Bloch, 1996) or public goods provision (Ray and Vohra, 2001), the sequential game of coalition formation yields coarser partitions than the singleton structure, but not the grand coalition. In these games, the first players to move typically tend to free-ride on the reduction of output or pollution abatement carried by the bigger groups that form later. This effect is not present in our case since if, for instance, one players forms a singleton, all the rest will do the same in order to avoid becoming inactive, resulting in a fierce contest.

6 Conclusion

The main objective of the present paper has been to gain insights into the reasons why confrontation erupts in some contexts and why coalitions form in such situations. It has explored a model of contests where agents first form groups and then compete over prizes by investing resources. Rivalry persists within coalitions once victory is attained, so prizes at this stage are assumed to be decreasing in the size of the group. We show that bigger groups tend to drop out of the contest and that coalition formation generates positive spillovers on non-members.

When coalitions form simultaneously, the contest among individual agents is the only stable structure if deviations are assumed to leave the rest of the structure intact. When, on the contrary, coalitions break apart completely after a member withdraws, more concentrated coalition structures, including the grand coalition, can be stable provided that intra-group rivalry is not too strong. On the other hand, the sequential game of coalition formation suggests that there exists a non-monotonic relationship between intra-group rivalry and social conflict: Whereas a two-sided contest emerges for low levels of rivalry, the grand coalition is likely to form for intermediate levels, and a fierce contest among individuals agents precipitates when intra-group rivalry is so strong that any kind of cooperation breaks up.

It is important to notice that our reduced-form formulation of prizes allows us to encompass several models of contests, as for instance those discussed in Example 1.1 by setting $\rho = 2$. Most of these contributions take the number of (asymmetric) groups as exogenously given. Our analysis suggests that this may not be a reasonable assumption given that stable structures tend to be symmetric, mostly the grand coalition and the singleton structure. If coalition formation were allowed in these models, groups would then either merge or break apart. These analysis are relevant though if there exist restrictions in the process of coalition formation due, for instance, to identity, ethnic belonging or ideology.

The private nature of the prize is also worth discussing. Esteban and Ray (2001) show that free-riding within groups is alleviated if the prize contains some public characteristics. One may wonder what would happen in our model if the Rivalry assumption were relaxed and, as in Katz et al. (1990), groups would compete for a public good. Our response is that, unless individual payoffs also violate the Anonymity assumption, contests over such good (even if excludable) make no sense. To see this, notice that if preferences are not group-specific, coalitions obtain the same good if they win the contest and under a peaceful agreement with the rest of groups, so there is no point in initiating a wasteful confrontation. In this case, group formation would always yield the grand coalition[9]. This may not be the case if Anonymity does not hold and individuals have different preferences. But this in turn would make coalition formation irrelevant unless a procedure to aggregate preferences within groups were explicitly introduced.

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[9]This would corroborate even further the non-monotonic relationship between rivalry and conflict.

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A Appendix

Lemma A *An active coalition $C_k \in C$ remains active after two coalitions in $C \backslash \{C_k\}$ merge if the newly formed coalition is active too.*

Proof. For this result to be true, it is enough to show that C_κ, the biggest coalition active in C, will still be active in C'. Note that the proof is trivial if the two merging coalitions are bigger than C_κ.

Suppose that only one of the two merging groups, denoted by C_l, is smaller (or of equal size) than C_κ. For it to become inactive, it must be that

$$\frac{\kappa - 1}{\sum_{j=1, j\neq l}^{\kappa} \frac{1}{\pi_j} + \frac{1}{\pi_l}} < \pi_\kappa < \frac{\kappa - 2}{\sum_{j=1, j\neq l}^{\kappa} \frac{1}{\pi_j}},$$

implying thus that

$$\pi_l < \frac{\kappa - 2}{\sum_{j=1, j\neq l}^{\kappa} \frac{1}{\pi_j}}.$$

At the same time, since C_l was active before the merger, it is true that

$$\pi_l > \frac{l - 2}{\sum_{j=1}^{l-1} \frac{1}{\pi_j}}.$$

These two statements are contradictory since for all coalitions with index $j > l$ it is true that $\frac{1}{\pi_j} < \frac{1}{\pi_{l-1}}$. Hence coalition C_κ must remain active.

Similarly, suppose that the two merging coalitions, indexed by l and m, are smaller than C_κ but that (with some abuse of notation) the resulting coalition $C_{l \cup m}$ is bigger than C_κ. Then it must be that,

$$\frac{\kappa - 1}{\sum_{j=1, j\neq l,m}^{\kappa} \frac{1}{\pi_j} + \frac{1}{\pi_l} + \frac{1}{\pi_m}} < \pi_\kappa < \frac{\kappa - 3}{\sum_{j=1, j\neq l,m}^{\kappa} \frac{1}{\pi_j}}.$$

These two bounds are compatible if and only if

$$
\begin{aligned}
2\sum_{j=1,j\neq l,m}^{\kappa}\frac{1}{\pi_j} &< (\kappa-3)(\frac{1}{\pi_l}+\frac{1}{\pi_m}), \\
&< (\kappa-3)\frac{1}{\pi_{l\,\cup\,m}}, \\
&< (\kappa-3)\frac{\sum_{j=1,j\neq l,m}^{\kappa}\frac{1}{\pi_j}+\sum_{j=\kappa}^{l\,\cup\,m}\frac{1}{\pi_j}}{(l\cup m)-1},
\end{aligned}
$$

where the second line derives from the fact that

$$\frac{1}{\pi_l}+\frac{1}{\pi_m}\leq\frac{l}{l+m}\frac{1}{\pi_{l\,\cup\,m}}+\frac{m}{l+m}\frac{1}{\pi_{l\,\cup\,m}}=\frac{1}{\pi_{l\,\cup\,m}}, \tag{15}$$

by applying Rivalry twice, and the third line comes from our assumption that coalition $C_{l\,\cup\,m}$ remains active after the merger. Since $|C_{l\,\cup\,m}|>|C_\kappa|$, we have reached a contradiction. Finally, suppose that $|C_{l\,\cup\,m}|\leq|C_\kappa|$. If C_κ becomes inactive it must be that

$$\frac{\kappa-1}{\sum_{j=1,j\neq l,m}^{\kappa}\frac{1}{\pi_j}+\frac{1}{\pi_l}+\frac{1}{\pi_m}}<\pi_\kappa<\frac{\kappa-2}{\sum_{j=1,j\neq l,m}^{\kappa}\frac{1}{\pi_j}+\frac{1}{\pi_{l\,\cup\,m}}}.$$

This implies thus that

$$(\kappa-2)(\frac{1}{\pi_l}+\frac{1}{\pi_m})>(\kappa-1)\frac{1}{\pi_{l\,\cup\,m}}+\sum_{j=1,j\neq l,m}^{\kappa}\frac{1}{\pi_j},$$

but notice that by applying (15) we have reached a contradiction again. Hence, we have proved that already active coalitions remain active after a merger of two other groups occurs. ■