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Effective Topology from Spacetime Tomography

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Abstract.
     We recover the effective topology of spacetime using the notion of record from the decoherent histories approach to Quantum Theory. From a series of (gedanken) experiments, we obtain the set of possible events, grouped into sub-sets that corresponds to histories, but with no other information such as (causal) order or any notion of proximity. This corresponds to tomography of the "effective" spacetime, that is done in an operational way. Making certain assumptions about these records, and using the existence of upper bound in the speed of transfer of matter and information, we recover the full partial (causal) order up to certain ambiguities. The partially ordered set of events corresponds to an "effective" causal set which is a discretized version of spacetime with the causal relation as defining feature. We conclude with a derivation of the topology of this effective discretized spacetime.

1. Motivation

Allowing the matter to be quantum, implies that we cannot have a direct (operational) way to speak about the metric or the topology of spacetime. We can do this approximately due to the phenomenon of decoherence, but this arises only in a sufficiently coarse-grained description. We may therefore claim that the only spacetime we may speak of is the "effective-spacetime" corresponding to the discretized spacetime that has as points equivalence classes of operationally indistinguishable "real" points, *i.e.*, we do not prescribe ontological status to anything that is finer grained than the scale of decoherence. We would like to use a set of data (records) in one (final) time to reconstruct an "arena" for a particular subsystem that we are interested in. This "arena" is the effective spacetime. The use of "final-time" records has a twofold motivation. First, in this way we are truly operational, since we can make statements about, say, the early universe, where clearly no observers were around. The second reason, relates with the "frozen-time" picture (*e.g.*, Ref.[1]). According to this, time is not something fundamental and arises only as a better way of organizing some "present" data. This philosophical belief is reinforced by the problem of time in quantum gravity where time disappear from any observable (*e.g.*, Ref.[2]).

In quantum theory, there is already a framework to speak of records, in the context of the decoherent histories approach. This is an alternative formulation of Quantum Theory developed by Griffiths [3], Omnès [4], and Gell-Mann and Hartle [5], design to deal with closed systems, and having as fundamental ingredients whole histories of the system, rather than single time propositions. For all the cases that we can assign probabilities to histories of closed systems, it can be shown that there exist a set of records in the final time, perfectly correlated with the corresponding history [6]. In our operational approach we will use exactly this property as starting point and do the inverse, *i.e.*, deduce the topology of the underlying spacetime given a set of records and certain assumptions about them [7, 8].

2. This paper
This contribution is largely based on Refs.[7, 8] done in collaboration with Raptis and Zapatrin. We will first state what assumption we will make about the records that we have, in Section 3. Then in section 4 we will revise some properties of the causal sets, that will be the effective spacetime that we will recover in Section 5. Finally we will derive the topology of the causal set in Section 6.

3. Records and Sub-Records
The records that we will get, are physically obtained, *i.e.*, measured in Copenhagen sense. In particular, we need to make certain assumptions about those records, in order to reconstruct the effective spacetime. We assume that the records ($C_i \in \mathcal{C}$) capture the spatiotemporal properties of our system, *i.e.*, correspond to (coarse-grained) trajectories. Furthermore, we assume that each of those records consists of sub-records ($p \in C_i$) that correspond to each (coarse-grained) event and the union of all these points is the totality of events ($\mathcal{P}$) in our effective spacetime. An important thing to note, is that the order the events of one trajectory is unknown in our set up. This bears a resemblance with a photograph of a trajectory ("frozen-picture"), where we do not know which way the particle crossed first[1]. The construction below will recover this order. Since our effective spacetime is discrete, it is natural to consider it as an effective causal set.

4. Causal Sets
Causal set is a version of discretized spacetime. It has been proposed as a more fundamental entity than the continuous spacetime manifold [9] and for that reason has been used as the basis for a quantum theory of gravity by its advocators. Here we are not going to make any statement about the ontology of this spacetime, and we will only use the fact that is a discrete version of spacetime since the effective spacetime we derive from our records is discrete. Another reason for this choice, is that we will claim that all the necessary information for deriving the effective spacetime is encoded in the causal relations.

A causal set, abbreviated as causet is a locally finite, partially ordered set (POSET). The points of the poset physically correspond to events and the order relation is the causal relation between the events. The reason it is believed that this corresponds to discrete version of spacetime stems from the following property that continuous Lorentzian manifolds have [10]. If we consider two different Lorentzian manifolds with a bijection that preserve the causal order, then we can recover 9 out of the 10 degrees of freedom of the metric. The remaining factor is a scale factor and the claim is that in a locally finite poset we can recover this factor by counting. Let us now proceed with some definitions.

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[1] In the photograph only the overall direction is unknown, while in our case the events are completely unordered.

Partially ordered sets.   A (weak) partially ordered set, is a set $\mathcal{P}$ endowed with a relation $\preceq$ having the following properties:

- Transitivity: $\forall \quad p, q, r \in \mathcal{P} \quad p \preceq q, \ q \preceq r \Rightarrow p \preceq r$.
- Reflexivity: $\forall \quad p \in \mathcal{P} \quad p \preceq p$.
- Antisymmetry: $\forall \quad p, q \in \mathcal{P} \quad p \preceq q, \ q \preceq p \Rightarrow p = q$.

We define future $J^+(p)$ and past $J^-(p)$ cones for each of the elements of the poset ($p \in \mathcal{P}$):

$$J^+(p) \ = \ \{q \in \mathcal{P} \mid p \preceq q\}$$
$$J^-(p) \ = \ \{r \in \mathcal{P} \mid r \preceq p\} \tag{1}$$

A poset is defined to be locally finite if

$$|J^-(p) \cap J^+(q)| < \infty, \quad \forall \quad p, q \in \mathcal{P} \tag{2}$$

where $|A|$ denotes set-cardinality.

   We will now review certain properties of locally finite posets (*i.e.*, causets). A subset $C \subset \mathcal{P}$ is called a CHAIN (also known as a LINEARLY ordered subset) if any pair of its points is ordered:

$$\forall \quad p, q \in C, \quad p \preceq q \quad or \quad q \preceq p \tag{3}$$

In the sequel, we shall consider maximal (that is, inextensible) chains in $\mathcal{P}$ and we will denote the set of all maximal chains by $\mathcal{C} \ = \ \{$ maximal chains of $\mathcal{P} \}$.

   In a similar way, we define an ANTICHAIN to be a subset S of $\mathcal{P}$ such that no pair of its points is ordered:

$$\not\exists \quad p, q \in S \mid p \preceq q \tag{4}$$

We shall need maximal antichains in $\mathcal{P}$, and denote the appropriate set by $\mathcal{S} \ = \ \{$ maximal antichains of $\mathcal{P} \}$. An element of a $\mathcal{P}$ is said to be minimal if $J^-(p) = \{p\}$, and maximal when $J^+(p) = \{p\}$. The notions of future and past cones can be extended to subsets of $\mathcal{P}$. For $A \subseteq \mathcal{P}$

$$J^+(A) \ = \ \{q \in \mathcal{P} \mid \exists \quad a \in A, \quad a \preceq q\}$$
$$J^-(A) \ = \ \{r \in \mathcal{P} \mid \exists \quad a \in A, \quad r \preceq a\} \tag{5}$$

In terms of posets, the chains stand for causal curves, while the antichains are reticular analogues spatial (hyper)surfaces. A foliation $\mathcal{F}$ is a partition of a causet $\mathcal{P}$ into spatial surfaces (*i.e.*, antichains) which respects the partial order $\preceq$ in $\mathcal{P}$.

   A linear foliation, is a foliation that all the antichains are linearly (as opposed to partially) ordered. It can be shown that any past finite causet (and this is the case we consider) admits a linear foliation. The following construction proves this statement:

- $A_0 := \{$minimal elements of $\mathcal{P}\}$
- $A_1 := \{$minimal elements of $\mathcal{P} \setminus A_0\}$
- $A_k := \left\{ \text{minimal elements of } \mathcal{P} \setminus \left(\bigcup\limits_{j=0}^{k-1} A_j \right) \right\}$

We will later use the existence of this "preferred" foliation. The rôle of spacelike surfaces in our approach is played by antichains in $\mathcal{P}$ that belong to $\mathcal{F}$.

5. Reconstruction of Casual Set

Let us remember what we have from our measurements. We have a set of records that correspond to coarse grained trajectories, *i.e.*, in the causal set vocabulary, the set of possible chains $C_i \in \mathcal{C}$. In particular in each of these records, we have assumed that we have sub-records corresponding to the effective events. By considering the union of the sub-records of all the records we get the set $\mathcal{P}$ of all the possible events. To summarize, we have the set of events $\mathcal{P}$ and a cover of this set by (unordered) subsets C_i. In this section we will recover the order of the events in each chain, in order to reconstruct the effective causal set. While we do not know the order between any two points, we do know whether they are causally related (exists a chain that they both belong) or not. Using this we can define an antichain S_i the following way:

$$S_i \subset \mathcal{P} \mid \forall \quad p, q \in S_i \quad \nexists \quad C_i \in \mathcal{C} \mid p \quad and \quad q \in C_i \tag{6}$$

In order to recover the full partial order, we give the following algorithm:

Step 1.   Pick a *maximal* collection of points $S^0 \in \mathcal{P}$ such that no pair of points $p, q \in S^0$ belong to a chain C^i, *i.e.*, a maximal antichain in the set $\mathcal{P}$. This S^0 will be the set of minimal elements. Assign $i := 1$.

Step 2.   Consider the set $\mathcal{P}^i = \mathcal{P} \setminus S^{i-1}$.

Step 3.   Pick a maximal collection of points $S^i \in \mathcal{P}^i$ such that no pair of points $p, q \in S^i$ belongs to a chain C^i, *i.e.*, a maximal antichain in the set $\mathcal{P}^i$. This S^i will be the ith layer, if it exists, and is assigned $i := i + 1$. Then go to Step 2. If such S^i does not exist, the branch fails and one should restart from Step 1.

Step 4.   Check for non-appearance of extra chains. If it turns out that the foliation involved gives rise to a new chain, the branch is rejected. Then return to Step 1, restarting with a different maximal antichain. To see an example of the appearance of an "new chain", see below.

Step 5.   If the set $\mathcal{P}$ is exhausted and all the causal chains can be reproduced without emergence of a new one (see an example below), then the collection $\mathcal{S} = \{S^i\}$ forms the foliation of $\mathcal{P}$, the latter regarded as a causet proper.

This algorithm is guaranteed to reach to an end, since all past finite causets can be linearly foliated. What we have effectively done in the foregoing is the following. We picked a partition of $\mathcal{P}$ into antichains. To define the antichains we used the set of causal chains—histories. We then chose randomly an order on these antichains. After that, we checked that our construction did not produce any new histories (extra chains, in other words). If it did, we restarted the procedure.

The way to derive the chains from our construction (in order to see if any new chains appear) is the following. We pick a point in S^0 and see which points are causally connected with it in S^1. Then, we continue with the point we chose from S^1 and do the same with S^2. Note that had we chosen the correct foliation we would not have any new chains, due to the transitivity of causal relations.

An example of "new" chain.   Here we show how new chains, not existing in the initial set of chains $\mathcal{C}$, may emerge during the reconstruction procedure described above in step 4. Consider the poset $\mathcal{P}$
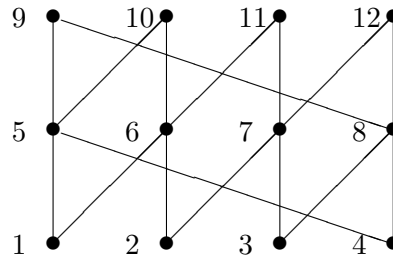
and try to restore the order starting from $\{5,6,7,8\}$ as the set of minimal elements. So, if 5 is a minimal element, then $J^+(5) = \mathcal{C}(5) = \{1,4,5,9,10\}$. Then take the element 1 (for which we deduced $5 < 1$). In the set $\mathcal{P} \setminus \{5,6,7,8\}$, consider $J^+(1) = \{1,9,10,11\}$, hence $1 < 11$. Thus, the chain $\{5,1,11\}$ must exist, but actually it does not(!), therefore we reject the initial supposition that the antichain $\{5,6,7,8\}$ is minimal.

Here we should stress that the above procedure does not give a unique result. For example two causal sets with reversed all their causal links would give the same result which naturally follows from our operational and "timeless"[2] assumptions. It can been shown [8] that all the ambiguities are essentially of this type.

6. Topology on a Causal Set

In order to speak about the topology of a spacelike surface of a causal set (*i.e.*, a maximal antichain) we need first to define neighborhoods on the surface that encode spatial proximity and then proceed to derive a topology on this finite (or countable) set that captures the topology of the underlying continuous one.

6.1. Proximity on an antichain

We could try to define some notion of distance on an antichain and then use this distance to get proximity.

We are given the set of points of the spacetime and their causal relations (causet). From this we can easily define a distance function for any two time-like separated events. For events p, q with $p \preceq q$

$$d_t(p,q) := \max |C_i \cap J^-(q) \cap J^+(p)|, \quad p, q \in C_i \tag{7}$$

where $|A|$ denotes the set-cardinality. In words this means that the timelike distance between two points is the maximum number of steps to go from the one to the other. This corresponds to the proper time between this two events. Things are different for points spacelike separated. The only way for an "inertial" observer in one point to know about its distance to another that is spacelike separated is by considering standard clocks and light beams. We would be therefore interested in the distance of a point from a geodesic corresponding to a chain following [11]. We consider a point p and a geodesic C_i such that w and z are points of C such that $w \preceq p \preceq z$. For point p, let $l(p)$ be the highest point in C which is below p, and $u(p)$ the lowest point of C_i that is above p. We then have

$$d_s(p, C_i) = d_t(l(p), u(p))/2 \tag{8}$$

Finally we would like to define the distance between points on the same antichain. We would like to consider the distance of point p from q by considering the cain that contains q and minimizes the distance $d_s(p, C_i)$ *i.e.*,

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[2] Here we refer to the frozen-time

$$d_s(p,q) := \min\left(d_s(p,C_i)\right) \mid q \in C_i \tag{9}$$

Note that this definition is positive definite and zero[3] when $p = q$. It is easily shown to be symmetric as well. Unfortunately, for a general causet, this definition fails to satisfy the triangular inequality and therefore it is not a real distance function. It can been shown though, that for a causet that can be embedded in a manifold, the triangular inequality is satisfied by this function on average. If one takes the point of view that causet is the fundamental reality, maybe the notion of distance of points on a spacelike surface need to be adjusted. On the other hand, the above definition, even if it fails to be a distance function, still captures some notion of proximity. This is exactly the ingredient we need in order to group the events into spatial neighborhoods and be able to derive the topology.

We would therefore form the neighborhoods in the following way. We pick sufficiently tuned (see below) length b. For each point p in the antichain S we make a neighborhood:

$$N_p = \{q \mid d_s(p,q) < b\} \tag{10}$$

We will define the set of all neighborhoods of the surface S_i as:

$$\mathcal{N} := \{N_p \quad \forall \quad p \in S_i\} \tag{11}$$

Major *et al.* in Ref.[13] followed a different method to get the neighborhoods, by considering a "thickened" antichain. In both approaches, we need to tune a parameter. This arises because our spacetime is discrete. Had we taken the finest description we would get neighborhoods with single points and that would lead to the discrete topology. On the other hand if we consider very large neighborhoods we will get the trivial topology (empty set and the total set). With the right tuning we avoid these two extremes.

6.2. Derivation of Topology

Sorkin in Ref.[12] dealt with the following question. If we have a continuous topological space, but we have access to only a particular finite open cover of this space, what can we say about the underlying continuous topology. The situation here is similar.

To review, we have an antichain S_i corresponding to the spacelike surface in question. Furthermore, we have a collection of subsets $\mathcal{N}$ of this surface corresponding to neighborhoods. If two points p,q belong to identically the same neighborhoods we will identify them as equivalent.

$$p \equiv q \quad if \quad p \in N_r \Leftrightarrow q \in N_r \quad \forall \quad N_r \in \mathcal{N} \tag{12}$$

We will consider the set of points in the spatial surface S_i modulo this equivalence[4]. To get the topology that captures the underlying continuous topology we do the following:

- We extend the set $\mathcal{N}$ to $\mathcal{N}^c$ to make it closed under intersection. This means that we will consider the set of all neighborhoods and their intersections. We will denote the elements of $\mathcal{N}^c$ as N_p as well.

[3] Provided that we used a "strict" poset, where the reflexive condition is replaced by irreflexive and thus $p \notin J^+(p)$.

[4] For notation simplicity we will use the same notation for the quotient space, but from now on we will be speaking of the latter.

- We construct a poset with elements the elements of $\mathcal{N}^c$ and order relation the set inclusion, *i.e.*, $N_p \preceq N_q$ if $N_p \subset N_q$.

- We take the Alexandrov topology of this poset. The Alexandrov topology of a poset is the topology where the open sets are future (with respect to the order relation) sets. A future set A of a partial order $\mathcal{P}$ is a set that contains its future, *i.e.*,

$$A \subset \mathcal{P} \mid J^+(A) \subset A \qquad (13)$$

- The topology that we get, captures the underlying continuous topology.

The first thing we should point out is what we do *not* do. We do not take the topology that is generated from considering intersections and unions of the neighborhoods N_p and take them as the open sets. The reason is that since we have a finite set, the above procedure would lead us to the discrete topology and thus it would not capture the continuous topology.

To compute topological invariants we could follow Major *et al.* in Ref.[13]. They considered the *nerve simplicial complex* in order to compute the homology. The nerve simplicial complex $\mathcal{NSC}(S_i)$ of S_i is constructed by mapping the elements of $\mathcal{N}$ to a vertex, every non-vanishing intersection $N_{p_0} \cap N_{p_1} \neq \emptyset$ to a 1-simplex, and in general every non-vanishing intersection $N_{p_0} \cap N_{p_1} \cdots N_{p_k} \neq \emptyset$ to a k-simplex.

An important thing to point out here, is that the nerve simplicial complex is not going to give in general a topology homeomorphic to the underlying continuous. We expect though that for suitably tuned neighborhoods (and sufficiently big antichain) it would essentially capture the main topological features, such as the homotopy and homology groups.

Consider the following example. Given an antichain $S_i = \{a, b, c, d\}$ and neighborhoods $\mathcal{N}$: $N_a = \{d, a, b\}$, $N_b = \{a, b, c\}$, $N_c = \{b, c, d\}$, $N_d = \{c, d, a\}$. This intuitively is a circle. The nerve simplicial complex, gives us a hollow tetrahedron which is homeomorphic to the two sphere S^2. It does not give a circle and moreover it gives different homotopy group (trivial, while $\pi_1(S^1) = \mathbf{Z}$) as well as homology groups.

We could say that one of the following two things went wrong. We had too large neighborhoods, or too small antichain. For the first, we could have considered smaller neighborhoods, that each of those contained only two points. The topology we would recover with the nerve simplicial complex in that case, would indeed be that of a circle. For the second we could have considered an antichain with five (rather than four) or even more points, keeping the size of the neighborhoods the same, *i.e.*, containing three consecutive points. We would then get a simplicial complex either homeomorphic to the Möbius strip (if the total number of points in the antichain is odd) or homeomorphic to the cylinder (for the even number case) and in both cases is *not* homeomorphic to the circle. Note though, that both the Möbius strip and the cylinder have indeed the same homotopy and homology groups with the circle since they are all homotopy equivalent. We could therefore claim that the outlined procedure (provided the tuning of the size of the neighborhood is correct) essentially captures the underlying topology, and in particular the homotopy and homology groups[5]. It is quite possible that if we consider the causal set as the most fundamental entity (or the only one that is operationally meaningful to speak of) we will not be able to make more precise statements about the continuous topology[6].

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[5] Possibly failing to identify the correct dimensionality. Note also, that the dimensionality is not a straight forward issue for a particular causet.

[6] In similar spirit with the realization that the concept of spatial distance is not in general well defined on a causal set.

7. Summary and Conclusions

We have derived the topology of the effective spacetime, based on a collection of final time records. The first part was to derive the causal order. This corresponds to the mathematical task of deducing the partial order of a set, given a collection of (unordered) subsets that will correspond to the (causal) chains when the order is recovered. This is of interest to the advocates of the "frozen-time" formalism, since the proposed scheme provides an example of how to recover the causal relations from a set of "frozen-pictures" at the final time, and thus have time "emerging" as a better way of organizing some collection of present records. The second part dealt with recovering spatial topology on a causal set. We first had to define some notion of proximity on an antichain and then using this to get a topology that captures the underlying continuous topology. The discussion in the second part is independent of the interpretation within the paper and has interest on its own right. Note though, that the discussion followed similar paths with Refs.[12, 13].

Acknowledgments

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