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Reasoning in Quantum Theory: Modus Ponens and the co-event interpretation

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Abstract.
Classical logic does not apply in quantum theory. However one would like to be able to reason according to some standard rules, for quantum systems as well (whether they are an electron or the universe itself). We examine what exactly is needed in order to be able to construct deductive arguments, and in particular to maintain the "Modus Ponens", which is the basic deductive rule of inference. It turns out that this requirement restricts the possible theories and in the context of the coevent interpretation, it results uniquely to the "multiplicative-scheme".

1. Introductory Remarks

Applying logic to (classical) physics, is something that is done so frequently and easily, that sometimes, one does not even notice it. Every time we wish to speak about the properties that a system has, we assign to those properties the truth value "True", and we expect the truth values we assign to behave in a Boolean way, and allow us to make deductions, using rules of inference such as the "Modus Ponens" (see below) and proofs by contradiction. In this contribution we will examine, what changes if one wishes to do the same in quantum theory, and what restrictions to a "quantum logic" we need to impose, in order to maintain the arguably vital ability to reason. In Section 2 we will see the general structure of applying logic to physics, in Section 3 we will apply this to classical physics and reason with Boolean logic. In Section 4 we see why this picture fails in quantum theory, and in Section 5 we deduce that requiring to have deductive reasoning requires, multiplicative maps which is central result of this contribution and was first proven by Fay Dowker and the author in [1]. In Section 6 we see what is implied by all this, in the co-event interpretation of quantum theory, which also serves as an extra motivation, for adapting this viewpoint. Finally we summarize and conclude in Section 7.

2. Logic and Physics - General Structure

There are three basic ingredients, when one wishes to use logic to assign truth values corresponding to physical questions.

(1) The first important structure, is the space of all finest grained descriptions Ω. In classical physics it is also the space of potential realities (as we will se below), however this picture does not carry over in quantum theory[1]. In probability theory, it is called sample space, while in histories formulation, is called *history space*. Each element of this space $h \in \Omega$, corresponds to a full description of the system, specifying every detail and property (for example, a fine grained history gives the exact position of the system along with the specification of any internal degree of freedom, for every moment of time). Along with Ω, we define the space $\mathcal{U}$, which consists of the collection of subsets $A \subseteq \Omega$ and the Boolean algebra associated with them (called events algebra), where the addition is defined as the symmetric difference between subsets (see Figure 1) $A+B := A \triangle B$ and the multiplication given by the intersection of subsets $A \cdot B := A \cap B$. Each subset $A \subseteq \Omega$ is called *event*, since all physical questions that one can ask, correspond to one of those subsets. If, for example, one wants to ask "was the system at the region Δ at time t?", it simply corresponds to the subset A defined as $\{A : h_i \in A \text{ iff } h_i(t) \in \Delta\}$, i.e. all histories that at time t are at the region Δ.

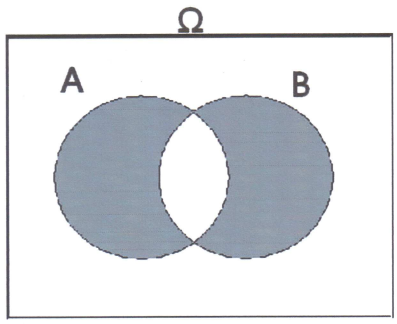

Figure 1. Symmetric Difference $A \triangle B$

(2) The second structure needed, is the space of truth values (for which we will use the notation $\mathcal{T}$), and the algebra associated with them. In the case of classical physics, the truth values are simply the two elements set $\mathcal{T} := \{\text{True, False}\}$ (or simply $\{1, 0\}$), and the Boolean algebra[2] associated (where $1 + 1 := 0, 0 + 0 := 0, 1 + 0 := 1, 0 \cdot 1 := 0, 1 \cdot 1 := 1, 0 \cdot 0 := 0$). As we will mention later, one could use as truth values a more general algebra, as for example a Heyting algebra.

(3) The third and final structure, is the space of possible *truth valuations maps* ($\phi : \mathcal{U} \to \mathcal{T}$). This corresponds, to an assignment of a truth value, to each of the possible questions (events). We shall use the notation $\mathcal{M}$ for this space.

However, the above is too general, to allow us to reason. In this description, there is no relation between the structure of $\mathcal{U}$ and that of the truth values assigned by the map ϕ. In particular, we would wish that the truth valuation map, respects the structure of the events algebra $\mathcal{U}$. For example, we wish to make arguments like: Statement 1: "All human, have two hands" , Statement 2: "Plato was human", therefore we deduce that "Deduction: Plato had

[1] We will see that reality can be thought of as a suitable co-event!
[2] called the truth values algebra

two hands". This follows from the fact that the event "all human, have two hands = A", is subset of the event "Plato having two hands = B", and if $\phi(A) = 1$ then to respect the events algebra, the valuation map ϕ needs to give truth value for the event B ($\phi(B) = 1$). In the subsequent section, we will formalize the above.

3. Classical Physics & Reasoning

To fully respect the events algebra, we restrict the allowed truth valuation maps ϕ to maps that are homomorphisms between the events algebra $\mathcal{U}$ and the truth values algebra $\mathcal{T}$ in other words we require the map ϕ to obey:

(a) Multiplicativity

$$\phi(A \cdot B) = \phi(A \cap B) = \phi(A)\phi(B) \tag{1}$$

(b) Additivity (Linearity)

$$\phi(A + B) = \phi(A \triangle B) = \phi(A) + \phi(B) \tag{2}$$

These requirements, are fulfilled in classical physics. An important observation, is that maps ϕ that are homomorphisms, are in a one-to-one correspondence with single elements h of the space of potential realities Ω. One defines maps that are the characteristic function for a particular history h, i.e.

$$\begin{aligned} \phi_h(A) &= 1 \text{ if } h \in A \\ \phi_h(A) &= 0 \text{ otherwise} \end{aligned} \tag{3}$$

One can see easily that such maps are indeed homomorphisms between the Boolean event algebra and the Boolean truth values algebra. Moreover, *all* homomorphisms are of this type one for each of the (single) elements h of Ω. In classical physics, what is realized (in other words what is real) is simply one single element h of the space Ω (e.g. a particular history of the system). Due to the one-to-one correspondence of homomorphic maps and single elements h, we could equally well, assume that reality is a homomorphic map ϕ_h between the event algebra $\mathcal{U}$ and the truth values algebra $\mathcal{T}$.

However, in order to reason, one needs rules of inference. We define $A \to B$ to be the *material implication* (or simply implication). It means "if event A then also event B". We notice, that if A is False, implication ($A \to B$) is trivially True. In the case that A is True, then the implication is also True, only when the reality lies at the part where $A \cap B$, as it can be seen at the Venn diagram (see Figure 2).

The most important rule of inference, that is needed in order to construct deductive proofs, is the "Modus Ponens" (MP) which is the short version of the Latin "Modus Ponendo Ponens" or in English "the way that affirms by affirming". Mathematically it is:

- The sentence (event), "if A then B" ($A \to B$) is True.
- The sentence (event) "A" is also True.
- Therefore the sentence (event) "B" is also True.

Its validity, in classical physics, is guaranteed by the fact that ϕ's allowed, are homomorphisms, however as we will see later, this is a sufficient but not a necessary requirement. MP is the basis for any *deductive* proof. For example: Event A "today is Monday", event B "I will go to

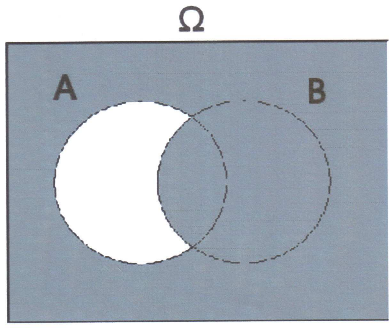

Figure 2. Material Implication $A \rightarrow B$

work". Event $A \rightarrow B$ ("if it is Monday I will go to work") is true. Event A is also true (it is Monday). Then we deduce, that I will go to work (event B going to work, is true).

Other than deductive proofs, one can construct proofs *by contradiction.* There are two types of such proofs, that both hold in classical physics (where allowed ϕ's are homomorphisms) and in that case, they are essentially the same:

(a) Event "A" is True. Its negation, event "$\neg A$", is False.

$$\phi(A) = 1 \Rightarrow \phi(\neg A) = 0 \tag{4}$$

(b) Event "A" is False. Its negation, event "$\neg A$" is True[3].

$$\phi(A) = 0 \Rightarrow \phi(\neg A) = 1 \tag{5}$$

While all this is obvious in classical physics, the above picture and consequently the straightforward way to reason, cannot be carried over to the case of quantum theory and quantum systems.

4. The trouble with Quantum Theory

Until this section we examined the potential realities (histories h or homomorphic maps ϕ), however nothing has been said about the *dynamics* or the *initial conditions* of the system. These are taken into account, by the existence of a measure (quantum measure for quantum systems[4]) μ defined on the space Ω, that gives the probability (or probability amplitude), that a particular event $A \subseteq \Omega$ occur. The least requirement that a map physical ϕ should satisfy is that any event with zero measure, gets the truth value "False", i.e.

$$\mu(A) = 0 \Rightarrow \phi(A) = 0 \tag{6}$$

and the map is called *preclusive.* This very simple requirement, along with requiring that ϕ is homomorphism, cannot be fulfilled in quantum theory. As a consequence of the Kochen

[3] This rule (the second type of contradiction), we shall refer to, as the NNR (No Name Rule) because it has no name and is important in the discussion we will do later.

[4] A quantum measure is a generalization of measure, where we weaken the Kolmogorov additivity condition. The reader is referred to the original paper [2].

and Specker theorem (original in [3] and in view of co-events and histories theories [4, 5]), we know that there exist quantum systems that the full history space Ω can be covered by (overlapping) zero quantum measure sets. A simple example of this arises if one considers a three-slits interference experiment. Consider a point at the screen, where crossing slit A destructively interferes with crossing through slit B and slit B destructively interferes with slit C, i.e. $\mu(\{h_1, h_2\}) = 0, \mu(\{h_2, h_3\}) = 0$ but $\mu(\{h_1, h_3\}) \neq 0$ (see Figure 3).

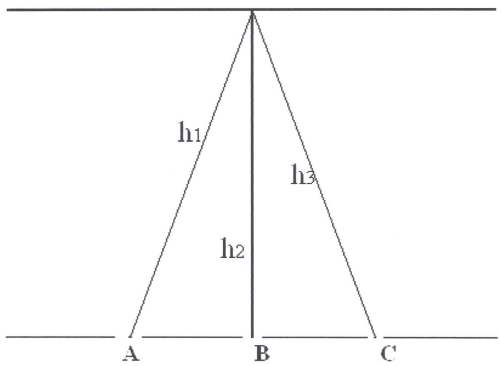

Figure 3. Three slits, h_1 is the history that passes from slit A and hits the central point on the screen, etc

If one requires a map ϕ that is homomorphic, then the above example leads to contradiction.

$$\phi(\{h_1, h_2\} \cap \{h_2, h_3\}) = \phi(\{h_1, h_2\}) \cdot \phi((\{h_2, h_3\}) \text{ implies } \phi(\{h_2\}) = 0 \tag{7}$$

$$\phi(\{h_1, h_2\} \triangle \{h_2, h_3\}) = \phi(\{h_1, h_2\}) + \phi((\{h_2, h_3\}) \text{ implies } \phi(\{h_1, h_3\}) = 0 \tag{8}$$

From Eqs 7 and 8 we get that $\phi(\{h_1, h_2, h_3\}) = 0$. However, since $\mu(\{h_1, h_2, h_3\}) \neq 0$ there are cases that the particle does hit the screen at the point we consider, while the above analysis would rule that out.

We therefore see that it is impossible to maintain the picture of classical physics and Boolean logic. There are three attempts to solve this problem, by changing one of the three structures analyzed in Section 2.

(i) Instead of considering the full set of possible questions the event algebra $\mathcal{U}$ (subsets of Ω), to somehow select a preferred set of classical questions, in other words a subalgebra of the event algebra (consistent/decoherent histories approach [6]). However, this approach (consistent histories) allows an arbitrary choice of which "Boolean subalgebra" to consider (see the critic in [7]) and in this sense, fails to give a satisfactory account of what actually occurs, unless some physical principle is discovered that selects a preferred classical domain.

(ii) Alter the space of truth values (and the algebra associated) $\mathcal{T}$. Instead of using a two values Boolean logic ({True, False}) one could use truth valuation maps that take truth values on a subobject classifier of a category, and the associated algebra is a Heyting algebras (see for example Isham, Butterfield and Doering in [8]). The resulting logic is intuitionistic logic which is deductive logic[5] but most importantly, contextual (the truth value, depends on the context/question asked).

(iii) Finally, one could alter the allowed maps ϕ from $\mathcal{U}$ to $\mathcal{T}$, while keeping both the event algebra and the truth values algebra, some as in classical physics. To do so we weaken the requirement that ϕ is homomorphism. In this line, is the co-event interpretation (see below). This cannot be done arbitrarily, but in a controlled way (keeping some of the structure) in order to maintain the ability to reason in the resulting logic. In the rest paper, we will see what restrictions are required and what they imply.

5. Modus Ponens and Multiplicativity

We therefore wish to weaken the requirement that the map is homomorphism, but in such a way that we are still able to reason. We therefore require that Modus Ponens holds and moreover, we require that the maps ϕ are unital, and from those two we derive restrictions. A map being unital means that $\phi(\Omega) = 1$ or in other words that the event "something happened" gets the answer "True".

It turns out that the following Theorem 1 holds, which will be proven in the rest part of this Section. This result is proven first in [1] by Fay Dowker and the author:

Theorem 1. *: If a valuation map ϕ, respects Modus Ponens, and it is unital, then necessarily it will be a multiplicative map, i.e. $\phi(A \cap B) = \phi(A)\phi(B)$.*

Theorem 2. *: (Filter property) Modus Ponens and unitality imply that "$\phi(A) = 1$ and $A \subseteq B \Rightarrow \phi(B) = 1$, i.e. in words, if event A is true, then any event containing A as subset, is also true.*

Proof. : The proof goes in four steps

(i) We know that: $A \to B = \neg(A \wedge \neg B)$

(ii) When $A \subseteq B$ from step (i) we get $A \to B = \Omega$.

(iii) Due to unitality, $\phi(\Omega) = 1$, and along with step (ii) we get $\phi(A \to B) = 1$

(iv) We can now apply MP (which by assumption holds) and using the fact that $\phi(A) = 1$ and the result of step (iii) deducts that that $\phi(B) = 1$.

□

Lemma 1. *: If $\phi(A) = 0$ for some event A, (and MP and unitality holds), then for* every *event $B \subseteq A$ it holds that $\phi(B) = 0$.*

Proof. If $\exists \quad B \subseteq A$ that $\phi(B) = 1$, then by Theorem 2 $\phi(A) = 1$, but by assumption $\phi(A) = 0$ and thus $\forall \quad B \subseteq A$, $\phi(B) = 0$. □

Theorem 3. *: If $\phi(A) = \phi(B) = 1$ and MP and unitality holds, then $\phi(A \cap B) = 1$.*

Proof. : The proof goes in three steps

(i) $B \subseteq (A \to B) = (A \to A \cap B)$

(ii) $\phi(B) = 1$ and thus by Theorem 2 and step (i): $\phi(A \to A \cap B) = 1$

(iii) $\phi(A) = 1$ and $\phi(A \to A \cap B) = 1$ so by MP (which by assumption holds) we get $\phi(A \cap B) = 1$

[5] Here again, as in the co-event interpretation, proofs by contradiction are not allowed, and only constructive/deductive proofs are acceptable.

□

From Lemma 1 and Theorem 3, Theorem 1 follows:

- If $\phi(A) = 0$ then by Lemma 1, $\phi(A \cap B) = 0$, and thus $\phi(A \cap B) = \phi(A)\phi(B)$. The same holds if $\phi(B) = 0$
- If $\phi(A) = \phi(B) = 1$ then by Theorem 3, $\phi(A \cap B) = 1$ and therefore in all cases $\phi(A \cap B) = \phi(A)\phi(B)$, which proves Theorem 1.

Theorem 1 is the key result, and we note that the additivity condition, Eq. (2), was not needed at all. Additivity is not necessary requirement for maps ϕ in order to have deductive proofs. However, to make proofs by contradiction something extra is needed, either additivity of the maps or the rule of Eq. (5). We do not expect to have proofs by contradiction in quantum theory. Even more interestingly, it seems that MP and multiplicativity are closely linked, since the (essentially) converse of Theorem 1 holds as well (see [1]):

Theorem 4. *: A valuation map ϕ that* is *multiplicative, obeys necessarily the Modus Ponens.*

Proof. We have $\phi(A)\phi(B) = \phi(A \cap B)$ and thus if any two of the three $\phi(A), \phi(B), \phi(A \cap B)$ are equal to 1, so is the third. To prove MP, let us assume that $\phi(A) = \phi(A \to B) = 1$. We get $\phi(A \cap (A \to B) \equiv A \cap B) = 1$. But now we have $\phi(A) = \phi(A \cap B) = 1$ and we are lead to the conclusion that $\phi(B)$ is also equal to 1, which completes the proof. □

6. The co-event Interpretation

It is an alternative interpretation of quantum theory, that was initiated and mainly developed by R. Sorkin [9, 10] , but also in [4, 5, 11] where one can find, other than developments of the interpretation, particular examples worked out. It is also called "Piombino Interpretation" because it was first presented in a talk by Sorkin at a D.I.C.E. meeting in Piombino 2006 (also known as anhomomorphic logic due to the anhomomorphic maps allowed). It is a histories theory and can be viewed as a development of the decoherent histories approach, though having many differences. The space Ω, is the space of possible histories h_i of the system, and a quantum measure μ is defined on it [2]. The quantum measure encodes the full dynamics and initial condition, and is related to standard quantum theory via the use of the decoherence functional. Potential realities are *co-events*. A co-event is defined to be a map ϕ from $\mathcal{U}$ to $\mathcal{T} = \{T, F\}$ obeying certain conditions. As analyzed earlier, maps cannot be homomorphic in general, if one wishes to describe quantum systems, and is unwilling to either restrict the possible questions or alter the space of truth values. Since, reasoning in the resulting theory is important, the requirement that the co-events are multiplicative maps, seems natural in light of the result of the previous section[6]. A multiplicative map (for finite Ω), corresponds to a characteristic function of some event $A \subseteq \Omega$ i.e.

$$\phi_A(B) = 1 \text{ iff } A \subseteq B \tag{9}$$

where A is called the support of the co-event ϕ_A. If A is a singleton (a single history event), then we are back to classical physics and the map is necessarily homomorphism. If it is not

[6] However, this was not the way it developed historically. Different relaxations of homomorphism, such as the additive-scheme was first considered. The multiplicative scheme, emerged as the main option, when other schemes contradicted some thought experiments with multiple-slits. Properties such as the unique classical domain and filter property, gave more arguments, while the present contribution, serves as one more very important reason to adopt the multiplicative scheme.

a singleton, then we are lead to some "paradoxes" stemming from anhomomorphisms. They appear when one asks a question B that intersects non-trivially both A and $\neg A$ and we get that both $\phi_A(B) = 0$ and $\phi_A(\neg B) = 0$. Due to this, it is apparent that we cannot use proofs by contradiction (at least no those that depend on Eq. (5))[7]. In principle, every event $A \in \mathcal{U}$ defines a multiplicative map, corresponding to Eq. (9). However, we impose further conditions for the physical co-events.

First, we require the possible maps ϕ to be preclusive, i.e. they have to obey Eq. (6). It is this point, that we (first) use the quantum measure, and thus dynamics enter the picture. Further, we would like to have co-events that are as close as possible (but not more) to classical co-events (singletons). The mathematical requirement for this, is that co-events are *primitive.* We say that a multiplicative co-event ψ dominates a multiplicative co-event ϕ if

$$\phi(A) = 1 \Rightarrow \psi(A) = 1 \quad \forall \quad A \in \mathcal{U} \tag{10}$$

A preclusive multiplicative co-event is *primitive* if it is not dominated by any other preclusive multiplicative co-event.

To sum up, possible realities, in the co-event interpretation, are maps ϕ from $\mathcal{U}$ to $\mathcal{T}$ called co-events, that are (i) multiplicative, (ii) preclusive and (iii) primitive. These are the conceivable realities. Remember, that in classical physics, where maps are homomorphisms, reality can be viewed as one single history h or in a dual manner, as the valuation (homomorphic) map ϕ_h. In the quantum case, we can still maintain this dual picture, where reality is either a multiplicative co-event ϕ_A (preclusive and primitive) or the event (coarse grained history) A. We now make few observations:

- In order to deal with probabilistic predictions, (where one assigns some chance of one of the co-events to be realized), we need to use once again the quantum measure μ and resort to the application of the " (weak) Cournot principle". The reader is referred to [11] for further details.
- An other very pleasing feature of the co-event interpretation, is that following the realist view point, that one co-event is actually realized, we can recover a unique, unambiguous, classical domain[8], which is given by the "Principle Classical Partition" (see Appendix A of [11]).
- Finally, we note that the resulting logic, which contains anhomomorphisms, depends on the set of preclusive sets, and thus on dynamics. It is justified, therefore, to say (in a bit provocative way) that we have a *dynamical logic.*

In the previous Sections of this contribution, we developed a full argument, why one would require for the valuation maps ϕ to be multiplicative if she/he wishes to be able to reason (Theorem 1). In this light, we see that choosing the physical co-events to be multiplicative maps, is not only well motivated, but we see that *any* other choice (different scheme), would not be suitable for applying deductive arguments (and use Modus Ponens).

7. Summary and Conclusions

We reviewed the use of logic in physics, which uses three structures: $\mathcal{U}$ the events algebra, $\mathcal{T}$ the truth values algebra and ϕ the maps between these algebras. In classical physics, the truth

[7] In this logic, we still have Eq.(4) holding and thus we can make this type of proofs by contradiction.

[8] c.f. in decoherent histories approach, the existence of many incompatible decoherent sets, introduced ambiguity in the interpretation of the approach and possibly, contextuallity.

valuation maps ϕ are homomorphisms between the algebras, and are into direct correspondence with fine grained histories h of Ω.

In quantum theory the above picture fails essentially due to interference, with more striking examples coming from three-slit example and Kochen Specker theorem. To deal with this, one has to change (at least) one of the above three structures described in Section 2. The present contribution, examined the case where one abandons homomorphisms as the only allowed maps.

In the main result of this contribution, we discovered the close link of Modus Ponens, and therefore of deductive logic, with the requirement that the maps are multiplicative (and unital). In particular, the additivity condition, was shown to be not necessary for having deductive logic (Theroem 1). This is an interesting result, on its own right, for logicians, and not directly tied to quantum theory.

Finally, we briefly introduced the co-event interpretation of quantum theory, and in the light of our previous result of Theorem 1 along with its converse, Theorem 4, justified why one selects as potential realities, maps/co-events that are multiplicative.

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