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On Pushouts of Partial Maps

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Abstract. The paper gives a sufficient condition for the existence of all pushouts in an arbitrary category of partial maps $\mathbb{C}_{*\mathcal{M}}$ that is necessary whenever the category of total maps $\mathbb{C} \subseteq \mathbb{C}_{*\mathcal{M}}$ has cocones of spans; the latter is the case in all slice categories of $\mathbb{C}$ and thus the condition is necessary locally. The main theorem is that, given an admissible class of monos $\mathcal{M}$ in a category $\mathbb{C}$ that has cocones of spans, the category of partial maps $\mathbb{C}_{*\mathcal{M}}$ has pushouts if and only if the category of total maps $\mathbb{C}$ has hereditary pushouts and right adjoints to inverse image functors (where both properties are w.r.t. $\mathcal{M}$). This result clarifies previous work by Kennaway on graph rewriting in categories of partial maps that implicitly assumed existence of cocones of spans in the category of total maps.

Introduction

The best-known approaches to algebraic graph transformation are single [1] and double [2] pushout rewriting (see also [3]). While the double pushout (DPO) approach has been studied extensively in a variety of categorical frameworks [4], all of which are variants of adhesive categories [5], the relation of single pushout (SPO) rewriting to adhesive categories has been much less extensively studied [6]. This is despite the fact that the work of Kennaway [7] has discussed the central concept of hereditary pushout, which is closely related to adhesive categories [8].

Kennaway's work [7] does not settle the question of what exactly is missing on top of hereditary pushouts to have all pushouts of partial maps. The answer to this question is the main contribution of this paper. Additionally, we identify the missing (implicit) assumption of cocones of spans in the statement of Theorem 3.2 of [7][3] which, taken literally, has a natural counterexample (see Example 3).

Motivated by the main theorem (stated in the abstract), we propose categories with hereditary pushouts and right adjoints to inverse image functors as the paradigmatic categorical framework for SPO rewriting, which is a class of categories that share key properties of the category of graphs (or of any quasi-topos) and provide enough structure to reason about the existence of pushouts of partial maps. This class includes most of the common categories of graph-like structures.

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[3] In the proof of Theorem 3.2(iii) in Ref. [7] on page 495, we can read "forming a commutative square in **C** with some arrows $N' \to D'$ and $O' \to D'$".

Finally, we describe how the encoding given in [9] of the Kappa Language [10] fits into this framework, making the connection to our motivating application for this work. We describe the categorical structure of the construction, which is reasonably easy (Lemma 4), mitigating the complexity on the concrete level of the category of so-called *pattern graphs*, which were introduced in [9] for the purpose of encoding. We reuse the category of pattern graphs as a key example, though, in principle, the usual category of hypergraphs (with term graphs as full subcategory [11, 12]) would be very similar; however, the concrete conditions for pushouts of partial maps of term graphs (with regular domains of definition) are non-trivial, and thus, the objective advantage of pattern graphs (with *coherent* pattern graphs as full subcategory) is the full treatment of all details in [9] – leaving the treatment of term graphs as future work.

Structure of the Paper We begin with a review of preliminary notions of category theory that are used to define categories of partial maps in Section 1, where we also define the category of pattern graphs [9], which we use as running example to illustrate the most important concepts. In Section 2, we develop and summarise results concerning pushouts of partial maps that will not only be essential to develop our main theorem but also serve to clarify our contribution. The main theorem itself is developed in Section 3; under a mild assumption, namely existence of cocones of spans, it gives a necessary and sufficient condition for the existence of pushouts of partial maps. Section 4 explains how the main theorem applies to the actual encoding of the Kappa language using pattern graphs, and we discuss further related work in Section 5 before we conclude.

1 Preliminaries

Assuming familiarity with basic concepts of category theory, we recall categories of partial maps based on admissible classes of monos [13]; we also define inverse and direct image functors. We shall reuse the authors' pattern graphs from Ref. [9] as running example category to illustrate the central concepts.

We use $\mathbb{C}, \mathbb{D}, \mathbb{X}$, etc. to range over categories, and $\mathbb{SET}$ is the category of sets and functions. We write $A \in \mathbb{C}$ if A is an object of the category $\mathbb{C}$ and $f \colon A \to B$ in $\mathbb{C}$ or $A -f\to B$ in $\mathbb{C}$ if f is a morphism in $\mathbb{C}$ with domain A and codomain B; finally, the identity on an object $A \in \mathbb{C}$ is denoted by id_A and $g \circ f$ is the composition of morphisms $f \colon A \to B$ and $g \colon B \to C$ in $\mathbb{C}$; we write $A' \rightarrowtail^m\to A$ if m is a mono. As usual, $\mathbb{C}(A, B)$ is the homset of morphisms with domain A and codomain B (assuming that $\mathbb{C}$ is locally small). We fix a category $\mathbb{C}$ to which all objects and morphisms belong, unless stated otherwise.
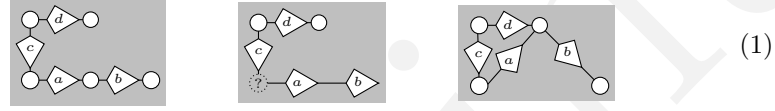
1.1 Pattern Graphs

A pattern graph is an edge labelled graph in which the targets of edges can be placeholders for nodes that satisfy a certain *specification*, represented by words over the set of edge labels; the idea is that a word $p = \lambda_1 \ldots \lambda_n$ of edge

labels λ_i $(i = 1, \ldots, n)$ stands for some node v that is at the start of a path $v = v_0, e_1, v_1 \ldots v_{n-1}, e_n, v_n$ where each edge e_i is labelled by λ_i $(i = 1, \ldots, n)$. Example 1 shows an example and the formal definition is as follows.

Definition 1 (Pattern Graph (PG)). *Let Λ be a fixed set of* labels*. We denote the set of prefix-closed languages over Λ by $\wp_{\le}(\Lambda^*) = \{\phi \subseteq \Lambda^* \mid pq \in \phi \text{ implies } p \in \phi\}$ where Λ^* is the monoid of words over Λ and $\varepsilon \in \Lambda^*$ is the empty word; elements of $\wp_{\le}(\Lambda^*)$ are* specifications.*

   A* pattern graph (PG) *is a pair $G = (V_G, E_G)$ where V_G is a set of* nodes *such that $V_G \cap \wp_{\le}(\Lambda^*) = \varnothing$ and $E_G \subseteq V_G \times \Lambda \times (V_G \cup \wp_{\le}(\Lambda^*))$ is a set of* edges. *A* basic graph *is a pattern graph (V_G, E_G) such that $E_G \subseteq V_G \times \Lambda \times V_G$.*

Example 1 (Pattern Graph). In the middle of (1),

(1)

we have illustrated a pattern graph with two nodes (drawn as white circles) and two edges (rendered as labelled kites) with labels c and d; the c-edge has the specification $\{\varepsilon, a, ab\}$ as target, which is drawn as a question mark with two consecutive kites with labels a and b. We can think of this pattern graph as a collection of basic graphs, including the ones shown on the left and the right.

Definition 2 (Semantics of Specifications). *Let $G = (V, E)$ be a pattern graph. A node $v \in V$ satisfies $p \in \Lambda^*$, written $v \models_G p$, if either p is the empty word ε or $p = \lambda p'$ (for some $\lambda \in \Lambda$ and $p' \in \Lambda^*$) and there exists $(v, \lambda, x) \in E$ such that either (i) $x \models_G p'$ and $x \in V$ or (ii) $p' \in x$ and $x \in \wp_{\le}(\Lambda^*)$. A node $v \in V$ satisfies $\phi \in \wp_{\le}(\Lambda^*)$, written $v \models_G \phi$, if $v \models_G p$ for all $p \in \phi$.*

   Pattern graphs congregate into a category where morphisms are functions between node sets that preserve the structure (w.r.t. suitable "instances" of specifications).

Definition 3 (Category of Pattern Graphs). *A* homomorphism *from a pattern graph G to a pattern graph H, denoted by $f\colon G \to H$, is a function $f\colon V_G \to V_H$ such that*

   (i) $(f(u), \lambda, f(v)) \in E_H$ holds whenever $(u, \lambda, v) \in E_G$ and $v \in V_G$; and
   (ii) for all edges $(u, \lambda, \psi) \in E_G$ with $\psi \in \wp_{\le}(\Lambda^*)$, there exists $x \in V_H \cup \wp_{\le}(\Lambda^*)$ such that $(f(u), \lambda, x) \in E_H$ and one of the following hold:
       (i) $x \in V_H$ and $x \models \psi$;
       (ii) $x \in \wp_{\le}(\Lambda^*)$ and $\psi \subseteq x$.

A homomorphism *$f\colon G \to H$ is an* inclusion *if $f(v) = v$ holds for all $v \in V_G$, in which case we write $G \subseteq H$ and call G a* subgraph *of H.*

   The category of pattern graphs*, denoted by $\mathbb{PG}$, has PGs as objects, homomorphisms as morphisms, the identity on a PG G is the function id_{V_G}, and composition of morphisms is function composition. Finally, $\mathbb{BG} \subseteq \mathbb{PG}$ is the full subcategory of basic graphs.*

1.2 Categories of Partial Maps

If $\mathbb{C}$ has pullbacks (along monos), we have an associated category of partial maps, which we denote by $\mathbb{C}_*$. It has the same objects as $\mathbb{C}$ and each homset $\mathbb{C}_*(A, B)$ contains partial maps, which are essentially pairs of a mono $A \leftarrow\!m\!\rightarrowtail A'$ in $\mathbb{C}$ and a morphism $A' -\!f\!\rightarrow B$ in $\mathbb{C}$ (quotiented up to isomorphism at A').

Definition 4 (Spans and Partial Maps). *A* span *is a diagram of the form $A \leftarrow\!m\!- X -\!f\!\rightarrow B$ in $\mathbb{C}$; such a span is a* partial map span *if m is a mono. A partial map from A to B, denoted by $(m, X, f)\colon A \rightharpoonup B$, (m, f), or $(\!(f, m)\!)$, is an isomorphism class of a partial map span, i.e.*

$$(m, X, f) = \left\{ A \xleftarrow{n} Y \xrightarrow{g} B \;\middle|\; \begin{array}{l} \textit{There exists an isomorphism } i\colon Y -\!\stackrel{\cong}{\rightarrow} X \\ \textit{such that } \begin{array}{c} m \swarrow X \searrow f \\ A \Leftarrow i\!\uparrow \Rightarrow B \\ n \nwarrow Y \nearrow g \end{array} \textit{ commutes.} \end{array} \right\}$$

for some representative partial map span $A \leftarrow\!m\!\rightarrowtail X -\!f\!\rightarrow B$. A partial map (m, f) is a total map *if m is an isomorphism.*

Partial maps in $\mathbb{SET}$ are essentially partial functions and a partial map from a PG G to a PG H corresponds to a pair of a subgraph $G' \subseteq G$ and a morphism from G' to H (where G' is the *domain of definition*); both correspondences amount to the standard choice of a representative span for each partial map.

Often one wants to restrict the class of monos that can be used in partial maps. For example, in [9], for the encoding of the Kappa language, it is crucial that the domains of definition in partial maps are *closed*.

Definition 5 (Closed Mono). *An inclusion $i\colon G -\!\subseteq\!\rightarrow H$ in $\mathbb{PG}$ is* closed *if $(v, \lambda, x) \in E_H$ and $v \in V_G$ imply $(v, \lambda, x) \in E_G$ (for all $v \in V_H$, $\lambda \in \Lambda$, and $x \in V_H \cup \wp_\leq(\Lambda^*)$); in this situation G is a* closed subgraph *of H. A mono $m\colon G' \rightarrowtail H$ is* closed *if it is isomorphic to a closed inclusion $i\colon G -\!\subseteq\!\rightarrow H$ (in $\mathbb{PG}/H$). The class of closed monos is denoted by $\mathcal{Cl}$.*

Thus, each node v in a closed subgraph $G \subseteq H$ has the same successors as v in H, where a successor of v is any node w for which $(v, \lambda, w) \in E_H$ holds for some $\lambda \in \Lambda$.

To obtain categories of partial maps where the left legs of all partial map spans belong to a certain class $\mathcal{M}$ (as detailed in Definition 7), one has to ensure that $\mathcal{M}$ is *admissible* [13].

Definition 6 (Admissible Classes of Monos). *Let $\mathcal{M}$ be a class of monos in $\mathbb{C}$, the elements of which are called $\mathcal{M}$-morphisms, and we write $A' \subset\!m\!\rightarrow A$ if $m \in \mathcal{M}$. The class $\mathcal{M}$ is* stable (under pullback) *if for each pair of morphisms $B -\!f\!\rightarrow A \leftarrow\!m\!\supset C$ with $m \in \mathcal{M}$ and each pullback $B \leftarrow\!m'\!- D -\!f'\!\rightarrow C$ of $B -\!f\!\rightarrow A \leftarrow\!m\!\supset C$, the mono m' belongs to $\mathcal{M}$.*

The class $\mathcal{M}$ of monos is admissible, *if*

 (i) the category $\mathbb{C}$ has pullbacks along $\mathcal{M}$-morphisms;
 (ii) the class $\mathcal{M}$ is stable under pullback;

(iii) *the class* $\mathcal{M}$ *contains all identities;*

(iv) *the class* $\mathcal{M}$ *is closed under composition: if* $(A \overset{m}{\hookrightarrow} B), (B \overset{n}{\hookrightarrow} C) \in \mathcal{M}$ *then* $(A \overset{n \circ m}{\hookrightarrow} C) \in \mathcal{M}$.

We now fix an admissible class $\mathcal{M}$ in $\mathbb{C}$. Examples of admissible classes (in any category) are regular monos and isomorphisms; open subspaces of topological spaces and downward closed subsets of partial orders induce more interesting examples, insofar as they are nontrivial proper subclasses of all monos, which we shall refer to as $\mathcal{M}$ono. Finally, closed monos are admissible.

Lemma 1 (Closed Monos are Admissible). *The class* $\mathcal{C}$l *is an admissible class of monos in* $\mathbb{PG}$.
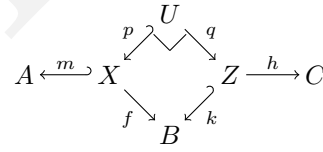
The definition of admissible classes of monos exactly captures the conditions of a well-defined category of $\mathcal{M}$-partial maps [13].

Definition 7 (Partial Map Categories). *The* category of $\mathcal{M}$-partial maps*, denoted by* $\mathbb{C}_{*\mathcal{M}}$*, has the same objects as the category* $\mathbb{C}$ *and the morphisms between two objects* $A, B \in \mathbb{C}_{*\mathcal{M}}$ *are the elements of*

$$\mathbb{C}_{*\mathcal{M}}(A, B) = \left\{ (m, X, f) \colon A \rightharpoonup B \mid A \overset{m}{\leftharpoonup} X \overset{f}{\rightarrow} B \ \& \ m \in \mathcal{M} \right\}$$

which contains all $\mathcal{M}$-partial maps *from A to B.*[4]

The identity on an object A is $(\mathrm{id}_A, A, \mathrm{id}_A)$*; given two* $\mathcal{M}$-partial maps $(m, X, f) \colon A \rightharpoonup B$ *and* $(k, Z, h) \colon B \rightharpoonup C$*, their composition is* $(k, Z, h) \circ (m, X, f) = (m \circ p, U, h \circ q)$ *where* $X \overset{p}{\leftharpoonup} U \overset{q}{\rightharpoonup} Z$ *is some arbitrary[5] pullback of* $X \overset{f}{\rightarrow} B \overset{k}{\leftharpoonup} Z$.

$$A \overset{m}{\leftharpoonup} X \quad \overset{\displaystyle U}{\overset{p \searrow \ \swarrow q}{\underset{f \searrow \ \swarrow k}{\underset{\displaystyle B}{}}}} \quad Z \overset{h}{\longrightarrow} C$$

The covariant embedding *of* $\mathbb{C}$*, denoted by* $\Gamma \colon \mathbb{C} \rightarrow \mathbb{C}_{*\mathcal{M}}$*, is the unique functor from* $\mathbb{C}$ *to* $\mathbb{C}_{*\mathcal{M}}$ *that maps each morphism* $f \colon A \rightarrow B$ *in* $\mathbb{C}$ *to the total map* $\Gamma f = (\mathrm{id}_A, f) \colon A \rightharpoonup B$ *in* $\mathbb{C}_{*\mathcal{M}}$ *(and thus satisfies* $\Gamma(A) = A$ *for all* $A \in \mathbb{C}$*).*

We shall call arrows in $\mathbb{C}$ *morphisms* and reserve *'map'* for arrows of $\mathbb{C}_{*\mathcal{M}}$.

We conclude this section with the definition of inverse image functions between meet-semilattices of $\mathcal{M}$-subobjects and a review of direct image functions. For this, recall that each $\mathcal{M}$-morphism $m \colon M \overset{m}{\hookrightarrow} A$ is a representative of the subobject $[m]$, i.e. its isomorphism class in the slice category $\mathbb{C}/A$. Note that a $\mathcal{M}$ono-subobject in $\mathbb{SET}$ is essentially a subset and closed subgraphs correspond to $\mathcal{C}$l-subobjects in $\mathbb{PG}$. We denote the poset of $\mathcal{M}$-subobjects over any object $A \in \mathbb{C}$ by $\mathrm{Sub}_\mathcal{M} A$; given $\mathcal{M}$-subobjects $[m], [n] \in \mathrm{Sub}_\mathcal{M} A$, the subobject $[m]$ is *included* in $[n]$, written $m \sqsubseteq n$, if there exists a morphism $i \colon m \rightarrow n$ in $\mathbb{C}/A$. For $A \in \mathbb{SET}$, $\mathrm{Sub}_{\mathcal{M}\text{ono}} A$ is isomorphic to the powerset $\wp(A)$ and the relation $\sqsubseteq$ is just the appropriate generalisation of inclusions of subsets. The meet $[m] \sqcap [n]$ is given by

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[4] This implies that the left leg of each representative partial map span is an $\mathcal{M}$-morphism.

[5] Arbitrary pullbacks suffice as they are unique up to isomorphism, thus avoiding unnecessary choices.

the diagonal of the pullback of m along n. Finally, inverse images are obtained by pulling back representatives of subobjects along morphisms, and for a partial map (n, f), its *domain of definition* is the subobject $[n]$.

Definition 8 (Inverse Images). *Let $f\colon A \to B$ in $\mathbb{C}$ be a morphism. The inverse image function $f^{-1}\colon \mathrm{Sub}_{\mathcal{M}}B \to \mathrm{Sub}_{\mathcal{M}}A$ maps each $[M \hookrightarrow^{m} B] \in \mathrm{Sub}_{\mathcal{M}}B$ to the subobject $f^{-1}([m])$ such that for all pullbacks $A \leftarrow^{m'} M' \to^{f_m} M$ of $A \to^{f} B \leftarrow^{m} M$ we have $f^{-1}([m]) = [m']$.*

For each $\mathcal{M}$-morphism $m\colon Y \hookrightarrow X$, *post-composition with m*, which maps $[y] \in \mathrm{Sub}_{\mathcal{M}}Y$ to $[m \circ y]$, is a monotone function; it is denoted by $\exists_m\colon \mathrm{Sub}_{\mathcal{M}}Y \to \mathrm{Sub}_{\mathcal{M}}X$ (as it is the lower adjoint to m^{-1}).

2 Pushouts of Partial Maps: the State of the Art

Pushouts of partial maps are at the heart of SPO rewriting [3], one of the standard approaches to graph rewriting; rules of rewriting can be arbitrary partial maps and applying rewriting rules amounts to taking pushouts of rules along a class of *matching morphisms*, which often are assumed to be total. Thus, the existence of pushouts of partial maps (along total maps) is pivotal. In this section, we therefore discuss results on the existence of certain pushouts in $\mathbb{C}_{*\mathcal{M}}$ and their corresponding diagrams in our fixed category $\mathbb{C}$ with its admissible class of monos $\mathcal{M}$. We have not found the following results formulated anywhere in the literature (despite closely related work [14, 15, 7]).

2.1 A Necessary Condition for Pushouts of Partial Maps

We begin with a discussion of the crucial role of right adjoints to inverse image functors, which appears to have been neglected in the literature; we use terminology from Galois connections as subobjects form posets to make clear that we are not discussing right adjoints to pullback functors.

Definition 9 (Upper Adjoints to Inverse Images). *Let $f\colon A \to B$ in $\mathbb{C}$, and let $f^{-1}\colon \mathrm{Sub}_{\mathcal{M}}B \to \mathrm{Sub}_{\mathcal{M}}A$ be its inverse image function. A $\sqsubseteq$-monotone function $\mathcal{U}\colon \mathrm{Sub}_{\mathcal{M}}A \to \mathrm{Sub}_{\mathcal{M}}B$ is an* upper adjoint *of f^{-1} if for all $n \in \mathrm{Sub}_{\mathcal{M}}B$ and all $m \in \mathrm{Sub}_{\mathcal{M}}A$, we have $f^{-1}(n) \sqsubseteq m$ if and only if $n \sqsubseteq \mathcal{U}(m)$; if an upper adjoint of f^{-1} exists, it is denoted by $\forall_f$ and we write $f^{-1} \dashv \forall_f$ or $\forall_f \vdash f^{-1}$.*[6]

An example of how the upper adjoint of a morphism in $\mathbb{PG}$ can act on subobjects is given in Example 2.

Proposition 1 (Necessity of Upper Adjoints to Inverse Images). *If $\mathbb{C}_{*\mathcal{M}}$ has all pushouts (along total maps), i.e. if for every morphism $f\colon A \to B$ in $\mathbb{C}$ and every map $\phi\colon A \to C$ in $\mathbb{C}_{*\mathcal{M}}$, there is a pushout of $C \leftarrow^{\phi} A \xrightarrow{(\mathrm{id}, f)} B$ in $\mathbb{C}_{*\mathcal{M}}$, then the upper adjoint $\forall_f \vdash f^{-1}$ exists for any morphism f in $\mathbb{C}$.*

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[6] Recall that upper adjoints are actually unique if they exist.

Thus, if we want $\mathbb{C}_{*\mathcal{M}}$ to have pushouts along total maps, we need upper adjoints of inverse image functions of $\mathcal{M}$-subobjects in $\mathbb{C}$. It is typically easy to check whether the latter exist; it suffices to show that for all morphisms $f\colon A \to B$ in $\mathbb{C}$ and every subobject $[m] \in \mathrm{Sub}_{\mathcal{M}}A$, the join $[m'] := \bigsqcup\{[n] \in \mathrm{Sub}_{\mathcal{M}}B \mid f^{-1}([n]) \sqsubseteq m\}$ exists and that setting $\forall_f([m]) := [m']$ yields $\forall_f \vdash f^{-1}$.

Lemma 2 (Upper Adjoint for Inverse Images of Closed Monos). *In $\mathbb{PG}$, for all $f\colon A \to B$, the upper adjoint $\forall_f\colon \mathrm{Sub}_{\mathrm{Cl}}A \to \mathrm{Sub}_{\mathrm{Cl}}B$ exists.*

Example 2 (Implicit Deletion in SPO Rewriting). In (2), we have a closed subgraph $m\colon K \hookrightarrow L$, a morphism $f\colon L \to G$, and $\forall_f([m])$ yields the result of applying the rule (m, id_K) at f using the SPO approach, i.e. the pushout of $G \leftharpoonup(f,\mathrm{id})- L -(m,\mathrm{id}_K)\rightharpoonup K$ in $\mathbb{PG}_{*\mathrm{Cl}}$ is $G -(m',\mathrm{id}_D)\rightharpoonup D \leftharpoonup(f',\mathrm{id})- K$ where $[m'] = \forall_f([m])$ and $f'\colon K \to D$ in $\mathbb{PG}$ is the unique morphism satisfying $f \circ m = m' \circ f'$ (cf. Proposition 1, Theorem 2). Roughly, to obtain $\forall_f([m]) \in \mathrm{Sub}_{\mathrm{Cl}}G$, we remove from G everything that is in L but not in K. Due to the choice of closed monos, removal of the node ③ forces the removal of the node ①, which would leave the "dangling edge" $\prec\!\langle d\rangle\!\succ$②, which is therefore also removed.

$$(2)$$
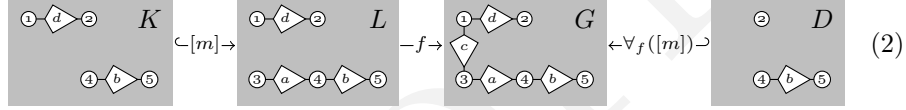
We now turn to our second condition for the existence of pushouts in $\mathbb{C}_{*\mathcal{M}}$, which is necessary if $\mathbb{C}$ is a slice category $\mathbb{C} = \mathbb{D}/T$.

2.2 A Locally Necessary Condition

One might expect that taking a pushout of a span of total maps in $\mathbb{C}_{*\mathcal{M}}$ yields a cospan of total maps; however, this is only true if spans in $\mathbb{C}$ have cocones, as implicitly assumed in [7]. This assumption implies that all pushouts in $\mathbb{C}$ are hereditary if $\mathbb{C}_{*\mathcal{M}}$ has pushouts.

Definition 10 (Hereditary Pushouts). *A pushout $B -b\to D \leftarrow c- C$ of a span $B \leftarrow f- A -g\to C$ in $\mathbb{C}$ is hereditary if $B -\Gamma b\to D \leftarrow \Gamma c- C$ is a pushout of the span $B \leftarrow \Gamma f- A -\Gamma g\to C$ in $\mathbb{C}_{*\mathcal{M}}$.*

Proposition 2 (Pushouts of Total Maps). *Suppose the category $\mathbb{C}$ has cocones of spans, i.e. for each span $C \leftarrow g- A -f\to B$, there exists a cospan $C -f'\to D \leftarrow g'- B$ such that $g' \circ f = f' \circ g$. If $\mathbb{C}_{*\mathcal{M}}$ has pushouts of partial maps (along total maps), then $\mathbb{C}$ has pushouts and the latter are hereditary.*

Proof. Let $C \leftarrow g- A -f\to B$ be a span in $\mathbb{C}$ with cocone $C -f'\to D \leftarrow g'- B$; moreover let $C -(m,M,h)\rightharpoonup E \leftharpoonup(k,N,n)- B$ be a pushout of $C \leftarrow \Gamma g- A -\Gamma f\to B$. By the universal property of the pushout in $\mathbb{C}_{*\mathcal{M}}$, there is a unique map $\phi\colon E \rightharpoonup D$ such that $\Gamma(f') = \phi \circ (m,h)$ and $\Gamma(g') = \phi \circ (n,k)$. The latter implies that $\mathrm{id}_C \sqsubseteq m$ and $\mathrm{id}_B \sqsubseteq n$ and thus both of n and m are isomorphisms. Now one can show that $C -h\circ m^{-1}\to E \leftarrow k\circ n^{-1}- B$ is a pushout of $C \leftarrow g- A -f\to B$ in $\mathbb{C}$ and that it is hereditary follows from $\Gamma(h \circ m^{-1}) = (m,h)$ and $\Gamma(k \circ n^{-1}) = (n,k)$. $\qquad\square$

Remark 1. As is well-known, pushouts are not hereditary, in general. The category of jungles [7] is one example; a very similar example occurs naturally for pattern graphs, namely $\Gamma\colon \mathbb{PG}\to\mathbb{PG}_{*\mathrm{Mono}}$ does not preserve pushouts. To see why, consider the span $①\!-\!\langle a\rangle\!-\!② \leftarrow\imath\!-\! ①\,② -\imath\to ①\!-\!\langle a\rangle\!-\!②$ where the morphism $\imath$ is the inclusion; this span has the pushout $①\!-\!\langle a\rangle\!-\!② -\mathrm{id}\to ①\!-\!\langle a\rangle\!-\!② \leftarrow\mathrm{id}\!-\! ①\!-\!\langle a\rangle\!-\!②$ in $\mathbb{PG}$. However, the embedding of this pushout into $\mathbb{PG}_{*\mathrm{Mono}}$ is not a pushout. To see this, note that the cospan $①\!-\!\langle a\rangle\!-\!② -(\imath,\imath)\to ①\!-\!\langle a\rangle\!-\!② \leftarrow\mathrm{id}\!-\! ①\!-\!\langle a\rangle\!-\!②$ is a cocone in $\mathbb{PG}_{*\mathrm{Mono}}$; moreover it is easy to show that there is no mediating morphism, making a case distinction on whether the edge is in the domain of definition or not. Thus, even if all pushouts exist, they need not be hereditary; the class of monos is crucial.

Thus, under mild assumptions on $\mathbb{C}$, having pushouts of partial maps (along total ones) implies that $\mathbb{C}$ has hereditary pushouts. The latter condition is often easy to check using the theorem that left adjoint functors preserve *all* colimits. Thus, to show that all pushouts (that exist) are hereditary, it suffices to establish a right adjoint to the covariant embedding $\Gamma\colon \mathbb{C}\to\mathbb{C}_{*\mathcal{M}}$.

Proposition 3 (Hereditary Pushouts of Pattern Graphs). *Pushouts of spans in the category* $\mathbb{PG}$ *are hereditary w.r.t.* $\Gamma\colon \mathbb{PG}\to\mathbb{PG}_{*\mathcal{Cl}}$.

Proof. Spelling out the definition of a right adjoint to Γ leads to the fact that it is enough to give, for each PG G, a closed inclusion $\tilde{g}\colon G\hookrightarrow G'$ such that for each partial map $(n,H',f)\colon H\rightharpoonup G$ there is a unique morphism $f'\colon H\to G'$ satisfying $[n] = f'^{-1}([\tilde{g}])$. In fact, taking $G' = \big(V_G\cup\{\bot\}, E_G\cup\{\bot\}\times\Lambda\times(V_G\cup\{\bot\}\cup \wp_\le(\Lambda^*))\big)$ we obtain the desired inclusion (cf. [8, Section 3.3]). □

Our main result will show that the discussed two conditions for the existence of pushouts of partial maps (which are necessary in the presence of cocones of spans) are in fact sufficient. To understand the main difficulty of this result, we discuss a peculiar fact about pushouts in $\mathbb{C}_{*\mathcal{M}}$ in terms of the underlying diagrams in $\mathbb{C}$.

2.3 Challenge for a Sufficient Condition

Our main theorem will establish that upper adjoints of inverse image functions and hereditary pushouts together are sufficient to obtain pushouts of all spans of partial maps. The crucial point in the proof is the construction of the domain of definition of the diagonal of a pushout candidate. The main difficulty is showing the existence of the join of subobjects illustrated in Figure 1 and spelled-out in the next proposition (following the proof idea of Theorem 3.2 of [7]).

Proposition 4 (Pushout Diagonal). *Assuming that* $\mathbb{C}$ *has cocones of spans, let* $C \leftharpoondown(g,N,n)- A -(m,M,f)\rightharpoondown B$ *be a span in* $\mathbb{C}_{*\mathcal{M}}$, *let* $C -(m',M',f')\rightharpoondown X \leftharpoondown(g',N',n')- B$ *be the pushout of the latter span in* $\mathbb{C}_{*\mathcal{M}}$, *and let* $(k,h) = (n',g')\circ(m,f)$ *be the* diagonal *of the resulting pushout square. Then* $[\boldsymbol{k}]\in\mathrm{Sub}_{\mathcal{M}}A$ *is the join of all those subobjects* $[x]\in\mathrm{Sub}_{\mathcal{M}}A$ *for which there are morphisms* $i'\colon x\to m$ *and* $j'\colon x\to n$ *in* $\mathbb{C}/A$ *that are representatives of inverse images of subobjects of* B *and* C, *respectively, i.e.* i' *and* j' *are subject to the additional condition that there exist* $[\tilde{n}]\in\mathrm{Sub}_{\mathcal{M}}B$ *and* $[\tilde{m}]\in\mathrm{Sub}_{\mathcal{M}}C$ *satisfying* $[i'] = f^{-1}([\tilde{n}])$ *and* $[j'] = g^{-1}([\tilde{m}])$ *(cf. Figure 1).*

In previous work, the existence of the join $[\boldsymbol{k}]$ in Figure 1 was either trivial [1] or assumed implicitly [7]; related assumptions are used for span-based rewriting, namely limits of small diagrams in [15] and the rather unwieldy final *triple diagrams* in [14]. In contrast, we shall *show* how existence of $[\boldsymbol{k}]$ follows from upper adjoints to inverse image functions and hereditary pushouts. Interestingly, this will involve the following characterisation of hereditary pushouts from [8] (see also Theorem B.4 of [16]).

Theorem 1 (Hereditary Pushout Characterisation [8]). *Let $\mathbb{C}$ be a category with pushouts and let $\mathcal{M}$ be an admissible class of monos in $\mathbb{C}$; let $B \leftarrow A \rightarrow C$ be a span with pushout $B \rightarrow D \leftarrow C$.*

The pushout is hereditary if and only if for every completion to a commutative cube as shown to the right, where the morphisms $B' \hookrightarrow B$ and $C' \hookrightarrow C$ are $\mathcal{M}$-morphisms and the back faces are pullback squares, the top face is a pushout if and only if the front faces are pullbacks and $d\colon D' \rightarrow D$ is an $\mathcal{M}$-morphism.
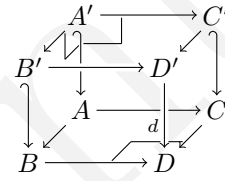
In the proof of our main theorem, we also shall use the following consequence from [8], generalising Lemma 2.3 of [5].

Lemma 3 (Pushouts along $\mathcal{M}$-morphisms [5, 8]). *Let $C \overset{m}{\leftarrowtail} A \overset{f}{\rightarrow} B$ be a span with $m \in \mathcal{M}$ and let $C \overset{g}{\rightarrow} D \overset{n}{\leftarrow} B$ be a pushout that is hereditary and assume $\mathbb{C}$ has pushouts. Then n is an $\mathcal{M}$-morphism, $[m] = f^{-1}[n]$, and $[n] = \forall_f([m])$.*

In particular, $\mathcal{M}$ is pushout stable and pushouts along $\mathcal{M}$ yield pullback squares.

3 Partial Map Pushouts by Inheritance

We now present our main contribution: a construction of pushouts of partial maps that uses only hereditary pushouts and upper adjoints of inverse image functions. Thus, the conditions from the previous section, which are necessary locally, turn out to be sufficient. As a direct consequence, our construction of partial map pushouts directly transfers to slice categories, which turns out to be surprisingly useful in practice [9].
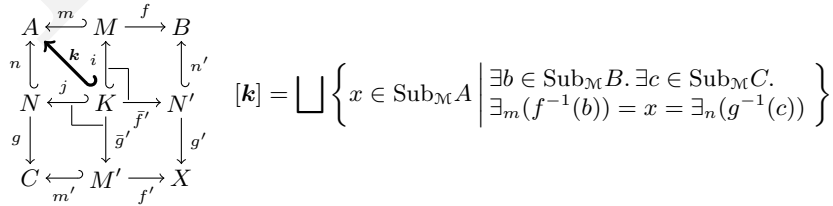
$$[\boldsymbol{k}] = \bigsqcup \left\{ x \in \operatorname{Sub}_{\mathcal{M}} A \;\middle|\; \begin{array}{l} \exists b \in \operatorname{Sub}_{\mathcal{M}} B.\, \exists c \in \operatorname{Sub}_{\mathcal{M}} C. \\ \exists_m(f^{-1}(b)) = x = \exists_n(g^{-1}(c)) \end{array} \right\}$$

Fig. 1. The domain of definition of the diagonal of a pushout of partial maps

Theorem 2 (Existence of Pushouts of Partial Maps). *Let $\mathbb{C}$ be a category with cocones of spans with an admissible class of monos $\mathcal{M}$. The partial map category $\mathbb{C}_{*\mathcal{M}}$ has pushouts if and only if $\mathbb{C}$ has hereditary pushouts and inverse image functions between $\mathcal{M}$-subobject posets have upper adjoints.*

Proof. The *only if*-part follows from Proposition 1 and Proposition 2.

For the converse, let $C \leftharpoondown\!\langle g, N, n\rangle\!- A -\!(m, M, f)\!\rightharpoonup B$ be a span in $\mathbb{C}_{*\mathcal{M}}$, assume that $\mathbb{C}$ has hereditary pushouts, and that for any morphism $h\colon Y \to Z$ in $\mathbb{C}$, the upper adjoint $\forall_h$ of the inverse image function $h^{-1}\colon \mathrm{Sub}_{\mathcal{M}}Z \to \mathrm{Sub}_{\mathcal{M}}Y$ exists. We shall first construct a suitable subobject $[k] \in \mathrm{Sub}_{\mathcal{M}}A$ (cf. Figure 1).

The Domain of Definition of the Diagonal Working in $\mathbb{C}$, we start by constructing the diagram on the left in (3).

$$
[k] = l^{-1}\bigl(\forall_l(m \sqcap n)\bigr)
\qquad
\begin{aligned}
&[q] = \forall_g([j]) \\
&[p] = \forall_f([i])
\end{aligned}
\tag{3}
$$

Thus, $C \hookrightarrow_{\bar{n}} G \leftharpoondown_{\bar{g}}- A$ is the pushout of $C \leftharpoondown_g N \hookrightarrow_n A$ and $A -_{\bar{f}} F \leftharpoondown_{\bar{m}} B$ is the pushout of $A \leftharpoondown_m M -_f B$; moreover, $G -_v W \leftharpoondown_u F$ is the pushout of $G \leftharpoondown_{\bar{g}} A -_{\bar{f}} F$ and $l = v \circ \bar{g} = u \circ \bar{f}$, which is the diagonal of the latter pushout in $\mathbb{C}$. Finally, we put $[k] := l^{-1}\bigl(\forall_l(m \sqcap n)\bigr)$.

Note that $[k] = l^{-1}\bigl(\forall_l(m \sqcap n)\bigr) \sqsubseteq (n \sqcap m)$. Hence, there are unique morphisms $j\colon k \to n$ and $i\colon k \to m$ (in $\mathbb{C}/A$), witnessing the respective inclusions $k \sqsubseteq n$ and $k \sqsubseteq m$ (which follow from $k \sqsubseteq (m \sqcap n) \sqsubseteq n$ and $k \sqsubseteq (m \sqcap n) \sqsubseteq m$).

The Construction of a Pushout Candidate Let $[q] = \forall_g([j])$ and $[p] = \forall_f([i])$. As illustrated in (3) on the right, we claim that there exist arrows $g'\colon (g \circ j) \to q$ in $\mathbb{C}/C$ and $f'\colon (f \circ i) \to p$ in $\mathbb{C}/B$, which then let us construct a pushout $Q -_{v'} X \leftharpoondown_{u'} P$ of $Q \leftharpoondown_{g'} K -_{f'} P$ (in $\mathbb{C}$) to obtain $C -\!(q, v')\!\rightharpoonup X \leftharpoondown\!\langle u', p \rangle\!- B$ as a pushout candidate, i.e. $\langle q, v'\rangle \circ \langle n, g\rangle = \langle p, u'\rangle \circ \langle m, f\rangle$ in $\mathbb{C}_{*\mathcal{M}}$. Thus, we first have to prove the following claim.

Claim. The equations $g^{-1} \circ \forall_g([j]) = [j]$ and $f^{-1} \circ \forall_f([i]) = [i]$ hold.

The relevant steps are two: first, we verify that $(\bar{g})^{-1}\bigl(\forall_{\bar{g}}([k])\bigr) = [k]$, and thus $\forall_{\bar{g}}([k]) \sqsubseteq [\bar{n}]$ (using Lemma 3); second, we show that $\forall_{\bar{g}}([k]) = [\bar{n} \circ q]$, whence the desired result follows.

Finally, one can verify the universal property of the pushout candidate. $\square$

Corollary 1. *If a category has cocones of spans of morphisms and pushouts of partial maps, the same is true for all of its slice categories.*

We give a name to categories that "inherit" partial map pushouts.

Definition 11 (Inherited Partial Map Pushouts). *A category* $\mathbb{C}$ *with an admissible class of monos* $\mathcal{M}$ *has* inherited $\mathcal{M}$-partial map pushouts *or is a* $\mathrm{M}\textsc{ipmap}$ category *if* $\mathbb{C}$ *has hereditary pushouts and upper adjoints to inverse image functions.*

Note that $\mathrm{M}\textsc{ipmap}$-categories are in particular vertical weak adhesive High Level Replacement Categories (cf. [4]) and partial map adhesive [8]. The category $\mathbb{PG}$ belongs to this class as does every (quasi-)topos (which directly follows from the definition of quasi-topos given in [17]).

4 On Pushouts in Full Subcategories

$\mathrm{M}\textsc{ipmap}$-categories share many properties with adhesive categories [5], are a development of recent generalisations [16, 8], and fit well with the theory of categorical frameworks for rewriting, surveyed in Ref. [4]. In particular, they allow the development of standard results of graph rewriting [3] that can be applied to a wide range of graph-like structures. However, some applications require restriction to a full subcategory of a $\mathrm{M}\textsc{ipmap}$-category: the case of *coherent* pattern graphs [9] is the motivation for the present section, but term graphs (being a full subcategory of hypergraphs [11, 12]) are another important example.

The approach taken in [9] to reason about existence of pushouts in a full subcategory $\mathbb{D} \subseteq \mathbb{C}_{*\mathcal{M}}$ of the partial map category of a $\mathrm{M}\textsc{ipmap}$-category $\mathbb{C}$ amounts to characterising the largest full subcategory $\mathbb{X} \subseteq \mathbb{C}_{*\mathcal{M}}$ that has $\mathbb{D}$ as reflective subcategory; then, all pushouts that exist in $\mathbb{D}$ can be *lifted* from $\mathbb{X}$ using the reflection. Finding a concrete description for the objects of $\mathbb{X}$ is usually non-trivial, and the full details for the case of *coherent* pattern graphs are quite involved (see [9]). We use a simplified example to illustrate the type of phenomena that have to be taken care of in the encoding of Kappa [9].

Example 3 (Branching-Free Graphs I). Let $\mathbb{B} \subseteq \mathbb{BG}$ be the full subcategory of all basic graphs that have at most one outgoing edge per node, i.e. in every graph $G \in \mathbb{B}$, any two edges (v, λ, u) and (v, λ', u') that share the same source node v are identical, i.e. $\lambda = \lambda'$ and $u = u'$. In this full subcategory $\mathbb{B} \subseteq \mathbb{BG}$, we have the following example of a span without cocone.

Note that if a cocone of this span would exist in $\mathbb{B}$, the image of node $\odot$ in the "tip" of the cocone would be the source of two different edges, namely one labelled a and one labelled b – a contradiction to branching-freeness.

In contrast, the embedding of this span into $\mathbb{B}_{*\mathcal{M}}$ has not only a cocone but we even have the pushout that is shown to the right. Note that both partial maps of the pushout cocone have $\{\odot\}$ as domain of definition and thus
are properly partial (cf. Proposition 2). To see that this square actually is a
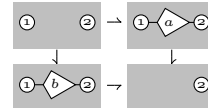
pushout square, we first observe that the maps of any cocone cannot have node ⊙ in the domain of definition as then both maps would also have the outgoing edge in the domain of definition, which in turn would imply that the "tip" of the cocone is not branching-free. The only remaining choice for a cocone is to either not contain node ⊚ in the domains of definition or that it is mapped to the same node by both morphisms. There is an obvious unique mediating morphism for both cases.

This example shows that pushouts in partial map categories are even more intricate if the category of total maps is not a MIPMAP-category. The concrete details of conditions for spans of partial maps that ensure the existence of a pushout can be rather complex; the motivating example is the situation of [9], but the same issues arise for term graphs [11, 12].[7] In general, we can show (non-constructively) that all pushouts that do exist in a full subcategory $\mathbb{D} \subseteq \mathbb{C}_{*M}$ can be *lifted* from a canonical subcategory $\mathbb{X} \subseteq \mathbb{C}_{*M}$.
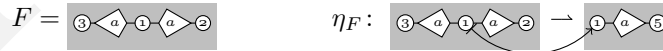
Lemma 4 (Pushout via Reflection). *Let $\mathbb{D} \subseteq \mathbb{E}$ be a full subcategory of an arbitrary category $\mathbb{E}$. There exists a greatest full subcategory $\mathbb{X} \subseteq \mathbb{E}$ such that $\mathbb{D} \subseteq \mathbb{X}$ is a reflective subcategory.*

Proof. Clearly, $\mathbb{D}$ is a reflective subcategory of itself. Moreover, a subcategory $\mathbb{X} \subseteq \mathbb{E}$ contains $\mathbb{D}$ as reflective subcategory if and only if for each object $X \in \mathbb{X}$ there exists a morphism $\eta_X \colon X \to \bar{X}$ in $\mathbb{E}$ with $\bar{X} \in \mathbb{D}$ such that for every other morphism $f \colon X \to D$ in $\mathbb{E}$ with $D \in \mathbb{D}$, there is a unique arrow $f^{\sharp} \colon \bar{X} \to D$ in $\mathbb{E}$ satisfying $f = f^{\sharp} \circ \eta_X$. Now, $\mathbb{X}$ is just the category that contains all objects $X \in \mathbb{X}$ for which there exist η_X as above, because these η_X define the unit of the reflection $\mathbb{D} \subseteq \mathbb{X}$. $\square$

This result allows to characterise when pushouts in $\mathbb{D}$ exist: a span $B \leftarrow f - A - g \rightarrow C$ in $\mathbb{D}$ has a pushout in $\mathbb{D}$ if, and only if, it has a pushout $B - g' \rightarrow X \leftarrow f' - C$ in $\mathbb{E}$ such that $X \in \mathbb{X}$. If such a pushout exists, then it can be lifted from $\mathbb{X}$ to $\mathbb{D}$, using the left adjoint $\mathcal{L}$ to the inclusion $\mathbb{D} \subseteq \mathbb{X}$, namely $B - \mathcal{L}(g') \rightarrow \mathcal{L}(X) \leftarrow \mathcal{L}(f') - C$ is the pushout of $B \leftarrow f - A - g \rightarrow C$; finally we have $\mathcal{L}(g') = \eta_X \circ g$ and $\mathcal{L}(f') = \eta_X \circ f$.

The category $\mathbb{X}$ of Lemma 4 can be non-trivial, i.e. $\mathbb{D} \neq \mathbb{X} \neq \mathbb{E}$, as in the example of branching-free graphs.
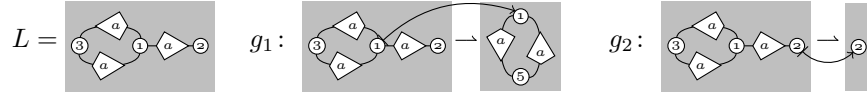
Example 4 (Branching-Free Graphs II). The greatest subcategory of $\mathbb{X} \subseteq \mathbb{BG}$ that contains the category of branching-free graphs $\mathbb{B}$ as reflective category is non-trivial. To see this, we first consider the *fork* graph F, below on the left.

$$F = \boxed{③ \triangleleft a \triangleleft ① \triangleright a \triangleright ②} \qquad \qquad \eta_F \colon \boxed{③ \triangleleft a \triangleleft ① \triangleright a \triangleright ②} \to \boxed{① \triangleright a \triangleright ⑤}$$

While F is clearly branching and $F \notin \mathbb{B}$, it is easy to verify that the map η_F above on the right is the universal way to make F branching-free, i.e. for any other $f \colon F \to F'$ such that F' is branching-free, there exists a unique $f^{\sharp} \colon \bar{F} \to F'$ such that $f = f^{\sharp} \circ \eta_F$.

In contrast, consider the situation in the *lollipop* L, below one the left.

---

[7] The illustration in Example 3 could equally well be seen as a pushout of partial maps of term graphs (requiring regular monos for the left legs of partial map spans).

$$L = \quad \boxed{\textcircled{3} \diamondsuit \overset{a}{\underset{a}{\diamondsuit}} \textcircled{1} \diamondsuit \textcircled{2}} \qquad g_1: \quad \boxed{\textcircled{3} \diamondsuit \overset{a}{\underset{a}{\diamondsuit}} \textcircled{1} \diamondsuit^a \textcircled{2}} \to \boxed{\diamondsuit^{\textcircled{1}}_{\textcircled{5}}} \qquad g_2: \quad \boxed{\textcircled{3} \diamondsuit \overset{a}{\underset{a}{\diamondsuit}} \textcircled{1} \diamondsuit^a \textcircled{2}} \to \boxed{\textcircled{2}}$$

There are essentially two ways to remedy the branching at node $\odot$: either $\odot$ is in the domain of definition, or not; the above partial maps g_1 and g_2 are examples for the respective cases. Now, suppose there was a universal arrow $\eta_L \colon L \to \bar{L}$ with $L \in \mathbb{B}$. If $\odot$ is in the domain of definition, then $\eta_L(\textcircled{2}) = \eta_L(\textcircled{3})$ by branching-freeness and closure of domains of definition; as a consequence, there does not exist any $g_2^\sharp$ such that $g_2 = g_2^\sharp \circ \eta_L$. Thus, the only possibility would be that $\odot$ is not in the domain of definition. However, in the latter case, there is no $g_1^\sharp$ such that $g_1 = g_1^\sharp \circ \eta_L$. In the end, we see that also $L \notin \mathbb{X}$, and thus $\mathbb{B} \neq \mathbb{X} \neq \mathbb{BG}$.

The encoding of the Kappa calculus into pattern graphs from Ref. [9] fits the situation of Lemma 4, using a full subcategory of a suitable slice category of pattern graphs (as discussed further in the next section). Similar situations arise for the category of term graphs (cf. Example 3 and Footnote 7).

5 Related and Future Work

The reference article for SPO rewriting using the algebraic approach is Ref. [1], which gives set-theoretic characterisations of pushouts; the idea of a categorical characterisation of pushouts of partial maps was first given in [7]. The present article gives a streamlined and rigorous account of (consequences of) results from [7], fixing minor omissions of the latter (see Footnote 3). Most importantly, our pushout construction in (3) does not involve any assumptions about existence of joins in subobject lattices (which again are assumed implicitly in [7]), and it only uses pushouts, pullbacks, and upper adjoints of inverse images in $\mathbb{C}$. This can be useful for applications as we can develop algorithms to construct pushouts in $\mathbb{C}_{*\mathcal{M}}$ using well-understood constructions in $\mathbb{C}$. Even in the case of algebras over a signature [1], our main results sheds new light on pushouts of partial maps.

The restriction to full subcategories in applications has an elegant theoretical solution (Lemma 4), even if the complexity of the details of the encoding of Kappa [9] as a full subcategory of $(\mathbb{PG}/T_\kappa)_{*\mathcal{M}}$ for a suitable type graph T_κ are daunting. Another example of a subcategory of an adhesive category has been used in [18] in combination with the double pushout approach (DPO) [2], which is a special case of SPO in the presence of hereditary pushouts (by Lemma 3).

For DPO rewriting, the literature contains a variety of categorical frameworks and here we comment only on those of the last decade that are surveyed in Ref. [4]. In proposing $\mathcal{M}$IPMAP-categories as a framework for SPO rewriting, we do not intend to replace any of these; $\mathcal{M}$IPMAP-categories are also not the most modest strengthening, as partial map adhesive categories with relatively pseudo-complemented subobject posets have already pushouts along monos in $\varGamma(\mathcal{M})$ (cf. [6]). $\mathcal{M}$IPMAP-categories are based on our main theorem, can be instantiated to many examples (including all quasi-topoi), and have additional properties

that are relevant for double pushout rewriting, e.g. the so-called Twisted-Triple-Pushout property (reusing the proof of Lemma 8.5 of [5]) without additional assumptions.

As future work, it remains to explore whether the results of the present paper can shed new light on term graph rewriting [11], making use of the categorical framework of MIPMAP-categories and complementing the study of term graphs as a (quasi-)adhesive category [12]. Moreover, guided by the idea that partial map adhesive categories are the natural weakening of adhesive categories when moving "down" from bi-categories of spans to categories of partial maps [8], it is natural to go "up" and study existence conditions for bi-pushouts of spans; a related goal is the characterisation of sesqui-pushout rewriting with monic matches [19] as a single bi-pushout, complementing existing work on span-based rewriting [15, 14].

Conclusion

The main result is a theorem of category theory that shows that upper adjoints of inverse images are necessary and sufficient for the existence of pushouts of partial maps, provided that spans have cocones. Based on this theorem, we propose MIPMAP-categories as a uniform framework for SPO and DPO rewriting. They are a natural strengthening of partial map adhesive categories [8], and even though there is scope for further generalisation, MIPMAP-categories are the first categorical framework that is relevant to both single and double pushout rewriting. A subtle point is the restriction to full subcategories. While it does not pose any theoretical problems (cf. Lemma 4), it adds an extra level of complexity to the pushout construction which can require substantial additional work in practice [9].

In summary, Theorem 2 justifies the categorical framework of MIPMAP-categories, distilling central ideas of [7]; moreover, Lemma 4 isolates the problems that one has to solve to characterise pushouts of (partial maps of) a full subcategory of a MIPMAP-category. The motivating example is the encoding of the rule-based modelling language Kappa of [9]; however, very similar problems arise in SPO rewriting of term graphs and jungle rewriting [7].

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A Basic Category Theory

Here, we give definitions of selected notions that are used in the paper and can be found in virtually any textbook on category theory.

Definition 12 (Full Subcategory). *Let $\mathbb{C}, \mathbb{D}$ be categories. The category $\mathbb{D}$ is a* subcategory *of $\mathbb{C}$ if each object $D \in \mathbb{D}$ is also an object of $\mathbb{C}$, i.e. $D \in \mathbb{C}$, and we have inclusions of homsets $\mathbb{D}(A, B) \subseteq \mathbb{C}(A, B)$ for all objects $A, B \in \mathbb{D}$; it is a* full subcategory *if moreover we have equalities of homsets $\mathbb{D}(A, B) = \mathbb{C}(A, B)$ for all objects $A, B \in \mathbb{D}$.*

Definition 13 (Slice Category). *Let $\mathbb{C}$ be a category and let $T \in \mathbb{C}$ be an object. The* category of objects over T *or the* slice category over T*, denoted by $\mathbb{C}/T$, has $\mathbb{C}$-morphisms with codomain T as objects, and for two objects $(A -^{t_A} \to T), (B -^{t_B} \to T) \in \mathbb{C}/T$, a morphism from t_A to t_B is an arrow $f \colon A \to B$ in $\mathbb{C}$ such that $t_B \circ f = t_A$. Identities and composition are taken from $\mathbb{C}$.*

Recall that a functor between categories $\mathcal{F} \colon \mathbb{C} \to \mathbb{D}$ maps each object $A \in \mathbb{C}$ to an object $\mathcal{F}(A) \in \mathbb{D}$ and preserves identities and composition, i.e. $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$ and $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

We shall use the following definition of an adjunction.

Definition 14 (Adjoint Functors). *Let $\mathcal{F} \colon \mathbb{C} \to \mathbb{D}$ and $\mathcal{G} \colon \mathbb{D} \to \mathbb{C}$ be functors. The functor $\mathcal{G}$ is* right adjoint *to $\mathcal{F}$ or $\mathcal{F}$ is* left adjoint *to $\mathcal{G}$ if for each $X \in \mathbb{C}$, there exists an arrow $\eta_X \colon X \to \mathcal{G} \circ \mathcal{F}(X)$ such that for each object $Y \in \mathbb{D}$ and each arrow $f \colon X \to \mathcal{G}(Y)$, there is a unique arrow $f^\sharp \colon \mathcal{F}(X) \to Y$ such that $f = \mathcal{G}(f^\sharp) \circ \eta_X$.*

Recall that left adjoints preserve all colimits in a category. As a special case of adjunctions, we have reflective subcategories.

Definition 15 (Reflective Subcategory). *Let $\mathbb{C}, \mathbb{D}$ be categories and let $\mathbb{D}$ be a subcategory of $\mathbb{C}$. It is* reflective *if the inclusion functor $\mathbb{D} \subseteq \mathbb{C}$ has a left adjoint.*

Another special case of adjuctions are adjoints between posets, as the latter can be seen as categories with at most one morphism in each homset. The definition in terms of posets is as follows.

Definition 16 (Upper and Lower Adjoints). *Let $(X, \sqsubseteq)$ and $(Y, \preceq)$ be posets; let $f \colon X \to Y$ be a monotone function, i.e. $x \sqsubseteq x'$ implies $f(x) \preceq f(x')$ for all $x, x' \in X$. A monotone function $g \colon Y \to X$ is an* upper *adjoint of f if for all $x \in X$ and $y \in Y$ we have $f(x) \preceq y$ if and only if $x \sqsubseteq g(y)$; it is a* lower *adjoint of f if for all $x \in X$ and $y \in Y$ we have $g(y) \sqsubseteq x$ if and only if $y \preceq f(x)$.*

Note that each monotone function has at most one upper or lower adjoint, which then is referred to as *the* respective adjoint.

B   On Necessary Conditions

We now show that upper adjoints of inverse image functions are a necessary condition for pushouts of partial maps (along total ones) and thus provide the proof for Proposition 1.

Let $f\colon A \to B$ be a morphism in $\mathbb{C}$, and we want to show that the upper adjoint to f^{-1} exists provided that $\mathbb{C}_{*\mathcal{M}}$ has pushouts (along total morphisms). It suffices to show that for each $\mathcal{M}$-morphism $m\colon A' \hookrightarrow A$, there is an $\mathcal{M}$-subobject $[n] \in \mathrm{Sub}_{\mathcal{M}} B$ such that for every $[p] \in \mathrm{Sub}_{\mathcal{M}} B$, $f^{-1}([p]) \sqsubseteq m$ if and only if $p \sqsubseteq n$, because putting $\forall_f([m]) := [n]$ yields the upper adjoint.

Thus, to show that a suitable $[n] \in \mathrm{Sub}_{\mathcal{M}} B$ exists, let $B -(n,i) \to D \leftarrow (g,j) - A'$ be the pushout of $B \leftarrow \Gamma f - A -(m,\mathrm{id}) \to A'$ in $\mathbb{C}_{*\mathcal{M}}$. Thus, in $\mathbb{C}$, we have a diagram as shown on the left in (4).

$$[n'] = f^{-1}([n]) \qquad\qquad [p'] = f^{-1}([p]) \quad (4)$$

Let $[p] \in \mathrm{Sub}_{\mathcal{M}} B$ be a subobject satisfying $p \sqsubseteq n$; we directly have the inclusion $f^{-1}([p]) \sqsubseteq f^{-1}([n]) = [n'] \sqsubseteq m$ using commutativity of the left hand diagram in (4) and that f^{-1} is monotone.

It remains to show that for each $[p] \in \mathrm{Sub}_{\mathcal{M}} B$ that satisfies $f^{-1}([p]) \sqsubseteq m$, we also have $p \sqsubseteq n$. Thus let $[p] \in \mathrm{Sub}_{\mathcal{M}} B$ be a subobject with $f^{-1}([p]) \sqsubseteq m$. There exist $h\colon Q \to P$ and $q\colon Q \to A'$ that yields the situation of the right diagram in (4), i.e. $A \leftarrow p' \rightarrow Q -h \to P$ is a pullback of $A -f \to B \leftarrow p \rightarrow P$ and also $p' = m \circ q$. This implies that $(p,\mathrm{id}) \circ \Gamma f = (q,h) \circ (m,\mathrm{id})$. Since $B -(n,i) \to D \leftarrow (g,j) - A'$ is a pushout, there exists a map $(k,r)\colon D \rightharpoonup P$ such that $(k,r) \circ (n,i) = (p,\mathrm{id})$. Hence, $p \sqsubseteq n$, and the proof is complete.

C Proving the Main Theorem

In this section, we complete the proof of Theorem 2. For this, we first give auxiliary results of inverse image functions and their upper adjoints that we shall use to finish the proof. Also recall the folklore Pullback Lemma.

Lemma 5 (Pullback Lemma). *Let the squares below on the left be commutative squares in an arbitrary category $\mathbb{C}$.*

$$
\begin{array}{ccc}
Q \xrightarrow{q} C \xrightarrow{\bar{n}} G \\
\end{array}
\Rightarrow
\left(
\begin{array}{ccc}
Q \xrightarrow{q} C \xrightarrow{\bar{n}} G & & Q \xrightarrow{q} C \xrightarrow{\bar{n}} G \\
\end{array}
\Leftrightarrow
\right)
$$

If $C \xleftarrow{g} N \xrightarrow{n} A$ is a pullback of $C \xhookrightarrow{\bar{n}} G \xleftarrow{\bar{g}} A$, then $Q \xleftarrow{g'} K \xrightarrow{j} N$ is a pullback of $Q \xrightarrow{q} C \xleftarrow{g} N$ if and only if $Q \xleftarrow{g'} K \xrightarrow{n \circ j} A$ is a pullback of $Q \xrightarrow{\bar{n} \circ q} G \xleftarrow{\bar{g}} A$.

C.1 Basic Properties of Upper Adjoints to Inverse Image Functions

Lemma 6 (Splitting Upper Adjoints). *Let $C \xhookrightarrow{\bar{n}} G \xleftarrow{\bar{g}} A$ be a cospan in $\mathbb{C}$ with pullback $C \xleftarrow{g} N \xhookrightarrow{n} A$ such that $\bar{n} \in \mathcal{M}$ and $[\bar{n}] = \forall_{\bar{g}}([n])$; moreover, let $q \colon Q \hookrightarrow C$ be an $\mathcal{M}$-morphism, let $Q \xleftarrow{g'} K \xhookrightarrow{j} N$ be the pullback of $Q \xhookrightarrow{q} C \xleftarrow{g} N$, and assume that $[\bar{n} \circ q] = \forall_{\bar{g}}([n \circ j])$.*

$$
\begin{array}{ccc}
Q \xhookrightarrow{q} C \xhookrightarrow{\bar{n}} G \\
\end{array}
\qquad [\bar{n}] = \forall_{\bar{g}}([n]) \quad [\bar{n} \circ q] = \forall_{\bar{g}}([n \circ j])
$$

Then $[q] = \forall_g[j]$.

Proof. Let $q' \colon Q' \hookrightarrow C$ be an $\mathcal{M}$-morphism. If $[q'] \sqsubseteq [q]$ holds then we derive $g^{-1}([q']) \sqsubseteq g^{-1}([q]) = [j]$, using that g^{-1} is monotone; conversely, assume that $g^{-1}([q']) \sqsubseteq [j]$. Using the Pullback Lemma, we obtain $\bar{g}^{-1}([\bar{n} \circ q']) \sqsubseteq [n \circ j]$, which in turn implies that $[\bar{n} \circ q'] \sqsubseteq \forall_{\bar{g}}([n \circ j]) = [\bar{n} \circ q]$ (where the latter equality is part of the assumptions). Finally, $[q'] \sqsubseteq [q]$ follows since $\bar{n}$ is a mono. $\square$

Lemma 7 (Composition of Inverse Image Functions). *For any pair of composable morphisms $f \colon A \to B$ and $g \colon B \to C$ we have*

$$
(g \circ f)^{-1} = f^{-1} \circ g^{-1}.
$$

Proof. This is a direct consequence of the Pullback Lemma (Lemma 5). $\square$

Lemma 8 (Counit of Upper Adjoints). *Let $h\colon Y \to Z$ be an arrow in $\mathbb{C}$ such that the upper adjoint $\forall_h \vdash h^{-1}$ exists; then, for all $y \in \mathrm{Sub}_{\mathcal{M}}Y$, we have*

$$h^{-1}\big(\forall_h(y)\big) \sqsubseteq y.$$

Proof. Given $y \in \mathrm{Sub}_{\mathcal{M}}Y$, we use the defining property of $\forall_h$ on the subobjects y and $\forall_h(y) \in \mathrm{Sub}_{\mathcal{M}}Z$, i.e. $h^{-1}\big(\forall_h(y)\big) \sqsubseteq y$ if and only if $\forall_h(y) \sqsubseteq \forall_h(y)$; now the desired follows, as the latter is trivially true. $\square$

C.2   Completing the Proof of the Main Theorem

We continue the proof of the Theorem 2. The proposed construction of a pushout candidate for a given span $C \leftharpoondown\!(g,N,n)\!- A -\!(m,M,f)\!\rightharpoondown B$ in $\mathbb{C}_{*\mathcal{M}}$ is shown in Figure 2. We first prove Claim 3, and we start with the equation $g^{-1} \circ \forall_g([j]) = [j]$. By
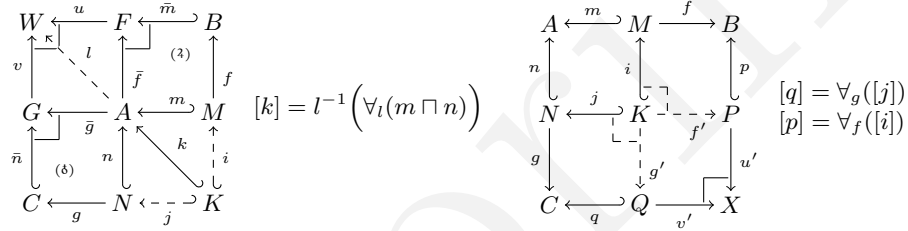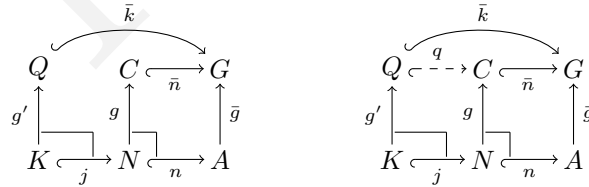

Fig. 2. Construction of pushouts of partial maps

definition, we have $[k] = l^{-1}\big(\forall_l([m] \sqcap [n])\big)$ and thus $[k] = (\bar{g})^{-1}\big(v^{-1}(\forall_l([m] \sqcap [n]))\big)$ (using Lemma 7). Now, let $\bar{k}\colon Q \hookrightarrow G$ be a representative $\mathcal{M}$-morphism of $[\bar{k}] = v^{-1}\big(\forall_l([m] \sqcap [n])\big)$; moreover, let $g'\colon K \to Q$ be the unique morphism that makes $Q \leftharpoondown g'\!- K \hookleftarrow k \rightharpoondown A$ a pullback of $Q \hookrightarrow \bar{k} \rightharpoondown G \leftharpoondown \bar{g}\!- A$, leading to the situation of the left one of the below diagrams.

Thus, as $(\bar{g})^{-1}([\bar{k}]) = [k] \sqsubseteq [n]$, we have $[\bar{k}] \sqsubseteq \forall_{\bar{g}}([n]) = [\bar{n}]$ (where the last equation follows from Lemma 3 and (δ) from Figure 2 being a pushout square); hence, there is a unique morphism $q\colon \bar{k} \to \bar{n}$ in $\mathbb{C}/G$. Now, we derive $[j] = g^{-1}([q])$ using the Pullback Lemma, and Lemma 6 implies $[q] = \forall_g([j])$; thus $[j] = g^{-1}([q]) = g^{-1} \circ \forall_g([j])$. This yields the first equation of Claim 3 and, *mutatis mutandis*, we derive $f^{-1} \circ \forall_f([i]) = [i]$. Thus, we have established

$$g^{-1} \circ \forall_g([j]) = [j] \text{ and } f^{-1} \circ \forall_f([i]) = [i], \tag{5}$$

and can in fact construct our candidate for a pushout as on the right in Figure 2, where $Q - v' \rightarrow X \leftarrow u' - P$ is the pushout of $Q \leftarrow g' - K - f' \rightarrow P$.

To establish the universal property of $C - (q,v') \rightarrow X \leftarrow (u',p) - B$, it will be appropriate to first characterise $[k]$ as a certain join (cf. Figure 1).

On the Domain of Definition of the Diagonal The crucial part of the proof is to show that $[k]$ can be characterised as the join $[k] = \bigsqcup \mathcal{A}$ where

$$\mathcal{A} = \left\{ x \in \mathrm{Sub}_{\mathcal{M}} A \left| \begin{array}{l} \exists c \in \mathrm{Sub}_{\mathcal{M}} C. \, \exists b \in \mathrm{Sub}_{\mathcal{M}} B. \\ \exists_m (f^{-1}(b)) = x = \exists_n (g^{-1}(c)) \end{array} \right. \right\}.$$

For this, we shall use that $G - v \rightarrow W \leftarrow u - F$ is a hereditary pushout of the span $G \leftarrow \bar{g} - A - \bar{f} \rightarrow F$ (as defined on the left in Figure 2).

It suffices to show that $[k]$ is a greatest element of $\mathcal{A}$. As $[k] \in \mathcal{A}$ follows from Equation (5) and $k = n \circ j = m \circ i$, it remains to show that it is an upper bound of $\mathcal{A}$. Thus let $[a \colon A' \hookrightarrow A] \in \mathcal{A}$; this means that there are $\mathcal{M}$-morphisms $b \colon B' \hookrightarrow B$, $c \colon C' \hookrightarrow C$ and pullbacks $M \leftarrow i' \rightarrow A' - f'' \rightarrow B'$, $N \leftarrow j' \rightarrow A' - g'' \rightarrow C'$ (of $M - f \rightarrow B \leftarrow b \rightarrow B'$ and $N - g \rightarrow C \leftarrow c \rightarrow C'$, respectively) such that $i' \colon a \rightarrow m$ and $j' \colon a \rightarrow n$ (as illustrated on the left in (6) where ($\mathfrak{A}$) and ($\mathfrak{O}$) are pullback squares).

$$\begin{array}{ccc}
\begin{array}{c}
C' \xleftarrow{g''} A' = A' = A' \xrightarrow{f''} B' \\
\scriptstyle c \downarrow \quad (\mathfrak{A}) \quad \downarrow j' \quad \downarrow a \quad i' \uparrow \quad (\mathfrak{O}) \quad \downarrow b \\
C \xleftarrow{g} N \hookrightarrow n \, A \hookleftarrow m \, M \xrightarrow{f} B
\end{array}
&
\begin{array}{c}
C' \xleftarrow{\bar{g}''} A' \xrightarrow{\bar{f}''} B' \\
\scriptstyle c \downarrow \quad (\mathfrak{A}) \qquad (\mathfrak{O}) \downarrow b \\
C \qquad\quad a \qquad B \\
\scriptstyle \bar{n} \downarrow \quad (\mathfrak{O}) \qquad (\mathfrak{A}) \uparrow \bar{m} \\
G \xleftarrow{\bar{g}} A \xrightarrow{\bar{f}} F
\end{array}
&
\begin{array}{c}
\quad A' \xrightarrow{f''} B' \\
g'' \quad\quad W' \quad \bar{m} \circ b \\
C' \downarrow \qquad A \quad w \quad F \\
\bar{n} \circ c \downarrow \qquad\qquad u \\
G \xrightarrow{v} W
\end{array}
\end{array}$$

$$(6)$$

Next, we paste pullback squares as illustrated in the middle of (6): by combining ($\mathfrak{A}$) with ($\mathfrak{O}$) and ($\mathfrak{O}$) with ($\mathfrak{A}$) (where ($\mathfrak{O}$) and ($\mathfrak{A}$) are taken from Figure 2), we obtain $C' \leftarrow g'' - A' \hookrightarrow a \rightarrow A$ and $A \leftarrow a \rightarrow A' - f'' \rightarrow B'$ as pullbacks of $C' \hookrightarrow \bar{n} \circ c \rightarrow G \leftarrow \bar{g} - A$ and $A - \bar{f} \rightarrow F \leftarrow \bar{m} \circ b \rightarrow B'$, respectively. Taking the pushout $C' - \bar{f}'' \rightarrow W' \leftarrow \bar{g}'' - B'$ of $C' \leftarrow g'' - A' - f'' \rightarrow B'$ yields a unique morphism $w \colon W' \rightarrow W$ such that $w \circ \bar{g}'' = u \circ \bar{m} \circ b$ and $w \circ \bar{f}'' = v \circ \bar{n} \circ c$ (as illustrated on the right in (6)). Using that pushouts are hereditary and Theorem 1, the spans $W' \leftarrow \bar{g}'' - B' \hookrightarrow \bar{m} \circ b \rightarrow F$ and $G \leftarrow \bar{n} \circ c \rightarrow C' - \bar{f}'' \rightarrow W'$ are pullbacks (of $W' - w \rightarrow W \leftarrow u - F$ and $G - v \rightarrow W \leftarrow w - W'$, respectively), and moreover w is an $\mathcal{M}$-morphism. This implies that $[a] = l^{-1}([w])$ as the "diagonal" of the right hand cube in (6) is a pullback square by the Pullback Lemma; clearly, $l^{-1}([w]) = [a] \sqsubseteq m \sqcap n$ and thus $[w] \sqsubseteq \forall_l (m \sqcap n)$, whence $[a] = l^{-1}([w]) \sqsubseteq l^{-1}\big(\forall_l (m \sqcap n)\big) = [k]$, using monotonicity of l^{-1}.

Existence and Uniqueness of Mediating Morphisms Existence and uniqueness of mediating morphisms are now a relatively easy consequence. Roughly, having the equation $[k] = \bigsqcup \mathcal{A}$, the mediating maps from our pushout candidate and their uniqueness are also "inherited" from the hereditary pushout $Q - v' \rightarrow X \leftarrow u' - P$ of the span $Q \leftarrow g' - K - f' \rightarrow P$ in Figure 2. The details are as follows.

Let $C -(r,d)\rightarrow X' \leftarrow(\!(e,s)\!)- B$ be a cospan with $(r,d) \circ (n,g) = (s,e) \circ (m,f) =:$ (k',h') in $\mathbb{C}_{*\mathcal{M}}$. Spelling out commutativity of the corresponding square in $\mathbb{C}_{*\mathcal{M}}$, we see that $\exists_n(g^{-1}([r])) = [k'] = \exists_m(f^{-1}([s]))$ and thus $[k'] \in \mathcal{A}$ and $[k'] \sqsubseteq [k] =$ $[n \circ j] = [m \circ i]$ (cf. Figure 2). Hence, we have $g^{-1}([r]) \sqsubseteq [j]$ and $f^{-1}[s] \sqsubseteq [i]$ as n and m are monos. Next, using that $[q] = \forall_g([j])$ and $[p] = \forall_f([i])$, we obtain $[r] \sqsubseteq [q]$ and $[s] \sqsubseteq [p]$ with respective inclusion morphisms $r' : r \rightarrow q$ and $s' : s \rightarrow p$.

$$
\begin{array}{ccccccccc}
C & \xleftarrow{q} & Q & \xrightarrow{v'} & X & \xleftarrow{u'} & P & \xrightarrow{p} & B \\
& \nwarrow{r} & \uparrow{r'} & & s' \uparrow & & \nearrow{s} & & \\
& & R & \xrightarrow{d} & X' & \xleftarrow{e} & S & &
\end{array}
\qquad
\begin{array}{ccc}
K & \xrightarrow{f'} & P \\
g' \downarrow & & \downarrow u' \\
Q & \xrightarrow{v'} & X
\end{array}
$$

In fact, $Q -(r',d)\rightarrow X' \leftarrow(\!(e,s')\!)- P$ is a cospan for $Q \leftarrow\Gamma(g')- K -\Gamma(f')\rightarrow P$ in $\mathbb{C}_{*\mathcal{M}}$ as $g'^{-1}([r']) = f'^{-1}([s'])$ (using that $[k'] \sqsubseteq [k]$ and the Pullback Lemma).

As $Q -v'\rightarrow X \leftarrow u'- P$ is a hereditary pushout of $Q \leftarrow g'- K -f'\rightarrow P$, there is a unique map $\phi : X \rightarrow X'$ such that

$$
(r',d) = \phi \circ \Gamma(v') \text{ and } (s',e) = \phi \circ \Gamma(u'),
$$

which implies $(r,d) = \phi \circ (q,v')$ and $(s,e) = \phi \circ (p,u')$, and thus ϕ is a mediating map. Any other map $\psi : X \rightarrow X'$ such that $(r,d) = \psi \circ (q,v')$ and $(s,e) = \psi \circ (p,u')$ satisfies

$$
(r',d) = \psi \circ \Gamma(v') \text{ and } (s',e) = \psi \circ \Gamma(u')
$$

(as the morphisms $r' : r \rightarrow q$ and $s' : s \rightarrow p$ are unique). Thus, using that $Q -\Gamma v'\rightarrow X$ and $X \leftarrow\Gamma u'- P$ are jointly epi (as they form a pushout), we derive $\psi = \phi$.