



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Theories as Categories

Citation for published version:

Fourman, MP & Vickers, S 1985, Theories as Categories. in Category Theory and Computer Programming, Tutorial and Workshop, Guildford, UK, September 16-20, 1985 Proceedings. vol. 240, Springer Berlin Heidelberg, pp. 434-448. DOI: 10.1007/3-540-17162-2_136

Digital Object Identifier (DOI):

[10.1007/3-540-17162-2_136](https://doi.org/10.1007/3-540-17162-2_136)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Peer reviewed version

Published In:

Category Theory and Computer Programming, Tutorial and Workshop, Guildford, UK, September 16-20, 1985 Proceedings

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.



Theories as Categories

Michael P. Fourman, Elec. Eng., Brunel University

michael@ee.brunel.ac.uk

and

Steven Vickers, Dept. of Computing, Imperial College.

This paper is not, and is not intended to be, original. Its purpose is to present a couple of examples from the folklore of topos theory, the theory of classifying topoi in particular. This theory and its applications developed initially without the benefit of widespread publication. Many ideas were spread among a relatively small group, largely by word of mouth. The result of this is that the literature does not provide an accessible introduction to the subject. Computer scientists studying the logic of computing have recently become interested in this area. They form our intended audience. In the space (and time) available we can only hope to provide a small selection of the many ideas missing from, or buried in, the literature. We attempt to give a perspective of the structure of the subject. Our viewpoint is, of necessity, idiosyncratic, and our treatment brief. We hope that missing technical details may be reconstructed from the literature. This may require some diligence.

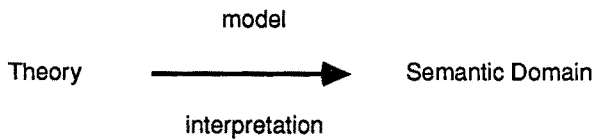
To apportion credit for the ideas presented here is difficult so long after the event. Lawvere and Joyal have a special position in this subject. Their intuitions have shaped it. Many others, who participated in the Peripatetic seminars in Europe, the New York Topos Theory Seminar (which also wandered) and the Category Theory meetings at Oberwolfach, contributed also. Their contributions are, in general, better reflected in their published works.

Finally, to apportion blame; this paper derives from notes taken by SJV of a talk by MPF. Any misrepresentations are the responsibility of the latter.

Introduction

Computer scientists have a far more flexible view of formalism and semantics than traditional logicians. What is regarded as a semantic domain at one moment may later be regarded as a formalism in need of semantics. A simple example of this phenomenon arises in the hierarchy of abstract machines which may be used to implement a high-level language. We aim, in this paper, to illustrate the use of category theory as a general setting for the study of theories and interpretations, which provides the kind of flexibility computer scientists are after. Unfortunately, the theory we present here does not seem to apply directly to computer science - for reasons we shall mention later. We hope it will provide an example, on which a theory of the logic of computation may later be modelled.

The theory we use as an example is variously known as topos theory, categorical logic or sheaves and logic. What this theory provides is a unified view of models, interpretations of a theory in a semantic domain,



and interpretations of one theory in another.

The unification is achieved by considering both sides of this arrow to be categories, the interpretation being a functor preserving some structure. With this basic conception: that categories may be viewed as theories, and certain functors between them as interpretations, many categories of category may be viewed from a logical perspective. We illustrate the presentation of theories as categories using examples of two extreme cases, a propositional theory and an algebraic theory.

For the benefit of those who have ventured into category theory already, we mention a few specific examples. Neophytes should first tackle the body of the paper, where explicit examples are presented in a more elementary way.

The category, \mathbf{Lim} , of (small) categories with finite limits, and functors which preserve these limits, is one example which embodies equational logic. Various equational theories are represented as categories with finite limits, relations between them are

expressed as limit preserving functors. One example we shall look at is the theory of Abelian groups, which is represented by the dual of the category of finitely presented Abelian groups.

The category, $\mathbf{Top}^{\mathbf{OP}}$, of (Grothendieck) topoi and the inverse image parts of geometric morphisms, is another which embodies geometric logic. Finally, $\mathbf{Loc}^{\mathbf{OP}}$, the category of locales with inverse image maps, embodies geometric propositional logic. We use these as (extremal) examples. Different categories correspond to different theories or semantic domains. Differing logics (or fragments of logic) correspond to different categories of category. In the current form of the theory, topoi and geometric morphisms play a special rôle. This is partly because they provide a natural extension of the usual set-based semantics: we can consider models "in a topos". Models in \mathbf{Set} , the category of sets and functions, provide classical semantics. Kripke models, Boolean-valued models, Heyting-valued models, permutation models and Beth models, can all be expressed as models in particular topoi.

Perhaps more importantly, other fragments of logic are related to geometric logic by adjunctions. For example, the obvious forgetful functor

$$\mathbf{Top}^{\mathbf{OP}} \longrightarrow \mathbf{Lim}$$

has a left adjoint: Given a (small) category \mathbf{C} with finite limits, the Yoneda embedding

$$\mathbf{C} \longrightarrow \mathbf{Set}^{\mathbf{C}^{\mathbf{OP}}}$$

provides us with a topos, $\mathbf{Set}^{\mathbf{C}^{\mathbf{OP}}}$, such that interpretations (limit preserving functors) of \mathbf{C} in a topos \mathbf{E} correspond naturally to interpretations of $\mathbf{Set}^{\mathbf{C}^{\mathbf{OP}}}$ in \mathbf{E} (geometric morphisms from \mathbf{E} to $\mathbf{S}^{\mathbf{C}^{\mathbf{OP}}}$). Similarly, the functor,

$$\mathbf{Top}^{\mathbf{OP}} \longrightarrow \mathbf{Loc}^{\mathbf{OP}}$$

which takes a topos to the locale of its subobjects of $\mathbf{1}$, has a left adjoint which associates to a locale Ω the topos, $\mathbf{Sh}(\Omega)$ of Ω -valued sets, or sheaves on Ω [Fourman & Scott, 79].

In general, the topos which represents a theory in this way is called the *classifying topos* of the theory. It is the topos *freely* generated by the theory, in the sense that the passage

$$\text{theory category} \quad \Big| \longrightarrow \text{classifying topos}$$

is left adjoint to the forgetful functor which treats a topos as a theory category. (This is what we mean here by *free*.)

Models for the theory, i.e. functors of a particular sort from the theory to a semantic domain, correspond to geometric morphisms from the semantic domain to the classifying topos; morphisms between theories correspond to geometric morphisms (in the other direction) between the classifying topoi. In particular, if we view the classifying topos as a semantic domain in its own right, the identity (geometric) morphism on the classifying topos corresponds to a model of the theory, its *generic model*.

The reader will probably already be aggrieved by the use of arrows in two conflicting directions: interpretations and models go one way, geometric morphisms the other. Historically this goes back to Grothendieck's dictum: *a topos is a generalised space*. This slogan should be taken literally, but not naïvely: it means that we can apply intuitions from the category of topological spaces and continuous maps to the category **Top**, and hence to logic. It does not mean that any space **is** a topos, in the naïve sense of "is". Formally, what we have is an adjunction relating **Top** to the category, **Esp**, of topological spaces. This provides a formal basis for the two, logical and geometric, views of topoi. Two views are better than one, and one learns to live with the arrows.

Propositional theories

The propositional logic we consider has finite conjunctions \wedge , arbitrary disjunctions \vee , true T and false \perp . We axiomatise a notion of entailment relation $S \vdash \phi$, (S entails ϕ) where ϕ is a proposition and S is a finite set of propositions, The logical laws governing these are expressed by the following proof rules:

$$\begin{array}{l}
 \text{reflexivity -} \\
 \text{monotonicity -} \\
 \text{transitivity -} \\
 \wedge - \\
 \vee - \\
 \\
 \text{ }
 \end{array}
 \begin{array}{c}
 \mathbf{s \vdash \phi} \\
 \frac{\mathbf{s \vdash \phi}}{\mathbf{s, \psi \vdash \phi}} \\
 \frac{\mathbf{s \vdash \phi} \quad \mathbf{s, \phi \vdash \psi}}{\mathbf{s \vdash \psi}} \\
 \frac{\mathbf{s \vdash \phi} \quad \mathbf{s \vdash \psi}}{\mathbf{s \vdash \phi \wedge \psi}} \\
 \frac{\mathbf{s, \psi_i \vdash \phi} \quad \mathbf{i \in I}}{\mathbf{s, \vee \psi_i \vdash \phi}} \\
 \\
 \mathbf{s \vdash \top} \quad \quad \mathbf{\perp \vdash \phi}
 \end{array}$$

Recall that on the left-hand side we have finite sets of formulae, so the occurrence there of a single formula, ϕ , should be interpreted as $\{\phi\}$, and the commas signify set union. Double lines signify rules which may be invoked in either direction.

This structure could also be presented algebraically by axiomatising the relation, $\phi \vdash \psi$, which is a preorder. Modulo equivalence (mutual entailment) the formulae form a distributive lattice, the locale of models of the theory. Locales are the Lindenbaum algebras of our propositional theories.

We elaborate a little on this viewpoint as it exemplifies the general view of topoi as both theories and spaces. A locale is a complete lattice with finite infs distributive over arbitrary sups. The prime example is the lattice, $\mathbf{O(X)}$, of open sets of a topological space, \mathbf{X} . For sober spaces, continuous maps $\mathbf{X} \rightarrow \mathbf{Y}$ correspond to inverse image maps $\mathbf{O(Y)} \rightarrow \mathbf{O(X)}$ which preserve finite infs and arbitrary sups (this may be taken as a definition of "sober"). Note the directions of the arrows! From the logical point of view these inverse image maps preserve the logical connectives of geometric propositional logic and correspond to interpretations of one theory in another. From the geometric point of view, the truth-value lattice is the locale of opens of the one-point space.

Models of a theory correspond to geometric morphisms from the one-point space to the corresponding locale. Thus points of the locale correspond to models of the theory.

We give an example to show how such propositional theories can be non-trivial:

Example - We take as elementary propositions, all symbols $P_{a,b}$ where a and b are rational numbers with $a < b$. We choose a real number, r , and associate to it a truth-valuation on these propositions:

$$\begin{aligned} \llbracket P_{a,b} \rrbracket = & \quad \{ \text{true if } a < r < b \\ & \quad \{ \text{false otherwise} \end{aligned}$$

This is a classical formulation, constructively the lattice of truth values should be viewed as the power set of the one-point space, $\{*\}$. We then write

$$\llbracket P_{a,b} \rrbracket = \quad \{ * \mid a < r < b \}.$$

This initially confusing notation indicates that we map to the top element, $T = \{*\}$ just in case r lies in the open interval (a,b) . (This is just a special case of the familiar set-theoretic notation $\{ x \mid P(x) \}$.) The lattice of truth values is also known as the *Sierpinski locale*.

We now write down some sequents which are valid under this interpretation. To keep the valuation uppermost in our mind, we write the truth condition $r \in (a,b)$ in place of the corresponding proposition, $P_{a,b}$. The validity of the sequent can then be observed directly, without a mental translation from the proposition to its truth condition.

$$\begin{aligned} r \in (a, b) \vdash r \in (a', b') & \quad \text{if } a' \leq a < b \leq b' \\ r \in (a, b) \wedge r \in (c, d) \vdash r \in (m, n) & \quad \text{if } m = \max(a, c), n = \min(b, d) \\ r \in (a, b) \wedge r \in (c, d) \vdash \perp & \quad \text{if } b \leq c \\ T \vdash \bigvee \{ r \in (a, b) \mid a < b \in \mathbf{Q} \} \\ r \in (a, b) \vdash r \in (a, d) \vee r \in (c, b) & \quad \text{if } a < c < d < b \\ r \in (a, b) \vdash \bigvee \{ r \in (a', b') \mid a < a' < b' < b \} \end{aligned}$$

Furthermore, every truth valuation making these sequents valid arises, in this way, from

a unique Dedekind real, r (exercise). Note, now, that in the Lindenbaum algebra of this theory, the basic propositions form a basis (the conjunction of two basic propositions is either a basic proposition or false). If we assume the completeness theorem, we can deduce that the Lindenbaum algebra is the lattice of open sets of the reals. Viewed as a locale, it is just the space of real numbers. The points of this space, the models of the theory, are just real numbers. However, we also see that the topology of the reals is intrinsic in the axiomatisation we have given.

To give another view of this example, consider the poset of finite non-trivial rational open intervals, ordered by inclusion ($I \leq J$ if I is contained in J). A truth valuation is a map from this poset to the Sierpinski locale. Now we characterise algebraically the truth valuations satisfying our axioms. The first axiom states that this truth valuation should be order-preserving:

- if $I \leq J$ then $f I \leq f J$

Order-preserving valuations satisfying the next three axioms may be characterised by the further requirements that:

- if $f I = T$ and $f J = T$ then for some K $K \leq I$, $K \leq J$ and $f K = T$
- for some I , $f I = T$

We say a map with these properties is *flat* (see below). (Note that, in particular that f preserves those finite meets which exist.)

To characterise the maps arising from reals among all flat maps, we have to introduce the idea of *covering* - a family of overlapping intervals covers its union. More precisely, a family $F = \{(a_i, b_i)\}$ of rational open intervals covers (c, d) iff in the real line we have $(c, d) = \bigcup_i (a_i, b_i)$. We can then characterise the maps we want as those which preserve covers, in the sense that if F covers K and $[[K]] = T$ then for some $J \in F$, we have $[[J]] = T$. Of course, we'd rather not talk about the real line until we've defined it, so we want more elementary ways of defining covers.

Firstly, we introduce an abstract notion of covering. A relation F covers K , between intervals, K , and families, F , of subintervals of K is a *Grothendieck topology* iff:

$\{K\}$ covers K

If F covers K and J is a subinterval of K , then $F \downarrow J = \{I \cap J \mid I \in F\}$ covers J .

If F covers K , and G is such that $G \downarrow I$ covers I , for each $I \in F$ then G covers K .

If F is a family of subintervals of K and $\{ I \mid I \subseteq J \text{ for some } J \in F \}$ covers K then so does F .

Clearly, the notion of covering by overlapping intervals introduced above gives an example of a Grothendieck topology. It may be characterised as the least Grothendieck topology (fewest covers) on the poset of rational open intervals such that:

$\{ (a, d), (c, b) \}$ covers (a, b) whenever $a \leq c < d \leq b$

$\{ (a', b') \mid a < a' < b' < b \}$ covers (a, b) .

These conditions correspond to our last two axioms. (The axioms for a Grothendieck topology find their logical reflection in the choice of the underlying logic.)

The reals are now the flat, cover-preserving maps.

(Of course, there are many other possible Grothendieck topologies, for example, if we omit the last condition, we get the notion of finite cover.)

We can recover the opens of the locale from this presentation as follows. A *crible* is a downwards closed family of rational open intervals, $(a, b) \in K$ and $a \leq a' < b' \leq b$ implies $(a', b') \in K$. A crible K is *closed* for a given topology iff whenever $K \downarrow I$ covers I , then $I \in K$. The closed cribles for the topology we have given correspond to the opens of the real line.

Note that if we want only to consider covering cribles, we can omit the final clause from our definition of Grothendieck topology. It says only that a family covers providing the crible it generates does.

To summarise briefly, we have presented a variety of views of a particular propositional theory. The logical, or syntactic, view gives rise to a Lindenbaum algebra which may be viewed geometrically as a locale. Models of the theory are points of this locale. The syntactic view has a more algebraic, but still recognisable presentation as a Grothendieck topology on a poset.

As we remarked earlier, we can reflect this discussion in the category **Top** by taking sheaves on the locale. It is also possible to construct the category of sheaves directly from the poset equipped with its Grothendieck topology.

Algebraic Theories

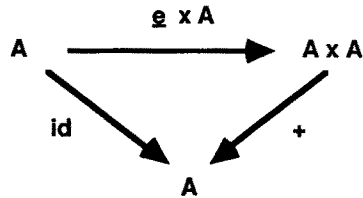
We consider Abelian groups as an example. From a categorical viewpoint, an Abelian group is an object A in **Set**, equipped with primitive operations represented as morphisms:

$$+: A^2 \rightarrow A$$

$$-: A \rightarrow A$$

$$e: 1 \rightarrow A$$

Such that certain diagrams commute. For example, to stipulate that e is a left identity, we ask that the diagram



should commute. (Here, e is the composition of e with the unique morphism from A to 1 .) In order to express this requirement, we need to use the product structure on our semantic domain, but it is clear how to interpret this definition in any category with finite products.

Lawvere showed how to view this more abstractly. The primitive operations, together with the morphisms that exist by virtue of the product structure, generate a subcategory of the semantic domain whose objects are the finite powers of A , and whose morphisms can be regarded as derived operations on A (or, rather, tuples of derived operations). Let T be the category with formal products of a generating object as objects, and tuples of formal derived operations as morphisms. (Formal derived operations may be regarded as terms modulo provable equivalence.) This category is identified as the algebraic theory, and models of the theory correspond to finite-product preserving functors to the semantic domain.

It is instructive to identify T concretely. We write n for the formal product of n copies of the generating object. Morphisms from n to m will be m -tuples of derived operations in

n variables. A formal derived operation in n variables can be represented as an element of the free Abelian group on n generators, \mathbb{Z}^n . Furthermore, an m -tuple of elements of a group, G , can be represented as a homomorphism from the free group $\mathbb{Z}^m \rightarrow G$. Thus the morphisms from n to m are given by group homomorphisms $\mathbb{Z}^m \rightarrow \mathbb{Z}^n$. In this way, we identify \mathbf{T}^{OP} as the category \mathbf{FFAb} of finitely generated, free Abelian groups. The functor to \mathbf{Set} corresponding to a particular group, G , is given by

$$n \quad \longmapsto \quad \text{Hom}[\mathbb{Z}^n, G]$$

Thus an algebraic theory is viewed as a category with finite products. Models of the theory are finite-product preserving functors to the semantic domain.

For technical reasons, it is convenient to take a slightly different view. Instead of just taking formal finite products in the construction of the theory, we take formal finite limits. Models are then represented as finite-limit preserving functors to a semantic domain with finite limits. The category of formal finite limits can be represented concretely as opposite of the category, \mathbf{FPAb} , of finitely presented Abelian groups. We can then appeal to the theorem mentioned earlier, that, if \mathbf{C} has finite limits, finite-limit preserving functors from \mathbf{C} to a topos \mathbf{E} can be represented by geometric morphisms $\mathbf{E} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{OP}}}$. Once the variances have been sorted out, this shows that models of our theory, Abelian groups, have been represented by geometric morphisms to the *presheaf* category $\mathbf{Set}^{\mathbf{FPAb}}$. How does this look if we view topoi as generalised spaces? Since the topos \mathbf{Set} corresponds to the one-point space, the points (morphisms from the one-point space) of our classifying topos are Abelian groups. Thus we regard this topos geometrically as the "space of abelian groups".

We may, of course, view $\mathbf{FPAb}^{\text{OP}}$ formally (syntactically) rather than concretely. We outline this construction of a *syntactic category* which is the analogue for predicate logic of the propositional Lindenbaum algebra. We take, as objects, presentations (by generators and relations) and, as morphisms, tuples of terms. Just as a judicious choice of notation helped us to see the validity of the axioms for a real, so here we write a presentation with generators x_1, \dots, x_n and relations r_1, \dots, r_m as $\{x_1, \dots, x_n \mid r_1, \dots, r_m\}$. This is really the notation for the set to which the presentation will be sent by a model, where the x_i then range over the underlying set of the model. A

morphism $\{ x_1, \dots, x_n \mid r_1, \dots, r_m \} \rightarrow \{ y_1, \dots, y_p \mid s_1, \dots, s_q \}$ is then given by a p-tuple τ_1, \dots, τ_p of terms in x_1, \dots, x_n such that

$$r_1, \dots, r_m \vdash s_1, \dots, s_q [\tau_i / y_i]$$

If the x_i satisfy the relations r_j , then the τ_k satisfy the relations s_l . Again, we have to quotient by an equivalence relation; provable equivalence under the assumption of the r_i . The corresponding homomorphism in **FPAb** is the one taking the generators, y , to the interpretations of the terms τ . The condition given guarantees that there is such a homomorphism. The category thus constructed will be equivalent to the concretely given version, **FPAb^{OP}**.

Taking stock, we have again given multiple views, formal and concrete, of a theory and shown how it may be represented as a topos.

Geometric theories

The notion that combines propositional and algebraic theories is that of *geometric theory*. Geometric logic is many-sorted predicate logic with equality, =, finite conjunction, \wedge, \top , arbitrary disjunction, \vee, \perp , and existential quantification \exists . The rules governing = and \exists are non-standard in that they cater for terms which are possibly undefined (see Scott [79], Fourman and Scott [79]), the rest are just as for propositional logic.

Firstly, we assume that our primitive relations and operations are *strict*:

$$\begin{array}{l} R(x) \vdash x = x \\ f(x) = f(x) \vdash x = x \end{array}$$

("x = x" is interpreted as "x exists").

Then we give the axioms for substitution, equality and existence:

$$\frac{S \vdash \phi}{S[\sigma/x] \vdash \phi[\sigma/x]}$$

$$\frac{S \vdash \phi[\tau/x]}{S, \tau = \sigma \vdash \phi[\sigma/x]}$$

$$\frac{S, x = x, \psi \vdash \phi}{S, \exists x.\psi \vdash \phi}$$

(where x is not free in ϕ)

Given a geometric theory, we construct a category \mathbf{C} , the *syntactic category* of the theory. Its objects are formulae, $\phi(x)$, (which might, more suggestively be written $\{x \mid \phi\}$), which are *strict*, in the sense that $\phi(x) \vdash x_i = x_i$. The bold faced variables here signify the list of free variables of the formula. A morphism from $\phi(x)$ to $\psi(y)$ is an equivalence class $[\theta(x, y)]$ where θ is "provably a function from ϕ to ψ ", i.e.

$$\theta(x, y) \vdash \phi(x) \wedge \psi(y)$$

$$\phi(x) \vdash \exists y. \theta(x, y)$$

$$\theta(x, y) \wedge \theta(x, y') \vdash y = y'$$

The equivalence is defined by $[\theta] = [\eta]$ iff θ and η are provably equivalent, i.e.

$$\theta(x, y) \vdash \eta(x, y) \quad \text{and} \quad \eta(x, y) \vdash \theta(x, y)$$

In the case of propositional theories, we introduced a notion of covering to capture disjunctive axioms. We do the same here. (Existential quantification is another form of disjunction.) The definition of a Grothendieck topology generalises directly to categories:

A *crible* of an object, X , is a set, K , of morphisms with codomain X , such that, if $f: Y \rightarrow X \in K$, and $g: Z \rightarrow Y$, then $f \circ g \in K$. A Grothendieck topology is specified by saying which cribles of X *cover* X , subject to the restrictions:

$$\{f \mid f: Y \rightarrow X\} \text{ covers } X$$

$$\text{If } K \text{ covers } X \text{ and } f: Y \rightarrow X, \text{ then } f^*K \text{ covers } Y$$

If K covers X and J is a crible of X such that f^*J covers Y , for each $f: Y \rightarrow X \in K$, then J covers X .

(Where $f^*K = \{ g \mid f \circ g \in K \}$.)

A category equipped with a Grothendieck topology is called a *site*. Sites are defined to allow us to construct sheaves on them. The category of sheaves on a site is a Grothendieck topos. Cribles are so-called because crible is French for an agricultural variety of sieve, a riddle, which is used to separate the germ from the stalks.

A functor between sites preserves covers iff the crible generated by the image of a covering crible is a covering crible. A Grothendieck topos has a canonical topology, generated by covering families of epimorphisms. The fundamental theorem of sheaf theory says that cover preserving functors, from a site \mathbf{C} with finite limits, to a Grothendieck topos, \mathbf{E} , correspond to geometric morphisms from \mathbf{E} to the category of sheaves, $\mathbf{Sh}(\mathbf{C})$.

We make \mathbf{C} into a site by defining $\{ [\theta_i(x_i, y)]: \{\phi_i(x_i)\} \rightarrow \{\psi(y)\} \}$ to cover $\{\psi(y)\}$ if the family $\{ [\theta_i(x_i, y)] \}$ is "provably epimorphic", i.e.

$$\psi(y) \vdash \bigvee_i \exists x_i. \theta_i(x_i, y)$$

This category has finite limits, and the models of the theory are the finite-limit and cover preserving functors to **Set**. In this case it is the construction of taking sheaves on a site which provides the left adjoint we need to reflect the theory in **Top**. The category of sheaves is thus a topos classifying models of the theory, as before.

Notions of truth.

One aspect of topos theory is difficult to convey in an introductory discussion. It is easy to show that topoi other than **Set** can be used as alternative semantic domains. It is more difficult to show why this might be profitable. Alternative semantics can have, at least, two distinct uses. The first is purely technical; alternative semantics can provide metamathematical results, the most celebrated example being Cohen's forcing. The second is, in part, philosophical; examples are Kripke's possible world semantics for

modal logic, and Beth's models for intuitionism. Topoi have been used to provide an extension of Beth's semantics which explicates Brouwer's conception of choice sequence (Fourman [82]). Work of Hyland [82] promises an explication of the logic of continuous functionals. A characteristic of these approaches is that a semantic domain is defined and then its intrinsic logic is studied. One approach to the development of a logic of computation should be to look for the appropriate semantic domain, and then study its intrinsic logic. Unfortunately, it appears that our examples can only be taken as parables in this endeavour: it is not straightforward to reconcile categories of domains, replete with fixed-points and solutions to domain equations, with the theory of topoi.

We conclude with a tabulation of the different views which may be taken of various notions:

<u>Logical</u>	<u>Categorical</u>	<u>Geometric</u>
Theory T	Classifying topos E_T	Space of models T
Trivial theory	Set	One-point space $*$
Model of T	Set $\rightarrow E_T$	Point of T
Interpretation	Geometric morphism ^{OP}	Continuous map ^{OP}

Bibliography

Fourman, Michael P.

[82] 'Continuous Truth I, Non-Constructive Objects', Logic Colloquium '82, eds G.Lolli, G.Longo and A.Marcja, North-Holland 1984.

Fourman, Michael P. and Dana S. Scott

[79] 'Sheaves and Logic', Applications of Sheaves, Springer LNM 753, pp. 302-401.

Hyland, J.M.E.

[82] 'The Effective Topos', The L.E.J. Brouwer Centenary Symposium, A.S. Troelstra and D. van Dalen eds., North-Holland, 1982.

Johnstone, Peter T.

[77] 'Topos theory', Academic Press, London.

[82] 'Stone Spaces', Cambridge Studies in Advanced Mathematics 3, Cambridge University Press, Cambridge.

Kock, Anders and Gonzalo E. Reyes

[77] 'Doctrines in Categorical Logic', Handbook of Mathematical Logic', North-Holland, 1977, pp. 283-313.

Lawvere, F. William

[69] 'Adjointness in Foundations', *Dialectica*, **23** (1969), pp. 281-196.

MacLane, Saunders

[71] 'Categories for the Working Mathematician', Graduate Texts in Mathematics 5, Springer-Verlag, New York.

Reyes, Gonzalo

[74] 'From Sheaves to Logic', Studies in Algebraic Logic, A.Daigneault, ed. MAA Studies in Math. vol. 9, Math. Assoc. America, 1974, pp. 143-134.

Scott, Dana S.

[79] 'Identity and Existence in Intuitionistic Logic', Applications of Sheaves, Springer LNM 753, pp. 660-696.

Tierney, M.

[76] 'On the spectrum of a ringed topos' and 'Forcing topologies and classifying topoi', both in 'Algebra, Topology and Category Theory: a collection of papers in honour of Samuel Eilenberg' (ed. A Heller and M.Tierney), Academic Press.

Wraith, Gavin

[73] 'Lectures on elementary topoi', from 'Model Theory and Topoi' (ed. F. W. Lawvere, C. Maurer and G. C. Wraith), Lecture Notes in Mathematics no. 445, Springer-Verlag, Berlin.