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Growth, entropy and commutativity of algebras satisfying prescribed relations

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Growth, entropy and commutativity of algebras satisfying prescribed relations

Agata Smoktunowicz[*]

Abstract

In 1964, Golod and Shafarevich found that, provided that the number of relations of each degree satisfy some bounds, there exist infinitely dimensional algebras satisfying the relations. These algebras are called Golod-Shafarevich algebras. This paper provides bounds for the growth function on images of Golod-Shafarevich algebras based upon the number of defining relations. This extends results from [33], [34]. Lower bounds of growth for constructed algebras are also obtained, permitting the construction of algebras with various growth functions of various entropies. In particular, the paper answers a question by Drensky [7] by constructing algebras with subexponential growth satisfying given relations, under mild assumption on the number of generating relations of each degree. Examples of nil algebras with neither polynomial nor exponential growth over uncountable fields are also constructed, answering a question by Zelmanov [40].

Recently, several open questions concerning the commutativity of algebras satisfying a prescribed number of defining relations have arisen from the study of noncommutative singularities. Additionally, this paper solves one such question, posed by Donovan and Wemyss in [8].

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Introduction

The Golod-Shafarevich theorem [12] is a beautiful result in noncommutative ring theory, which shows that Golod-Shafarevich algebras are infinite dimensional. The Golod-Shafarevich theorem has been used to solve many important questions, ranging from group theory and noncommutative ring theory, to number theory. For example, Golod used it to show that there exist infinite torsion groups which are finitely generated, to solve the General Burnside Problem [11], and, in collaboration with Shafarevich, to solve the Class Field Problem in number theory [12]. However, despite these successes, many inspiring open questions remain. For example, Anick asked whether the converse of the Golod-Shafarevich theorem is true, and he also asked whether the minimal possible Hilbert series of an algebra defined by a given number of relations given by the Golod-Shafarevich theorem is attainable [1] (Anick managed to solve this latter question in some special cases [1]). Interesting results related to Anick's questions were obtained by Wisliceny [39], and more recently by Iyudu and Shakarin [13], [14].

This paper explores the growth and entropy of algebras satisfying prescribed relations. Information on growth often contains important information about the properties of algebras; for example, Jategoankar showed that domains with subexponential growth satisfy the Ore condition, and hence have Ore rings of fractions ([18], pp. 48), and Zhang and Stephenson found that noetherian graded algebras cannot have exponential growth [38] (for other interesting results see [15], [10], [18], [29], [30]). Assumptions about the growth of algebras are often used in studies related to noncommutative (projective) algebraic geometry (see for example [2], [37], [31]). The growth

of algebras provides information about the extended centroids of the given rings [32], [37]. Combinatorial methods also find applications in results related to the growth of algebras, as the growth of algebras is an analogue of word growth in combinatorics. Connections between the growth of groups and related rings have also been investigated. A celebrated result of Gromov theorem says that if G is a finitely generated group and the group algebra $K[G]$ has finite Gelfand-Kirillov dimension, then G is nilpotent by finite (see [18], pp 139). The growth of groups has been extensively studied, but there are some basic open questions remaining in this area. One of the most tantalizing is Grighorhuk's question of whether there exist groups with growth between polynomial and $exp\sqrt{n}$. Analogous questions can also be asked about algebras, although here the situation is completely different, and the question of which functions are growth functions of associative algebras also remains open [33]. Related questions have been studied by Bell and Young [6], where they constructed nil algebras with subexponential growth over uncountable fields (notice that nil rings with polynomial growth were previously constructed in [21]). However, they did not give the lower bounds of the growth functions of the constructed algebras, and it was still possible that their algebras had polynomial growth. In Corollary 4.5 we use Theorem 1.1 to construct the first example of a nil algebra with growth strictly between polynomial and exponential over an arbitrary field, answering a question of Zelmanov [40].

In a connection with dynamical systems, Newman, Shneider and Shalev introduced a notion of an entropy of an associative algebra. They defined the entropy of a graded associative algebra A as $H(A) = \limsup_{n\to\infty} a_n^{1/n}$ where $a_n = \dim A(n)$ is the vector space dimension of the $n-th$ homogeneous component [28]. Notice that we can use Theorem 1.1 to construct algebras with arbitrarily small but nonzero entropy satisfying prescribed relations (if the number of relations is not too big).

Inspiring open questions on the growth of algebras satisfying prescribed relations were asked by Zelmanov in [41]. He also suggested composing results

from [21] (where examples of nil algebras with polynomial growth were found) and the Golod-Shafarevich theorem to attack these problems, which turned out to be an inspiring idea. These ideas were later investigated by a student of Zelmanov, Alexander Young, who obtained several interesting results on the growth of factor algebras A/I where I is a regimented regimented ideal, that is a special type of ideal containing repeated paterns, in some sense resembling nil ideals. For example, a regimented ideal generated by a single element $f_1 \in A$ has a form $\bigcup_{1 \leq i \leq \deg f} \sum_{j=0,2,\dots} A((\deg f)j)fA$. In [33] it was shown that similar results can be obtained without assumption that the ideal is regimented, answering a question by Zelmanov. In more detail, it was shown in [33] that Golod-Shafarevich algebras with a polynomial number of defining relations of sparse degrees can be mapped onto algebras with polynomial growth, under mild assumption on the number of generating relations of each degree. The main objective of this paper is to show that a more general result holds, and to bound the growth function from below.

Theorem 1.1 (Theorem A). *Let K be an algebraically closed field, and let A be the free noncommutative algebra generated in degree one by elements x, y. Let I denote the ideal generated in A by homogeneous elements $f_1, f_2, \dots \in A$. Let r_n denote the number of elements among $f_1, f_2, \dots$ with degrees larger than 2^n and not exceeding 2^{n+1}. Suppose that $r_i = 0$ for $i < 8$. Denote $Y = \{n : r_n \neq 0\}$.*

- *Suppose that there are no elements among $f_1, f_2, \dots$ with degree k if $k \in [2^n - 2^{n-3}, 2^n + 2^{n-2}]$ for some n.*

- *Suppose that for all $m, n \in Y \cup \{0\}$ with $m < n$ we have*
  $2^{3n+4} r_m^{33} < r_n < 2^{2^{n-m-3}}$ *and* $r_n < 2^{2^{n/2-4}}$.

Then there is an infinitely dimensional graded algebra R, such that the following holds:

1. *Algebra A/I can be homomorphically mapped onto algebra R.*

2. *If n is a natural number, and k is maximal such that $k \leq 2log(n)$ and $k \in Y$ then*

$$\dim R_n \leq 8n^4 r_k^{33},$$

where $R_n \subseteq R$ is the K-linear space consisting of elements of degrees not exceeding k.

3. *Moreover, for all $j \in Y$, $j \leq log(n)$ we have*

$$\frac{1}{2}r_j^4 \leq dim R_n.$$

We note that the assumption on the number of relations is necessary, in fact there are Golod-Shafarevich algebras without non trivial homomorphic images of subexponential or polynomial growth [36]. An analogous result holds for Golod-Shafarevich groups, though the proof is completely different [9].

Observe that Theorem 1.1 allows us to construct algebras with various growth functions, for example with a subexponential growth, which satisfy given relations. This answers a question of Vesselin Drensky about when there exist algebras of subexponential (and larger than polynomial) growth satisfying prescribed relations [7]. Observe that if the algebra R in Theorem 1.1 is not Jacobson radical, then we can construct a primitive algebra satisfying the prescribed relations using the fact that the Jacobson radical is the intersection of all primitive ideals in the given ring. Notice also that if the number of relations $f_1, f_2, \ldots$ is finite, under the assumptions of Theorem 1.1 the algebra A/I can be mapped onto prime Noetherian algebra with linear growth (by using the main result of [34] and Lemma 4.3). Notice that Theorem 1.1 allows us to bound the entropy of the constructed algebra R, with entropy defined as in [28]. Observe that some of the elements $f_1, f_2, \ldots$ may be zero, which allows us to apply Theorem 1.1 to constuct algebras with various growth functions of various entropies (for some related results see [17]).

Recently many open questions have arisen about algebras satisfying a prescribed number of relations in the area of resolutions of noncommutative

singularities. The following question related to equivalences of the derived category of 3-folds in algebraic geometry was posed in [8], Remark 5.3: Suppose that F is the formal free algebra in two variables, and consider two relations f_1, f_2 such that if we write both f_1 and f_2 as a sum of words, each word has degree two or higher. Denote I to be the two sided ideal generated by f_1 and f_2. Is it true that if F/I is finite dimensional, it cannot be commutative? In chapter 4 we answer this question and prove a slightly more general result, namely:

Theorem 1.2. *Let K be a field. Let F be either the free associative K-algebra on the set of free generators $X = \{x_1, x_2, \ldots, x_n\}$ over the field K or F be the formal free power series algebra over K in n variables $x_1, \ldots, x_n$. Let $d = 2$ for $n = 2$, and let $d = \frac{n(n+1)}{2} - 2$ for $n > 2$. Consider d relations $f_1, f_2, \ldots, f_d \in F$ such that if we write each of the $f_1, \ldots, f_d$ as a sum of words then each word has degree two or higher. If F/I is finite dimensional then it cannot be commutative.*

Notice that relations $f_1, f_2, \ldots$ in Theorem 1.2 need not be homogeneous. Wemyss constructed several not commutative finitely dimensional algebras defined by f_1 and f_2 as above. These examples also show that under assumptions of Theorem 1.2 it is possible to obtain finitely dimensional algebra F/I which is not commutative. Let F be a free algebra in n generators $x_1, \ldots, x_n$, and let I be the ideal generated by $\frac{n(n+1)}{2}$ relations $x_i x_j - x_j x_i$ and x_i^2 for $i \neq j$, $i, j \leq n$, $n > 2$. Then F/I is finitely dimensional and commutative. On the other hand Theorem 1.2 assures that the algebra F/I with I generated by less than $\frac{n(n+1)}{2} - 1$ relations cannot be commutative and finitely dimensional. We don't know if there is finitely dimensional, commutative algebra F/I with I generated by $\frac{n(n+1)}{2} - 1$ relations. For some recent results related to power series rings and polynomial rings with a small number of defining relations, see [23, 24, 25, 26, 27, 42].

1.1 Notation and definition of the Golod-Shafarevich algebras

We use the same notation as in [33], [34]. In what follows, K is a countable, algebraically closed field and A is the free K-algebra in two non-commuting indeterminates x and y. By a graded algebra we mean an algebra graded by the additive semigroup of natural numbers. The set of monomials in $\{x, y\}$ is denoted by M and, for each $k \geq 0$, its subset of monomials of degree k is denoted by $M(k)$. Thus, $M(0) = \{1\}$ and for $k \geq 1$ the elements in $M(k)$ are of the form $x_1 \cdots x_k$ with all $x_i \in \{x, y\}$. The span of $M(k)$ in A is denoted by $A(k)$; its elements are called *homogenous polynomials of degree k*. More generally, for any subsetset X of a graded algebra, we denote by $X(k)$ its subset of homogeneous elements of degree k. The *degree* $\deg f$ of an element $f \in A$ is the least $k \geq 0$ such that $f \in A(0) + \cdots + A(k)$. Any $f \in A$ can be uniquely written in the form $f = f_0 + f_1 + \cdots + f_k$ with each $f_i \in A(i)$. The elements f_i are the *homogeneous components* of f. A (right, left, two-sided) ideal of A is *homogeneous* if it is spanned by its elements' homogeneous components. If V is a linear space over K, we denote by $\dim V$ the dimension of V over K (for information about the growth of algebras, see [18]). All logarithms are in base 2.

We now recall the definition of Golod-Shafarevich algebras. Let A be a free associative algebra in d generators $X = \{x_1, x_2, \ldots, x_d\}$, with generators x_i of degree 1. Suppose we are given homogeneous elements $f_1, f_2, \ldots \in A$, with r_i elements of degree i among $f_1, f_2, \ldots$ (and $r_1 = 0$). Let I be the ideal of A generated by $f_1, f_2, \ldots$. Denote $R = A/I$. Recall that $H_R(t) = \sum_{i=0}^{\infty} \dim R(i) t^i$ is called the Hilbert series of algebra R. Golod and Shafarevich proved in [12] that $H_R(t)(1 - mt + \sum_{i=2}^{\infty} r_i t^i) \geq 1$ holds coëfficient-wise. An algebra admitting such a presentation is called a Golod-Shafarevich algebra. In a survey by Zelmanov [40] it was noted that if there exists a number $t_0 \in (0, 1)$ such that $1 - mt + \sum_{i=2}^{\infty} r_i t^i$ converges at t_0 and $1 - mt_0 + \sum_{i=2}^{\infty} r_i t_0^i < 0$, then A/I is infinite dimensional.

Notice that if for infinitely many natural numbers m, there is a number

$t_m > 0$ *such that* $1 - dt_m + \sum_{i=2}^{m} r_i t_m^{\ i} < 0$ *then algebra* A/I *is infinitely dimensional.* Although this is an easily found consequence of Zelmanov's observation, we did not find it in the literature, so we provide a short proof below for the convenience of the reader:

Proof. Fix number m and let I_m be the ideal of A generated by all elements f_i which have degrees not exceeding m. If there is $t_m > 0$ such that $1 - dt_m + \sum_{i=2}^{m} r_i t_m^{\ i} < 0$ then by the Golod-Shafarevich theorem and the previous remarks from [40] A/I_m is infinitely dimensional.

Since I is a graded ideal and $I = \bigcup_{i=2}^{\infty} I_m$ it follows that A/I is infinite dimensional. Indeed a graded and finite dimensional algebra is nilpotent. If A/I is finitely dimensional them $A(n) \in I$ for some n, and since I is graded $A(n) \subseteq I_n$, so A/I_n is finitely dimensional, a contradiction. $\qquad\square$

2   General construction

The general construction is similar to that in [21], [33], [34]. We will use the notation from [34]. In this chapter we will review some basic concepts from chapter 1 in [34].

Let K be a field and A be a free K-algebra generated in degree one by two elements x, y. Suppose that subspaces $U(2^m), V(2^m)$ of $A(2^m)$ satisfy, for every $m \geq 1$, the following properties:

1. $V(2^m)$ is spanned by monomials;

2. $V(2^m) + U(2^m) = A(2^m)$ and $V(2^m) \cap U(2^m) = 0$;

3. $A(2^{m-1})U(2^{m-1}) + U(2^{m-1})A(2^{m-1}) \subseteq U(2^m)$;

4. $V(2^m) \subseteq V(2^{m-1})V(2^{m-1})$, where for $m = 0$ we set $V(2^0) = Kx + Ky$, $U(2^0) = 0$.

Following [21], define a graded subspace E of A by constructing its homogeneous components $E(k)$ as follows. Given $k \in N$, let $n \in N$ be such that

$2^{n-1} \le k < 2^n$. Then $r \in E(k)$ precisely if, for all $j \in \{0, \ldots, 2^{n+1} - k\}$, we have $A(j)rA(2^{n+1} - j - k) \subseteq U(2^n)A(2^n) + A(2^n)U(2^n)$. More concisely,

$$E(k) = \{r \in A(k) \mid ArA \cap A(2^{n+1}) \subseteq U(2^n)A(2^n) + A(2^n)U(2^n)\}. \quad (1)$$

Set then $E = \bigoplus_{k \in N} E(k)$.

Lemma 2.1 (Lemma 1.1, [34]). *The set E is an ideal in A. Moreover, if all sets $V(2^n)$ are nonzero, then algebra A/E is infinite dimensional over K.*

Notice that the proof of the first claim is the same as in Theorem 5 [21].

In the next chapters we will construct appropriate sets $U(2^n)$, $V(2^n)$, so that the algebra $R = A/E$ will satisfy thesis of Theorem 1.1.

Let $k \in N$ be given. We write it as a sum of increasing powers of 2, namely $k = \sum_{i=1}^{t} 2^{p_i}$ with $0 \le p_1 < p_2 < \ldots < p_t$. Following [33], [34] we set

$$U^<(k) = \sum_{i=0}^{t} A(2^{p_1} + \cdots + 2^{p_{i-1}})U(2^{p_i})A(2^{p_{i+1}} + \cdots + 2^{p_t}), \quad (2)$$

$$V^<(k) = V(2^{p_1}) \cdots V(2^{p_t}), \quad (3)$$

$$U^>(k) = \sum_{i=0}^{t} A(2^{p_t} + \cdots + 2^{p_{i+1}})U(2^{p_i})A(2^{p_{i-1}} + \cdots + 2^{p_1}), \quad (4)$$

$$V^>(k) = V(2^{p_t}) \cdots V(2^{p_1}). \quad (5)$$

The following lemma corresponds to Lemma 1.4 in [34].

Lemma 2.2 ([21], pp. 993–994). *For all $k \in N$ we have $A(k) = U^<(k) \oplus V^<(k) = U^>(k) \oplus V^>(k)$.*

For all $k, \ell \in N$ we have $A(k)U^<(\ell) \subseteq U^<(k + \ell)$ and $U^>(k)A(\ell) \subseteq U^>(k + \ell)$.

The following proposition corresponds to Proposition 1.1 in [34].

Proposition 2.1 (Theorem 11, [21]). *For every $k \in N$ we have*

$$\dim A(k)/E(k) \le \sum_{j=0}^{k} \dim V^<(k - j) \dim V^>(j),$$

where we set $\dim V^>(0) = \dim V^<(0) = 1$.

Proof. The proof is the same as the proof of Theorem 11 in [21], or the proof of Theorem 5.2 in [22]. □

3   Chapters 3, 4, 5 and 6 from [34]

In this section we assume that r_i and $Y, f_1, f_2, \dots$ are as in Theorem A. We moreover assume that there are natural numbers $\{e_i\}_{i \in Y}$ which satisfy the following conditions: for all $n \in Y$: $1 \le e(n) \le n-1$, sets $S_n = \{n-1-e(n), n-1\}$ are disjoint and $r_n \le 2^{t(n)}$ where $t_n = 2^{e(n)-1} - 3n - 4 - \sum_{k \in Y, k < n} 2^{e(k)+2}$. Let $S = \bigcup_{n \in Y} \{n-1-e(n), n-1\}$.

We will use some results from [34] and assume that the reader is familiar with chapters 3, 4 and 5 of it. Also our Section 2 corresponds with section 1 in [34], but we omitted Theorems 1.2 and 1.3 from [34], as they are not relevant to this paper. Because we are using the same notation, we can use chapters $3, 4, 5$ in [34] without modification (similar results with different notation were also studied in [33]).

We have slightly weaker assumptions on elements f_i in Theorem A than those in Theorem 0.1 in [34], namely that we can have an infinite number of nonzero f_i in Theorem A and only assume that there are no f_i of degree k if $k \in [2^n - 2^{n-3}, 2^n + 2^{n-2}]$ for some n. In [34] it was assumed that the number of f_i is finite and that there are no f_i of degree k if $k \in [2^n - 2^{n-3}, 2^n + 2^{n-1}]$ for some n. However, this does not influence the proofs in chapters 3, 4 and 5 in [34]. On the other hand, our assumption in Theorem A that for all $n, m \in Y \cup \{0\}$ with $m < n$ we have $2^{3n+4} r_m^{33} < r_n < 2^{2^{n-m-3}}$ is a stronger assumption than $2^{3n+4} \prod_{i < n, i \in Y} r_i^{32} < r_n < 2^{2^{n-m-3}}$ in Theorem 0.1 [34] (see Lemma 4.3 for an explanation), hence we can use the results from [34].

We slightly modify results from section 6 in [34] as follows, to suit our new assumptions. We begin with the following lemma, which also generalizes Lemma 4.1 from [33].

Lemma 3.1 (Modified Lemma 6.1, [34])**.** *Let n be a natural number. Suppose that, for all $m < n$ with $m \in Y$, we constructed sets $F(2^m)$, and for*

every $m < n$ we are given subspaces $V(2^m)$, $U(2^m)$ of $A(2^m)$ which satisfy the following properties (with Y, $\{e(i)\}_{i \in Y}$ defined as at the beginning of this section):

1. *$\dim V(2^m) = 2$ if $m \notin S$;*

2. *$\dim V(2^{m-e(m)-1+j}) = 2^{2^j}$ for all $m \in Y$ and all $0 \leq j \leq e(m)$;*

3. *$V(2^m)$ is spanned by monomials;*

4. *$F(2^m) \subseteq U(2^m)$ for every $m \in Y$;*

5. *$V(2^m) \oplus U(2^m) = A(2^m)$;*

6. *$A(2^{m-1})U(2^{m-1}) + U(2^{m-1})A(2^{m-1}) \subseteq U(2^m)$:*

7. *$V(2^m) \subseteq V(2^{m-1})V(2^{m-1})$.*

Consider all $f \in A(k) \cap \{f_1, f_2, \ldots\}$ with $2^n + 2^{n-2} \leq k \leq 2^n + 2^{n-1} + 2^{n-2}$. Then there exists a linear K-space $F'(2^n) \subseteq A(2^n)$ with the following properties:

-   *$0 < \dim F'(2^n) \leq \frac{1}{2} \dim V(2^{n-1})^2$;*

-   *for all $i, j \geq 0$ with $i + j = k - 2^n$ and for every $f \in A(k) \cap \{f_1, f_2, \ldots\}$, we have $f \in A(i)F'(2^n)A(j) + U^<(i)A(k-i) + A(k-j)U^>(j)$ with the sets $U^<(i), U^>(i)$ defined in Section 2.*

Proof. The proof is the same as the proof of Lemma 6.1, [34]. □

Observe that properties $1-7$ in Lemma 3.1 correspond to properties $1-7$ in Theorem 4.1 [34].

We note that there is a small typo in Theorem 4.1 (2) [34], and it should state " $m \in Y$" instead of "$m \in Z$." Theorem 4.1 (1)-(7) [34] was invented and used in [21] but with a different notation.

Lemma 3.2 (Modified Lemma 6.2, [34]). *Suppose that sets* $U(2^m), V(2^m)$ *were already constructed for all* $m < n$, *and sets* $F(2^m)$ *were already constructed for all* $m < n$ *with* $m \in Y$, *and satisfy the conditions of Theorem A. Let* $n \in Y$. *Let* $F = \{f_1, f_2, \ldots\}$ *be as in Theorem A. Define a K-linear subspace* $Q \subseteq A(2^{n+1})$ *as follows:*

$$Q = \sum_{f \in F: 2^n + 2^{n-2} \leq \deg f \leq 2^n + 2^{n-1} + 2^{n-2}} \sum_{i+j = 2^{n+1} - \deg f} V^>(i) f V^<(j).$$

Then $\dim Q \leq \frac{1}{4}(\frac{1}{2} \dim V(2^{n-1})^2 - 2)$.

Proof. (modified to suit our paper) By Lemma 5.3 in [34], the inner sum has dimension at most

$$2^{2n} \Big(\prod_{k<n, k \in Y} 2^{2^{e(k)+1}} \Big)^2 [V(2^{n-1})]^2 / 2^{2^{e(n)-1}}.$$

Summing over all $i + j = 2^{n+1} - \deg f$ multiplies by a factor of at most 2^n (because $2^{n+1} - \deg f \leq 2^{n-1}$); summing over all $f \in \{f_1, f_2, \ldots\}$ with degrees between $2^n + 2^{n-2}$ and $2^n + 2^{n-1} + 2^{n-2}$ multiplies by r_n. Therefore,

$$\dim Q \leq r_n 2^{3n} \Big(\prod_{k<n, k \in Y} 2^{2^{e(k)+1}} \Big)^2 [V(2^{n-1})]^2 / 2^{2^{e(n)-1}}.$$

By assumption that $r_n \leq 2^{t_n}$ from the beginning of this section, we get $\dim Q \leq \frac{1}{16} \dim V(2^{n-1})^2$.

Observe now that $\frac{1}{4} \dim V(2^{n-1})^2 \leq \frac{1}{2} \dim V(2^{n-1})^2 - 2$, because $V(2^{n-1})^2 = (2^{2^{e(n)}})^2 \geq 2^{2^2} \geq 16$. We get $\dim Q \leq \frac{1}{4}(\frac{1}{2} \dim V(2^{n-1})^2 - 2)$ as required. $\square$

We will also use Proposition 6.1 [34] and repeat it for the convenience of the reader. This proposition previously appeared with different notation as Proposition 3.5 in [33].

Proposition 3.1 (Proposition 6.1, [34]). *Let* K *be an algebraically closed field. With notation as in Lemma 3.1 there is a linear K-space* $F(2^n) \subseteq A(2^n)$ *satisfying* $\dim F(2^n) \leq \dim V(2^{n-1})^2 - 2$ *and*

$$F'(2^n) \subseteq F(2^n) + U(2^{n-1})A(2^{n-1}) + A(2^{n-1})U(2^{n-1}).$$

Moreover, for all $f \in \{f_1, f_2, \ldots\}$ with $\deg f \in \{2^n + 2^{n-2}, \ldots, 2^n + 2^{n-1} + 2^{n-2}\}$ we have

$$AfA \cap A(2^{n+1}) \subseteq A(2^n)F(2^n) + F(2^n)A(2^n)$$
$$+ A(2^{n-1})U(2^{n-1})A(2^n) + A(2^n)U(2^{n-1})A(2^{n-1})$$
$$+ U(2^{n-1})A(2^n + 2^{n-1}) + A(2^n + 2^{n-1})U(2^{n-1}).$$

Proof. (modified to suit our paper) Proof of this proposition under the assumptions of our section is the same, but we use modified Lemma 6.2 (Lemma 3.1) as above, instead of using Lemma 6.2 from [34]. Similarly we use modified Lemma 6.2 (Lemma 3.2) instead of Lemma 6.2. $\square$

We will also use Proposition 5.1 from [34].

Proposition 3.2 (Proposition 5.1, [34]). *Let $\alpha = 2^{p_1} + \cdots + 2^{p_t}$ be a natural number in the binary form. Then $[V^>(\alpha)] < 2\alpha \prod_{i \leq m, i \in Y} 2^{2^{e(i)+1}}$, where m is maximal such that $\sum_{p_i \in \{m-e(m)-1,\ldots,m-1\}} 2^{p_i}$ is nonzero.*

4 Dimensions of images of Golod-Shafarevich algebras

In this section we will prove Theorem A. We also obtain the following result, which is more general than Theorem A.

Theorem 4.1. *Assume that the assumptions of Theorem A hold and that we use the same notation as in Theorem A. Assume that, for each $n \in Y$, there is a natural number $1 \leq e(n) < \frac{n}{2} - 1$ such that, for all $n \in Y$, sets $S_n = \{n - 1 - e(n), n - 1\}$ are disjoint and*

$$r_n 2^{3n+4} \prod_{k < n, k \in Y} 2^{2^{e(k)+2}} \leq 2^{2^{e(n)-1}}$$

(where we put $\prod_{k<n,k\in Y} 2^{2^{e(k)+2}} = 1$ if there are no $k < n$ with $k \in Y$). Then A/I can be mapped onto an infinite dimensional, graded algebra R which satisfies $\dim R(n) \leq 8n^3 \prod_{i \in Y, i \leq 2log(n)} 2^{2^{e(i)+2}}$.

Proof. Let $e(i)$, Y be as in the assumptions of this theorem. We will construct sets $U(2^n)$, $V(2^n)$, $F(2^n)$ satisfying properties $1-7$ from Theorem 4.1 [34] (or equivalently properties $1-7$ from Lemma 3.1) applied for $e(n)$ as above. We start the induction with $U(2^0) = F(2^0) = 0$ and $V(2^0) = Kx + Ky$. Then, assuming that we constructed $U(2^m), V(2^m)$ for all $m < n$, if $n \in Y$ we construct $F(2^n)$ using Proposition 3.1, and if $n \notin Y$ we set $F(2^n) = 0$. We then construct $U(2^n)$, $V(2^n)$ using Theorem 4.1 (1)-(7) in [34]. Let E be defined as in Section 2. By Lemma 2.1, the set E is an ideal in A and A/E is an infinite dimensional algebra. Because E is homogeneous, A/E is graded. We will now show that $R = A/E$ is a homomorphic image of A/I. We need to show that $I \subseteq E$, that is that elements $f_1, f_2, \ldots \in E$. Let $f \in A(k)$ be one of these elements. By Lemma 3.1 and Proposition 3.1, and because $F'(2^n) \subseteq F(2^n) + U(2^{n-1})A(2^{n-1}) + A(2^{n-1})U(2^{n-1}) \subseteq U(2^n)$, we get that f satisfies the assumptions of Theorem 3.1 [34]. Therefore, and by thesis of Theorem 3.1 [34], we have $f \in E$, as required.

Recall that we assumed that $e(n) \leq \frac{n}{2} - 1$. We will apply Proposition 3.2 for $\alpha = n$. Let $n = \sum_{i=1}^{t} 2^{p_i}$, and let m be as in Proposition 3.2. Recall that m is maximal such that $\sum_{p_i \in \{m-e(m)-1,\ldots,m-1\}} 2^{p_i} \neq 0$. We will show that $m \leq 2log(n)$. Note that $log(n) \geq p_t$ and so $m \leq 2log(n)$, because otherwise $m > 2p_t$ would imply $m - e(m) - 1 > p_t$, as $e(m) \leq \frac{m}{2} - 1$ by assumption, so $\sum_{p_i \in \{m-e(m)-1,\ldots,m-1\}} 2^{p_i} = 0$, a contradiction. Therefore, $m \leq 2log(n)$.

Clearly, $\dim R(n) = \dim A(n)/E(n)$. By Lemma 2.1 we have

$$\dim A(n)/E(n) \leq \sum_{j=0}^{n} \dim V^<(n-j)\dim V^>(j).$$

By Proposition 3.2, we get $\dim A(n)/E(n) \leq (n+1)(2n\prod_{i \in Y, i \leq 2log(n)} 2^{2^{e(i)+1}})^2 \leq 8n^3 \prod_{i \in Y, i \leq 2log(n)} 2^{2^{e(i)+2}}$, as required. $\square$

Lemma 4.2. *With the assumptions as in Theorem 4.1 we in addition have* $\dim R_n > \frac{1}{2}2^{2^{e_l}}$, *for all* $l \in Y$, $l \leq log(n)$, *where R_n is the linear space of elements with degrees not exceeding n in R.*

Proof. Let n be a natural number and let $l \in Y$, $l \leq log(n)$. By the construction from Theorem 4.1 [34] we get that $V(2^{l-1})$ has $2^{2^{e(l)}}$ elements. Indeed

by Theorem 4.1 (2) [34] we get $\dim V(2^{m-e(m)-1+j}) = 2^{2^j}$ for all $m \in Y$ and all $0 \leq j \leq e(m)$ (we note that there is a small typo in the statement of Theorem 4.1 (2) [34], saying $m \in Z$ instead of $m \in Y$). Therefore, and since $l \in Y$ we get $\dim V(2^{l-1}) = \dim V(2^{l-e(l)-1+e(l)}) = 2^{2^{e(l)}}$. By assumption $e(l) \geq 1$ hence $\dim V(2^l) \geq 4$. By Theorem 4.1 (1) and (2) [34], we get $V(2^{l-2})V(2^{l-2}) = V(2^{l-1})$ and $\dim V(2^{l-2}) = \dim V(2^{l-e(l)-1+e(l)-1}) = 2^{2^{e(l)-1}}$ (it is also true if $e(l) = 1$ by Theorem 4.1 (1) [34]). Set $V(2^{l-1})$ is generated by monomials, so at least half of these monomials end with x or at least half of these monomials ends with y. We can assume without restricting the generality that the former holds. Let $v_1, \ldots, v_p \in A(2^{l-1} - 1)$ be monomials such that $v_1 x, \ldots v_p x \in V(2^{l-1})$ and $p \geq \frac{1}{2} \dim V(2^{l-1})$. We claim that images of elements $v_1, \ldots, v_p$ in A/E are linearly independent over K. As $A/E = R$, and $p \geq \frac{1}{2} \dim V(2^{l-1}) = \frac{1}{2} 2^{2^{e(l)}}$, this statement implies the thesis of our lemma. It remains to show that if $v = \sum_{i=1}^p \alpha_i v_i$ for some $\alpha_i \in K$ then $v \notin E$ (unless all α_i are zero). Suppose on the contrary that $v \in E$. Let $w \in V(2^{l-2})$. By the definition of E, and since $v \in E$, we get $wvxw \in U(2^{l-1})A(2^{l-1}) + A(2^{l-1})U(2^{l-1})$. Write $v_i x = u_i z_i$ for $u_i, z_i \in V(2^{l-2})$. Now $wu_i, z_i w \in V(2^{l-1})$ (because $V(2^{l-2})V(2^{l-2}) = V(2^{l-1})$), and so $wvxw \in V(2^{l-1})V(2^{l-1})$. By the property 2 from the beginning of chapter 1 we get that $V(2^{l-1})V(2^{l-1}) \cap (U(2^{l-1})A(2^{l-1}) + A(2^{l-1})U(2^{l-1})) = 0$. It follows that $wvxw = 0$ in A, and so $vx = 0$, hence all $\alpha_i = 0$, which proves our claim. $\qquad \square$

Lemma 4.3. *Suppose that elements r_i are as in Theorem A. Then*

$$2^{3n+4} \prod_{i<n, i \in Y} r_i^{32} < r_n < 2^{2^{n-m-3}}.$$

Moreover for all $m, n \in Y \cup \{0\}$, with $m < n$ we have $r_n < 2^{2^{n-m-3}}$.

Proof. Observe first that by the assumptions of Theorem A we have $2^{3n+4} r_m^{33} < r_n < 2^{2^{n-m-3}}$. By a similar argument we get that $r_m > r_{m'}^{33}$ for $m' < m$, with $m, m' \in Y$. Consequently, for any $m_1 > m_2 > m_3 \ldots$ with $m_1, m_2, \ldots \in Y$ the following holds: $r_{m_1}^{33} = (r_{m_1}^{32}) r_{m_1} > r_{m_1}^{32} r_{m_2}^{33} > r_{m_1}^{32} r_{m_2}^{32} r_{m_2} > r_{m_1}^{32} r_{m_2}^{32} r_{m_3}^{33} \geq$

…. It follows that

$$2^{3n+4} \prod_{i<n, i\in Y} r_i^{32} < r_n < 2^{2^{n-m-3}}.$$

$\square$

Lemma 4.4. *Let Y be a subset of the set of natural numbers and let $\{r_i\}_{i\in Y}$ be a sequence of natural numbers which satisfy the assumptions of Theorem A. Then there are natural numbers $\{e(n)\}_{n\in Y}$ such that for all $n \in Y$: $1 \le e(n) \le \frac{n}{2} - 1$ and sets $S_n = \{n - 1 - e(n), n - 1\}$ are disjoint and $r_n 2^{3n+4} \prod_{k<n, k\in Y} 2^{2^{e(k)+2}} \le 2^{2^{e(n)-1}}$. Moreover $2^{2^{en}} \ge r_n^4$.*

Proof. The proof of the first statement is the same as proof of Lemma 7.2 in [34] (because by Lemma 4.3 elements $\{r_i\}_{i\in Y}$ satisfy assumptions of Theorem 0.1 [34]). Recall, that in particular for each i, $e(i)$ is such that $2^{2^{e(i)-3}} \le r_i < 2^{2^{e(i)-2}}$. Observe that then $e(n) < \frac{n}{2} - 1$ because by assumptions of Theorem A, $r_n < 2^{2^{n/2-4}}$, and hence $2^{2^{e(n)-3}} \le r_n < 2^{2^{n/2-4}}$ which implies $e(n) \le \frac{n}{2} - 1$.

To prove the second statement observe that $2^{2^{en}} = 2^{2^{(en-2)+2}} = 2^{2^{en-2}4} = (2^{2^{en-2}})^4$. Recall that $2^{2^{e(i)-2}} \ge r_i$. Consequently $2^{2^{en}} \ge r_n^4$, as required. $\square$

Proof of Theorem 1.1 By Lemma 4.4, we can find $e(i)$ satisfying the assumptions of Theorem 4.1, and by the thesis of Theorem 4.1 [34] we get that A/I can be mapped onto an infinite dimensional, graded algebra R which satisfies $\dim R(n) \le 8n^3 \prod_{i\in Y, i\le 2\log(n)} 2^{2^{e(i)+2}}$. Recall that in Lemma 4.4 we assume that $2^{2^{e(i)-3}} \le r_i < 2^{2^{e(i)-2}}$, for each $i \in Y$. Consequently, $2^{2^{e(i)+2}} = 2^{2^{(e(i)-3)+5}} = 2^{2^{e(i)-3}32} = (2^{2^{e(i)-3}})^{32} \le r_i^{32}$. Therefore, $\prod_{i\in Y, i\le 2\log(n)} 2^{2^{e(i)+2}} \le \prod_{i\in Y, i\le 2\log(n)} r_i^{32}$.

This implies that $\dim R(n) \le 8n^3 \prod_{i\in Y, i\le 2\log(n)} r_i^{32}$. Recall that $R(n)$ is the homogeneous space of elements with degree n, and R_n is the subspace of R consisting of all elements with degrees not exceeding n. It follows that $\dim R_n \le n(\sup_{k\le n} \dim R(k))$. Consequently, $\dim R_n \le 8n^4 \prod_{i\in Y, i\le 2\log(n)} r_i^{32}$, as required.

To prove the lower bound observe that $\dim R_n > \frac{1}{2} 2^{2^{e(l)}}$ for all $l \le \log(n)$, by Lemma 4.2. Consequently, by the last part of Lemma 4.4, we get $\dim R_n > \frac{1}{2} 2^{2^{e(l)}} \ge \frac{1}{2} r_l^4$ for all $l \le \log(n)$.

Corollary 4.5. *Let K be a field. Then there is a graded nil algebra with neither polynomial nor exponential growth. Moreover, this algebra is generated by two elements of degree one.*

Proof. Let A be the free noncommutative algebra generated by two generators x and y of degree one. Let M be the set of monomials in A. Let n be a natural number. Fix $(a_1, \ldots, a_n)$ with all $a_i \in M$. Let $L(a_1, \ldots, a_n)^q$ be the linear space spanned by elements s^q where $s \in Ka_1 + Ka_2 + \ldots Ka_n$. Observe that $L(a_1, \ldots, a_n)^q$ is spanned as a linear K-space by its q^n elements (this can be seen by writing each of these elements as a sum of homogeneous components).

Fix $(a_1, \ldots, a_n)$ with all $a_i \in M$. Let $t > \deg a_i$, $r > 3^{6t}$, $w = 2r$. By Theorem 2 [35] there exists a set $F \subseteq A(r)$ such that

1. $L(a_1, \ldots, a_n)^{20r}$ is contained in the ideal generated by F in A.

2. cardinality of $F = F(a_1, \ldots, a_n)$ is less than $(20r)^n(r3^{2t}t^2)$.

(Note that the fact that the algebra A in [35] is generated by three monomials does not influence the proof.) Observe that the cardinality of F is less than $(20r)^{n+1}3^{4t} < (20r)^{n+1}r < (20r)^{n+2}$. If $r = 2^m + 2^{m-1}$ we get that the cardinality of F is less that 40^{8nm}.

The set S of all tuples $(a_1, a_2, \ldots, a_n)$ with $a_i \in M$, $n \geq 1$ is countable. Therefore, there exists an injective function $F : S \to N$ (N is the set of natural numbers). We can assume that

- $f(a_1, a_2, \ldots, a_n) = 2^s + 2^{s-1}$ for some $s \in N$.

- If $2^m + 2^{m-1} = f(a_1, a_2, \ldots, a_n)$ and $2^{m'} + 2^{m'-1} = f(b_1, b_2, \ldots, b_{n'})$ for some $m < m'$ then $200m^3 < m'$, $n < m$, $n' < m'$, $m^3 < 2^{m/2-4}$ and $m > 100$.

Observe then that $r_m^{33}2^{3m'+4} < r_{m'}$, where $r_m = (40)^{8m^3}$ and $r(m') = (40)^{8(m')^3}$. Observe also that $r_{m'} < 2^{2^{m'/2-4}} < 2^{2^{m'-m-3}}$ since $m > 100$ and $200m^3 < (m')$.

Let $f(a_1, \ldots, a_n) = 2^m + 2^{m-1} = r$. Then

$$L(a_1, \ldots, a_n)^{20r} = L(a_1, \ldots, a_n)^{20f(a_1, \ldots, a_n)}$$

is in the ideal generated by some set $F(a_1, \ldots, a_n) \subseteq A(r)$, with cardinality $(40)^{8m^3}$. Denote the cardinality of this set as r_m. Denote $Y = \{n : r_n \neq 0\}$, and notice that Y, r_n satisfy the assumptions of Theorem 1.1 (Theorem A) with relations f_i from all the sets $F(a_1, \ldots, a_n)$. Then, by Theorem 1.1 there is algebra R in which all elements from sets $F(a_1, \ldots, a_n)$ are zero and hence this K-algebra is nil (even if K is uncountable). We claim that the growth of this algebra is neither polynomial nor exponential.

By Theorem 1.1 (Theorem A), for infinitely many m, $\dim R_{2^{m+1}} > r_m^4 > 40^{8m^2}$ therefore R cannot have polynomial growth.

On the other hand $\dim R_n < 8n^4 r_k^{33}$ for some $k < 2log(n)$. Observe then that $r_k < 40^{32log(n)^3} < 2^{200log(n)^3}$. Consequently $\dim R_n < 8n^4 2^{200(log(n))^3}$ for all n. If R had exponential growth then for some $c > 1$ we would have $\dim R_n > c^n$. This would imply $c^n < 2^{400(log(n))^3}$, a contradition. It follows that R has smaller growth than exponential, as required. $\qquad\square$

5   Applications in algebraic geometry

Here we prove Theorem 1.2 In the proof we will use the following lemmas.

Lemma 5.1. *Let K be a field and let F be either the free commutative K-algebra on the set of free generators $X = \{x_1, x_2, \ldots, x_n\}$ or F be the formal commutative power series algebra over K in n variables $x_1, x_2, \ldots, x_n$ (without the identity element). Let t be a natural number and let J be an ideal in F such that $x_i^t \in F^{tn} + J$, for all $i \leq n$. Then there is an element $z \in F$ such that the algebra $\frac{F}{zF + J}$ is nilpotent and hence finitely dimensional.*

Proof. With a slight abuse of notation let x_i denote the image of x_i in F/J. Let $v = [x_1^t, x_2^t, \ldots, x_n^t]$. Observe that there is an n by n matrix M with coefficients in F/J such that $Mv^T = v^T$. Let I denote the identity matrix.

By the same argument as in Theorem 75 [16] we get that $(I-M)v^T = 0$, hence the determinant of $I - M$ annihilates all coefficients of v. This determinant is of the form $1 + z$ with $z \in F/J$. It follows that $zx_i^t = x_i^t$ for all $i \leq n$. Therefore the ideal $zF + J$ contains all elements x_i^t for $i \leq n$. It follows that $F/zF + J$ is a nilpotent, finitely dimensional algebra, as required. $\qquad \square$

The following lemma is well known and follows from the dimension theory.

Lemma 5.2. *Let K be a field, and let $\bar{F}$ be either the free commutative algebra on the set of free generators $X = \{x_1, \ldots, x_n\}$ or $\bar{F}$ be the formal commutative power series algebra over K in n variables $x_1, x_2, \ldots, x_n$ (without the identity element). Let I be an ideal in $\bar{F}$ generated by less than n elements. Then $\frac{\bar{F}}{I}$ is not a nilpotent algebra.*

Proof. In [3] pp. 9, section 5.4 we read that the dimension of a noetherian commutative ring R is the minimum over all ideals I of definition of R, of the numbers of generators of i. Moreover J is an ideal of definition iff R/J is Artinian. Recall that the dimension of $\bar{F}$ is n, and $\bar{F}$ is noetherian. Hence, $\bar{F}/I$ is not Artinian, and hence it is not finitely dimensional, as required. $\quad \square$

We are now ready to prove Theorem 1.2.

Proof. (Proof of Theorem 1.2) Assume that F is the free algebra in n free variables $x_1, \ldots, x_n$. The proof in the case when F is the power series ring is the same. Recall that $x_1, \ldots, x_n$ have degree one. Suppose on the contrary that F/I is commutative and finitely dimensional. Then all elements of the form $x_i x_j - x_j x_i$ for $i \neq j$ are in I. As each f_i is a sum of homogeneous elements of degrees two or higher, it follows that each element of the form $x_i x_j - x_j x_i$ is the homogeneous component of the smallest degree of some element $f \in I$. Without restricting the generality, we can assume that for each $i < j$, $i, j \leq n$ the element $x_i x_j - x_j x_j$ is the lowest term of f_k for some $k \leq \frac{n(n-1)}{2}$. Then $g_k = f_k - (x_i x_j - x_j x_i)$ has all homogeneous components of degree 3 or higher.

Let S be the linear K-space spanned by terms of degree two in elements f_k for $k > \frac{n(n-1)}{2}$, and L be the linear K space spanned by terms of degree two in elements f_k for $k \le \frac{n(n-1)}{2}$. Without restricting generality we can assume that $S \cap L = 0$.

Observe now that the ideal I equals the ideal $\bar{I}$ of F generated by elements f_i for $i > \frac{n(n-1)}{2}$ and $g_1, g_2, \ldots, g_{\frac{n(n-1)}{2}}$ and all elements $x_i x_j - x_j x_i$ for $i, j \le n$. Indeed as all elements $x_i x_j - x_j x_i$ are in I it follows that $g_k \in I$ and hence $\bar{I} \subseteq I$. On the other hand $f_k = g_k + (x_i x_j - x_j x_i) \in \bar{I}$ for all $k \le \frac{n(n-1)}{2}$ (and for appropriate i, j depending on k) hence $I \subseteq \bar{I}$. It follows that $I = \bar{I}$.

Let J be an ideal generated by elements $x_i x_j - x_j x_i$ for $i, j \le n$ and by elements f_i for $i > \frac{n(n-1)}{2}$. Denote $G = K g_1 + K g_2 + \ldots + K g_{\frac{n(n-1)}{2}}$.

We claim that for each $j \ge 1$ we have $G \subseteq F^j G + J$. Observe that F/J is commutative, hence $GF \subseteq FG + J$. We will first show that $G \subseteq FG + J$. Observe that $g_k \in I$ hence $g_k \in \sum_{k=1}^{\frac{n(n-1)}{2}} \alpha_k f_k + Q$ for some $\alpha_i \in K$, where $Q = IF + FI + FIF + J$.

By comparing elements of degree two in the left hand side and right hand side we get $0 = \sum_{k=1}^{\frac{n(n-1)}{2}} \alpha_k t_k + s$ for some $s \in S$, where t_k denotes the component of degree 2 in f_k for all $k \le \frac{n(n-1)}{2}$. Recall that $\sum_{k=1}^{\frac{n(n-1)}{2}} \alpha_k t_k \in L$. It follows that $\sum_{k=1}^{\frac{n(n-1)}{2}} \alpha_k t_k \in L \cap S = 0$. Observe that elements t_k are linearly independent over K (because each element t_k corresponds with exactly one element $x_i x_j - x_j x_i$ with $i < j$). It follows that $\alpha_k = 0$ for all $k \le \frac{n(n-1)}{2}$.

Therefore, $g_k \in FI + IF + FIF + J$. Observe that $I = \bar{I}$ hence $g_k \in F\bar{I} + \bar{I}F + F\bar{I}F + J$. Recall that $\bar{I}$ is generated by elements from G and from J. It follows that $g_k \in FG + GF + FGF + J = FG + J$ as required.

As it holds for all $k \le d - 1$ we get $G \subseteq FG + J$. By applying this observation again we get $G \subseteq FFG + J$. Continuing in this way we get that $F^i G + J$ for all j, as required.

We will now show that there is $m > 0$ such that $x_1^m \in F^\alpha + J$ for every $\alpha > 0$. Since F/I is a finitely dimensional algebra then for some $\beta_i \in K$ and some numbers m, k we have $x_1^m - \sum_{i>m}^k \beta_i x_1^i \in I$. Recall that $I = \bar{I} = KG + FG + GF + FGF + J = FG + J$. By our previous observation

it follows that $I \subseteq F^\alpha G + J$. It follows that $x_1^m \in \sum_{i>m}^k \beta_i x_1^i + F^\alpha G + J$. Note that $\sum_{i>m}^k \beta_i x_1^i = x_1^m r$ hence $x_1^m \in x_1^m r + F^\alpha + J \subseteq x_1^m r^2 + F^\alpha + J$. Continuing in this way (by substituting $x_1^m r + F^\alpha + J$ into x_1^m) we get $x_1^m \subseteq F^\alpha + J$.

Similarly, there is $t > 0$ such that $x_i^t \in F^\alpha + J$ for all $i \leq n$ and all $\alpha > 0$, hence assumptions of Lemma 5.1 are satisfied. Therefore, there is $z \in F$ such that $F/J + zF$ is nilpotent.

Let M be the ideal of F generated by all elements $x_i x_j - x_j x_i$ for $i, j \leq n$. Recall that $J + zF = M + T$ where T is an ideal generated by elements f_i for $i > \frac{n(n-1)}{2}$ and by z, hence by $n - 1$ elements. It follows that $\bar{F} = F/M$ has an ideal Q generated by less than n elements such that $\bar{F}/Q$ is nilpotent, a contradiction with Lemma 5.2, thus proving the case $n > 2$ of our theorem.

Let now $n = 2$ and $d = 2$. We know that $x_1^t + c \in J$ and $x_2^t + c' \in J$ for some t and some $c, c' \in F^\alpha G$. Let $\alpha > tn$. Observe that J is generated by $x_1 x_2 - x_2 x_1$ and f_2. Let M be the ideal of F generated by all elements $x_1 x_2 - x_2 x_1$. With a slight abuse of notation let f_2, x_1, x_2 denote the images of elements f_2, x_1, x_2 in $\bar{F} = F/M$. Then $x_1^m + c = f_2 h$ and $x_2^t + c = f_2 g$ for some $h, g \in F$. Let q be the lowest degree term of f_2 in $\bar{F} = F/M$. Then q divides both x_1^m and x_2^m in $\bar{F}$, a contradiction. $\qquad\square$

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