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THE RING OF EVENLY WEIGHTED POINTS ON THE LINE

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Abstract. Let $M_w = (\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$ denote the geometric invariant theory quotient of $(\mathbb{P}^1)^n$ by the diagonal action of SL_2 using the line bundle $\mathcal{O}(w_1, w_2, \ldots, w_n)$ on $(\mathbb{P}^1)^n$. Let R_w be the coordinate ring of M_w. We give a closed formula for the Hilbert function of R_w, which allows us to compute the degree of M_w. The graded parts of R_w are certain Kostka numbers, so this Hilbert function computes stretched Kostka numbers. If all the weights w_i are even, we find a presentation of R_w so that the ideal I_w of this presentation has a quadratic Gröbner basis. In particular, R_w is Koszul. We obtain this result by studying the homogeneous coordinate ring of a projective toric variety arising as a degeneration of M_w.

1. Introduction

The study of the ring of invariants for the action of the automorphism group of $\mathbb{P}^1$ on n points on $\mathbb{P}^1$ goes back to the 19th century. In 1894 Kempe [20] proved that this ring is generated by the invariants of lowest degree. More than a century later Howard, Millson, Snowden, and Vakil [17] were finally able to describe the ideal of relations between Kempe's generators, when the characteristic of the ground field $\mathbf{k}$ is zero or $p > 11$.

More generally, for $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n$, let $L_w = \mathcal{O}_{(\mathbb{P}^1)^n}(w_1, \ldots, w_n)$. Assume that all w_i are positive, so that L_w is very ample. The group $\mathrm{S}L(2)$ acts diagonally on $(\mathbb{P}^1)^n$ and the line bundle L_w admits a unique linearization. Let

$$R_w = \left(\bigoplus_{d \geq 0} H^0\left((\mathbb{P}^1)^n, L_w^d \right) \right)^{\mathrm{S}L(2)}$$

denote the corresponding ring of invariant sections, and let $M_w = (\mathbb{P}^1)^n /\!/ \mathrm{S}L(2)$ denote the GIT quotient. When $w_i = 1$ for $1 \leq i \leq n$, we write $w = 1^n$.

In [16, Theorem 2.3] the authors show that R_w is generated by the invariants of lowest degree for arbitrary w and in [17, Theorem 1.1] that, in characteristic zero or $p > 11$, the ideal of relations I_w is generated by quadratic polynomials in the generators unless $w = 1^6$, in which case there is an essential cubic relation. Moreover, in [16, Section 2.15], the authors obtain a recursive formula for the degree of M_w.

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Our first theorem is an extension of Howe's formula [18, 5.4.2.3] for the Hilbert function of R_w in the case $w = 1^n$ to arbitrary w. In particular, we obtain a closed formula for the degree of M_w.

Theorem 1.1. *Let $[n] = \{1, \ldots, n\}$, and for $J \subseteq [n]$, set $|w_J| = \sum_{j \in J} w_j$, $w_\emptyset = 0$ and $|w| = w_1 + \cdots + w_n$.*

(1) The Hilbert function for R_w is given by

$$h(d) = \sum_{\substack{J \subseteq [n] \\ |w_J| < |w|/2}} (-1)^{|J|} \binom{d\,(|w|/2 - |w_J|) + n - |J| - 2}{n-2}$$

if $d|w|$ is even, and zero otherwise.

(2) For $|w|$ even, the degree of M_w is

$$\frac{1}{n-2} \left(\sum_{\substack{J \subseteq [n] \\ |w_J| < |w|/2}} (-1)^{|J|} \left(|w|/2 - |w_J|\right)^{n-3} \left(\sum_{i=0}^{n-3} n - |J| - 2 - i \right) \right).$$

Let $K(\lambda, \mu)$ be the Kostka number counting semistandard Young tableaux of shape λ with filling μ. The dimension of the d-th graded part $(R_w)_d$ of R_w is equal to the *stretched* Kostka number $K(d\lambda, d\mu)$ where $\lambda = (|w|/2, |w|/2)$ and $\mu = w$ (see 2.1). We give a closed formula for Kostka numbers of this form in Proposition 3.2. It was shown in [22] and [2] that for partitions λ and μ the function $K(d\lambda, d\mu)$ is a polynomial in d. Thus the Hilbert function gives a closed formula for the polynomials $K(d\lambda, d\mu)$ in this special case.

In [37] Jakub Witaszek studies the multigraded Poincaré-Hilbert series of the homogeneous coordinate ring of the Plücker embedding of the Grassmannian $G(2, n)$ for a certain $\mathbb{N}^n$-grading. We obtain a closed formula for the multigraded Hilbert function and for the Poincaré-Hilbert series, see Remark 3.3.

For a field $\mathbf{k}$, recall that a graded $\mathbf{k}$-algebra R is *Koszul* if $\mathbf{k}$ admits a linear free resolution as an R-module. If $R = \mathbf{k}[X_0, \ldots, X_N]/I$, then the existence of a quadratic Gröbner basis for I implies that R is Koszul, which in turn implies that I is generated by quadratic equations. In [19], Keel and Tevelev show that the section ring of the log-canonical line bundle on $\overline{M}_{0,n}$ is Koszul. However, while for $w = 1^8$, I_w is generated by quadratic equations, we show in Example 3.9 that R_w is not Koszul.

In general, high enough Veronese subrings of graded rings are Koszul [1, 14], and up to a linear transformation, they admit a quadratic Gröbner basis [8]. We show that for R_w already the second Veronese subring satisfies these properties.

Theorem 1.2. *Assume $w \in (2\mathbb{Z})^n$. Then I_w admits a squarefree quadratic Gröbner basis. In particular, R_w is Koszul.*

Both theorems apply in all characteristics, and in fact more generally over the integers. Note that Theorem 1.2 implies that for w with $|w|$ odd, I_w admits a squarefree quadratic Gröbner basis, since in this case $R_w = R_{2w}$ by Proposition 2.1. In particular, if $w = 1^n$ with n odd, then I_w admits a quadratic Gröbner basis. However, it is not known whether for $w = 1^{10}$, R_w is Koszul, see Remark 3.10.

As in [16, 17], our proof is based on a toric degeneration. This toric degeneration is a SAGBI degeneration, and we show that this toric degeneration admits a quadratic Gröbner basis. After we shared our result with Manon, he was able to extend it to more general polytopes that arise as degenerations of the coordinate rings of the moduli stack of quasi-parabolic $\mathrm{S}L(2, \mathbb{C})$ principal bundles on a generic marked projective curve in [24, Theorem 1.10], see Remark 5.12.

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2. The coordinate ring R_w of M_w

In this section we set up basic notation and describe the invariant ring R_w in terms of certain semistandard Young tableaux. Let $\mathbf{k}$ be a field, and let

$$S = \mathbf{k}[x_1, y_1, x_2, y_2, \ldots, x_n, y_n],$$

which we view as the set of polynomial functions on the space $\mathbb{A}^{2\times n}$ of $2 \times n$ matrices with entries in the field $\mathbf{k}$:

$$\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}.$$

The polynomial ring S is graded by $\mathbb{N}^n$, where the degree of the monomial $\prod_{i=1}^n x_i^{a_i} y_i^{b_i}$ is equal to $(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$. Given $r = (r_1, \ldots, r_n) \in \mathbb{N}^n$, let S_r denote the r-th graded part of S. Let $L_r = \mathcal{O}(r_1, \ldots, r_n)$. Viewing x_i, y_i as homogeneous coordinates for the i'th point in $(\mathbb{P}^1)^n$, we have $H^0\left((\mathbb{P}^1)^n, L_r\right) = S_r$. The line bundle L_r admits a linearization for the diagonal action of $\mathrm{SL}(2, \mathbf{k})$ on $(\mathbb{P}^1)^n$, (see [28, Chapter 3.1]) such that the induced action on the section ring $R(L_r)$ is given by matrix multiplication on the left. It is easy to see that for $i < j$ the polynomials

$$p_{ij} = \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$$

are invariant under the $\mathrm{SL}(2, \mathbf{k})$ action. Note that $p_{i,j}$ are the Plücker coordinates on the Grassmannian $G(2, n)$. The First Fundamental Theorem of Invariant Theory says that they generate the ring of invariants $S^{\mathrm{SL}(2,\mathbf{k})}$, [6, Theorem 2.1]. Note that $S^{\mathrm{SL}(2,\mathbf{k})}$ is the homogeneous coordinate ring of $G(2, n)$ in the Plücker embedding.

For our purposes, it is most convenient to study this invariant ring using tableaux of shape (k,k),

$$\tau = \begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline j_1 & j_2 \\ \hline \end{array} \cdots \begin{array}{|c|}\hline i_k \\ \hline j_k \\ \hline \end{array}, \tag{2.1}$$

where $1 \le i_\ell, j_\ell \le n$. A tableau τ is called *semistandard* if its entries are weakly increasing in the rows and strictly increasing in the columns, i.e., in our setting we have $i_1 \le \cdots \le i_k, j_1 \le \cdots, \le j_k$, and $i_1 < j_1, \ldots, i_k < j_k$. The *content* of a tableau τ is the n-tuple $w(\tau) = (w(\tau)_1, \ldots, w(\tau)_n)$, where $w(\tau)_i$ denotes the number of times i occurs in τ.

To a semistandard tableau τ as in (2.1) we associate the polynomial

$$s_\tau = p_{i_1,j_1} p_{i_2,j_2} \cdots p_{i_k,j_k} \in S^{\mathrm{SL}(2,\mathbf{k})}.$$

Note that s_τ is homogeneous of degree $w(\tau)$.

The following proposition is an algebraic incarnation of the Gel'fand-MacPherson correspondence [11]. Let R_w be the invariant ring of the introduction and let $|w| = w_1 + \cdots + w_n$.

Proposition 2.1. *The polynomials s_τ, where τ ranges over all semistandard tableaux of shape $(\frac{d|w|}{2}, \frac{d|w|}{2})$ with content dw for some $d \in \mathbb{N}$, form a vector space basis for the invariant ring R_w.*

Proof. Let $X = (\mathbb{P}^1)^n$. The torus $T_w = \{\mathrm{diag}(t_1, \ldots, t_n) \in \mathrm{GL}(n, \mathbf{k}) \mid t_1^{w_1} t_2^{w_2} \cdots t_n^{w_n} = 1\}$ acts on the right of $\mathbb{A}^{2\times n}$ by matrix multiplication inducing an action on S such that $S^{T_w} = \oplus S_{dw}$. Thus we obtain

$$R_w = \left(\bigoplus_d H^0\left(\left(\mathbb{P}^1\right)^n, L_w^d \right) \right)^{\mathrm{SL}(2,\mathbf{k})} = \left(\bigoplus_d S_{dw} \right)^{\mathrm{SL}(2,\mathbf{k})} = \left(S^{T_w}\right)^{\mathrm{SL}(2,\mathbf{k})}.$$

Since the actions of $\mathrm{SL}(2,\mathbf{k})$ and T_w commute, we have $(S^{T_w})^{\mathrm{SL}(2,\mathbf{k})} = (S^{\mathrm{SL}(2,\mathbf{k})})^{T_w}$. Now, $S^{\mathrm{SL}(2,\mathbf{k})}$ has a vector space basis consisting of s_τ where τ ranges over semistandard Young tableaux of shape (k,k), see for example [6, Theorem 2.3]. Let $f = \sum_\tau a_\tau s_\tau \in S^{\mathrm{SL}(2,\mathbf{k})}$, where τ runs over semistandard tableaux. Since the action of T_w is linear, and the s_τ are linearly independent, f is invariant under T_w if and only if every s_τ is invariant. Note that for a tableau τ with content $w(\tau)$, we have $t \cdot s_\tau = t_1^{w(\tau)_1} \cdots t_n^{w(\tau)_n} s_\tau$. In particular, s_τ is invariant if and only if there is d with $w(\tau) = dw$. The claim follows. ☐

In particular, when $w = 1^n$ and $n = 2m$ even, the dimension of the space of lowest degree invariants in R_w is the Catalan number C_m.

Remark 2.2. One can view R_w as a multigraded Veronese subring of the Plücker algebra. The Plücker algebra $S^{\mathrm{SL}(2,\mathbf{k})}$ admits a $\mathbb{N}^n$-grading determined by the weight under the action of the diagonal torus $T = \mathrm{diag}(t_1, \ldots, t_n)$ via $t \cdot p_{i,j} = t_i t_j p_{i,j}$. Then for a tableau τ with content $w(\tau)$, we have $t \cdot s_\tau = t^{w(\tau)} s_\tau$ and thus we can conclude as in the proof of Proposition 2.1 that

$$\left(S^{\mathrm{SL}(2,\mathbf{k})}\right)_w = \langle \{ s_\tau \mid \tau \text{ is semistandard of shape } (|w|/2, |w|/2) \text{ with filling } w(\tau) = w \} \rangle.$$

Here $\langle \cdots \rangle$ denotes the span as a vector space over $\mathbf{k}$. In particular, $\left(S^{\mathrm{SL}(2,\mathbf{k})}\right)_w = 0$ if $|w|$ is odd. It then follows that $R_w = \bigoplus_{d\in\mathbb{N}} \left(S^{\mathrm{SL}(2,\mathbf{k})}\right)_{dw}$.

Definition 2.3. To a tableau τ of shape (k,k) is associated a partition $\nu_\tau = (\nu_1,\dots,\nu_n)$ of d, the content of the first row of τ, i.e., $\nu_i = |\{j \mid i_j = i\}|$.

The following Lemma can be easily deduced from the discussion in [15, Section 3]. We include a sketch of the proof for the convenience of the reader.

Lemma 2.4. *The semistandard Young tableaux of shape $(|w|/2, |w|/2)$ with filling w are in bijection with partitions $\nu = (\nu_1,\dots,\nu_n)$ of $|w|/2$ satisfying*

$$0 \le \nu_\ell \le w_\ell \text{ and} \tag{2.2}$$

$$2(\nu_1 + \cdots + \nu_{\ell-1}) + \nu_\ell \ge w_1 + \cdots + w_\ell \tag{2.3}$$

for $1 \le \ell \le n$. These conditions imply $\nu_1 = w_1$ and $\nu_n = 0$.

Proof. To a tableau with filling w and increasing rows is associated a partition of $|w|/2$ by Definition 2.3. Conversely, to a partition ν satisfying (2.2), we associate a tableau τ of shape $(\frac{|w|}{2}, \frac{|w|}{2})$ by filling the first row with ν_1 1's, ν_2 2's, etc., and the second row with $(w_1-\nu_1)$ 1's, $(w_2-\nu_2)$ 2's, etc. By construction the rows of this tableau are increasing and it has filling w. These associations are inverse to each other. Moreoever, τ is semistandard if and only if $\nu_1 + \cdots + \nu_{\ell-1} \ge (w_1 - \nu_1) + \cdots + (w_\ell - \nu_\ell)$ for $1 \le \ell \le n$. This condition is equivalent to (2.3). $\square$

3. The Hilbert polynomial and degree of M_w

In this section we will prove the formulas for the Hilbert function of R_w and the degree of M_w of Theorem 1.1. Our techniques are similar to those of Howe [18, 5.4.2.3.] who computed the case when $w = 1^n$.

Let $\lambda = (|w|/2, |w|/2)$ and $\mu = w$. For partitions λ and μ the Kostka numbers $K(\lambda,\mu)$ are defined to be the number of semistandard Young tableaux of shape λ and content μ. For a partition $\lambda = (\lambda_1,\dots,\lambda_s)$, we let $d\lambda = (d\lambda_1,\dots,d\lambda_s)$. Note that by Proposition 2.1 the dimension of $(R_w)_d$ is equal to $K(d\lambda, d\mu)$.

In order to compute the Hilbert polynomial, we give a formula for these particular Kostka numbers. The main step in proving this formula is to set up a relationship between the Kostka numbers and numbers of the corresponding partitions of Definition 2.3. We let $\Pi(n,\infty,k) = \{(\nu_1,\dots,\nu_n) \mid 0 \le \nu_i \text{ for } i \in [n] \text{ and } \nu_1 + \cdots + \nu_n = k\}$, and let

$$\pi(n,\infty,k) = |\Pi(n,\infty,k)| = \binom{n-1+k}{n-1}. \tag{3.1}$$

Let $w = (w_1,\dots,w_n) \in \mathbb{N}^n$. We let $\Pi(n,w,k) = \{(\nu_1,\dots,\nu_n) \mid 0 \le \nu_i \le w_i \text{ for } i \in [n] \text{ and } \nu_1 + \cdots + \nu_n = k\}$ and let $\pi(n,w,k) = |\Pi(n,w,k)|$. For a subset $I \subseteq [n] = \{1,\dots,n\}$, we let $\Pi(n,w_I,k) = \{(\nu_1,\dots,\nu_n) \in \Pi(n,\infty,k) \mid 0 \le \nu_i \le w_i \text{ for } i \in I\}$ and $\pi(n,w_I,k) = |\Pi(n,w_I,k)|$.

Lemma 3.1. *For any $I \subseteq [n]$,*

$$\begin{aligned} \pi(n, w_I, k) &= \sum_{J \subseteq I} (-1)^{|J|} \pi\left(n, \infty, k - (|w_J| + |J|)\right) \\ &= \sum_{J \subseteq I} (-1)^{|J|} \binom{n-1+k-(|w_J|+|J|)}{n-1}. \end{aligned}$$

Proof. We proceed by induction on the cardinality of I. When $I = \emptyset$, the above claim is immediate. If $I \neq \emptyset$, let $j \in I$. Since

$$\pi(n, w_I, k) = \pi(n, w_{I \setminus \{j\}}, k) - \pi(n, w_{I \setminus \{j\}}, k - w_j - 1),$$

the claim follows from the induction hypothesis. The last equality follows from (3.1). □

Proposition 3.2. *Let $w \in \mathbb{N}^n$ and assume that $|w|$ is even. Then for $\lambda = (|w|/2, |w|/2)$ and $\mu = w$, we have*

$$\begin{aligned} K(\lambda, \mu) &= \pi\left(n, w, |w|/2\right) - \pi\left(n, w, |w|/2 - 1\right) \\ &= \sum_{\substack{J \subseteq [n] \\ |w_J| < |w|/2}} (-1)^{|J|} \binom{|w|/2 - |w_J| + n - |J| - 2}{n-2} \end{aligned}$$

Proof. For the first equality, we need to express the Kostka numbers in terms of partitions. For $\nu \in \mathbb{R}^n$, we define a function $f_\nu \colon [n] \to \mathbb{R}$ by

$$f_\nu(i) = 2\nu_1 + \cdots + 2\nu_{i-1} + \nu_i - (w_1 + \cdots + w_i) \tag{3.2}$$

for $1 \leq i \leq n$. Then Lemma 2.4 implies that for $w \in \mathbb{Z}^n$ with $|w|$ even, $\lambda = (|w|/2, |w|/2)$, and $\mu = w$ we have

$$K(\lambda, \mu) = |\{\nu \in \Pi\left(n, w, |w|/2\right) \mid f_\nu(i) \geq 0 \text{ for all } 1 \leq i \leq n\}|. \tag{3.3}$$

For $\nu \in \mathbb{Z}^n$ we let $m_\nu = \min\{f_\nu(j) \mid j \in [n]\}$ and $i_\nu = \max\{j \mid f_\nu(j) = m_\nu\}$ and define

$$\begin{aligned} \phi \colon \{\nu \in \Pi\left(n, w, |w|/2\right) \mid \exists\, i \text{ such that } f_\nu(i) < 0\} &\to \Pi\left(n, w, |w|/2 - 1\right) \\ (\nu_1, \ldots, \nu_n) &\mapsto (\nu_1, \ldots, \nu_{i_\nu} - 1, \ldots, \nu_n). \end{aligned}$$

We claim that ϕ is well-defined and gives a bijection. Note that the first equality then follows from this claim together with (3.3).

The following equalities follow easily from the definition of f_ν:

$$f_\nu(i+1) = f_\nu(i) + \nu_i - w_{i+1} + \nu_{i+1} \tag{3.4}$$

$$f_\nu(n) = 2|\nu| - |w| - \nu_n. \tag{3.5}$$

To see that ϕ is well defined, we have to show that $\nu_{i_\nu} > 0$. When $i_\nu < n$ then $f_\nu(i+1) > f_\nu(i)$ and (3.4) implies $\nu_i > w_{i+1} - \nu_{i+1} \geq 0$. If $i_\nu = n$, then $f_\nu(n) = m_\nu < 0$, since there exists i such that $f_\nu(i) < 0$ and $m_\nu \leq f_\nu(i)$. Since $|\nu| = |w|/2$, $f_\nu(n) = -\nu_n$ by (3.5), so we have $\nu_n > 0$.

To see that ϕ is a bijection, we exhibit the inverse map. For $\nu' \in \Pi\left(n, w, |w|/2 - 1\right)$, we let $j_{\nu'} = \min\{i \mid f_{\nu'}(i) = m_{\nu'}\}$ and define

$$\psi \colon (\nu'_1, \ldots, \nu'_n) \mapsto (\nu'_1, \ldots, \nu'_{j_{\nu'}} + 1, \ldots \nu'_n).$$

We have

$$f_{\psi(\nu')}(i) = \begin{cases} f_{\nu'}(i) & \geq & m_{\nu'}+1 & \text{if } i < j_{\nu'} \\ f_{\nu'}(i)+1 & = & m_{\nu'}+1 & \text{if } i = j_{\nu'} \\ f_{\nu'}(i)+2 & \geq & m_{\nu'}+2 & \text{if } i > j_{\nu'}. \end{cases} \tag{3.6}$$

We have to show that ψ is well-defined. If $j_{\nu'} > 1$, then $f_{\psi(\nu')}(j_{\nu'}) = f_{\psi(\nu')}(j_{\nu'}-1) + \nu'_{j_{\nu'}-1} - w_{j_{\nu'}} + \nu'_{j_{\nu'}} + 1$ by (3.4). Plugging in the values from (3.6), we see that $w_{j_{\nu'}} \geq \nu_{j_{\nu'}}+1$. If $j_{\nu'} = 1$, then $f_{\psi(\nu')}(1) = \nu'_1 + 1 - w_1 = m_{\nu'}+1$. However, note that $m_{\nu'} \leq f_{\nu'}(n) \leq -2$ by 3.5, and so it follows that $\nu'_1 + 2 \leq w_1$. Moreover, (3.6) implies that $i_{\psi(\nu')} = j_{\nu'}$. One can check similarly that $j_{\phi(\nu)} = i_\nu$ and it follows that ϕ and ψ are inverse to each other.

For the second equality, note that Lemma 3.1 implies that

$$\begin{aligned} &\pi(n,w,|w|/2) - \pi(n,w,|w|/2-1) \\ &\quad = \sum_{J\subseteq[n]} (-1)^{|J|} \left[\binom{n-1+|w|/2-(|w_J|+|J|)}{n-1} - \binom{n-2+|w|/2-(|w_J|+|J|)}{n-1} \right] \end{aligned}$$

Using the identity $\binom{m}{n} - \binom{m-1}{n} = \binom{m-1}{n-1}$, one obtains the formula in the statement. Note that if $|w_J| \geq |w|/2$, the expression in the top of the binomial coefficient is less than $n-2$, so it suffices to sum over those $J \subseteq [n]$ such that $|w_J| < |w|/2$. □

Proof of Theorem 1.1. Fix $w \in \mathbb{N}^n$, let $\lambda = (|w|/2, |w|/2)$, and let $\mu = w$. It follows from Proposition 2.1 that $\dim(R_w)_d = 0$ if $d|w|$ is odd and that $\dim(R_w)_d = K(d\lambda, d\mu)$ if $d|w|$ is even. The formula for $h(d)$ then follows from Proposition 3.2.

The formula for the degree of M_w is obtained by computing the coefficient of d^{n-3} in the Hilbert polynomial and multiplying by $(n-3)!$. □

Remark 3.3. Our formula also implies a closed formula for the multigraded Hilbert function and Poincaré-Hilbert series of the coordinate ring of the Grassmannian $G(2,n)$ in the Plücker embedding with the multigrading described in Remark 2.2. In [37, Theorem 3.4.3] Jakub Witaszek gives a recursive formula for the multigraded Poincaré-Hilbert series $\sum_{w\in\Lambda} \dim \left(S^{\mathrm{SL}(2,\mathbf{k})}\right)_w z^w$ of the Plücker algebra for the multigrading described in Remark 2.2. He also obtains a combinatorial formula for the Poincaré-Hilbert series.

Let $\Lambda = \{w \in \mathbb{Z}^n \mid |w| \in 2\mathbb{Z}\}$. By Remark 2.2, we have $\dim \left(S^{\mathrm{SL}(2,\mathbf{k})}\right)_w = K(\lambda,\mu)$, where $\lambda = (|w|/2, |w|/2)$ and $\mu = w$. It follows that the support of the multigraded Hilbert function $\mathbf{h}$ is $\{w \in \Lambda \mid |w|/2 \geq w_j \text{ for all } 1 \leq j \leq n\}$. Then Proposition 3.2 implies that for $w \in \Lambda$,

$$\mathbf{h}(w) = \dim \left(S^{\mathrm{SL}(2,\mathbf{k})}\right)_w = \sum_{\substack{J\subseteq[n] \\ |w_J|<|w|/2}} (-1)^{|J|} \binom{|w|/2 - |w_J| + n - |J| - 2}{n-2}.$$

So we obtain a closed formula for the multigraded Poincaré-Hilbert series.

A point $p = (p_1, \ldots, p_n) \in (\mathbb{P}^1)^n$ is stable (resp. semistable) for the $\mathrm{SL}(2,\mathbf{k})$-linearization of L_w if for all subsets of indices of colliding points $J = \{j \in [n] \mid p_j = p \text{ for some } p\}$ we have $|w_J| < |w|/2$ (resp. $|w_J| \leq |w|/2$), see [28, Chapter 3], [36, Section

6], or [13, Section 8]. Identifying $\mathbb{N}^n$ with the effective divisors on $(\mathbb{P}^1)^n$, we see that the fact that in the formula for the multigraded Hilbert function we sum over those $J \subseteq [n]$ such that $|w_J| < |w|/2$ reflects the chamber structure for the GIT chambers whose walls are given by $|w_J| = |w|/2$. In particular, the multigraded Hilbert function is piecewise polynomial in $w \in \Lambda$ and the domains of polynomiality agree with the GIT chambers.

Remark 3.4. In the formula for the Hilbert polynomial of Theorem 1.1, the terms of degree $(n-2)$ cancel out since $\dim M_w = n-3$. Thus we obtain the following identity:

$$\sum_{J \subseteq [n]} (-1)^{|J|} (|w|/2 - |w_J|)^{n-2} = 0.$$

Remark 3.5. Let $g(d)$ denote the Hilbert function of $G(2,n)$ in the Plücker embedding. Recall the multigraded Hilbert function $\mathbf{h}$ for the Plücker embedding from Remark 3.3. Then we have

$$g(d) = \sum_{\substack{w \in \mathbb{N}^n \\ |w| = 2d}} \mathbf{h}(w),$$

implying the identity

$$\binom{n+d-1}{d}^2 - \binom{n+d}{d+1}\binom{n+d-2}{d-1} = \sum_{\substack{w \in \mathbb{N}^n \\ |w|=2d}} \sum_{\substack{J \subseteq [n] \\ |w_J| < |w|/2}} (-1)^{|J|} \binom{|w|/2 - |w_J| + n - |J| - 2}{n-2}. \tag{3.7}$$

Remark 3.6. While our formula counts semistandard tableaux, it contains negative signs. It would be nice to have a formula with positive coefficients. In fact, King, Tollu and Toumazet conjecture that for arbitrary λ, μ the coefficients of the polynomial $K(d\lambda, d\mu)$ are positive in [21, Conjecture 3.2].

Remark 3.7. Narayanan shows in [29, Theorem 1] that the problem of computing Kostka numbers $K(\lambda, \mu)$ is $\#P$-complete. Note that for our formula, one has to compute first all subsets of $[n]$, where n is the length of μ.

Example 3.8. The formula in Theorem 1.1 shows that $\deg(M_4) = 1$, $\deg(M_6) = 3$, $\deg(M_8) = 40$, $\deg(M_{10}) = 1225$, $\deg(M_{12}) = 67956$, $\deg(M_{14}) = 5986134$, $\deg(M_{16}) = 769550496$, so this sequence agrees with A012250 on Sloane's online encyclopedia of integer sequences [32], compare [16, Section 2.15]. Similarly, when $w = (2, \ldots, 2)$, the degrees of M_w are $\deg(M_{2^4}) = 2$, $\deg(M_{2^5}) = 5$, $\deg(M_{2^6}) = 24$, $\deg(M_{2^7}) = 154$, $\deg(M_{2^8}) = 1280$, $\deg(M_{2^9}) = 13005$, $\deg(M_{2^{10}}) = 156800$, $\deg(M_{2^{11}}) = 2189726$ which agrees with sequence A012249 [33].

The following example shows that while for $w = 1^8$, I_w is generated by quadratic equations by [17], the ring of invariants R_w is not Koszul.

Example 3.9. Recall that for a graded algebra R the Hilbert series is given by $H(z) = \sum_{d=0}^{\infty} \dim(R_d) z^d$. Similarly, the Poincaré series is given by $P(z) = \sum_{i=0}^{\infty} \dim \mathrm{Tor}_i^R(\mathbf{k}, \mathbf{k}) z^i$. Let $\mathbf{P}(u, v) = \sum_{i=0}^{\infty} \dim \mathrm{Tor}_i^R(\mathbf{k}, \mathbf{k})_j u^i v^j$. Then we have $H(z)\mathbf{P}(-1, z) = 1$ by [31, Chapter

2, Proposition 2.1]. If R is Koszul, then the minimal free graded resolution of $\mathbf{k}$ over R is linear, and so $P(uv) = \mathbf{P}(u,v)$. Then $H(z)P(-z) = 1$ if and only if R is Koszul, see [10, Theorem 1]. In particular, the power series $H(-z)^{-1} = P(z)$ must have positive coefficients.

Note that for $w = 1^8$, the Hilbert function is given by

$$h(d) = \sum_{j=0}^{3} (-1)^j \binom{8}{j} \binom{d(4-j)+6-j}{6},$$

so the Hilbert series is

$$\begin{aligned} H(z) &= 1 + 14z + 91z^2 + 364z^3 + 1085z^4 + 2666z^5 + 5719z^6 \\ &\quad + 11096z^7 + 19929z^8 + O(z^9) \\ &= \frac{1 + 8z + 22z^2 + 8z^3 + z^4}{(1-z)^6}. \end{aligned}$$

Then

$$\begin{aligned} H(-z)^{-1} = 1 + 14z + 105z^2 + 560z^3 + 2296z^4 + 6880z^5 + 8904z^6 \\ - 62320z^7 - 641704z^8 + O(z^9). \end{aligned} \tag{3.8}$$

Remark 3.10. For $w = 1^{10}$ it is not known whether R_w is Koszul or whether the ideal of relations between the generators admits a quadratic Gröbner basis. One can check that the first 800 coefficients of $H(-z)^{-1}$ are positive, however it is not known whether the relation $H(-z)P(z) = 1$ is satisfied. In general one might hope that for n large and $w = 1^n$ the ring R_w is Koszul or has other nice properties, such as Green's property N_p, see for example [23, Section 1.8.D].

4. A SAGBI degeneration of R_w

A crucial part in the proof of [17, Theorem 1.1] is the existence of toric degenerations of M_w indexed by trivalent trees on n leaves, see [17, Section 3.3]. The existence of one of these toric degenerations had been established by Foth and Hu in [9, Theorem 3.2]. In fact, the toric degenerations are torus quotients of the toric degenerations of the Grassmannian $G(2,n)$ studied by Sturmfels and Speyer [34] and by Gonciulea and Lakshmibai in [12]. Manon mentions in [26, Theorem 1.3.6] that the ring of invariants R admits a SAGBI degeneration. In this section we will give an explicit description of this SAGBI degeneration, by taking the torus invariants of the SAGBI degeneration described in [27, Section 14.3].

Let R be a finitely generated subalgebra of the polynomial ring $S = \mathbf{k}[x_1, \ldots, x_n]$, and let $\prec$ be a term order on S. Let $\mathrm{in}_\prec(R)$ be the subalgebra of S generated by the inital terms of the elements of R. Assume this subalgebra is finitely generated. (Note that this is rarely the case). A set of generators $\{f_1, \ldots, f_r\}$ is called a *SAGBI basis* for R with respect to $\prec$ if $\mathrm{in}_\prec f_1, \ldots, \mathrm{in}_\prec f_r$ generate $\mathrm{in}_\prec(R)$. Let I be the ideal of relations between the generators of R. Then $\mathrm{in}_\prec(R) \cong R/\mathrm{in}_\prec I$, see [4, Proof of Corollary 2.1]. Since $\mathrm{in}_\prec(R)$ is a monomial algebra, it is toric, and therefore the existence of a SAGBI basis for an algebra implies the existence of a flat degeneration of this algebra to a toric algebra [7, Theorem 15.17]. Note that the toric algebra need not be normal.

Now let $S = \mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and R the ring of invariants as in Section 2. Let $\prec$ be the the purely lexicographic term order with $x_1 \succ \cdots \succ x_n \succ y_1 \succ \cdots \succ y_n$.

Recall the polynomial s_τ associated to a $2 \times n$ tableau $\tau = \begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline j_1 & j_2 \\ \hline\end{array} \cdots \begin{array}{|c|}\hline i_r \\ \hline j_r \\ \hline\end{array}$. We also associate a monomial $m_\tau = x_{i_1} \cdots x_{i_r} y_{j_1} \cdots y_{j_r} \in S$.

Lemma 4.1. *A monomial m is a leading monomial of an element in R if and only if $m = m_\tau$, where τ is semistandard with filling dw, for some d. In particular, the set of these m_τ is a vector space basis for* $\mathrm{in}_\prec(R)$.

Proof. By Proposition 2.1, the polynomials s_τ, where τ ranges over semistandard Young tableaux of shape $2 \times \frac{d|w|}{2}$ with content dw form a vector space basis of R. Note that every monomial occurring in s_τ is of the form $m_{\tau'}$, where τ' is a (not necessarily semistandard) tableau with the same shape and same content as τ. Among those monomials, the largest with respect to the term order $\prec$ is m_τ. In particular, $m_\tau \in \mathrm{in}_\prec(R)$. Moreover, the leading monomial of an element $\sum_{\tau \in T} a_\tau s_\tau \in R$ is $m_{\tau'}$, where $m_{\tau'}$ is the largest monomial in $\{m_\tau \mid \tau \in T\, a_\tau \neq 0\}$ with respect to the term order $\prec$. Since to distinct semistandard tableaux are associated distinct monomials, the m_τ are also linearly independent. See also [27, Lemma 14.13]. □

Definition 4.2. Let $w \in \mathbb{N}^n$. Let $Q_w \subset \mathbb{R}^n$ be the polytope defined by $\nu_1 + \cdots + \nu_n = \frac{|w|}{2}$ and the inequalities (2.2) and (2.3).

Example 4.3. Let $w = (2, 2, 2, 2, 2)$. Then the inequalities for Q_w imply $\nu_1 = 2, \nu_5 = 0$, and $\nu_4 = 3 - \nu_2 - \nu_3$. Thus Q_w is the 2-dimensional polytope given by the inequalities $0 \leq \nu_2, \nu_3 \leq 2, 0 \leq 3 - \nu_2 - \nu_3 \leq 2$, and $2\nu_2 + \nu_3 \geq 2$. From this we can see that Q_w is isomorphic to the convex hull of $\langle (1, 0), (2, 0), (2, 1), (1, 2), (0, 2) \rangle$.

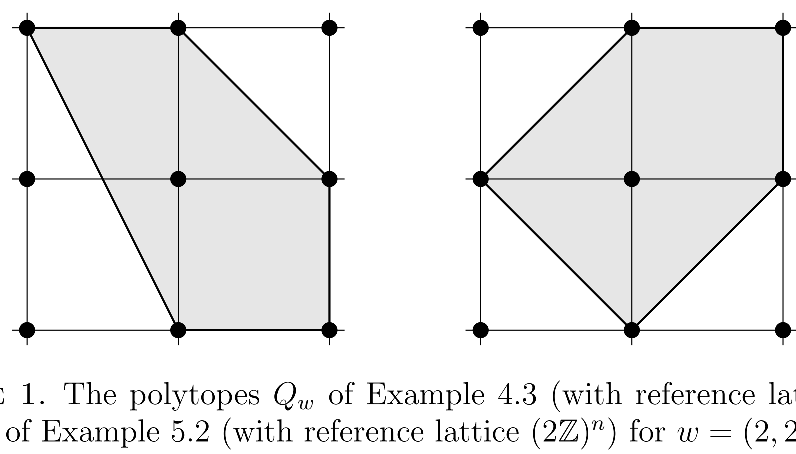

FIGURE 1. The polytopes Q_w of Example 4.3 (with reference lattice $\mathbb{Z}^n$) and P_w of Example 5.2 (with reference lattice $(2\mathbb{Z})^n$) for $w = (2, 2, 2, 2, 2)$.

Note that when $w = (1, 1, 1, 1, 1)$, Q_w contains no lattice points; in particular, it is not a lattice polytope. However, it follows from Lemma 5.3 and Lemma 5.4 that $2Q_w$ is a lattice polytope for all w.

Remark 4.4. Note that Q_w is a Gelfand-Tsetlin polytope, see for example [5].

Recall that to a rational polytope P is associated a graded monoid $S_P = \{(\mathbf{u}, d) \mid \mathbf{u} \in dP \cap \mathbb{Z}^n, d \in \mathbb{N}\}$ and an algebra $\mathbf{k}[S_P] = \langle x^{\mathbf{u}} z^d \mid (\mathbf{u}, d) \in S_P \rangle$.

Proposition 4.5. *The subalgebra* $\text{in}_\prec R$ *is isomorphic to the polytopal semigroup algebra* $\mathbf{k}[S_{Q_w}]$*. In particular,* $\text{in}_\prec R$ *is finitely generated.*

Proof. By Lemma 4.1, $\text{in}_\prec(R)$ is generated as a vector space by m_τ where τ runs over all semistandard Young tableaux of shape $2 \times \frac{d|w|}{2}$ with content dw. Note that for such a semistandard Young tableau τ we have $m_\tau = x^\nu y^{w-\nu}$, where ν is the partition associated to τ as in Definition 2.3. Let $\phi\colon \text{in}_\prec R \to \mathbf{k}[x_1, \ldots, x_n, z]$ be the homomorphism induced by letting $\phi(m_\tau) = x^\nu z^d$, when τ has shape $2 \times \frac{d|w|}{2}$. Since ν determines τ, this homomorphism is injective. It follows from Lemma 2.4 that it is surjective onto $\mathbf{k}[S_{Q_w}]$. Note that for any rational polytope, the associated polytopal semigroup algebra $\mathbf{k}[S_{Q_w}]$ is finitely generated. □

Remark 4.6. When $w = 1^n$, one can show that this toric degeneration is a degeneration of Fano varieties, and the corresponding line bundle the anticanonical line bundle. As this seems well known, we omit the proof.

5. The quadratic Gröbner basis

We now assume that $w \in (2\mathbb{Z})^n$. The goal of this section is to show that in this case, the polytopal semigroup algebra $\mathbf{k}[S_{Q_w}]$ is generated in degree 1, and admits a presentation such that the ideal of relations has a quadratic Gröbner basis. It then follows from general properties of SAGBI degenerations that R_w also admits such a presentation.

Instead of showing these properties directly for the polytopes Q_w, we will show them for a family of isomorphic polytopes P_w. The latter ones exhibit more symmetry that we will exploit later on. We denote the i'th component of a vector $u \in \mathbb{R}^{n-3}$ by $u(i+1)$ instead of u_i.

Definition 5.1. We say that a point $(x, y, z) \in \mathbb{R}^3$ *satisfies the triangle inequalities* if $x + y \geq z, x + z \geq y$, and $y + z \geq x$. To $w \in (2\mathbb{N})^n$ is associated a polytope $P_w \subset \mathbb{R}^{n-3}$ consisting of $(u(2), \ldots, u(n-2))$ such that

$$(w_1, w_2, u(2)), (u(n-2), w_{n-1}, w_n), \text{ and } (u(i-1), w_i, u(i))$$

satisfy the triangle inequalities for $3 \leq i \leq n-2$.

Example 5.2. When $w = (2, 2, 2, 2, 2)$, the polytope P_w is given by the inequalities $0 \leq u(2) \leq 4,\ 0 \leq u(3) \leq 4,\ u(2) + u(3) \geq 2, u(2) + 2 \geq u(3), u(3) + 2 \geq u(2)$. See Figure 4.3.

It follows from Lemma 5.4 that P_w is a lattice polytope for the lattice $M := (2\mathbb{Z})^{n-3}$.

Lemma 5.3. *The polytopal semigroups* $S_{Q_w} = \{(\nu, d) \mid d \in \mathbb{N}, \nu \in dQ_w \cap \mathbb{Z}^n\}$ *and* $S_{P_w} = \{(u, d) \mid d \in \mathbb{N}, u \in dP_w \cap M\}$ *are isomorphic.*

Proof. Let $V = \{(\nu_1, \dots, \nu_n, d) \in \mathbb{R}^n \times \mathbb{R} \mid \nu_1 = dw_1, \nu_n = 0, \text{ and } \nu_1 + \cdots + \nu_n = \frac{d|w|}{2}\}$, an affine subspace of $\mathbb{R}^n \times \mathbb{R}$. Then $S_{Q_w} \subset V$. We identify V with $\mathbb{R}^{n-3} \times \mathbb{R}$ via $(\nu_1, \dots, \nu_n, d) \mapsto (\nu_2, \dots, \nu_{n-2}, d)$, with inverse $(\nu_2, \dots, \nu_{n-2}, d) \mapsto (dw_1, \nu_2, \dots, \nu_{n-2}, \frac{d|w|}{2} - dw_1 - \nu_2 - \cdots - \nu_{n-2}, 0, d)$. Since $|w|$ is even, this identification respects the lattices $V \cap (\mathbb{Z}^n \times \mathbb{Z})$ and $\mathbb{Z}^{n-3} \times \mathbb{Z}$. Let

$$\phi\colon \mathbb{R}^{n-3} \times \mathbb{R} \to \mathbb{R}^{n-3} \times \mathbb{R}, (\nu_2, \dots, \nu_{n-2}, d) \mapsto (u(2), \dots, u(n-2), d),$$

where $u(\ell) = 2(\nu_2 + \cdots + \nu_\ell) - d(w_2 + \cdots + w_\ell - w_1)$ for $2 \leq \ell \leq n-2$. Then ϕ has an inverse given by letting

$$\nu_2 = \frac{u(2) - dw_1 + dw_2}{2} \text{ and } \nu_\ell = \frac{u(\ell) - u(\ell-1) + dw_\ell}{2}$$

for $3 \leq \ell \leq n-2$. So ϕ is an isomorphism. Moreover, it induces an isomorphism between $\mathbb{Z}^{n-3} \times \mathbb{Z}$ and $M \times \mathbb{Z}$.

We claim that $\phi(dQ_w \times \{d\}) = dP_w \times \{d\}$. Using the fact that for $(\nu_1, \dots, \nu_n) \in dQ_w$ we have $\nu_1 = dw_1, \nu_n = 0$, and $\nu_{n-1} = \frac{d|w|}{2} - (dw_1 + \nu_2 + \cdots + \nu_{n-2})$, the inequalities for $dQ_w \cap \mathbb{R}^{n-3}$ are in the left column below, where $3 \leq \ell \leq n-2$. The corresponding inequalities for dP_w are on the right.

$$\begin{aligned}
\nu_2 &\geq 0 & u(2) &\geq dw_1 - dw_2\\
\nu_2 &\leq dw_2 & u(2) &\leq dw_1 + dw_2\\
\nu_2 &\geq dw_2 - dw_1 & u(2) &\geq dw_2 - dw_1\\
\nu_\ell &\geq 0 & u(\ell-1) - u(\ell) &\leq dw_\ell\\
\nu_\ell &\leq dw_\ell & u(\ell) - u(\ell-1) &\leq dw_\ell\\
2(\nu_2 + \cdots + \nu_{\ell-1}) + \nu_\ell &\geq d(w_2 + \cdots + w_\ell - w_1) & u(\ell) + u(\ell-1) &\geq dw_\ell\\
\nu_2 + \cdots + \nu_{n-2} &\leq \frac{d|w|}{2} - dw_1 & u(n-2) &\leq dw_{n-1} + dw_n\\
\nu_2 + \cdots + \nu_{n-2} &\geq \frac{d|w|}{2} - dw_1 - dw_{n-1} & u(n-2) &\geq dw_n - dw_{n-1}\\
\nu_2 + \cdots + \nu_{n-2} &\geq \frac{d|w|}{2} - dw_1 - dw_n & u(n-2) &\geq dw_{n-1} - dw_n
\end{aligned}$$

where the last inequality on the left follows from (2.3) for $\ell = n-1$. It is now easy to check that the inequalities on the left correspond to the inequalities on the right under ϕ, so ϕ induces the required isomorphism of semigroups. □

Lemma 5.4. *We have the following properties of P_w.*

(i) The polytope P_w is normal with respect to M, i.e., every lattice point in $mP_w \cap M$ is a sum of m lattice points in $P_w \cap M$.
(ii) For $v, v' \in P_w \cap M$ there is $u, u' \in P_w \cap M$ such that $v + v' = u + u'$ and $|u(i) - u'(i)| \leq 2$ for all $2 \leq i \leq n-2$.

Proof. The proof of (i) closely follows [16, Lemma 7.3] and is essentially the proof of Lemma 6.4 in [15]. We first need to introduce some notation. Let $\sigma \in \{+, -\}$ and let e^σ denote rounding to the nearest even integer, where for $a \in \mathbb{Z}$, we let $e^+(2a+1) = 2a+2$

and $e^{-}(2a+1)=2a$. For $r=(r(2),\ldots,r(n-2))\in mP_w\cap M$, we say that a sequence of signs $\sigma(i)\in\{+,-\}$ for $2\le i\le n-3$ is (r,m)-admissible if it satisfies $\sigma(i+1)=-\sigma(i)$ if and only if $\frac{r(i)}{m}$ and $\frac{r(i+1)}{m}$ are odd integers and $\frac{r(i)}{m}+\frac{r(i+1)}{m}=w_{i+1}$. Such a sequence exists and is unique up to a global sign change.

We claim that if $\sigma(i)$ is (r,m)-admissible, then

$$u_r=(u(2),\ldots,u(n-2))=\left(e^{\sigma(2)}\left(\frac{r(2)}{m}\right),\ldots,e^{\sigma(n-2)}\left(\frac{r(n-2)}{m}\right)\right),$$

is a lattice point in P_w.

The following properties of e^{σ} for $\sigma\in\{+,-\}$, $a\in 2\mathbb{Z}$ and $x,y\in\mathbb{R}$ will be useful:

(1) e^{σ} is increasing.
(2) $e^{\sigma}(x+a)=e^{\sigma}(x)+a$.
(3) If $a\ge x$ then $a\ge e^{\sigma}(x)$.
(4) If $x+y\ge a$ then $e^{\sigma}(x)+e^{-\sigma}(y)\ge a$.
(5) $e^{\sigma}(x)+e^{\sigma}(y)\ge x+y-2$.
(6) $e^{\sigma}(-x)=-e^{-\sigma}(x)$.

That $(w_1,w_2,u(2))$ and $(u(n-2),w_{n-1},w_n)$ satisfy the triangle inequalities follows from the assumption that $\left(w_1,w_2,\frac{r(2)}{m}\right)$ and $\left(\frac{r(n-2)}{m},w_{n-1},w_n\right)$ satisfy the triangle inequalities and (1), (2) and (3). For example, we have $w_1+u(2)=w_1+e^{\sigma}\left(\frac{r(2)}{m}\right)=e^{\sigma}\left(w_1+\frac{r(2)}{m}\right)\ge e^{\sigma}(w_2)=w_2$ and $w_1+w_2\ge e^{\sigma}\left(\frac{r(2)}{m}\right)=u(2)$.

When $2\le i\le n-3$, we have to show

$$u(i)+w_{i+1}\ge u(i+1) \tag{5.1}$$

$$u(i+1)+w_{i+1}\ge u(i) \tag{5.2}$$

$$u(i)+u(i+1)\ge w_{i+1}. \tag{5.3}$$

We consider two cases. We first assume that $\sigma(i)=\sigma(i+1)$, and we let $\sigma=\sigma(i)=\sigma(i+1)$. Then $\frac{r(i)}{m}$ and $\frac{r(i+1)}{m}$ are not both odd integers or $\frac{r(i)}{m}+\frac{r(i+1)}{m}>w_{i+1}$. To see (5.2), note that $u(i+1)+w_{i+1}=e^{\sigma}\left(\frac{r(i+1)}{m}\right)+w_{i+1}=e^{\sigma}\left(\frac{r(i+1)}{m}+w_{i+1}\right)\ge e^{\sigma}\left(\frac{r(i)}{m}\right)=u(i)$ by (1), (2) and the assumption that $\left(\frac{r(i)}{m},\frac{r(i+1)}{m},w_{i+1}\right)$ satisfy the triangle inequalities. (5.1) follows similarly. For (5.3), if both $\frac{r(i)}{m}$ and $\frac{r(i+1)}{m}$ are odd integers then $\frac{r(i)}{m}+\frac{r(i+1)}{m}>w_{i+1}$ by assumption. Hence, by (5), $u(i)+u(i+1)\ge\frac{r(i)}{m}+\frac{r(i+1)}{m}-2>w_{i+1}-2$ which implies (5.3) since $u(i),u(i+1)$ and w_{i+1} are even. Otherwise there exists $x\in\left\{\frac{r(i)}{m},\frac{r(i+1)}{m}\right\}$ that is not an odd integer. Then $e^{\sigma}(x)=e^{-\sigma}(x)$ and now (5.3) follows from (4).

Suppose now that $\sigma(i+1)=-\sigma(i)$. Then $\frac{r(i)}{m}$ and $\frac{r(i+1)}{m}$ are odd integers and $\frac{r(i)}{m}+\frac{r(i+1)}{m}=w_{i+1}$. Then (5.3) follows from (4). If $\sigma(i)=+$ and $\sigma(i+1)=-$ then (5.1) follows easily. Since $\frac{r(i+1)}{m}$ is a positive odd integer, we have $2\frac{r(i+1)}{m}>0$, and using the assumption $\frac{r(i)}{m}+\frac{r(i+1)}{m}=w_{i+1}$, we obtain $u(i+1)+w_{i+1}=\frac{r(i+1)}{m}-1+w_{i+1}>\frac{r(i)}{m}-1=$

$u(i)-2$. Since $u(i+1), u(i)$ and w_{i+1} are even integers, (5.2) follows. The case $\sigma(i)=-$ and $\sigma(i+1)=+$ follows analogously.

To show (i), we proceed by induction on m. For $m=1$ there is nothing to show. Assume that $m \geq 2$. For $r \in mP \cap M$, let $u=u_r$ as in the claim. Then by the claim, u lies in $P_w \cap M$. Let $v=(v(2), \ldots, v(n-2))$, where $v(i)=e^{-\sigma(i)}\left(\frac{m-1}{m} r(i)\right)$. Note that it follows from (4) and (6) that $u(i)+v(i)=r(i)$, so $u+v=r$. Replacing u by v, σ by $-\sigma$, w_i by $(m-1) w_i$ and $\frac{r(i)}{m}$ by $\frac{m-1}{m} r(i)$, and noting that $\left(\frac{m-1}{m} r\right) \in(m-1) P_w$ and that $\frac{r(i)}{m}$ is odd if and only if $\frac{(m-1) r(i)}{m}$ is odd, the same arguments as in the proof of the claim show that $v=r-u \in(m-1) P_w \cap M$. But v is a sum of $m-1$ lattice points in P_w by induction.

For (ii), we apply the claim to $r=v+v'$ and let $(\sigma(2), \ldots, \sigma(n-2))$ be a $(r, 2)$-admissible sequence of signs. Then by the claim we have that

$$\begin{aligned} u &= \left(e^{\sigma(2)}\left(\frac{r(2)}{2}\right), \ldots, e^{\sigma(n-2)}\left(\frac{r(n-2)}{2}\right)\right) \text{ and } \\ u' &= \left(e^{-\sigma(2)}\left(\frac{r(2)}{2}\right), \ldots, e^{-\sigma(n-2)}\left(\frac{r(n-2)}{2}\right)\right) \end{aligned}$$

are lattice points in P_w. The assertion follows. □

Let J be the toric ideal associated to the polytope P_w, i.e., J is the kernel of the map $\mathbf{k}[X_u \mid u \in P_w \cap M] \rightarrow \mathbf{k}[S_{P_w}]$, where $X_u \mapsto (u(2), u(3), \ldots, u(n-2), 1)$. Since P_w is normal, the line bundle associated to P_w induces a projectively normal embedding of the toric variety X_{P_w} associated to P_w, with homogeneous coordinate ring $\mathbf{k}[X_u]/J$. By Proposition 4.5 and Lemma 5.3, the toric variety X_{P_w} is isomorphic to $\operatorname{Proj}(\operatorname{in}_{\prec}(R))$.

Definition 5.5. Let $m=\prod_{t=1}^{\ell} X_{u_t}$ be a monomial in $\mathbf{k}[X_u]$. We define the *norm* of m to be

$$N(m)=\sum_{t=1}^{\ell} \| u_t \|^2=\sum_{t=1}^{\ell} \sum_{i=2}^{n-2} u_t(i)^2 .$$

Definition 5.6. We say that a monomial m is *norm-minimal*, if for all m' with $m'-m \in J$, we have $N(m') \geq N(m)$.

The following lemma characterizes norm-minimal monomials.

Lemma 5.7. *A monomial m is norm-minimal if and only if for all $X_v X_{v'}$ dividing m, we have $|v(i)-v'(i)| \leq 2$ for all $2 \leq i \leq n-2$.*

Proof. To prove the Lemma, we will need the following fact:

($\star$) Let $a_i, b_i \in 2\mathbb{Z}$ with $a_1+\cdots+a_{\ell}=b_1+\cdots+b_{\ell}$, and $|a_i-a_j| \leq 2$ for all i, j. Then $\sum_{i=1}^{\ell} a_i^2 \leq \sum_{i=1}^{\ell} b_i^2$. Moreoever, equality holds if and only if $\{a_1, \ldots, a_{\ell}\}=\{b_1, \ldots, b_{\ell}\}$.

Given ($\star$), assume that m is norm-minimal, but that there exists a quadratic factor $X_v X_{v'}$ of m and $2 \leq i \leq n-2$ such that $|v(i)-v'(i)|>2$. By Lemma 5.4, there are

$u, u' \in P_w \cap M$ with $v + v' = u + u'$ and $|u(i) - u'(i)| \leq 2$ for all i. So $X_u X_{u'} - X_v X_{v'} \in J$, and by ($\star$), $N(X_u X_{u'}) < N(X_v X_{v'})$. Thus for $m' = m \frac{X_u X_{u'}}{X_v X_{v'}}$, we have $m' - m \in J$, but $N(m') < N(m)$, a contradiction. The converse follows immediately from ($\star$).

For ($\star$), note that since $|a_i - a_j| \leq 2$ for all i, j, there exists α such that $a_i \in \{\alpha, \alpha+2\}$ for all i. After renumbering, we may assume $a_1 = \cdots = a_p = \alpha$ and $a_{p+1} = \cdots = a_\ell = \alpha + 2$. If we set $b_i = a_i + k_i$, then $\sum k_i = 0$, and we have $\sum_{i=1}^{\ell} b_i^2 - \sum_{i=1}^{\ell} a_i^2 = \sum_{i=1}^{\ell} k_i^2 + \sum_{i=p+1}^{\ell} 4k_i$. Note that when $k_i \leq -4$, then $k_i^2 + 4k_i \geq 0$, and if $k_i \geq 0$, then $4k_i \geq 0$, so it suffices to show that if $\sum k_i \geq 0$, $k_i \in 2\mathbb{Z}$, and $k_i = -2$ for $p+1 \leq i \leq \ell$, then $\sum_{i=1}^{\ell} k_i^2 + \sum_{i=p+1}^{\ell} 4k_i \geq 0$. This in turn follows from the fact that for $k \in 2\mathbb{N}$ we have $k^2 \geq 2k$, so for $k_i \in 2\mathbb{N}$ with $\sum_{i=1}^{p} k_i \geq 2(\ell - p)$, we have $\sum_{i=1}^{p} k_i^2 \geq \sum_{i=1}^{p} 2k_i \geq 4(\ell - p)$. Now suppose equality holds, so $\sum_{i=1}^{\ell} k_i^2 + \sum_{i=p+1}^{\ell} 4k_i = 0$ where $k_i \in 2\mathbb{Z}$ and $\sum k_i = 0$. Note that $k_i^2 + 4k_i$ is non-negative unless $k_i = -2$. If $R = \{i \mid k_i = -2\}$ and $T = \{i \mid k_i > 0\}$, then we must have $\sum_{i \in T} k_i \geq 2|R|$ and $\sum_{i \in T} k_i^2 + \sum_{i \in T, i \geq p+1} 4k_i \leq 4|R|$, but the only situation when this holds is when $k_i = 2$ for all $i \in T$, no element in T is larger than p, and $|T| = |R|$. The claim follows. □

We now proceed to define a term order on the monomials in the variables X_u, $u \in P_w$.

We first use the standard lexicographic ordering $<_{\text{lex}}$ on $M \cong \mathbb{Z}^{n-3}$ to order the variables $X_u, u \in P_w$. Let $\prec_{\text{grevlex}}$ be the graded reverse lexicographic order on $k[X_u \mid u \in P_w \cap M]$ induced by this ordering of the variables, i.e., $m' \prec_{\text{grevlex}} m$ if $\deg(m') < \deg(m)$ or $\deg(m') = \deg(m)$ and for the smallest variable where the exponents of m and m' differ the exponent of m' is larger than the exponent of m.

We define $m' \prec m$ iff

- $\deg(m') < \deg(m)$, or
- $\deg(m') = \deg(m)$ and $N(m') < N(m)$, or
- $\deg(m') = \deg(m)$, $N(m') = N(m)$, and $m' \prec_{\text{grevlex}} m$.

We shall consider two types A, B of quadratic binomial relations.

Definition 5.8. (Type A) The type A relations are relations $X_v X_{v'} - X_u X_{u'} \in J$, where $N(X_v X_{v'}) > N(X_u X_{u'})$.

Definition 5.9. (Type B at position j) Suppose that $u, v \in P_w \cap M$, and $3 \leq j \leq n-2$. Suppose that $(u(j-1), w_j, v(j))$ and $(v(j-1), w_j, u(j))$ satisfy the triangle inequalities. Let

$$u' = (u(2), \ldots, u(j-1), v(j), \ldots, v(n-2)),$$

$$v' = (v(2), \ldots, v(j-1), u(j), \ldots, u(n-2)).$$

We call $X_u X_v - X_{u'} X_{v'}$ a type B relation at position j.

Note that $u', v' \in P_w$ and $u + v = u' + v'$, so a type B relation is well-defined. Moreover, $N(X_u X_v) = N(X_{u'} X_{v'})$ for any relation $X_u X_v - X_{u'} X_{v'}$ of type B.

Theorem 5.10. *The relations of type A and B form a quadratic Gröbner basis for the ideal J.*

Proof. It suffices to show that

$$\mathrm{in}_{\prec}(J) = \langle \mathrm{in}_{\prec} f \mid f \text{ is a type A or a type B relation} \rangle.$$

Let $m \in \mathrm{in}_{\prec} J$. Suppose m is not norm-minimal. By Lemma 5.7 there exists a quadratic factor $X_v X_{v'}$ dividing m such that $|v(i) - v'(i)| > 2$ for some i. By Lemma 5.4 there is $u, u' \in P_w \cap M$ such that $X_v X_{v'} - X_u X_{u'} \in J$, and $|u(i) - u'(i)| \leq 2$ for all i. By Lemma 5.7, $X_u X_{u'}$ is norm-minimal, and $X_v X_{v'}$ is not, so $X_v X_{v'} - X_u X_{u'}$ is a type A relation and $m \in \langle X_v X_{v'} \rangle \subset \langle \mathrm{in}_{\prec} f \mid f \text{ is a type A relation} \rangle$.

Suppose m is norm-minimal. Let $f \in J$ be such that $\mathrm{in}_{\prec} f = m$. Since J is a homogeneous binomial ideal, we may assume that $f = m - m'$, where m' is a norm-minimal monomial of the same degree as m. Then $N(m') = N(m)$, but $m' \prec_{grevlex} m$. Let $m = X_{u_1} X_{u_2} \cdots X_{u_\ell}$, where $u_1 \leq_{lex} u_2 \leq_{lex} \cdots \leq_{lex} u_\ell$ and $m' = X_{v_1} X_{v_2} \cdots X_{v_\ell}$ where $v_1 \leq_{lex} v_2 \leq_{lex} \cdots \leq_{lex} v_\ell$. Note that since $m - m' \in J$, we have

$$\sum_{s=1}^{\ell} u_s = \sum_{s=1}^{\ell} v_s. \tag{5.4}$$

Factoring out the largest common multiple of m and m', we may assume $v_1 <_{lex} u_1$. Let j be the first index where $v_1(j) < u_1(j)$. Note that if $i < j$ then $u_1(i) = v_1(i)$.

It follows from Lemma 5.7 and from (5.4) that for every $2 \leq k \leq n-2$ there exists an even integer α_k such that for every X_u dividing m or m', $u(k) \in \{\alpha_k, \alpha_k + 2\}$.

This implies that $v_1(j) = \alpha_j$ and $u_1(j) = \alpha_j + 2$. It follows from (5.4) that there exists some $t > 1$ such that $u_t(j) = v_1(j) = \alpha_j$. Note that, since $u_1 <_{\mathrm{lex}} u_t$ and $u_1(j) > u_t(j)$, there is $i' < j$ such that $u_1(i') < u_t(i')$ and $u_1(i) = u_t(i)$ for $i < i'$. In particular, $j > 2$. Now $(u_1(j-1), w_j, u_t(j)) = (v_1(j-1), w_j, v_1(j))$ satisfy the triangle inequalities, since $v_1 \in P_w$. If $(u_t(j-1), w_j, u_1(j))$ satisfy the triangle inequalities, then there exists a type B relation at position j of the form $X_{u_1} X_{u_t} - X_{u_1'} X_{u_t'}$. Since $u_1' <_{\mathrm{lex}} u_1 <_{\mathrm{lex}} u_t$, we have $X_{u_1} X_{u_t} \succ_{grevlex} X_{u_1'} X_{u_t'}$. Therefore, $m \in \mathrm{in}_{\prec}\{f \mid f \text{ is a type B relation}\}$.

Now suppose $(u_t(j-1), w_j, u_1(j))$ do not satisfy the triangle inequalities. This implies that whenever $v_s(j-1) = u_t(j-1)$, then $v_s(j) \neq u_1(j)$, since $(v_s(j-1), w_j, v_s(j))$ satisfy the triangle inequalities and so $v_s(j) = v_1(j)$. Note however that $v_1(j-1) = u_1(j-1) \neq u_t(j-1)$ by assumption. In particular,

$$\{s \mid 1 \leq s \leq \ell, v_s(j-1) = u_t(j-1)\} \subsetneq \{s \mid 1 \leq s \leq \ell, v_s(j) = v_1(j)\}.$$

Since $\sum u_s(j-1) = \sum v_s(j-1)$, we have

$$\#\{s \mid 1 \leq s \leq \ell, v_s(j-1) = u_t(j-1)\} = \#\{s \mid 1 \leq s \leq \ell, u_s(j-1) = u_t(j-1)\}$$

and similarly

$$\#\{s \mid 1 \leq s \leq \ell, v_s(j) = v_1(j)\} = \#\{s \mid 1 \leq s \leq \ell, u_s(j) = v_1(j)\}.$$

Using $u_t(j) = v_1(j)$, we obtain

$$\#\{s \mid 1 \leq s \leq \ell, u_s(j-1) = u_t(j-1)\} < \#\{s \mid 1 \leq s \leq \ell, u_s(j) = u_t(j)\}.$$

Pick an s such that $u_s(j-1) \neq u_t(j-1)$ but $u_s(j) = u_t(j)$. Since $u_1(j) \neq u_t(j)$, $s \neq 1$. By assumption, $u_1(j-1) \neq u_t(j-1)$, hence $u_s(j-1) = u_1(j-1)$ since $\{u_s(j-1), u_1(j-1), u_t(j-1)\} \subset \{\alpha_{j-1}, \alpha_{j-1}+2\}$. So $(u_s(j-1), w_j, u_1(j)) = (u_1(j-1), w_j, u_1(j))$ satisfy the triangle inequalities. On the other hand, since $u_s(j) = u_t(j) = v_1(j)$, we have that $(u_1(j-1), w_j, u_s(j)) = (v_1(j-1), w_j, v_1(j))$ satisfy the triangle inequalities. So there exists a type B relation $X_{u_1}X_{u_s} - X_{u_1'}X_{u_s'}$ at position j. Note that $X_{u_1}X_{u_s} \succ_{\text{grevlex}} X_{u_1'}X_{u_s'}$, so $m \in \text{in}_\prec\{f \mid f \text{ is a type B relation}\}$. □

Proposition 5.11. *The initial ideal* $\text{in}_\prec(J)$ *is radical.*

Proof. Since the initial ideal is generated by quadratic monomials, we only need to check there is no perfect square $X_v^2 \in \text{in}_\prec(J)$. Suppose that $X_v^2 - X_uX_w \in J$ is a non-zero relation for some u, v, w. Then $2v = u + w$, and so $N(X_v^2) < N(X_uX_w)$, since by Lemma 5.7 X_v^2 is norm-minimal, but X_uX_w is not. So X_v^2 is no leading term of any binomial in J. □

Note that this implies that P_w admits a regular unimodular triangulation, see [35, Corollary 8.9]

Proof of Theorem 1.2. By Kempe's theorem [16, Theorem 2.3], R_w is generated by its lowest degree invariants. Note that for $w \in (2\mathbb{Z})^n$ this also follows from Lemma 5.4, Proposition 4.5, and Lemma 5.3. Recall that I_w denotes the ideal of relations between these generators. Since the polytopal semigroup algebra $\mathbf{k}[S_{P_w}]$ is a SAGBI degeneration of R_w by Proposition 4.5, and since the toric ideal J associated to P_w has a square free quadratic initial ideal by Theorem 5.10 and Proposition 5.11, the claim follows from [4, Corollary 2.2]. The Koszul property follows for example from [8, Proposition 3]. □

Remark 5.12. In [24, Theorem 1.10] Manon generalizes Theorem 1.2 to certain subpolytopes of P_w. For $L \geq 0$, the polytope P_L is given by the adding to the inequalities of Definition 5.1 the inequalities $w_1 + w_2 + u(2) \leq 2L, u(n-2) + w_{n-1} + w_n \leq 2L, u(i-1) + w_i + u_i \leq 2L$. Note that when L is large, then $P_w = P_L$. For $L = 1$, the polytopes P_L are slices of the polytopes studied in [3].

Remark 5.13. It is easy to see that for $w = 1^n$, P_w is reflexive and that S_{P_w} is Gorenstein. This implies that the toric variety V associated to P_w is arithmetically Gorenstein and Fano. In particular, V has canonical singularities. However, when $w = 1^6$, then V does not have terminal singularities. Compare also [30, Proposition 1.4]. The Gorenstein property for the toric varieties arising as degenerations of R_w corresponding to arbitrary trivalent trees was studied by Manon in [25].

References

[1] Jörgen Backelin and Ralf Fröberg. Koszul algebras, Veronese subrings and rings with linear resolutions. *Rev. Roumaine Math. Pures Appl.*, 30(2):85–97, 1985. 2

[2] Sara Billey, Victor Guillemin, and Etienne Rassart. A vector partition function for the multiplicities of $\mathfrak{sl}_k\mathbb{C}$. *J. Algebra*, 278(1):251–293, 2004. arXiv:math/0307227 [math.CO]. 2

[3] Weronika Buczyńska and Jarosław A. Wiśniewski. On geometry of binary symmetric models of phylogenetic trees. *J. Eur. Math. Soc. (JEMS)*, 9(3):609–635, 2007. arXiv:math/0601357 [math.AG]. 17

[4] Aldo Conca, Jürgen Herzog, and Giuseppe Valla. Sagbi bases with applications to blow-up algebras. *J. Reine Angew. Math.*, 474:113–138, 1996. 9, 17

[5] Jesús A. De Loera and Tyrrell B. McAllister. Vertices of Gelfand-Tsetlin polytopes. *Discrete Comput. Geom.*, 32(4):459–470, 2004. arXiv:math/0309329 [math.CO]. 11

[6] Igor Dolgachev. *Lectures on invariant theory*, volume 296 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003. 3, 4

[7] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. 10

[8] David Eisenbud, Alyson Reeves, and Burt Totaro. Initial ideals, Veronese subrings, and rates of algebras. *Adv. Math.*, 109(2):168–187, 1994. arXiv:alg-geom/9310007. 2, 17

[9] Philip Foth and Yi Hu. Toric degenerations of weight varieties and applications. In *Travaux mathématiques. Fasc. XVI*, Trav. Math., XVI, pages 87–105. Univ. Luxemb., Luxembourg, 2005. arXiv:math/0406329 [math.AG]. 9

[10] R. Fröberg. Koszul algebras. In *Advances in commutative ring theory (Fez, 1997)*, volume 205 of *Lecture Notes in Pure and Appl. Math.*, pages 337–350. Dekker, New York, 1999. 9

[11] I. M. Gel'fand and R. D. MacPherson. Geometry in Grassmannians and a generalization of the dilogarithm. *Adv. in Math.*, 44(3):279–312, 1982. 4

[12] N. Gonciulea and V. Lakshmibai. Degenerations of flag and Schubert varieties to toric varieties. *Transform. Groups*, 1(3):215–248, 1996. 9

[13] Brendan Hassett. Moduli spaces of weighted pointed stable curves. *Adv. Math.*, 173(2):316–352, 2003. 8

[14] Milena Hering. Multigraded regularity and the Koszul property. *J. Algebra*, 323(4):1012–1017, 2010. arXiv:0712.2251 [math.AG]. 2

[15] Benjamin Howard, John Millson, Andrew Snowden, and Ravi Vakil. The projective invariants of ordered points on the line. arXiv:math/0505096 [math.AG], 2005. 5, 12

[16] Benjamin Howard, John Millson, Andrew Snowden, and Ravi Vakil. The equations for the moduli space of n points on the line. *Duke Math. J.*, 146(2):175–226, 2009. 1, 3, 8, 12, 17

[17] Benjamin Howard, John Millson, Andrew Snowden, and Ravi Vakil. The ideal of relations for the ring of invariants of n points on the line. *J. Eur. Math. Soc. (JEMS)*, 14(1):1–60, 2012. arXiv:0909.3230 [math.AG]. 1, 3, 8, 9

[18] Roger Howe. The classical groups and invariants of binary forms. In *The mathematical heritage of Hermann Weyl (Durham, NC, 1987)*, volume 48 of *Proc. Sympos. Pure Math.*, pages 133–166. Amer. Math. Soc., Providence, RI, 1988. 2, 5

[19] Sean Keel and Jenia Tevelev. Equations for $\overline{M}_{0,n}$. *Internat. J. Math.*, 20(9):1159–1184, 2009. arXiv:math/0507093 [math.AG]. 2

[20] Alfred Kempe. On regular difference terms. *Proc. London Math. Soc.*, 259:343–350, 1894. 1

[21] R. C. King, C. Tollu, and F. Toumazet. Stretched Littlewood-Richardson and Kostka coefficients. In *Symmetry in physics*, volume 34 of *CRM Proc. Lecture Notes*, pages 99–112. Amer. Math. Soc., Providence, RI, 2004. 8

[22] A. N. Kirillov and N. Yu. Reshetikhin. The Bethe ansatz and the combinatorics of Young tableaux. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 155(Differentsialnaya Geometriya, Gruppy Li i Mekh. VIII):65–115, 194, 1986. 2

[23] Robert Lazarsfeld. *Positivity in algebraic geometry. I.* Springer-Verlag, Berlin, 2004. 9

[24] Christopher Manon. Coordinate rings for the moduli stack of $SL_2(\mathbb{C})$ quasi-parabolic principal bundles on a curve and toric fiber products. *J. Algebra*, 365:163–183, 2012. arXiv:1105.2045 [math.AC]. 3, 17

[25] Christopher Manon. Gorenstein semigroup algebras of weighted trees and ordered points on the projective line. *J. Algebra*, 354:110–120, 2012. arXiv:0810.1353 [math.AC]. 17

[26] Christopher A. Manon. *Presentations of semigroup algebras of weighted trees*. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–University of Maryland, College Park. 9

[27] Ezra Miller and Bernd Sturmfels. *Combinatorial Commutative Algebra*, volume 227 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. 9, 10

[28] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994. 3, 7

[29] Hariharan Narayanan. On the complexity of computing Kostka numbers and Littlewood-Richardson coefficients. *J. Algebraic Combin.*, 24(3):347–354, 2006. 8

[30] Benjamin Nill. Gorenstein toric fano varieties. *Manuscripta Math.*, 116(2):183–210, 2005. arXiv:math/0405448 [math.AG]. 17

[31] Alexander Polishchuk and Leonid Positselski. *Quadratic algebras*, volume 37 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2005. 9

[32] N. J. A. Sloane. The on-line encyclopedia of integer sequences. Published electronically at http://oeis.org, Sequence A012250, 2012. 8

[33] N. J. A. Sloane. The on-line encyclopedia of integer sequences. Published electronically at http://oeis.org, Sequence A012249, 2012. 8

[34] David Speyer and Bernd Sturmfels. The tropical Grassmannian. *Adv. Geom.*, 4(3):389–411, 2004. arXiv:math/0304218 [math.AG]. 9

[35] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996. 17

[36] Michael Thaddeus. Geometric invariant theory and flips. *J. Amer. Math. Soc.*, 9(3):691–723, 1996. 8

[37] Jakub Witaszek. The degeneration of the Grassmannian into a toric variety and the calculation of the eigenspaces of a torus action. arXiv:1209.3689 [math.AG], 2012. 2, 7

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