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# Asymptotics of Some Convolutional Recurrences

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## Abstract

We study the asymptotic behavior of the terms in sequences satisfying recurrences of the form  $a_n = a_{n-1} + \sum_{k=d}^{n-d} f(n, k)a_k a_{n-k}$  where, very roughly speaking,  $f(n, k)$  behaves like a product of reciprocals of binomial coefficients. Some examples of such sequences from map enumerations, Airy constants, and Painlevé I equations are discussed in detail.

## 1 Main results

There are many examples in the literature of sequences defined recursively using a convolution. It often seems difficult to determine the asymptotic behavior of such sequences. In this note we study the asymptotics of a general class of such sequences. We prove

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subexponential growth by using an iterative method that may be useful for other recurrences. By subexponential growth we mean that, for every constant  $D > 1$ ,  $a_n = o(D^n)$  as  $n \rightarrow \infty$ . Thus our motivation for this note is both the method and the applications we give.

Let  $d > 0$  be a fixed integer and let  $f(n, k) \geq 0$  be a function that behaves like a product of some powers of reciprocals of binomial coefficients, in a general sense to be specified in Theorem 1. We deal with the sequence  $a_n$  for  $n \geq d$  where  $a_d, a_{d+1}, \dots, a_{2d-1} \geq 0$  are arbitrary and, when  $n \geq 2d$ ,

$$a_n = a_{n-1} + \sum_{k=d}^{n-d} f(n, k) a_k a_{n-k}. \quad (1)$$

Without loss of generality,

$$\text{we assume that } f(n, k) = f(n, n - k)$$

since we can replace  $f(n, k)$  and  $f(n, n - k)$  in (1) with  $\frac{1}{2}(f(n, k) + f(n, n - k))$ .

Theorem 1 proves subexponential growth. Theorem 2 provide more accurate estimates under additional assumptions. In Section 2, we apply the corollary to some examples.

**Theorem 1 (Subexponential growth)** *Let  $a_n$  be defined by recursion (1) with  $a_d > 0$ . Suppose there is a function  $R(x)$  defined on  $(0, 1/2]$ , an  $\alpha > 0$  and an  $r$  such that*

- (a)  $0 < R(x) < r < 1$ ,
- (b)  $\lim_{x \rightarrow 0^+} R(x) = 0$ ,
- (c)  $0 \leq f(n, k) = O(n^{-\alpha} R^{k-d}(k/n))$  uniformly for  $d \leq k \leq n/2$ .

*Then  $a_n$  grows subexponentially; in fact,*

$$a_n = (1 + O(n^{-\alpha})) a_{n-1}. \quad (2)$$

*Proof:* We first note that the  $a_n$  are non-decreasing when  $n \geq 2d - 1$ .

Our proof is in three steps. We first prove that  $a_n = O(C^n)$  for some constant  $C > 2$ . We then prove that  $C$  can be chosen very close to 1. Finally we deduce (2) and subexponential growth.

**First Step:** Since the bound in (c) is bounded by some constant times the geometric series  $n^{-\alpha} r^{k-d}$  with ratio less than 1,  $\sum_{k=d}^{n-d} f(n, k) = O(n^{-\alpha})$ . Hence we can choose  $M$  so large that  $\sum_{k=d}^{n-d} f(n, k) < 1/4$  when  $n > M$ . Next choose  $C \geq 2$  so large ( $C = \max\{a_d, a_{d+1}, \dots, a_{2d-1}, a_M, 2\}$  will do) that  $a_n < 2C^n$  for  $n \leq M$ . By induction, using the recursion (1), we have for  $n > M$

$$a_n < 2C^{n-1} + (1/4)4C^n \leq C^n + C^n = 2C^n.$$

**Second Step:** By (b) there is a  $\lambda$  in  $(0, 1/2)$  such that  $R(x) < \frac{1}{2C}$  for  $0 < x < \lambda$ . Fix any  $D \leq C$  such that  $a_n = O(D^n)$ , which is true for  $D = C$  by the First Step.

Split the sum in (1) into  $\lambda n \leq k \leq (1 - \lambda)n$  and the rest, calling the first range of  $k$  the “center” and the rest the “tail”. Noting  $r < 1$ , the center sum is bounded by

$$2 \sum_{k=\lambda n+1}^{n/2} f(n, k) a_k a_{n-k} = O\left(D^n \sum_{k=\lambda n+1}^{n/2} r^{k-d}\right) = O((r^\lambda D)^n). \quad (3)$$

Since  $a_j$  are increasing, the tail sum is bounded by

$$\begin{aligned} 2 \sum_{k=d}^{\lambda n} f(n, k) a_k a_{n-k} &= O(n^{-\alpha}) a_{n-1} \sum_{k=d}^{\lambda n} R(x)^{k-d} D^k \\ &= O(n^{-\alpha}) a_{n-1} \sum_{k=d}^{\lambda n} (DR(x))^{k-d} = O(n^{-\alpha} a_{n-1}), \end{aligned} \quad (4)$$

where the last equality follows from the fact that  $DR(x) < 1/2$ . Combining (3) and (4),

$$a_n = (1 + O(n^{-\alpha})) a_{n-1} + O((r^\lambda D)^n). \quad (5)$$

When  $r^\lambda D > 1$ , induction on  $n$  easily leads to  $a_n = O((r^\lambda D')^n)$  for any  $D' > D$ , an exponential growth rate no larger than  $r^\lambda D'$ .

Since  $r^\lambda$  has a fixed value less than one, we can iterate this process, replacing  $D$  by  $r^\lambda D'$  at the start of the Second Step. We finally obtain a growth rate  $D > 1$  with  $r^\lambda D < 1$ . This completes the second step.

**Third Step:** With the value of  $D$  just obtained, the last term in (5) is exponentially small and hence is  $O(n^{-\alpha} a_{n-1})$ . Thus we obtain (2) which immediately implies subexponential growth of  $a_n$ , since  $1 + O(n^{-\alpha}) < D$  for any  $D > 1$  and sufficiently large  $n$ . ■

To say more than (2), we need additional information about the behavior of the  $f(n, k)$ . When  $f(n, k)/f(n, d)$  is small for each  $k$  in the range  $d + 1 \leq k \leq n - d - 1$ , the first and last terms dominate the sum. The following theorem is based on this observation.

**Theorem 2 (Asymptotic behavior)** *Assume (a)–(c) of Theorem 1 hold. Suppose further that there is a  $\beta > 0$  such that*

$$\frac{f(n, k)}{f(n, d)} = O(n^{-\beta} r^{k-d-1}) \quad \text{uniformly for } d + 1 \leq k \leq n/2. \quad (6)$$

Then

$$\log a_n = 2a_d \sum_{k=2d+1}^n f(k, d) + O\left(\sum_{k=2d}^n f(k, d) (k^{-\alpha} + k^{-\beta})\right). \quad (7)$$

*Proof:* We assume  $n > 2d$ . Remove the  $k = d$  and  $k = n - d$  terms from the sum in (1). We first deal with the remaining sum. Theorem 1 gives  $a_k = O(D^k)$  for all  $D > 1$ , so we can assume  $D < 1/r$ . Using (6)

$$\begin{aligned} \sum_{k=d+1}^{n-d-1} f(n, k) a_k a_{n-k} &= O\left(f(n, d) n^{-\beta} a_{n-1}\right) \sum_{k=d+1}^{n/2} r^{k-d-1} D^k \\ &= O\left(f(n, d) n^{-\beta} a_{n-1}\right). \end{aligned}$$

Combining this with (1), we obtain

$$\begin{aligned} a_n &= a_{n-1} + 2a_d f(n, d) a_{n-d} + f(n, d) O(n^{-\beta}) a_{n-1} \\ &= a_{n-1} \left(1 + 2a_d f(n, d) + \{O(n^{-\alpha}) + O(n^{-\beta})\} f(n, d)\right), \end{aligned}$$

Taking logarithms and noting for expansion purposes that  $f(n, d) = O(n^{-\alpha})$ , we obtain

$$\log a_n - \log a_{n-1} = 2a_d f(n, d) + O\left((n^{-\alpha} + n^{-\beta}) f(n, d)\right).$$

Sum over  $n$  starting with  $n = 2d + 1$ . The theorem follows immediately when we note that the constant terms can be incorporated into the  $O(\cdot)$  in (7) since the sum therein is bounded below by a nonzero constant. ■

**Corollary 1** *Assume the conditions of Theorem 2 hold and  $f(n, d) = \Theta(n^{-\alpha})$ .*

- *If  $\alpha < 1$ , then  $a_n = \exp(\Theta(n^{1-\alpha}))$ .*
- *If  $\alpha > 1$ , then  $a_n = K + O(n^{1-\alpha})$  for some constant  $K$ .*
- *If  $f(n, d) - A/n$  are the terms of a convergent series, then  $a_n \sim Cn^{2Aa_d}$  for some positive constant  $C$ .*

*Proof:* Since  $\alpha > 0$  and  $\beta > 0$ , (7) gives  $\log a_n = \Theta(\sum_{k=2d+1}^n k^{-\alpha})$ . The case  $\alpha < 1$  follows immediately; for  $\alpha > 1$ , we see that  $a_n$  is bounded and nondecreasing and therefore has a limit  $K$ . For  $m > n$ , (2) gives  $\log(a_m/a_n) = O(n^{1-\alpha})$  uniformly in  $m$ . Letting  $m \rightarrow \infty$ , we obtain the claim regarding  $\alpha > 1$ .

For  $\alpha = 1$ , the first sum in (7) is  $A \log n + B + o(1)$  for some constant  $B$ , and the last sum in (7) converges. ■

## 2 Examples

We apply Theorem 2 and Corollary 1 to some recursions which arise from combinatorial applications. In our examples,  $f(n, k)$  behaves like a product of the reciprocal of binomial coefficients, which satisfies the conditions of Theorems 1 and 2. A more general case of interest is when  $f(n, k)$  takes the form of the product of functions like

$$g(n, k) = \frac{[a]_k [a]_{n-k}}{[a]_n}$$

for some constant  $a > 0$ , where  $[x]_k = x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}$ , the rising factorial. We note that when  $a = 1$ ,  $g(n, k) = \binom{n}{k}^{-1}$ .

We begin with some useful bounds. When  $a > 0$  and  $1 \leq k \leq n/2$ ,

$$\begin{aligned} g(n, k) &= \prod_{j=0}^{k-1} \frac{a+j}{a+n-k+j} < \left(\frac{a+k}{a+n}\right)^k \\ &\leq (k/n)^k \left(\frac{1+a/k}{1+a/n}\right)^k = O((k/n)^k) = O(n^{-1}(3k/2n)^{k-1}) \end{aligned} \tag{8}$$

since  $k(2/3)^{k-1}$  is bounded. So  $g$  satisfies the condition on  $f$  in Theorem 1(c), with  $\alpha = 1$ . Similarly, when  $a > 0$  and  $d \leq k \leq n/2$ ,

$$\frac{g(n, k)}{g(n, d)} = \prod_{j=0}^{k-d-1} \frac{a+d+j}{a+n-k+d+j} = O(n^{-1}(3k/2n)^{k-d-1}). \tag{9}$$

This is in accordance with (6) with  $\beta = 1$ .

**Example 1 (Map enumeration constants)** There are numbers  $t_n$  appearing in the asymptotic enumeration of maps in an orientable surface of genus  $n$ , whose value does not concern us here. Define  $u_n$  by

$$t_n = 8 \frac{[1/5]_n [4/5]_{n-1}}{\Gamma\left(\frac{5n-1}{2}\right)} \left(\frac{25}{96}\right)^n u_n.$$

Then  $u_1 = 1/10$  and  $u_n$  satisfies the following recursion [3]

$$u_n = u_{n-1} + \sum_{k=1}^{n-1} f(n, k) u_k u_{n-k} \quad \text{for } n \geq 2, \tag{10}$$

where

$$f(n, k) = \frac{[1/5]_k [1/5]_{n-k} [4/5]_{k-1} [4/5]_{n-k-1}}{[1/5]_n [4/5]_{n-1}}.$$

From the observations above, the conditions of Theorem 2 are satisfied with  $d = 1$ ,  $R(\lambda) = (3\lambda/2)^2$  and  $\alpha = \beta = 2$ . Hence,  $u_n \sim K$  for some constant  $K$ . Unlike the proof in [3], this does not depend on the value of  $u_1$ . ■

**Example 2 (Airy constants)** The *Airy constants*  $\Omega_n$  are determined by  $\Omega_1 = 1/2$  and the recurrence [7]

$$\Omega_n = (3n-4)n\Omega_{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} \Omega_k \Omega_{n-k} \quad \text{for } n \geq 2.$$

Let  $\Omega_n = n! [2/3]_{n-1} 3^n a_n$ . Then  $a_n$  satisfies (1) with  $d = 1$  and

$$f(n, k) = \frac{[2/3]_{k-1} [2/3]_{n-k-1}}{[2/3]_{n-1}}.$$

Theorem 2 applies with  $d = 1$ ,  $R(\lambda) = 3\lambda/2$  and  $\alpha = \beta = 1$ . Since

$$f(n, 1) = \frac{1}{n - 4/3} = \frac{1}{n} + \frac{4/3}{n(n - 4/3)}$$

and  $a_1 = 1/6$ , we have  $a_n \sim Cn^{1/3}$  for some constant  $C$ .

We note that it is possible to apply the result of Olde Daalhuis [13] to obtain a full asymptotic expansion for  $\Omega_n$ . Let

$$A_n = \frac{\Omega_n}{3^n n!}.$$

Then the recursion for  $\Omega_n$  becomes

$$A_n = (n - 4/3) A_{n-1} + \sum_{k=1}^{n-1} A_k A_{n-k}, \quad n \geq 2.$$

It follows that the formal series

$$F(z) = \sum_{n \geq 1} \frac{A_n}{z^n}$$

satisfies the Riccati equation

$$F'(z) + \left(1 + \frac{1}{3z}\right) F(z) - F^2(z) - \frac{1}{6z} = 0.$$

It then follows from the result of Olde Daalhuis [13] that

$$A_n \sim \frac{1}{2\pi} \sum_{k=0}^{\infty} b_k \Gamma(n - k), \quad \text{as } n \rightarrow \infty,$$

where  $b_0 = 1$  and  $b_k$  can be computed using the recursion

$$b_k = \frac{-2}{k} \sum_{j=2}^{k+1} b_{k+1-j} A_j, \quad k \geq 1.$$

In particular, we have

$$\Omega_n \sim \frac{1}{2\pi} \Gamma(n) 3^n n! = \frac{1}{2\pi n} (n!)^2 3^n, \quad \text{as } n \rightarrow \infty.$$

It is well known that solutions to the Riccati equation have infinitely many singularities, hence  $F(z)$  (via its Borel transform [2]) cannot satisfy a linear ODE with polynomial coefficients. This implies that the sequence  $A_n$  (and hence  $\Omega_n$ ) is not holonomic. ■

**Example 3** The following recursion, with  $\ell > 0$  and  $\ell \neq 1/2$ , appeared in [6]. The Airy constants are the special case  $\ell = 1$ . The case  $\ell = 2$  corresponds to the recursion studied in [9, 10], which arises in the study of the Wiener index of Catalan trees. We have  $C_1 = \frac{\Gamma(\ell-1/2)}{\sqrt{\pi}}$  and, for  $n \geq 2$ ,

$$C_n = n \frac{\Gamma(n\ell + (n/2) - 1)}{\Gamma((n-1)\ell + (n/2) - 1)} C_{n-1} + \frac{1}{4} \sum_{k=1}^{n-1} \binom{n}{k} C_k C_{n-k}. \quad (11)$$

Define  $a_n$  by  $C_n = n! g(n) a_n$  where  $g(1) = 1$  and

$$g(m) = \prod_{k=2}^m \frac{\Gamma(k\ell + (k/2) - 1)}{\Gamma((k-1)\ell + (k/2) - 1)}.$$

Then (11) becomes

$$a_n = a_{n-1} + \sum_{k=1}^{n-1} \frac{g(k)g(n-k)}{4g(n)} a_k a_{n-k},$$

so  $f(n, k) = g(k)g(n-k)/4g(n)$ .

With  $a$  fixed and  $x \rightarrow \infty$  and using 6.1.47 on p.257 of [1] (or using Stirling's formula), we have

$$\begin{aligned} \frac{\Gamma(x+a)}{\Gamma(x)} &= x^a \left( 1 + \frac{a(a-1)}{2x} + O(1/x^2) \right) \\ &= x^a \left( 1 + \frac{a-1}{2x} \right)^a \left( 1 + O(1/x^2) \right) \end{aligned} \quad (12)$$

$$= \left( x + \frac{a-1}{2} \right)^a \left( 1 + O(1/x^2) \right). \quad (13)$$

When  $m > 1$ , (13) gives us

$$\begin{aligned} g(m) &= \prod_{k=2}^m \left( \frac{(2\ell+1)k - \ell - 3}{2} \right)^\ell \left( 1 + O(1/k^2) \right) \\ &= \Theta(1) \left( (\ell+1/2)^m \prod_{k=2}^m \left( k - \frac{\ell+3}{2\ell+1} \right) \right)^\ell \\ &= \Theta(1) ((\ell+1/2)^m [a]_{m-1})^\ell, \quad \text{where } a = \frac{3\ell-1}{2\ell+1}. \end{aligned}$$

Hence

$$f(n, k) = \Theta(1) \left| \frac{[a]_{k-1} [a]_{n-k-1}}{[a]_{n-1}} \right|^\ell.$$

where the absolute values have been introduced to allow for  $a < 0$ . A slight adjustment of the argument leading to (8) and (9) leads to

$$f(n, k) = O(n^{-\ell} (3k/2n)^{\ell(k-1)}) \quad \text{and} \quad \frac{f(n, k)}{f(n, 1)} = O(n^{-\ell} (3k/2n)^{\ell(k-d-1)})$$



for  $1 \leq k \leq n/2$ . Hence Theorem 2 applies with  $\alpha = \beta = \ell$ , and  $a_n$  converges to a constant when  $\ell > 1$  by Corollary 1. ■

It is interesting to note that there is a simple relation between the sequence  $u_n$  in Example 1 and the sequence  $a_n$  in Example 3 with  $\ell = 2$ . It is not difficult to check that the  $f(n, k)$  defined in Example 3 is exactly five times the  $f(n, k)$  in Example 1: since  $a_1 = 5u_1$ , we have  $a_n = 5u_n$  for all  $n \geq 1$ . This simple relation suggests a relationship between the number of maps on an orientable surface of genus  $g$  and the  $g$ th moment of a particular toll function on a certain type of trees. Using a bijective approach, Chapuy [4] recently found an expression for  $t_g$  as the  $g$ th moment of the labels in a random well-labelled tree.

### 3 A convolutional recursion arising from Painlevé I

The following is recursion (44) in [11].

$$\alpha_n = (n-1)^2 \alpha_{n-1} + \sum_{k=2}^{n-2} \alpha_k \alpha_{n-k}, \quad n \geq 1, \quad n \geq 3. \quad (14)$$

It follows from Proposition 14 of [11] that, for  $0 < \alpha_1 < 1$  and  $\alpha_2 = \alpha_1 - \alpha_1^2$ ,

$$\alpha_n = c(\alpha_1)((n-1)!)^2 \left( 1 - \frac{2\alpha_2(n-3)}{3(n-1)^2(n-2)^2} + \delta_n \right), \quad (15)$$

where  $c(\alpha_1)$  depends only on  $\alpha_1$ , and

$$\delta_n = O(1/n^4).$$

We note that  $\alpha_n$  for  $n \geq 3$  depends only on  $\alpha_2$ . The proof of (15) relies on the fact that  $0 < \alpha_2 < 1/4$  for  $0 < \alpha_1 < 1$ . It is conjectured in [11] that the asymptotic expression (15) actually holds for a wider range of values of  $\alpha_1$ .

For  $n \geq 1$ , let

$$p_n = \frac{\alpha_n}{((n-1)!)^2}.$$

Then, as shown in [11],  $p_n$  satisfies recursion (1) with  $d = 2$  and

$$f(n, k) = \left( \frac{(n-k-1)!(k-1)!}{(n-1)!} \right)^2.$$

We note here  $f(n, 2) = O(n^{-4})$ . It follows from Theorem 2 that

$$p_n = p(1 + \epsilon_n) \quad \text{for any } \alpha_2 > 0,$$

where  $p = p(\alpha_2)$  is a positive constant and  $\epsilon_n = O(1/n^3)$ .

It is also interesting to note that, with  $\alpha_1 = 1/50$ ,  $\alpha_2 = 49/2500$ , the sequence  $\alpha_n$  is related to the sequence  $u_n$  in Example 1 by

$$\alpha_n = [1/5]_n [4/5]_{n-1} u_n.$$

As mentioned in [11], the formal series  $v(t) = \sum_{n \geq 1} \alpha_n t^{-n}$  satisfies

$$t^2 v'' + t v' - (t + 2\alpha_1)v + t v^2 + \alpha_1 = 0, \tag{16}$$

and hence, with

$$t = \frac{8\sqrt{6}}{25} x^{5/2},$$

$y(x) = (x/6)^{1/2}(1 - 2v(t))$  satisfies the following Painlevé I:

$$y'' = 6y^2 - x.$$

This connection with Painlevé I is used in [8] to show that the sequence  $\alpha_n$  is not holonomic (It follows that  $u_n$  and  $t_n$  in Example 1 are also not holonomic). The proof uses the fact that solutions to Painlevé I have infinitely many singularities and hence cannot satisfy a linear ODE with polynomial coefficients.

In the following we apply the techniques of [14] to prove that (15) holds for any complex constant  $\alpha_1$ . It is convenient to introduce the formal series

$$u_0(z) = v(z^2) = \sum_{n=2}^{\infty} b_n z^{-n} = \sum_{n=1}^{\infty} \alpha_n z^{-2n}.$$

It follows from (16) that  $u = u_0(z)$  is a formal solution to the differential equation

$$\frac{1}{4}u'' + \frac{1}{4z}u' - \left(1 + \frac{2\alpha_1}{z^2}\right)u + u^2 + \frac{\alpha_1}{z^2} = 0.$$

The Stokes lines for this differential equation are the positive and the negative real axes. When the negative real axis is crossed the Stokes phenomenon switches on a divergent series

$$u_1(z) = K e^{2z} z^{-1/2} \sum_{n=0}^{\infty} c_n z^{-n},$$

in which the Stokes multiplier  $K$  is a constant (depending on the constant  $\alpha_1$ ). To determine the coefficients  $c_n$  we observe that  $u_1$  is a solution of the linear differential equation

$$\frac{1}{4}u_1'' + \frac{1}{4z}u_1' - \left(1 + \frac{2\alpha_1}{z^2} - 2u_0\right)u_1 = 0.$$

Hence, for the coefficients  $c_n$  we can take  $c_0 = 1$  and for the others we have

$$n c_n = \frac{1}{4} \left(n - \frac{1}{2}\right)^2 c_{n-1} + 2 \sum_{k=4}^{n+1} b_k c_{n+1-k}, \quad n \geq 1.$$

The first five coefficients are

$$c_0 = 1, \quad c_1 = \frac{1}{16}, \quad c_2 = \frac{9}{512}, \quad c_3 = \frac{75}{8192} + \frac{2}{3}\alpha_2, \quad c_4 = \frac{3675}{524288} + \frac{13}{24}\alpha_2.$$

In a similar manner it can be shown that when the positive real axis is crossed the Stokes phenomenon switches on a divergent series

$$u_2(z) = iK e^{-2z} z^{-1/2} \sum_{n=0}^{\infty} (-1)^n c_n z^{-n}.$$

This is all the information that is needed to conclude that

$$\alpha_n = b_{2n} \sim \frac{K}{\pi} \sum_{k=0}^{\infty} (-1)^k c_k \frac{\Gamma(2n - k - \frac{1}{2})}{2^{2n-k-(1/2)}}, \quad \text{as } n \rightarrow \infty.$$

By taking the first 4 terms in this expansion we can verify that (15) holds for any complex constant  $\alpha_1$ .

For more details see [12], [13] and [14]. (It's best to get the version of the first reference on the website <http://www.maths.ed.ac.uk/adri/public.htm>.)

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