# Fitting quotients of finitely presented abelian-by-nilpotent groups

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**Abstract.** We show that every finitely generated nilpotent group of class 2 occurs as the quotient of a finitely presented abelian-by-nilpotent group by its largest nilpotent normal subgroup.

## 1   Introduction

Brookes showed in [2] that a finitely presented abelian-by-polycyclic group is virtually an extension of a nilpotent group by a nilpotent group of class at most 2. In this paper we show the following.

**Theorem.** *Let $Q$ be a finitely generated nilpotent group of class* 2*. Then there exists a finitely presented group $G$ with an abelian Fitting subgroup $A$ so that the quotient $G/A$ is isomorphic to $Q$.*

If $Q$ were a finitely generated abelian group, then this result would be a very straightforward one. But the arguments needed here cannot be a simple extension of those straightforward techniques. In [3], Brookes and the first author have proved a theorem which implies, amongst other things, that if a finitely presented abelian-by-nilpotent group is subdirectly irreducible, then the quotient by the Fitting subgroup is virtually a central product of generalized Heisenberg groups. (By a *generalized Heisenberg* group we mean a group which is torsion-free nilpotent of class at most 2 and has cyclic centre; this includes the infinite cyclic group.) Thus our arguments here will require groups which are subdirect products in a nontrivial way.

The basic idea of the proof is straightforward. We express the nilpotent group $Q$ as a subdirect product of subdirectly irreducible groups $Q_i$ and for each of those groups, we produce a module $B_i$ so that the extension of this module by $Q_i$ is finitely presented. The group $G$ is then a split extension of $Q$ by the direct sum $A$ of the modules $B_i$. The heart of the paper is in showing how to choose these modules so that the group $G$ is finitely presented. We do this with the

aid of a theory, developed by the second author, of $\Sigma$ invariants for modules over nilpotent groups.

## 2 Subdirect products of nilpotent groups

The following result is presumably well known.

**Proposition 2.1.** *Let $H$ be a finitely generated nilpotent group of class at most* 2.

(i) *$H$ is a subdirect product of a finite group and a finite number of torsion-free groups having infinite cyclic centre.*

(ii) *If $H$ has infinite cyclic centre then, for some $k \geq 0$, it has a presentation with generators $\{x_1, \ldots, x_k, y_1, \ldots, y_k, z\}$ and with relations*

$$
\begin{aligned}
[x_i, x_j] = [y_i, y_j] = 1 & \quad \textit{for all } 1 \leq i < j \leq k, \\
[x_i, y_j] = z^{\delta_{ij} m_i} & \quad \textit{for all } 1 \leq i, j \leq k, \\
[x_i, z] = [y_i, z] = 1 & \quad \textit{for all } 1 \leq i \leq k
\end{aligned}
\tag{2.1}
$$

*for suitable non-zero integers $m_i$.*

A group which is presented as above will be called a *generalized Heisenberg group of rank $k$* and the set $\{x_1, y_1, \ldots, x_k, y_k\}$ will be called a *symplectic basis* for $H$. Note that for $k = 0$, such groups are infinite cyclic.

*Proof.* (i) Because $H$ is finitely generated nilpotent, it has a finite torsion subgroup $T$ and it is residually finite. Thus there is a normal subgroup $N$ of finite index which meets $T$ trivially and so $H$ is the subdirect product of the torsion-free group $Q = H/T$ and $H/N$. The centre $Z$ of $Q$ is torsion-free, hence free abelian, say of rank $r$. There exist therefore $r$ direct summands $N_i$ of $Z$ having rank $r - 1$ and intersecting in $\{1\}$. Put $Q_i = Q/N_i$. Then $Q$ is a subdirect product of the torsion-free quotient groups $Q_i$. Since the Hirsch length of each $Q_i$ is smaller than that of $Q$, we may assume inductively that each quotient $Q_i$ is a subdirect product of subdirectly irreducible torsion-free quotient groups $Q_{ij}$. So $Q$ is the subdirect product of all the $Q_{ij}$ whence $H$ is the subdirect product of the subdirectly irreducible torsion-free quotient groups $Q_{ij}$ and the finite quotient group $H/N$. Notice that the previous argument shows that the centre of a subdirectly irreducible torsion-free quotient is infinite cyclic.

(ii) Suppose that $H$ has cyclic centre $Z$ generated by $z$. Then there is a symplectic form on $H/Z$ given by, for $x, y \in H$,

$$
(xZ, yZ) = m \quad \text{exactly when } [x, y] = z^m.
$$

It is a standard result (see, for example, [1, Section 5]) that $H/Z$ then has a symplectic basis as $\mathbb{Z}$-module. Lifting this symplectic basis to $H$, we arrive at exactly the type of presentation described in the proposition. □

Observe that there is room for choice of symplectic basis in the proof of the second part of the proposition. We will make use of this later.

## 3 Modules for generalized Heisenberg groups

A similar construction to the one described here appears in the thesis of Dugal Ure [5]. We have included details here both for completeness and because we know of no easily accessible source. Suppose that $H$ is a generalized Heisenberg group of rank $k \geq 0$. Choose a generating set $\{x_1, \ldots, x_k, y_1, \ldots, y_k, z\}$ for $H$ and suppose that $H$ then has presentation (2.1). We shall describe the construction of a module $A$ for $H$ so that the split extension $G$ of $A$ by $H$ is finitely presented and $A$ is the Fitting subgroup of $G$.

We, in fact, use HNN extensions to construct inductively groups $G_0, \ldots, G_k$ so that $G_k$ is the required group $G$ and we will recover $A = A_k$ as a normal subgroup of $G_k$.

Let $L$ be the free abelian group with basis $\{x_1, \ldots, x_k\}$; let $A_{-1}$ be the free $\mathbb{Z}L$-module with basis $\mathcal{B} = \{a_1, a_2\}$ and let $G_{-1}$ be the split extension of $A_{-1}$ by $L$. Then $G_{-1}$ has a presentation with generators $\{x_1, \ldots, x_k, a_1, a_2\}$ and relations

$$[x_i, x_j] = 1 \ (1 \leq i < j \leq k), \quad [a', a^w] = 1 \ (a, a' \in \mathcal{B}, w \in L). \quad (3.1)$$

Using this presentation, it is easy to see that the assignments $x_i \mapsto x_i, a_1 \mapsto a_1 a_2$, $a_2 \mapsto a_1 a_2^2$ extend to an endomorphism of $G_{-1}$. It is equally simple to write down an inverse for this endomorphism and so it is an automorphism $\phi_0$ of $G_{-1}$. Let $G_0$ be the HNN extension (in this case, split extension) of $G_{-1}$ corresponding to $\phi_0$ and with stable letter $z$. Thus $G_0$ has an extra generator $z$ and extra relations

$$[x_i, z] = 1 \ (1 \leq i \leq k), \quad a_1^z = a_1 a_2, \quad a_2^z = a_1 a_2^2. \quad (3.2)$$

Suppose now that we have constructed, by a series of HNN extensions, a group $G_{r-1}$ with the generators of $G_0$ together with new generators $y_1, \ldots, y_{r-1}$ and the relations of $G_0$ together with new relations

$$[x_i, y_j] = z^{\delta_{ij} m_i}, \quad [y_j, z] = 1, \quad [y_j, y_l] = 1, \quad a^{y_j} = a a^{x_j} \quad (3.3)$$

where $1 \leq i \leq k$, $1 \leq j, l \leq r - 1$ and $a \in \mathcal{B}$.

We now construct $G_r$. Consider the assignment $\phi_r$ that fixes $z, y_1, \ldots, y_{r-1}$ and each $x_i \in L$ except $x_r$, and satisfies $x_r \mapsto x_r z^{m_r}$ and $a \mapsto aa^{x_r}$ for $a \in \mathcal{B}$. It is easily verified that $\phi_r$ preserves the relations of $G_{r-1}$ as given in (3.1), (3.2) and (3.3). Thus $\phi_r$ induces an endomorphism of $G_{r-1}$. To see that $\phi_r$ is injective, observe firstly that $G_{r-1}$ is obtained by a series of ascending HNN extensions with stable letter $z$ or some $y_i$, each of which is fixed by $\phi_r$. Thus it will suffice to establish that the restriction of $\phi_r$ to $G_{-1}$ is injective. Now $\phi_r$ induces an automorphism of the group $L$, hence a ring automorphism of the group ring $\mathbb{Z}L$; on the other hand, since $\mathbb{Z}L$ is an integral domain, the assignments $a_1 \mapsto a_1 a_2$ and $a_2 \mapsto a_1 a_2^2$ induce an injective endomorphism of the $L$-module $A_{-1}$. It follows that $\phi_r \colon G_{r-1} \to G_{r-1}$ is injective.

Thus we can again construct an HNN extension $G_r$ corresponding to the assignment $\phi_r$ and with stable letter $y_r$. This group $G_r$ has a presentation with generators $\{x_1, \ldots, x_k, y_1, \ldots, y_r, z, a_1, a_2\}$ and relations (3.1) and (3.2) together with (3.3), but with $r - 1$ replaced by $r$. Repeating this process, we eventually arrive at the group $G_k$ with the generating set $\{x_1, \ldots, x_k, y_1, \ldots, y_k, z, a_1, a_2\}$ and relations given by (3.1) and (3.2) together with (3.3), but with $r - 1$ replaced by $k$. Let $A = A_k$ denote the normal closure of $\mathcal{B}$ in $G = G_k$. The group $A$ contains the subgroup $A_{-1}$ which is abelian; as $A$ is obtained from $A_{-1}$ by a sequence of localizations, $A$ itself is abelian. From the presentation, it is easy to verify that $G/A \cong H$. Also, if there were a nilpotent normal subgroup of $G$ larger than $A$, then some non-trivial element of the centre of $H$ would have to act nilpotently on $A$. But it is clear from the definition that no non-trivial power of $z$ acts nilpotently on $A$. Thus $A$ is the Fitting subgroup of $G$. The final step, the justification that $G$ is finitely presented, will be deferred to the next section.

## 4 Geometric invariants for modules over nilpotent groups

### 4.1 Definition of the invariants

We shall give only a very brief description of the invariants here. The subject is treated in detail and with proofs in a paper of the second author [4].

Let $Q$ be a finitely generated group. By a *character* of $Q$ we mean a homomorphism $Q \to \mathbb{R}$; that is, an element of the dual $Q^*$ of $Q$. Then $Q^*$ is a finite dimensional real vector space.

The usual version of the invariant we want to use is obtained by identifying characters which differ by multiplication by a positive real and so replacing the vector space $Q^*$ by a sphere. Most of the arguments in this paper will require a vector space, however, and so we will translate results and terminology from [4] accordingly.

Let $A$ denote a finitely generated $\mathbb{Z}Q$-module. The subset $\Delta(Q; A)$ of $Q^*$ consists of the equivalence classes of the complement $S(G) \smallsetminus \Sigma^0(Q; A)$ of the set $\Sigma^0(Q; A)$ defined in [4]. More directly, if $\chi \in Q^*$, then $\chi \in \Delta(Q; A)$ if and only if, either $A$ is not finitely generated as $\mathbb{Z}Q_\chi$-module, or $\chi = 0$. Here $Q_\chi$ is the submonoid of $Q$ consisting of the elements with non-negative $\chi$-value.

In [4], the term *tame* is defined with respect to an arbitrary central series of a group. We shall use it, however, only for the very simple case of the lower central series of a group of class 2. We therefore define it only in this special case.

**Definition 4.1.** Suppose that $Q$ is a nilpotent group of class 2 and $A$ is a $\mathbb{Z}Q$-module with finite generating set $\mathcal{A}$. We say that $A$ is *tame* (for the lower central series) if both $\Delta(Q; A)$ and $\Delta(Q'; \mathcal{A} \cdot \mathbb{Z}Q')$ contain no lines (equivalently, no diametrically opposite points).

The key theorem we shall use is the following, which is a special case of a much more general result appearing as [4, Theorem 4.1].

**Theorem 4.2.** *With the notation above, if $A$ is tame, then the split extension of $A$ by $Q$ is finitely presented.*

## 4.2 Calculating the invariant of the examples

We now turn to showing that the group $G$, constructed in the previous section, is finitely presented. Clearly, by Theorem 4.2, it suffices to show that the module $A$ constructed there for the subdirectly irreducible group $H$ is tame. We begin with the trivial cases. If $H$ is finite or infinite cyclic, then $A$ is finitely generated as abelian group and so $\Delta(H; A)$ consists only of $\{0\}$. (In fact, in this case, the extension of $A$ by $H$ is polycyclic.)

We shall use the following criterion for calculating $\Delta$ which has been adapted from the more general situation of [4].

> Let $A$ be a cyclic module with generator $a$. Then $\chi \notin \Delta(Q; A)$ if and only if there exists $\alpha \in \mathbb{Z}Q$ so that $a = a\alpha$ and $\chi(q) > 0$ for each $q$ in the support of $\alpha$.

In the case when $H$ is a generalized Heisenberg group of rank $k > 0$, then we observe that either element $a$ of $\{a_1, a_2\}$ is a generator for $A$ as $\mathbb{Z}Q$-module and that the defining relations of $H$ imply that $a$ is annihilated by each of the elements

$$1 + x_1 - y_1, \ 1 + x_2 - y_2, \ \ldots, \ 1 + x_k - y_k. \tag{4.1}$$

(Note that we regard $A$ as *right* $\mathbb{Z}Q$-module.) These annihilating elements and the previously quoted criterion allow us to find an upper bound for $\Delta(H; A)$ as follows.

Fix an index $i \in \{1, \ldots, k\}$ and let $\chi \colon H \to \mathbb{R}$ be a character that assumes its minimum only once on the support $\{1, x_i, y_i\}$ of $1 + x_i - y_i$. If the minimum occurs at 1, then we rewrite the equation $a \cdot (1 + x_i - y_i) = 0$ in the form $a = a(-x_i + y_i)$ and deduce from the criterion that $\chi \notin \Delta(H; A)$ if $\chi(x_i) > 0$ and $\chi(y_i) > 0$. If the minimum is taken on $x_i$, we use that $a$ is annihilated by $(1 + x_i - y_i) \cdot x_i^{-1} = 1 + x_i^{-1} - y_i x_i^{-1}$ and infer that $\chi \notin \Delta(H; A)$ if $\chi(x_i^{-1}) > 0$ and $\chi(y_i x_i^{-1}) > 0$; if it is taken on $y_i$, we argue similarly. We conclude that $\chi$ can only belong to $\Delta(H; A)$ if it assumes its minimum at least twice on the support of each of the elements $1 + x_i - y_i$.

It follows that $\Delta(H; A)$ is contained in the union of $3^k$ subsets. To describe these sets concisely, we introduce characters $\chi_i, \psi_i, \phi_i \in H^*$ defined by

$$\chi_i(x_i) = 1, \quad \psi_i(y_i) = 1, \quad \phi_i(x_i) = \phi_i(y_i) = -1,$$

all values on other generators $x_j$ or $y_j$ being zero. The subsets are then as described by the first statement of the next proposition; the second statement is a simplified version of the first which will be used later.

**Proposition 4.3.** *Let $H$ and $A$ be as defined in Section 2 and let $\chi_i$, $\psi_i$ and $\phi_i = -\chi_i - \psi_i$ in $H^*$ be as before. Then:*

(i) $\Delta(H; A)$ *lies in the union of the $3^k$ sets, each defined as the convex cone generated by a set of the form $\{\lambda_1, \ldots, \lambda_k\}$ where each $\lambda_i$ is an element of $\{\chi_i, \psi_i, \phi_i\}$.*

(ii) $\Delta(H; A)$ *lies in the union of the $3^k$ subspaces each spanned by a set of the form $\{\lambda_1, \ldots, \lambda_k\}$ where each $\lambda_i$ is an element of $\{\chi_i, \psi_i, \phi_i\}$.*

### 4.3 The invariant for the module in the theorem

The $Q$-module $A$ occurring in the statement of our theorem will be obtained by combining three constructions. To begin with, the given group $Q$ is expressed as a subdirect product of a finite group $Q_1$ and generalized Heisenberg groups $Q_2, \ldots, Q_m$. For each factor $Q_i$ one constructs a tame $Q_i$-module $B_i$, pulls its action back to $Q$ and then chooses $A$ to be the direct sum $B_1 \oplus \cdots \oplus B_m$. To calculate $\Delta(Q, A)$, one then uses the direct sum formula

$$\Delta(Q; A_1 \oplus A_2) = \Delta(Q; A_1) \cup \Delta(Q; A_2); \tag{4.2}$$

it is valid for every couple $A_1$, $A_2$ of finitely generated $Q$-modules (cf. [4, Lemma 2.2]). This leaves us with the calculation of the invariants $\Delta(Q; B_i)$. Each projection $\pi_i \colon Q \twoheadrightarrow Q_i$ induces an injective linear map $\pi_i^* \colon Q_i^* \rightarrowtail Q^*$ and

$$\Delta(Q; B_i) = \pi_i^*(\Delta(Q_i; B_i)); \tag{4.3}$$

(cf. [4, Lemma 2.5]). The calculation of $\Delta(Q; A)$ can thus be reduced to that of the subsets $\Delta(Q_1; B_1), \ldots, \Delta(Q_m; B_m)$.

Formula (4.3) implies that $\Delta(Q; B_i)$ contains no line if $\Delta(Q_i; B_i)$ has this property; but such a conclusion cannot be drawn from formula (4.2): if $\Delta(Q; A_1)$ and $\Delta(Q; A_2)$ both contain a half-line, their union can be a line.

If $Q_i$ is finite or infinite cyclic, this problem does not arise, since $B_i$ can then be chosen so that $\Delta(Q_i; B_i)$ is reduced to the origin. If, however, $Q_i$ is a Heisenberg group of rank $k > 0$, no easy solution seems available. The solution proposed in Section 6 will involve the choice of a symplectic basis of a normal subgroup $P$ of $Q_i$ of finite index and the module $B_i$ will be an induced module $B \otimes_P \mathbb{Z}Q_i$ with $B$ a $P$-module of the kind constructed in Section 3. The set $\Delta(Q_i; B \otimes_P \mathbb{Z}Q_i)$ can then be computed with the help of the formula

$$\iota^*(\Delta(Q_i; B \otimes_P \mathbb{Z}Q_i)) = \Delta(P; B) \cap \operatorname{im} \iota^*. \tag{4.4}$$

In the above $\iota \colon P \leq Q_i$ denotes the inclusion and $\iota^*$ is the induced linear map $Q_i^* \rightarrowtail P^*$ (see [4, Lemma 2.4]).

## 5  Symplectic spaces

Let $\mathbb{F}$ be a subfield of $\mathbb{R}$ and $(V, \beta)$ be a finite dimensional vector space over $\mathbb{F}$ equipped with a non-degenerate skew-symmetric bilinear form $\beta \colon V \times V \to \mathbb{F}$. In the sequel, $(V, \beta)$ will be referred to as a *symplectic space*. We shall make use of the following basic properties of $(V, \beta)$:

(a) $V$ is even dimensional, say $\dim_{\mathbb{F}} V = 2k$,

(b) $V$ admits a basis $\{e_1, f_1, \ldots, e_k, f_k\}$, called *symplectic*, such that

$$\beta(e_i, e_j) = \beta(f_i, f_j) = 0 \quad \text{and} \quad \beta(e_i, f_j) = \delta_{i,j} \quad \text{for all } i, j,$$

(c) for every subspace $U$ the *annihilator*

$$U^{\perp} = \{v \in V \mid \beta(v, u) = 0 \text{ for all } u \in U\}$$

has dimension $2k - \dim U$ and $(U^{\perp})^{\perp} = U$

(see, for instance, [6, Corollary 5.60, Theorem 5.59 and Proposition 5.43]).

A subspace $U$ of $V$ with $U \subseteq U^{\perp}$ is called *isotropic*; it is said to be *Lagrangian* if it is isotropic and not contained in a larger isotropic subspace. It is clear that every isotropic subspace is contained in a Lagrangian subspace; moreover, the formula $\dim U^{\perp} = \dim V - \dim U$ (see (c) above) implies that every Lagrangian subspace is of dimension $\frac{1}{2} \dim V$. If $\{e_1, f_1, \ldots, e_k, f_k\}$ is a symplectic basis of the symplectic space $(V, \beta)$, the subspace $U$ spanned by $e_1, \ldots, e_k$ is Lagrangian; conversely, every Lagrangian subspace is of this form (easy induction on $k$).

As before, let $(V, \beta)$ be a symplectic space of dimension $2k$ over a subfield $\mathbb{F}$ of $\mathbb{R}$ and let $\Omega$ be a finite collection of subspaces of $V$ each having dimension at most $k$. In the arguments of this section, we will use without further comment the well-known fact that a finite dimensional vector space over an infinite field cannot be the union of a finite collection of proper subspaces.

**Lemma 5.1.** *There is a Lagrangian subspace $L$ of $V$ so that $L \cap W = \{0\}$ for each $W \in \Omega$.*

*Proof.* We shall prove the result by induction on $k$; if $k = 1$, it is immediate. We can clearly suppose that each $W \in \Omega$ is non-zero; as $\beta$ is non-degenerate, this implies that each annihilator $W^\perp$ is a proper subspace. Thus we can find $u \in V$ so that, for each $W \in \Omega$, the vector $u$ belongs to neither $W$ nor $W^\perp$. Let $U$ denote the subspace spanned by $u$ and let $U^\perp$ be its annihilator. Then $U$ has dimension 1 and $U^\perp$ has dimension $2k - 1$. Further, as $u \notin W^\perp$ it follows that each $W \in \Omega$ is not a subspace of $U^\perp$. Thus $W \cap U^\perp$ will have dimension $\dim(W) - 1$ and its image $\hat{W}$ in $U^\perp/U$ will also have dimension $\dim(W) - 1$.

The symplectic form $\beta$ induces a skew-symmetric form on the $2(k-1)$-dimensional space $U^\perp/U$; since $(U^\perp)^\perp = U$ by property (c), the induced form is non-degenerate. Moreover, the set $\hat{\Omega} = \{\hat{W} \mid W \in \Omega\}$ is a finite set of subspaces of dimension at most $k - 1$. Thus the inductive hypothesis provides a Lagrangian subspace $\hat{L}$ of $U^\perp/U$ so that $\hat{L} \cap \hat{W} = \{0\}$. Let $L$ be the lift of $\hat{L}$ in $U^\perp$. If $W \in \Omega$, then $L \cap W$ must lie in $U$ and so must be zero as $u \notin W$. $\square$

The next lemma is the technical heart of this section. Let $L$ be a Lagrangian subspace of $V$ and suppose that each element of $\Omega$ has trivial intersection with $L$. Choose a basis $\{e_1, \ldots, e_k\}$ for the subspace $L$ and extend it to a symplectic basis $\{e_1, f_1, \ldots, e_k, f_k\}$ for $V$.

Given $\mu = (\mu_1, \ldots, \mu_k) \in [0, 1]^k$, let $K_\mu$ be the span of the vectors

$$\{(1 - \mu_1)e_1 + \mu_1 f_1, \ldots, (1 - \mu_k)e_k + \mu_k f_k\}.$$

Thus if $\mu = (0, \ldots, 0)$, then $K_\mu = L$, and if $\mu = (1, \ldots, 1)$, then $K_\mu$ is the span of $\{f_1, \ldots, f_k\}$.

**Lemma 5.2.** *The family $K_\mu$ of subspaces has the following properties:*

(i) *for every $\mu \in [0, 1]^k$ the subspace $K_\mu$ is Lagrangian,*

(ii) *for every $\mu \in (0, 1]^k$ the subspace $K_\mu$ is transversal to $L$,*

(iii) *if every subspace in $\Omega$ has dimension $k$, then for every $\mu$ sufficiently close to $(0, \ldots, 0)$ the subspace $K_\mu$ is transversal to every subspace in $\Omega$.*

*Proof.* (i)–(ii) For every index $i$, the vectors $e_i$ and $f_i$ span a symplectic plane; these planes are orthogonal to each other. It follows that each subspace $K_\mu$ is isotropic and of dimension $k$, hence Lagrangian. Moreover, if $\mu \in (0, 1]^k$, the subspace $L + K_\mu$ contains a basis of $V$ whence the Lagrangian subspaces $L$ and $K_\mu$ must be transversal.

(iii) Let $W$ be a subspace in $\Omega$ and let $w_1, \ldots, w_k$ a basis of $W$. Consider the function

$$g : [0, 1]^k \to \mathbb{R},$$
$$\mu \mapsto \det((1 - \mu_1)e_1 + \mu_1 f_1, \ldots, (1 - \mu_k)e_k + \mu_k f_k, w_1, \ldots, w_k).$$

This function is continuous and is non-zero at $(0, 0, \ldots, 0)$ (since $W$ is a complement of $L$). Hence there exists an interval $[0, \varepsilon_W]$ such that $g$ does not vanish on $[0, \varepsilon_W]^k$. Set $\varepsilon = \min\{\varepsilon_W \mid W \in \Omega\}$. For every $\mu$ with $0 < \mu_i \le \varepsilon$ for $1 \le i \le k$, the subspace $K_\mu$ is then a Lagrangian subspace that is transversal to $L$ and to every subspace in $\Omega$. □

**Definition 5.3.** Let $\mathcal{C} = \{\chi_1, \psi_1, \ldots, \chi_k, \psi_k\}$ be a symplectic basis for $V$. Then the *associated subspaces* corresponding to the basis $\mathcal{C}$ are all subspaces of the form $\langle \lambda_1, \ldots, \lambda_k \rangle$ where each $\lambda_i$ is an element of $\{\chi_i, \psi_i, \chi_i + \psi_i\}$.

Note that, in this language, the set of subspaces described in (ii) of Proposition 4.3 is the set of subspaces associated to the basis $\{\chi_1, \psi_1, \ldots, \chi_k, \psi_k\}$ of $H^*$ which is dual to the basis $\{x_1, y_1, \ldots, x_k, y_k\}$ of $H$. (Here $\mathbb{F} = \mathbb{R}$.)

**Proposition 5.4.** *There exists a symplectic basis $e_1, \hat{f}_1, \ldots, e_k, \hat{f}_k$ such that every subspace associated to it has trivial intersection with each subspace in $\Omega$.*

*Proof.* Observe that it is sufficient to prove the theorem in the case that each subspace in $\Omega$ is replaced by a subspace of dimension $k$ which contains it. Let $L$ be the symplectic basis $\{e_1, f_1, \ldots, e_k, f_k\}$ and let the family $K_\mu$ be as described in Lemma 5.2. Fix a positive integer $p$ so that $K_\mu$ is transversal to every subspace in $\Omega$ for each $\mu \in [0, \frac{1}{p}]^k$. Set $\hat{f}_i = p(e_i + \frac{1}{p} f_i)$. Then the set $\{e_1, \hat{f}_1, \ldots, e_k, \hat{f}_k\}$ is also a symplectic basis for $V$.

Further, any subspace associated to this basis is spanned by a set of the form $\{u_1, \ldots, u_k\}$ where $u_i$ is one of the vectors

$$e_i, \quad \hat{f}_i = p\left(e_i + \frac{1}{p} f_i\right) \text{ or } e_i + \hat{f}_i = (p + 1)\left(e_i + \frac{1}{(p + 1)} f_i\right).$$

Thus each associated subspace is of the form $K_\mu$ with $\mu \in [0, \frac{1}{p}]^k$ and so meets every subspace $W \in \Omega$ trivially. □

## 6 Proof of the theorem

Suppose that $Q$ is a finitely generated nilpotent group of class 2. Write $Q$ as a subdirect product with factors $Q_1$, …, $Q_m$ where $Q_1$ is finite and the remaining factors are torsion-free groups with infinite cyclic centre and so generalized Heisenberg groups (see Proposition 2.1). We then have epimorphisms $\pi_i \colon Q \to Q_i$ and $\bigcap_{i=1}^m \ker(\pi_i)$ is trivial.

We shall construct the group $G$ by forming a module $A$ for $Q$ and then defining $G$ as a split extension of $A$ by $Q$. We will take, for each $i$, a suitable module $B_i$ for $Q_i$ and then regard $B_i$ as a $Q$-module via $\pi_i$. The module $A$ will then be the direct sum of the $B_i$. We must exercise considerable care, however, to ensure that the resulting module $A$ is tame and hence that $G$ is finitely presented.

The factor $Q_1$ is finite; order the remaining factors so that for $i \le j$ the rank of $Q_i$, as generalized Heisenberg group, is less than or equal to the rank of $Q_j$. For the finite factor $Q_1$ the module $B_1$ will be the group ring of $Q_1$; note that $\Delta(Q_1; B_1)$ is reduced to 0. In all other cases, the module will be a modified version of the modules $A$ constructed for generalized Heisenberg groups in Section 3. If $Q_i$ is cyclic, we can use such a module for $B_i$ without further adjustment. In this case, the module $B_i$ is finitely generated as abelian group and the corresponding $\Delta(Q_i; B_i)$ is again reduced to the origin. For the other $B_i$ we will need to take more care to ensure that the resulting direct sum of the modules $B_i$ is tame.

Suppose that, for some integer $l$ with $1 \le l \le m$, we have constructed modules $B_1, \dots, B_{l-1}$ for $Q$ so that the set

$$\Delta(Q; B_1 \oplus \cdots \oplus B_{l-1}) = \Delta(Q; B_1) \cup \cdots \cup \Delta(Q; B_{l-1})$$

contains no lines and is contained in a finite collection $\Omega$ of subspaces of $Q^*$, each of dimension no more than the rank $k$ of $H = Q_l$ and spanned by at most $k$ characters with images in $\mathbb{Q}$. In order to describe our construction of the module $B_l$, we need to consider symplectic forms on $H$ and on $\mathrm{Hom}(H, \mathbb{Q})$; we shall denote $\mathrm{Hom}(H, \mathbb{Q})$ by $H^\#$. The projection $\pi = \pi_l \colon Q \twoheadrightarrow H = Q_l$ yields an embedding $H^* \rightarrowtail Q^*$ and the inclusion of the rational numbers in the reals yields a further embedding $H^\# \rightarrowtail H^*$. We shall treat these embeddings as inclusions.

We have observed in the proof of Proposition 2.1 that the quotient $H/Z$ by the centre has a natural symplectic form given by commutation. This yields an embedding $\rho \colon H/Z \to H^\#$ defined by $\rho(h_1 Z)[h_2] = (h_1 Z, h_2 Z)$ and the image of $\rho$ in $H^\#$ is a lattice in $H^\#$ of full rank; that is, it contains a basis of $H^\#$. It follows that $H^\#$ inherits a symplectic form $\beta$ satisfying

$$\beta(\rho(h_1 Z), \rho(h_2 Z)) = (h_1 Z, h_2 Z) = s \quad \text{exactly when} \quad [h_1, h_2] = z^s$$

where $z$ is the chosen generator of $Z$.

Let $\hat{\Omega} = \{W \cap H^{\#} \mid W \in \Omega\}$. Then $\hat{\Omega}$ is a finite collection of subspaces of $H^{\#}$ each of which has dimension no more than $k$ (recall that each $W \in \Omega$ is spanned by at most $k$ elements of $H^{\#}$). Thus we can apply Proposition 5.4 (with $\mathbb{F} = \mathbb{Q}$) to find a symplectic basis $\mathcal{C}$ for $V = H^{\#}$ so that each associated subspace of this symplectic basis avoids each element of $\hat{\Omega}$ and hence avoids each element of $\Omega$.

We shall now show how to construct the module $B_l$. As $\rho(H/Z)$ is a full lattice in $H^{\#}$, there is a positive integer $j$ so that $j\mathcal{C} \subseteq \rho(H/Z)$. Thus there is a subset $\mathcal{D} \subseteq H$ so that $\rho$ maps $\{dZ \mid d \in \mathcal{D}\}$ bijectively onto $j\mathcal{C}$. It follows easily that $\mathcal{D}$ is a symplectic basis for a normal subgroup $P$ of finite index in $Q_l = H$. We then proceed with the construction of Section 3 to produce a module $B$ for $P$ so that the set $\Delta(B) \subseteq P^{*} = Q_l^{*}$ will lie in the union of the subspaces associated to the symplectic basis $\mathcal{D}$ of the Heisenberg group $P$; because $\mathcal{C} = j^{-1}\rho(\mathcal{D})$ these associated subspaces coincide with the subspaces associated to the symplectic basis $\mathcal{C}$ of the vector space $V = H^{\#}$. We then induce $B$ from $P$ to $Q_l$ to obtain the module $B_l$. It follows from formula (4.4) and the fact that $\iota \colon P \le H$ induces an isomorphism $\mathrm{im}^{*} \colon H^{*} \to P^{*}$ that $\Delta(Q_l; B_l)$ lies in the union of the subspaces associated to the symplectic basis $\mathcal{C}$.

We have now constructed a module $B_l$ for $Q_l$ so that $\Delta(Q_l; B_l)$ contains no lines. We pull $B_l$ back to $Q$; formula (4.3) then shows that $\Delta(Q; B_l)$ contains no lines either and that it lies in a finite union of subspaces each of which intersects the subspaces in $\Omega$ trivially. Formula (4.2) then implies that

$$\Delta(Q; B_1 \oplus \cdots \oplus B_l) = \Delta(Q; B_1) \cup \cdots \cup \Delta(Q; B_l)$$

contains no lines. Moreover, our construction shows that $\Delta(Q; B_1 \oplus \cdots \oplus B_l)$ lies in the finite union of subspaces, each either a member of $\Omega$ or one of the associated subspaces of the symplectic basis $\mathcal{C}$, and generated by at most $k$ elements of $Q^{\#}$.

We have thus completed the inductive step and so we can now deduce that, if $A = B_1 \oplus \cdots \oplus B_m$, then $A$ is finitely generated as $\mathbb{Z}Q$-module and $\Delta(Q; A)$ contains no lines. Further, if $\mathcal{A}$ is a finite generating set for $A$, then $Q'$ acts on $A$ in such a way that the split extension of $A$ by $Q'$ is locally polycyclic. Thus $\Delta(Q'; \mathcal{A} \cdot \mathbb{Z}Q')$ is reduced to the origin and so certainly contains no lines. Thus, recalling Definition 4.1, $A$ is tame.

The group $G$ required by the theorem is the split extension of $A$ by $Q$ and we deduce from Theorem 4.2 that it is finitely presented. The final step is to observe that, if $A$ were not the Fitting subgroup of $G$, then some non-trivial element of the centre of $Q$ would act nilpotently on $A$. The image of this element in some $Q_i$ would be non-trivial and still central and would act nilpotently on $B_i$. But the construction of the action of the centre of each $Q_i$ makes it clear that, in all cases, no non-trivial element of the centre acts nilpotently on $B_i$. The proof is complete.

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