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Analysis of Discrete Fractional Operators and Discrete Fractional Rheological Models

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ANALYSIS OF DISCRETE FRACTIONAL OPERATORS AND DISCRETE FRACTIONAL RHEOLOGICAL MODELS

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This thesis is comprised of two main parts: Monotonicity results on discrete fractional operators and discrete fractional rheological constitutive equations. In the first part of the thesis, we introduce and prove new monotonicity concepts in discrete fractional calculus. In the remainder, we carry previous results about fractional rheological models to the discrete fractional case. The discrete method is expected to provide a better understanding of the concept than the continuous case as this has been the case in the past. In the first chapter, we give brief information about the main results. In the second chapter, we present some fundamental definitions and formulas in discrete fractional calculus. In the third chapter, we introduce two new monotonicity concepts for nonnegative or nonpositive valued functions defined on discrete domains, and then we prove some monotonicity criteria based on the sign of the fractional difference operator of a function. In the fourth chapter, we emphasize the rheological models: We start by giving a brief introduction to rheological models such as Maxwell and Kelvin-Voigt, and then we construct and solve discrete fractional rheological constitutive equations. Finally, we finish this thesis by describing the conclusion and future work.

Chapter 1

INTRODUCTION

In recent decades the field of discrete fractional calculus has attracted the interest of researchers from several areas including mathematics, biology, physics, chemistry, engineering and even finance and social sciences [**15**], [**16**], [**18**], [**19**]. Particularly, in the area of viscoelasticity, a significant effort has been made in describing more closely the behavior of materials by using fractional mathematical model.

The concept of monotonicity is very important in mathematics. Unfortunately, in theory and applications of fractional calculus, researchers faced a lack of monotonicity results for fractional operators. Dahal and Goodrich in [**10**] and Goodrich in [**12**] initiated monotonicity analyse of the discrete fractional operators. However, they did not consider monotonicity results for fractional orders between zero and one. Therefore, in the first part of this thesis we concentrate on non-integer orders which lead us to introduce new definitions of monotonicity concepts.

The mechanical properties of biomaterials are often represented by linear differential equations developed from physical models of springs and dashpots. However, it was recognized that biological tissues exhibit more complex behavior such as hysteresis, fatigue, and memory that could not be accounted for by using combinations of ideal springs and dashpots [**16**]. Since the tissues within the human body are viscoelastic in nature, it is important to apply proper viscoelastic relations when investigating the mechanics of deformation [**13**]. Testing and modeling of the mechanical properties of biological soft tissues presents a unique set of challenges. Fractional calculus is used to construct stress-strain relationships for viscoelastic materials.

It has been recognized that rheological constitutive equations with fractional derivatives have long played an important role in the description of the properties of viscoelastic materials [**19**]. In the rheological constitutive equations this requires the replacement of the first-order derivatives by fractional order derivatives. Because the fractional derivative of a function depends on its whole time history and not on its instantaneous behavior, they are perfectly suited for the description of materials with memory, such as polymers or tissues [**8**]. The above given reasons led us to construct discrete fractional rheological models. Discrete models characterize a material by a finite number of springs and dashpots.

Chapter 2

PRELIMINARIES

The main purpose of this chapter is presenting some fundamental definitions and formulas in discrete fractional calculus so that the thesis is self-contained.

The forward difference operator (Δ), for a function $f : \mathbb{N}_a \to \mathbb{R}$ is defined by

$$\Delta f(t) = f(t+1) - f(t),$$

where $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$.

If the function f is defined on a product space, $X \times Y$, where X is a discrete space, then the definition of forward difference operator becomes

$$\Delta_t f(t,s) = f(t+1,s) - f(t,s).$$

2.1. Falling and Rising Factorial Power Functions

In this section, we focus on the falling factorial power function since the definition of the discrete fractional difference and sum operators involve them. In addition to that, we give the definition and some properties of the rising factorial power function which plays an important role in the monotonicity results of the following chapter.

2.1.1. Falling Factorial. The falling factorial power function is defined by [**14**], which is

$$t^{(r)} = t(t-1)(t-2)\dots(t-r+1), r \in \mathbb{N}.$$

Let Γ denote the usual special Gamma function and let $t > 1$. For $\mu \in \mathbb{R}$, the falling factorial power function is

$$t^{(\mu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}.$$

Throughout, we assume that if $t+1-\mu \in \{0,-1,\dots,-k,\dots\}$, then $t^{(\mu)} = 0$.

We consider a map $t \to t^{(\mu)}$ from the set $\{t \in \mathbb{R} : t$ and $t+\nu$ do not belong to $\mathbb{Z}^- \cup \{0\}\}$ to the set of real numbers $\mathbb{R}$.

THEOREM 2.1.1. [**1**] *Let* $\mu \in \mathbb{R}$ *and* $t \in \mathbb{R} - \{\dots,-2,-1,0\}$. *Then*

(i) $\Delta t^{(\mu)} = \mu t^{(\mu-1)}$,

(ii) $\mu^{(\mu)} = \Gamma(\mu+1)$.

2.1.2. Rising Factorial. The rising factorial power function is defined by [**?, 5**], which is

$$t^{\overline{r}} = t(t+1)(t+2)\dots(t+r-1), r \in \mathbb{N}$$

and $t^{\overline{0}} = 1$.

Let ν be any real number. Then the rising factorial power function is

$$t^{\overline{\nu}} = \frac{\Gamma(t+\nu)}{\Gamma(t)}.$$

We note that the Gamma function is not defined at zero and negative integers. Therefore we consider a map $t \to t^{\overline{\nu}}$ from the set $\{t \in \mathbb{R} : t$ and $t+\nu$ do not belong to $\mathbb{Z}^- \cup \{0\}\}$ to the set of real numbers $\mathbb{R}$.

LEMMA 2.1.2. [**3**] *Let* α *and* β *be any real numbers and* $t \in \mathbb{R} - \{\dots, -2, -1, 0\}$*. Then*

(i) $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$,

(ii) $\displaystyle\sum_{n=0}^{k} \frac{t^{\overline{n}}}{\Gamma(n+1)} = \frac{(t+1)^{\overline{k}}}{\Gamma(k+1)}$.

2.2. Fractional Sum and Difference Operators

The fractional sum operator of a function $f(t)$ is denoted by $\Delta_a^{-\nu} f(t)$ and the fractional difference operator of a function $f(t)$ is denoted by $\Delta_a^{\nu} f(t)$ with arbitrary positive real order ν, starting at a. We consider the forward fractional sum as defined by Miller and Ross [**17**]

$$\Delta_a^{-\nu} f(t) = \sum_{s=a}^{t-\nu} \frac{(t-\sigma(s))^{(\nu-1)}}{\Gamma(\nu)} f(s), \tag{2.1}$$

where $\nu \geq 0$, $a \in \mathbb{R}$, and $\sigma(s) = s+1$. Define $\mathbb{N}_{t_0} = \{t_0, t_0+1, t_0+2, \dots\}$ and note that $\Delta_a^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$. Further, we shall consider the Riemann-Liouville fractional difference

$$\Delta_a^{\mu} f(t) = \Delta_a^{m-\nu} f(t) = \Delta^m(\Delta_a^{-\nu} f(t)),$$

where $\mu > 0$, $m - 1 < \mu \leq m$, m denotes a positive integer, and $-\nu = \mu - m$.

There are several ways to establish (2.1). Here we exemplify a simple process of defining the fractional summation as indicated in [**19**].

Let $\mathbb{T}$ be a time scale, which is an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$ the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := sup\{s \in \mathbb{T} : s < t\}.$$

We say that t is left-scattered if $\rho(t) < t$. The set $\mathbb{T}^\kappa$ is derived from the time scale $\mathbb{T}$ as follows:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} - \{M\}, & M \textit{ is the } left - scattered\ maximum \\ \mathbb{T}, & Otherwise. \end{cases}$$

THEOREM 2.2.1. [**7**] *Let $a \in \mathbb{T}^\kappa$, and assume $f : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ is continuous at (a,t), where $t \in \mathbb{T}^\kappa$ with $t > a$. Also assume that $f^\Delta(t,.)$ is rd-continuous on $[a, \sigma(t)]$, where f^Δ denotes the derivative of f with respect to the first variable. Then*

$$g(t) := \int_a^t f(t,\tau)\Delta\tau \quad implies \quad g^\Delta(t) = \int_a^t f^\Delta(t,\tau)\Delta\tau + f(\sigma(t), t). \tag{2.2}$$

As a special case, taking $\mathbb{T} = \mathbb{Z}$ in Theorem 2.2.1 we have,

THEOREM 2.2.2. *Let $a \in \mathbb{Z}$, and assume $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is a function, where $t \in \mathbb{Z}$ with $t > a$. Then*

$$g(t) = \sum_{s=a}^{t-1} f(t,s)$$

implies

$$\Delta_t g(t) = \sum_{s=a}^{t-1} \Delta_t f(t,s) + f(t+1,t).$$

PROOF. By definition of the forward difference operator, we have

$$\Delta_t g(t) = \sum_{s=a}^{t} f(t+1,s) - \sum_{s=a}^{t-1} f(t,s)$$

$$= \sum_{s=a}^{t-1} [f(t+1,s) - f(t,s)] + f(t+1,t)$$

$$= \sum_{s=a}^{t-1} \Delta_t f(t,s) + f(t+1,t).$$

$\square$

Let F_1 and F_2 be defined on $\mathbb{Z}$ and $a, t \in \mathbb{Z}$ with $t \geq a$. Then, by Theorem 2.2.2 we have

$$\Delta_t \left(\sum_{s=a}^{t-1} F_1(t-\sigma(s))F_2(s) \right) = \sum_{s=a}^{t-1} \Delta F_1(t-\sigma(s))F_2(s) + F_1(a)F_2(t). \tag{2.3}$$

To find a general rule for the summation procedure of (2.3) we take the difference $(n-1)$ times and the function F_1 is chosen in such a manner that the function itself and its first $(n-1)$ differences vanishe at $t = a$ (i.e., $\Delta^{n-1}F_1(a) = 0$). Now, apply the difference on both sides of (2.3), we get

$$\Delta_t^2 \left(\sum_{s=a}^{t-1} F_1(t-\sigma(s))F_2(s) \right) = \Delta_t \left(\sum_{s=a}^{t-1} \Delta F_1(t-\sigma(s))F_2(s) \right) + \Delta F_1(a)F_2(t)$$

$$= \sum_{s=a}^{t-1} \Delta^2 F_1(t-\sigma(s))F_2(s) + 2\Delta F_1(a)F_2(t).$$

If we apply the difference on both sides to the above equation one more time, we obtain

$$\Delta_t^3 \left(\sum_{s=a}^{t-1} F_1(t-\sigma(s))F_2(s) \right) = \Delta_t \left(\sum_{s=a}^{t-1} \Delta^2 F_1(t-\sigma(s))F_2(s) + 2\Delta F_1(a)F_2(t) \right)$$

$$= \sum_{s=a}^{t-1} \Delta^3 F_1(t-\sigma(s))F_2(s) + 3\Delta^2 F_1(a)F_2(t).$$

In order to obtain the $n-th$ difference, we follow the above procedure. Therefore we have

$$\Delta_t^n\left(\sum_{s=a}^{t-1}F_1(t-\sigma(s))F_2(s)\right)=\sum_{s=a}^{t-1}\Delta^n F_1(t-\sigma(s))F_2(s)+n\Delta^{n-1}F_1(a)F_2(t)$$
$$=\sum_{s=a}^{t-1}\Delta^n F_1(t-\sigma(s))F_2(s),$$

where $n\in\mathbb{N}_0$. Choose $F_1(t)=\dfrac{t^{(n)}}{n!}$. Hence, the above equality becomes

$$\Delta_t^n\left(\sum_{s=a}^{t-1}\frac{(t-\sigma(s))^{(n)}}{n!}F_2(s)\right)=\sum_{s=a}^{t-1}\Delta^n\left(\frac{(t-\sigma(s))^{(n)}}{n!}\right)F_2(s)=\sum_{s=a}^{t-1}F_2(s). \tag{2.4}$$

Now, if we choose $f(t)=\Delta_t\left(\sum_{s=a}^{t-1}F_2(s)\right)$ and apply the difference on both sides of (2.4) we obtain

$$\Delta_t^{n+1}\left(\sum_{s=a}^{t-1}\frac{(t-\sigma(s))^{(n)}}{n!}F_2(s)\right)=\Delta_t\left(\sum_{s=a}^{t-1}F_2(s)\right)=f(t).$$

Next, taking the $(n+1)$-fold summation of $f(t)$ we have

$$\underbrace{\sum_{s=a}^{t-1}\sum_{s=a}^{t-1}\dots\sum_{s=a}^{t-1}}_{n+1\quad term}f(t)=\sum_{s=a}^{t-1}\frac{(t-\sigma(s))^{(n)}}{n!}F_2(s)=F(t).$$

Therefore,

$$F(t)=\Delta_a^{-(n+1)}f(t)=\frac{1}{\Gamma(n+1)}\sum_{s=a}^{t-1}(t-\sigma(s))^{(n)}F_2(s).$$

Now, it is possible to define an operator $\Delta^{-\nu}$ even when the positive number ν is not necessarily an integer, namely by the expression (2.1).

We would like to close this section by giving a geometric illustration of discrete fractional derivatives. In order to do that we use given data, then plot it for different

α values between zero and one in mathematica (see in Appendix). Therefore, we obtain

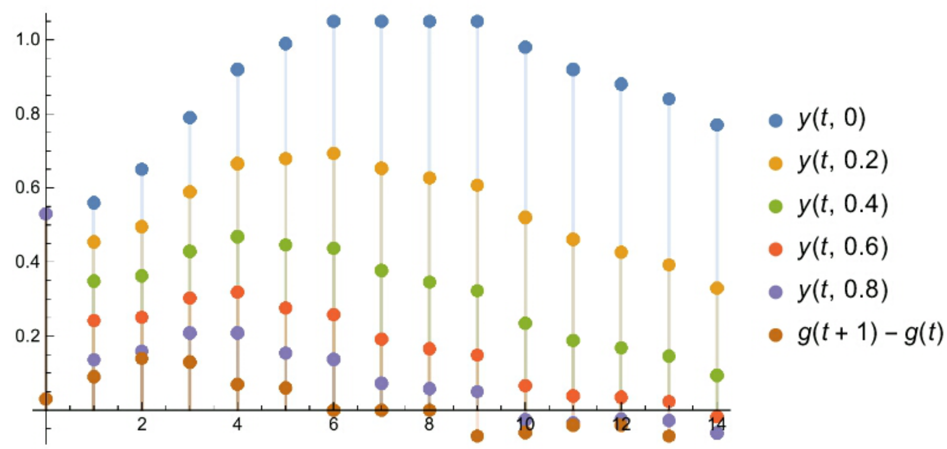

FIGURE 2.2.1. A geometric illustration of discrete fractional derivatives.

By looking at this graph, one observes that

- As α approaches to 1, the discrete fractional derivative "$\Delta_0^\alpha g(t)$" gets closer to the difference operator "$\Delta g(t)$",
- As α approaches to 0, the discrete fractional derivative "$\Delta_0^\alpha g(t)$" gets closer to the function "$g(t)$".

2.3. A Discrete Transform Method of Solution

In this section we will define the discrete transform (R-transform) by

$$R_{t_0}(f(t))(s) = \sum_{t=t_0}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} f(t). \tag{2.5}$$

$R_{t_0}(f(t))(s)$ is the Laplace transform on the time scale of integers [**7, 11**].

The R-transform plays an important role by solving discrete fractional difference equations. We will need this method to be able to solve discrete rheological models. Therefore, we will state some properties of the R-transform for the convenience of the reader.

LEMMA 2.3.1. [**1**] *For any* $\nu \in \mathbb{R} \setminus \{\ldots -2, -1, 0\}$,

(i) $R_{\nu-1}(t^{(\nu-1)})(s) = \dfrac{\Gamma(\nu)}{s^{\nu}}$

(ii) $R_{\nu-1}(t^{(\nu-1)}\alpha^t)(s) = \dfrac{\alpha^{\nu-1}\Gamma(\nu)}{(s+1-\alpha)^{\nu}}$.

We shall also need to make use of a convolution theorem given in [**1**]. Let $h(t) = t^{(\nu-1)}$ and $g(t) = \alpha^t$. Define the convolution

$$(h * g)(t) = \sum_{s=0}^{t-\nu} (t - \sigma(s))^{(\nu-1)} g(s),$$

where $\nu \in \mathbb{R} \setminus \{\ldots -2, -1, 0\}$.

Now we obtain a standard property for $R_{\nu}(h(t))(s)R_0(g(t))(s)$.

LEMMA 2.3.2. [**1**] *For any* $\nu \in \mathbb{R} \setminus \{\ldots -2, -1, 0\}$, *we have*

$$R_{\nu}((h * g)(t))(s) = R_{\nu-1}(h(t))(s)R_0(g(t))(s). \tag{2.6}$$

The above lemma was proven for $g(t) = \alpha^t$ in [**1**]. One can easily notice that the proof can be carried to where g is any function for $t \in \mathbb{N}_0$.

We introduce a few more properties of the R-transform. Treat $\Delta^{\nu} f(t)$ as a convolution and apply Lemma 2.3.2 to obtain

$$R_{a+\nu}(\Delta^{-\nu} f(t))(s) = s^{-\nu} R_a(f(t))(s). \tag{2.7}$$

LEMMA 2.3.3. [**1**] *For $0 < \nu < 1$ and the function f defined for $\nu-1, \nu, \nu+1, \ldots,$ we have*

$$R_0(\Delta^{\nu} f(t))(s) = s^{\nu} R_{\nu-1}(f(t)) - f(\nu-1). \tag{2.8}$$

Chapter 3

MONOTONICITY RESULTS ON DISCRETE FRACTIONAL OPERATORS

In the papers [**10, 12**], the authors obtained the following monotonicity and convexity results based on the sign of the fractional difference operator of a nonnegative real valued function defined on $\mathbb{N}_0$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Theorem 3.0.4. [**10**] *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a nonnegative function satisfying $y(0) = 0$. Fix $\nu \in (1, 2)$ and suppose that $\Delta_0^\nu y(t) \geq 0$ for each $t \in \mathbb{N}_{2-\nu}$. Then y is increasing on $\mathbb{N}_0$.*

Theorem 3.0.5. [**12**] *Fix $\mu \in (N-1, N)$, for $N \in \mathbb{N}_3$ given, and let $y : \mathbb{N}_0 \to \mathbb{R}$ be a given function satisfying $\Delta^j y(0) = 0$ for each $j \in \{0, 1, 2, \ldots, N-3\}$, $\Delta^{N-2} y(0) \geq 0$, and $\Delta_0^\mu y(t) \geq 0$ for each $t \in \mathbb{N}_{N-\mu}$. Then $\Delta^{N-1} y(t) \geq 0$, for each $t \in \mathbb{N}_0$.*

We note that the above two results do not include the case where ν is between zero and one.

The main purpose of this thesis is to obtain monotonicity results for $\nu \in (0, 1)$. First we introduce ν-increasing and ν-decreasing functions for any positive real number ν. We give some restrictions on ν to compare these new monotonicity concepts with the traditional ones. We restate the following monotonicity criterion of the discrete calculus in the discrete fractional calculus:

Let $f : \mathbb{N}_0 \to \mathbb{R}$.

f is monotone increasing on $\mathbb{N}_0$ if and only if $\Delta f(t) \geq 0$ for all $t \in \mathbb{N}_0$.

For this purpose we consider a forward fractional difference operator of Riemann-Liouville type as in the papers [2 – 7]. We then prove some monotonicity criteria for a function f which is defined on $\mathbb{N}_0$ and has a sign (positive or negative) for $\Delta^\nu f$ when ν is between 0 and 1.

We introduce two new monotonicity concepts. Let ν be any positive real number.

Definition 3.0.6. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) \geq 0$. y is called a ν-increasing function on $\mathbb{N}_0$, if*

$$y(a+1) \geq \nu y(a) \ for\ all\ a \in \mathbb{N}_0.$$

Note that if y is increasing on $\mathbb{N}_0$ and $0 < \nu < 1$, then y is ν-increasing on $\mathbb{N}_0$. Also, if y is ν-increasing on $\mathbb{N}_0$ and $\nu \geq 1$, then y is increasing on $\mathbb{N}_0$. If $\nu = 1$, then y is increasing on $\mathbb{N}_0$ if and only if y is ν-increasing on $\mathbb{N}_0$.

Definition 3.0.7. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) \leq 0$. y is called a ν-decreasing function on $\mathbb{N}_0$, if*

$$y(a+1) \leq \nu y(a) \ for\ all\ a \in \mathbb{N}_0.$$

Note that if y is decreasing on $\mathbb{N}_0$ and $0 < \nu < 1$, then y is ν-decreasing on $\mathbb{N}_0$. Also, if y is ν-decreasing on $\mathbb{N}_0$ and $\nu \geq 1$, then y is decreasing on $\mathbb{N}_0$. If $\nu = 1$, then y is decreasing on $\mathbb{N}_0$ if and only if y is ν-decreasing on $\mathbb{N}_0$.

Example 3.0.8. *We note that any increasing function on $\mathbb{N}_0$ with positive initial point is ν-increasing.*

Example 3.0.9. *Consider $g(t) = e^{-t}$ on $\mathbb{N}_0$. We claim that the function g is ν-increasing when $\nu \in (0, 1/e]$. This can be easily verified. In fact, we multiply each side of the inequality $0 < \nu \leq 1/e$ by e^{-t}. This implies that $0 < \nu e^{-t} \leq e^{-(1+t)}$. Therefore, by Definition 3.0.6, $g(t) = e^{-t}$ is ν-increasing on $\mathbb{N}_0$.*

3.1. ν-Increasing Functions

Theorem 3.1.1. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) \geq 0$. Fix $\nu \in (0, 1)$ and suppose that*

$$\Delta_0^\nu y(t) \geq 0 \text{ for each } t \in \mathbb{N}_{1-\nu}.$$

Then, y is ν-increasing on $\mathbb{N}_0$.

Proof. We will prove that y is ν-increasing by mathematical induction. First, we observe that

$$\Delta_0^\nu y(t) = \Delta\Delta_0^{-(1-\nu)} y(t) = \Delta\left[\frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{t-(1-\nu)} (t - \sigma(s))^{(-\nu)} y(s)\right] \geq 0.$$

Let $s(t) = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{t-(1-\nu)} (t-\sigma(s))^{(-\nu)} y(s)$. Since $\Delta s(t) \geq 0$, $s(t)$ is an increasing function on $\mathbb{N}_{1-\nu}$. This implies that

$$\begin{aligned} s(2-\nu) - s(1-\nu) &= \frac{1}{\Gamma(1-\nu)}(1-\nu)^{(-\nu)} y(0) + \frac{1}{\Gamma(1-\nu)}(-\nu)^{(-\nu)} y(1) - \frac{1}{\Gamma(1-\nu)}(-\nu)^{(-\nu)} y(0) \\ &= \frac{1}{\Gamma(1-\nu)}\left[\Delta(-\nu)^{(-\nu)} y(0) + (-\nu)^{(-\nu)} y(1)\right] \\ &= \frac{1}{\Gamma(1-\nu)}\left[-\nu \frac{\Gamma(1-\nu)}{\Gamma(2)} y(0) + \frac{\Gamma(1-\nu)}{\Gamma(1)} y(1)\right] \geq 0. \end{aligned}$$

Therefore, we have

$$y(1) \geq \nu y(0).$$

Now, let us assume that the induction hypothesis is valid up to $n = k-1$. Hence we have

$$y(k) \geq \nu y(k-1) \geq \nu^2 y(k-2) \geq \ldots \geq \nu^k y(0) \geq 0. \tag{3.1}$$

We want to prove that for $n = k$, the inequality

$$y(k+1) \geq \nu y(k) \tag{3.2}$$

is valid. To prove (3.2) we first calculate,

$$s(k+1-\nu) = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k} (k+1-\nu-\sigma(s))^{(-\nu)} y(s)$$

and

$$s(k+2-\nu) = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k+1} (k+2-\nu-\sigma(s))^{(-\nu)} y(s).$$

Since $s(t)$ is increasing, we have

$$
\begin{aligned}
s(k+2-\nu) - s(k+1-\nu)& \\
&= \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k+1} (k+1-\nu-s)^{(-\nu)} y(s) \\
&\quad - \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k} (k-\nu-s)^{(-\nu)} y(s) \geq 0.
\end{aligned}
$$

Performing the sum operations above, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\nu)} \Big[(k+1-\nu)^{(-\nu)} y(0) + (k-\nu)^{(-\nu)} y(1) \\
&+ (k-1-\nu)^{(-\nu)} y(2) + \ldots + (2-\nu)^{(-\nu)} y(k-1) + (1-\nu)^{(-\nu)} y(k) + (-\nu)^{(-\nu)} y(k+1) \Big] \\
&- \frac{1}{\Gamma(1-\nu)} \Big[(k-\nu)^{(-\nu)} y(0) + (k-1-\nu)^{(-\nu)} y(1) + (k-2-\nu)^{(-\nu)} y(2) \\
&+ \ldots + (1-\nu)^{(-\nu)} y(k-1) + (-\nu)^{(-\nu)} y(k) \Big] \geq 0.
\end{aligned}
$$

Then grouping the like terms we obtain the following inequality:

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\nu)} \Big[\big[(k+1-\nu)^{(-\nu)} - (k-\nu)^{(-\nu)} \big] y(0) + \big[(k-\nu)^{(-\nu)} - (k-1-\nu)^{(-\nu)} \big] y(1) \\
&+ \big[(k-1-\nu)^{(-\nu)} - (k-2-\nu)^{(-\nu)} \big] y(2) \\
&+ \ldots + \big[(2-\nu)^{(-\nu)} - (1-\nu)^{(-\nu)} \big] y(k-1) + \big[(1-\nu)^{(-\nu)} - (-\nu)^{(-\nu)} \big] y(k) + (-\nu)^{(-\nu)} y(k+1) \Big] \geq 0.
\end{aligned}
$$

Next we rewrite the coefficients of $y(0), y(1), \ldots y(k+1)$ with Δ operator as follows

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\nu)} \Big[\Delta(k-\nu)^{(-\nu)} y(0) + \Delta(k-1-\nu)^{(-\nu)} y(1) + \Delta(k-2-\nu)^{(-\nu)} y(2) \\
&+ \ldots + \Delta(1-\nu)^{(-\nu)} y(k-1) + \Delta(-\nu)^{(-\nu)} y(k) + (-\nu)^{(-\nu)} y(k+1) \Big] \geq 0.
\end{aligned}
$$

After performing the Δ operator using the power rule in Theorem 2.1.1 (i), we obtain

$$\frac{1}{\Gamma(1-\nu)}\Big[(-\nu)(k-\nu)^{(-\nu-1)}y(0)+(-\nu)(k-1-\nu)^{(-\nu-1)}y(1)+(-\nu)(k-2-\nu)^{(-\nu-1)}y(2)$$
$$+\ldots+(-\nu)(1-\nu)^{(-\nu-1)}y(k-1)+(-\nu)(-\nu)^{(-\nu-1)}y(k)+(-\nu)^{(-\nu)}y(k+1)\Big]\geq 0.$$

Then using the definition of falling factorial power, we have

$$\frac{1}{\Gamma(1-\nu)}\Big[(-\nu)\frac{\Gamma(k-\nu+1)}{\Gamma(k-\nu+1+\nu+1)}y(0)+(-\nu)\frac{\Gamma(k-1-\nu+1)}{\Gamma(k-1-\nu+1+\nu+1)}y(1)$$
$$+(-\nu)\frac{\Gamma(k-2-\nu+1)}{\Gamma(k-2-\nu+1+\nu+1)}y(2)+\ldots+(-\nu)\frac{\Gamma(1-\nu+1)}{\Gamma(1-\nu+1+\nu+1)}y(k-1)$$
$$+(-\nu)\frac{\Gamma(-\nu+1)}{\Gamma(-\nu+1+\nu+1)}y(k)+\Gamma(-\nu+1)y(k+1)\Big]\geq 0.$$

Next we simplify the above expression as the following

$$y(k+1)+\frac{(-\nu)}{\Gamma(1-\nu)}\Big[\frac{\Gamma(k-\nu+1)}{\Gamma(k+2)}y(0)+\frac{\Gamma(k-\nu)}{\Gamma(k+1)}y(1)+\frac{\Gamma(k-1-\nu)}{\Gamma(k)}y(2)$$
$$+\ldots+\frac{\Gamma(2-\nu)}{\Gamma(3)}y(k-1)+\frac{\Gamma(1-\nu)}{\Gamma(2)}y(k)\Big]\geq 0.$$

So, we have

$$y(k+1)+\frac{(-\nu)}{\Gamma(1-\nu)}\Big[\frac{(k-\nu)(k-1-\nu)\ldots(2-\nu)(1-\nu)\Gamma(1-\nu)}{\Gamma(k+2)}y(0)$$
$$+\frac{(k-1-\nu)(k-2-\nu)\ldots(2-\nu)(1-\nu)\Gamma(1-\nu)}{\Gamma(k+1)}y(1)$$
$$+\frac{(k-2-\nu)(k-3-\nu)\ldots(2-\nu)(1-\nu)\Gamma(1-\nu)}{\Gamma(k)}y(2)$$
$$+\ldots+\frac{(1-\nu)\Gamma(1-\nu)}{\Gamma(3)}y(k-1)+\frac{\Gamma(1-\nu)}{\Gamma(2)}y(k)\Big]\geq 0.$$

Only by one simple algebraic step we obtain

$$y(k+1) \geq \frac{(k-\nu)(k-1-\nu)\dots(2-\nu)(1-\nu)\nu}{\Gamma(k+2)}y(0)$$
$$+\frac{(k-1-\nu)(k-2-\nu)\dots(2-\nu)(1-\nu)\nu}{\Gamma(k+1)}y(1)+\frac{(k-2-\nu)(k-3-\nu)\dots(2-\nu)(1-\nu)\nu}{\Gamma(k)}y(2)$$
$$+\dots+\frac{(1-\nu)\nu}{\Gamma(3)}y(k-1)+\frac{\nu}{\Gamma(2)}y(k).$$

By the induction assumption (3.1), we have

$$y(k+1)-\nu y(k) \geq \frac{(k-\nu)(k-1-\nu)\dots(2-\nu)(1-\nu)\nu}{\Gamma(k+2)}y(0)$$
$$+\frac{(k-1-\nu)(k-2-\nu)\dots(2-\nu)(1-\nu)\nu}{\Gamma(k+1)}y(1)+\frac{(k-2-\nu)(k-3-\nu)\dots(2-\nu)(1-\nu)\nu}{\Gamma(k)}y(2)$$
$$+\dots+\frac{(1-\nu)\nu}{\Gamma(3)}y(k-1) \geq 0.$$

Hence, we conclude that for each $k \in \mathbb{N}$,

$$y(k+1)-\nu y(k) \geq 0.$$

□

In the proof of the next theorem, the rising factorial power function plays an important role.

THEOREM 3.1.2. *Let* $y : \mathbb{N}_0 \to \mathbb{R}$ *be a function satisfying* $y(0) \geq 0$. *Fix* $\nu \in (0,1)$ *and assume that* y *is an increasing function on* $\mathbb{N}_0$. *Then,*

$$\Delta_0^\nu y(t) \geq 0 \ for\ each\ t \in \mathbb{N}_{1-\nu}.$$

Proof. We want to show that

$$\Delta_0^{\nu} y(t) = \Delta\Delta_0^{-(1-\nu)} y(t) = \Delta\left[\frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{t-(1-\nu)} (t-\sigma(s))^{(-\nu)} y(s)\right] \geq 0.$$

Similarly, let

$$s(t) = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{t-(1-\nu)} (t-\sigma(s))^{(-\nu)} y(s).$$

To complete the proof, we need to show that $s(t)$ is increasing on $\mathbb{N}_{1-\nu}$.

For any natural number k with $k \geq 1$ we show that

$$s(k+1-\nu) - s(k-\nu) \geq 0,$$

is valid. In fact, we have

$$\begin{aligned}
&s(k+1-\nu) - s(k-\nu) \\
&\quad = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k} (k+1-\nu-\sigma(s))^{(-\nu)} y(s) - \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k-1} (k-\nu-\sigma(s))^{(-\nu)} y(s) \\
&\quad = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k-1} \Delta_k (k-\nu-\sigma(s))^{(-\nu)} y(s) + \frac{1}{\Gamma(1-\nu)} (-\nu)^{(-\nu)} y(k) \\
&\quad = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k-1} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)} y(s) + y(k) \\
&\quad = y(k) - \nu y(k-1) + \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k-2} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)} y(s) \\
&\quad = y(k) - \nu y(k-1) + \frac{\nu}{\Gamma(1-\nu)} \sum_{s=0}^{k-2} (k-\nu-\sigma(s))^{(-\nu-1)} (y(k-1) - y(s)) \\
&\qquad\quad + \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k-2} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)} y(k-1) \\
&\quad \geq y(k) - \nu y(k-1) + \frac{y(k-1)}{\Gamma(1-\nu)} \sum_{s=0}^{k-2} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)}
\end{aligned}$$

$$= y(k) - y(k-1) + y(k-1) + \frac{y(k-1)}{\Gamma(1-\nu)} \sum_{s=0}^{k-1} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)}$$

$$\geq y(k-1)\left(1 + \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k-1} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)}\right).$$

Then using Theorem 2.1.1 (ii) and the definitions of falling and rising factorial power functions, we have

$$s(k+1-\nu) - s(k-\nu) \geq y(k-1)\frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k} (-\nu)(k-\nu-\sigma(s))^{(-\nu-1)}$$

$$= y(k-1) \sum_{s=0}^{k} \frac{(-\nu)^{\overline{s}}}{s!}.$$

By Theorem 2.1.2 (ii), we have $\sum_{s=0}^{k} \frac{(-\nu)^{\overline{s}}}{s!} = \frac{(1-\nu)^{\overline{k}}}{k!}$. Therefore, we obtain

$$s(k+1-\nu) - s(k-\nu) \geq y(k-1)\frac{(1-\nu)^{\overline{k}}}{k!}.$$

Choosing $\alpha = k-1$ and $\beta = 1$ in Theorem 2.1.2 (i), we rewrite the notation

$$\frac{(1-\nu)^{\overline{k}}}{k!} = \frac{(1-\nu)^{\overline{k-1}}}{(k-1)!}\frac{(1-\nu+k-1)^{\overline{1}}}{k}.$$

By the definition of the rising factorial power function, we say $\frac{(1-\nu)^{\overline{k-1}}}{(k-1)!} \geq 0$. Hence, we have

$$s(k+1-\nu) - s(k-\nu) \geq y(k-1)\frac{(k-\nu)}{k} \geq 0.$$

This completes the proof. □

As a conclusion of the above obtained results, with $y(0) \geq 0$ and for $0 < \nu < 1$ we have the following list of statements:

- If y is increasing function on $\mathbb{N}_0$ if and only if $\Delta y(t) \geq 0$ for all $t \in \mathbb{N}_0$.

- If y is increasing function on $\mathbb{N}_0$ then y is ν-increasing function on $\mathbb{N}_0$.
- If y is increasing function on $\mathbb{N}_0$ then $\Delta_0^\nu y(t) \geq 0$ for each $t \in \mathbb{N}_{1-\nu}$.
- If $\Delta_0^\nu y(t) \geq 0$ for each $t \in \mathbb{N}_{1-\nu}$ then y is ν-increasing function on $\mathbb{N}_0$.
- If $\Delta y(t) \geq 0$ for all $t \in \mathbb{N}_0$ then y is ν-increasing function on $\mathbb{N}_0$.
- If $\Delta y(t) \geq 0$ for all $t \in \mathbb{N}_0$ then $\Delta_0^\nu y(t) \geq 0$ for each $t \in \mathbb{N}_{1-\nu}$.

We can summarize all these statements by using the diagram below for $0 < \nu < 1$,

$$\begin{tikzcd}
y(t)\ is\ increasing \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \Delta y(t) \geq 0 \\
\Delta_0^\nu y(t) \geq 0 \arrow[r, Rightarrow] & y(t)\ is\ \nu - increasing
\end{tikzcd}$$

The above proof can be easily carried over to the proof of the following result.

Theorem 3.1.3. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) > 0$. Fix $\nu \in (0,1)$ and assume that y is a strictly increasing function on $\mathbb{N}_0$. Then,*

$$\Delta_0^\nu y(t) > 0 \ for\ each\ t \in \mathbb{N}_{1-\nu}.$$

Corollary 3.1.4. *Let $h : [1, +\infty)_{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ be a nonnegative, continuous function, and let A be a nonnegative real number. Then the unique solution of the discrete fractional IVP*

$$\Delta_0^\nu y(t) = h(t + \nu - 1, y(t + \nu - 1)),\ t \in \mathbb{N}_{1-\nu}$$

$$y(0) = A,$$

y is ν-increasing and nonnegative.

3.2. ν-Decreasing Functions

In a similar way, the above results can be obtained for the function which takes a negative value at the initial point of its domain. For the convenience of the reader we would like to include the proof of the next theorem.

THEOREM 3.2.1. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) \le 0$. Fix $\nu \in (0,1)$ and suppose that*

$$\Delta_0^{\nu} y(t) \le 0 \ for\ each\ t \in \mathbb{N}_{1-\nu}.$$

Then, y is ν-decreasing on $\mathbb{N}_0$.

PROOF. We will prove that y is a ν-decreasing function on $\mathbb{N}_0$ by mathematical induction. We observe that

$$\Delta_0^{\nu} y(t) = \Delta\Delta_0^{-(1-\nu)} y(t) = \Delta\left[\frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{t-(1-\nu)} (t-\sigma(s))^{(-\nu)} y(s)\right] \le 0.$$

Let $s(t) = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{t-(1-\nu)} (t-\sigma(s))^{(-\nu)} y(s)$. Since $\Delta s(t) \le 0$, $s(t)$ is a decreasing function on $\mathbb{N}_{1-\nu}$. This implies that

$$s(2-\nu) - s(1-\nu) = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{1} (2-\nu-\sigma(s))^{(-\nu)} y(s) - \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{0} (1-\nu-\sigma(s))^{(-\nu)} y(s)$$

$$= \frac{1}{\Gamma(1-\nu)} (1-\nu)^{(-\nu)} y(0) + \frac{1}{\Gamma(1-\nu)} (-\nu)^{(-\nu)} y(1) - \frac{1}{\Gamma(1-\nu)} (-\nu)^{(-\nu)} y(0)$$

$$= \frac{1}{\Gamma(1-\nu)} \left[\Delta(-\nu)^{(-\nu)} y(0) + (-\nu)^{(-\nu)} y(1)\right]$$

$$= \frac{1}{\Gamma(1-\nu)} \left[-\nu \frac{\Gamma(1-\nu)}{\Gamma(2)} y(0) + \frac{\Gamma(1-\nu)}{\Gamma(1)} y(1))\right] \le 0.$$

This implies that

$$y(1) \leq \nu y(0).$$

Now, we assume that the induction hypothesis is valid up to $n = k - 1$. Thus we have

$$y(k) \leq \nu y(k-1) \leq \nu^2 y(k-2) \leq \ldots \leq \nu^k y(0) \leq 0. \tag{3.3}$$

We want to show that for $n = k$, the inequality

$$y(k+1) \leq \nu y(k) \tag{3.4}$$

is valid. Now, in order to prove (3.4) we calculate

$$s(k+1-\nu) = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k} (k-\nu-s)^{(-\nu)} y(s)$$

and

$$s(k+2-\nu) = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k+1} (k+1-\nu-s)^{(-\nu)} y(s).$$

Since $s(t)$ is decreasing, we have

$$\begin{aligned} &s(k+2-\nu) - s(k+1-\nu) \\ &\qquad = \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k+1} (k+1-\nu-s)^{(-\nu)} y(s) \\ &\qquad\quad - \frac{1}{\Gamma(1-\nu)} \sum_{s=0}^{k} (k-\nu-s)^{(-\nu)} y(s) \leq 0. \end{aligned}$$

By performing the sum operations above, we obtain

$$\frac{1}{\Gamma(1-\nu)}\Bigg[(k+1-\nu)^{(-\nu)}y(0)+(k-\nu)^{(-\nu)}y(1)$$

$$+(k-1-\nu)^{(-\nu)}y(2)+\ldots+(2-\nu)^{(-\nu)}y(k-1)+(1-\nu)^{(-\nu)}y(k)+(-\nu)^{(-\nu)}y(k+1)\Bigg]$$

$$-\frac{1}{\Gamma(1-\nu)}\Bigg[(k-\nu)^{(-\nu)}y(0)+(k-1-\nu)^{(-\nu)}y(1)+(k-2-\nu)^{(-\nu)}y(2)$$

$$+\ldots+(1-\nu)^{(-\nu)}y(k-1)+(-\nu)^{(-\nu)}y(k)\Bigg]\leq 0.$$

Then by grouping the terms we get the following inequality:

$$\frac{1}{\Gamma(1-\nu)}\Bigg[\big[(k+1-\nu)^{(-\nu)}-(k-\nu)^{(-\nu)}\big]y(0)+\big[(k-\nu)^{(-\nu)}-(k-1-\nu)^{(-\nu)}\big]y(1)$$

$$+\big[(k-1-\nu)^{(-\nu)}-(k-2-\nu)^{(-\nu)}\big]y(2)$$

$$+\ldots+\big[(2-\nu)^{(-\nu)}-(1-\nu)^{(-\nu)}\big]y(k-1)+\big[(1-\nu)^{(-\nu)}-(-\nu)^{(-\nu)}\big]y(k)+(-\nu)^{(-\nu)}y(k+1)\Bigg]\leq 0.$$

Now, we rewrite the coefficients of $y(0), y(1), \ldots y(k+1)$ with Δ operator as follows

$$\frac{1}{\Gamma(1-\nu)}\Bigg[\Delta(k-\nu)^{(-\nu)}y(0)+\Delta(k-1-\nu)^{(-\nu)}y(1)+\Delta(k-2-\nu)^{(-\nu)}y(2)$$

$$+\ldots+\Delta(1-\nu)^{(-\nu)}y(k-1)+\Delta(-\nu)^{(-\nu)}y(k)+(-\nu)^{(-\nu)}y(k+1)\Bigg]\leq 0.$$

After performing the Δ operator using the power rule in Theorem 2.1.1 (i) and the definition of the falling factorial power, we obtain

$$\frac{1}{\Gamma(1-\nu)}\Big[(-\nu)(k-\nu)^{(-\nu-1)}y(0)+(-\nu)(k-1-\nu)^{(-\nu-1)}y(1)+(-\nu)(k-2-\nu)^{(-\nu-1)}y(2)$$
$$+\ldots+(-\nu)(1-\nu)^{(-\nu-1)}y(k-1)+(-\nu)(-\nu)^{(-\nu-1)}y(k)+(-\nu)^{(-\nu)}y(k+1)\Big]$$
$$=\frac{1}{\Gamma(1-\nu)}\Big[(-\nu)\frac{\Gamma(k-\nu+1)}{\Gamma(k-\nu+1+\nu+1)}y(0)+(-\nu)\frac{\Gamma(k-1-\nu+1)}{\Gamma(k-1-\nu+1+\nu+1)}y(1)$$
$$+(-\nu)\frac{\Gamma(k-2-\nu+1)}{\Gamma(k-2-\nu+1+\nu+1)}y(2)+\ldots+(-\nu)\frac{\Gamma(1-\nu+1)}{\Gamma(1-\nu+1+\nu+1)}y(k-1)$$
$$+(-\nu)\frac{\Gamma(-\nu+1)}{\Gamma(-\nu+1+\nu+1)}y(k)+\Gamma(-\nu+1)y(k+1)\Big]\le 0.$$

Next we simplify the above expression, then we have

$$y(k+1)+\frac{(-\nu)}{\Gamma(1-\nu)}\Big[\frac{(k-\nu)(k-1-\nu)\ldots(2-\nu)(1-\nu)\Gamma(1-\nu)}{\Gamma(k+2)}y(0)$$
$$+\frac{(k-1-\nu)(k-2-\nu)\ldots(2-\nu)(1-\nu)\Gamma(1-\nu)}{\Gamma(k+1)}y(1)$$
$$+\frac{(k-2-\nu)(k-3-\nu)\ldots(2-\nu)(1-\nu)\Gamma(1-\nu)}{\Gamma(k)}y(2)$$
$$+\ldots+\frac{(1-\nu)\Gamma(1-\nu)}{\Gamma(3)}y(k-1)+\frac{\Gamma(1-\nu)}{\Gamma(2)}y(k)\Big]\le 0.$$

By a simple algebraic step and the induction assumption (3.3), we have

$$y(k+1)-\nu y(k)\le\frac{(k-\nu)(k-1-\nu)\ldots(2-\nu)(1-\nu)\nu}{\Gamma(k+2)}y(0)$$
$$+\frac{(k-1-\nu)(k-2-\nu)\ldots(2-\nu)(1-\nu)\nu}{\Gamma(k+1)}y(1)+\frac{(k-2-\nu)(k-3-\nu)\ldots(2-\nu)(1-\nu)\nu}{\Gamma(k)}y(2)$$
$$+\ldots+\frac{(1-\nu)\nu}{\Gamma(3)}y(k-1)\le 0.$$

Therefore, we conclude that for each $k \in \mathbb{N}$,

$$y(k+1) \leq \nu y(k)$$

which means that y is a ν-decreasing function for all n on $\mathbb{N}_0$. □

Since following theorem can be proved in a similar way as in Theorem 3.1.2, so we omit the proof.

THEOREM 3.2.2. *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $y(0) \leq 0$. Fix $\nu \in (0,1)$ and assume that y is decreasing function on $\mathbb{N}_0$. Then,*

$$\Delta_0^\nu y(t) \leq 0 \; for \; each \; t \in \mathbb{N}_{1-\nu}.$$

COROLLARY 3.2.3. *Let $h : [1,+\infty)_{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ be a nonpositive, continuous function, and let A be a nonpositive real number. Then the unique solution of the discrete fractional IVP*

$$\Delta_0^\nu y(t) = h(t+\nu-1, y(t+\nu-1)), \;\; t \in \mathbb{N}_{1-\nu}$$

$$y(0) = A,$$

y is ν-decreasing and nonpositive.

Chapter 4

DISCRETE FRACTIONAL RHEOLOGICAL MODELS

The plan of this chapter is the following. We recall the essential notions of linear viscoelasticity in order to present our notations for the mechanical models. We limit our attention to the basic mechanical models, characterized by two and four parameters.

We will consider a range of approaches to the linear theory of viscoelasticity from integer-order models to discrete fractional calculus models.

4.1. Integer-Order Rheological Models

The word viscoelastic is derived from the words " viscous " and " elastic "; a viscoelastic material exhibits both viscous and elastic behavior– a bit like a fluid and a bit like a solid. One can build up a model of linear viscoelasticity by considering combinations of the linear elastic spring and the linear viscous dash-pot. These are known as rheological models.

4.1.1. The Linear Elastic Spring and Viscous Dash-Pot. The springs are assumed to obey Hooke's Law, which responds as a linear elastic spring of constant stiffness E as:

$$\sigma(t) = E\epsilon(t) \tag{4.1}$$

where σ is the stress and ϵ is the strain that occurs under the given stress.

The dash-pots are assumed to obey Newton's Fluid Law, which responds with a strain-rate proportional to stress:

$$\sigma(t) = \eta \frac{d\epsilon(t)}{dt} \tag{4.2}$$

where η is the constant coefficient of viscosity.

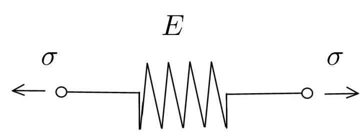

FIGURE 4.1.1. The linear elastic spring.

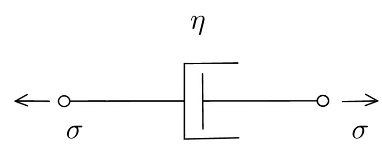

FIGURE 4.1.2. The linear viscous dash-pot.

Hooke's Law represents the rheological constitutive equation of an ideal solid, whereas Newton's Law corresponds to an ideal fluid. Therefore, relations (4.1) and (4.2) are not universal laws, but only mathematical models for an ideal solid material and for an ideal fluid, neither of which exists in the real world. In fact, real materials combine properties of those two limit cases and lie somewhere between ideal solids and ideal fluids, if materials are sorted with respect to their firmness [**18**].

In order to derive the rheological constitutive equations for viscoelastic materials, one starts by combining a small number of springs and dashpots in series or parallel.

4.1.2. The Maxwell Model. The serial connection of the two basic elements gives Maxwell's Model of viscoelasticity. Therefore, the sum of the strain of spring (ϵ_1) and dash-pot (ϵ_2) is equal to the total strain (ϵ). Also, equilibrium requires that the stress (σ) be the same in both elements. One thus has the following three equations in four unknowns:

$$\epsilon_1(t) = \frac{1}{E}\sigma(t), \quad \frac{d\epsilon_2(t)}{dt} = \frac{1}{\eta}\sigma(t), \quad \epsilon(t) = \epsilon_1(t) + \epsilon_2(t)$$

To eliminate ϵ_1 and ϵ_2, differentiate the first and third equations, and put the first and second into the third. We obtain the continuous Maxwell model which is written as

$$\frac{d\sigma(t)}{dt} + \frac{1}{\tau}\sigma(t) = E\frac{d\epsilon(t)}{dt} \tag{4.3}$$

where $\tau = \dfrac{\eta}{E}$ has the units of time. This constitute equation has been put in what is known as the standard form– stress on left, strain on right.

In this model, the stress relaxes or decays with time constant τ following a unit step function in applied strain. This stress relaxation behavior is characteristic of a viscoelastic material.

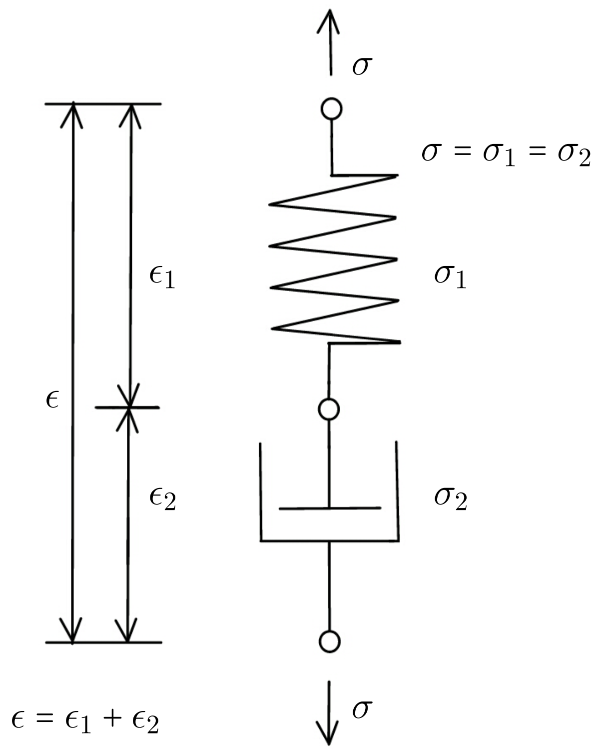

FIGURE 4.1.3. The Maxwell model.

4.1.3. The Kelvin-Voigt Model. The parallel connection of the two basic elements gives Kelvin-Voigt's Model. This model assumes that there is no bending in this type of parallel arrangement, so that the strain experienced by the spring is the same as that experienced by the dash-pot. Therefore, we have

$$\epsilon(t) = \frac{1}{E}\sigma_1(t), \quad \frac{d\epsilon(t)}{dt} = \frac{1}{\eta}\sigma_2(t), \quad \sigma(t) = \sigma_1(t) + \sigma_2(t)$$

where σ_1 is the stress in the spring and σ_2 is the stress in the dash-pot.

Eliminating σ_1 and σ_2 leaves the constitutive law of the Kelvin-Voigt model, which is written as

$$\sigma(t) = E\epsilon(t) + \eta\frac{d\epsilon(t)}{dt}. \tag{4.4}$$

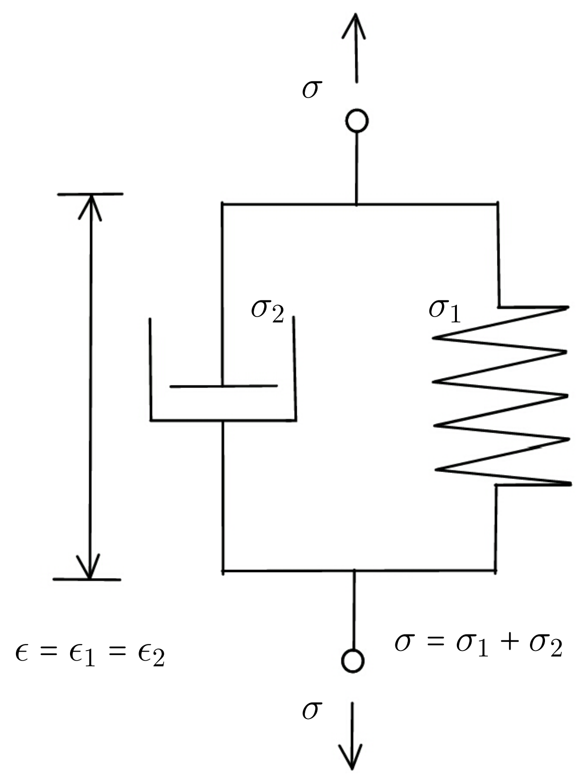

FIGURE 4.1.4. The Kelvin-Voigt model.

4.2. The Discrete Fractional Order Elements in Constitutive Models of Viscoelasticity

The next step in developing fractional viscoelasticity models is to combine a fractional order element with other mechanical components. The behavior of these systems can be described by fractional order differential constitutive equations. In order to show how the fractional order models behave in a number of usual dynamic experiments on viscoelastic solids, Smith and Vries in [**19**] used various conventional experiments, such as constant-rate-of-deformation test, relaxation test, creep test, hysteresis experiment and oscillatory experiment to compare fractional models with integer-order models.

The goal is to develop models whose solutions are more appropriate for describing complex physical or physiological systems, as in [**16**].

This fractional order element is characterized by the fractional order differential equation

$$\sigma(t) = E\tau^{\gamma}\frac{d^{\gamma}\epsilon(t)}{dt^{\gamma}} \tag{4.5}$$

and includes three parameters (γ, E, τ).

Since our purpose is constructing discrete fractional models, instead of using (4.5) we need discrete fractional order elements, which give the fractional difference equation

$$\sigma(t) = E\tau^{\gamma}\Delta^{\gamma}\epsilon(t), \tag{4.6}$$

where $0 < \gamma \leq 1$.

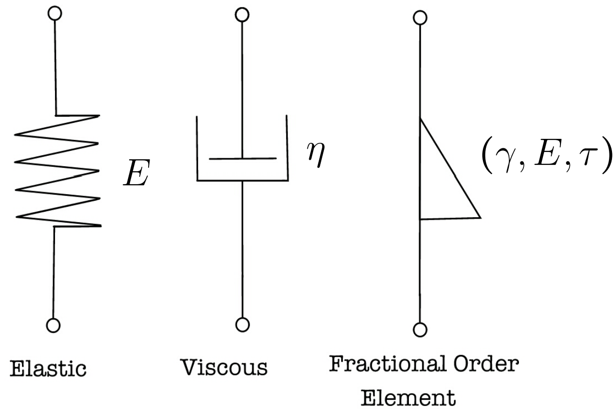

FIGURE 4.2.1. Elastic, viscous, and fractional order viscoelastic models for ideal elastic ($\gamma = 0, \tau = 1$), viscous ($\gamma = 1, E\tau = \eta$), and fractional order (γ, E, τ) elements.

Before we define discrete fractional models, we state and prove a theorem which will help us to solve the equations of these new models.

THEOREM 4.2.1. *Assume* $0 < \nu < 1$ *and* f *is defined on* $\mathbb{N}_{\nu-1}$ *for* $t \in \mathbb{N}_0$. *The discrete fractional difference equation*

$$\Delta^\alpha z(t) + \lambda z(t + \alpha - 1) = f(t + \alpha - 1) \tag{4.7}$$

has the general solution

$$z(t) = z(\alpha - 1)E(t, -\lambda, \alpha) + \left[f(t + \alpha - 1) * E(t, -\lambda, \alpha)\right], \tag{4.8}$$

where $E(t, -\lambda, \alpha) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma((n+1)\alpha)}(t + n(\alpha - 1))^{((n+1)\alpha - 1)}$.

Proof. Applying the R-transform on both sides of (4.7), we obtain

$$R_0\left(\Delta^\alpha z(t)\right)(s) + \lambda R_0\left(z(t+\alpha-1)\right)(s) = R_0\left(f(t+\alpha-1)\right)(s).$$

By Lemma (2.3.3), we have

$$s^\alpha R_{\alpha-1}\left(z(t)\right) - z(\alpha-1) + \lambda R_0\left(z(t+\alpha-1)\right)(s) = R_0\left(f(t+\alpha-1)\right).$$

Also, by (2.5) we have

$$R_0\left(z(t+\alpha-1)\right) = \left(\frac{1}{s+1}\right)^{1-\alpha} R_{\alpha-1}\left(z(t)\right).$$

Therefore, we obtain

$$s^\alpha R_{\alpha-1}\left(z(t)\right) - z(\alpha-1) + \frac{\lambda}{(s+1)^{1-\alpha}} R_{\alpha-1}\left(z(t)\right) = R_0\left(f(t+\alpha-1)\right).$$

Only by one simple algebraic step we obtain

$$R_{\alpha-1}\left(z(t)\right)\left[s^\alpha + \frac{\lambda}{(s+1)^{1-\alpha}}\right] = R_0\left(f(t+\alpha-1)\right) + z(\alpha-1).$$

We solve this equation for $R_{\alpha-1}\left(z(t)\right)$, we get

$$R_{\alpha-1}\left(z(t)\right) = \frac{1}{s^\alpha + \frac{\lambda}{(s+1)^{1-\alpha}}}\left[R_0\left(f(t+\alpha-1)\right) + z(\alpha-1)\right]. \tag{4.9}$$

Now, we need to arrange $\dfrac{1}{s^\alpha + \frac{\lambda}{(s+1)^{1-\alpha}}}$ as

$$\frac{1}{s^\alpha + \frac{\lambda}{(s+1)^{1-\alpha}}} = \frac{1}{s^\alpha}\frac{1}{1 + \frac{\lambda}{s^\alpha (s+1)^{1-\alpha}}} = \frac{1}{s^\alpha}\frac{1}{1 + \frac{\lambda (s+1)^{\alpha-1}}{s^\alpha}}.$$

In fact, we have the following series expansion

$$\frac{1}{s^{\alpha}+\frac{\lambda}{(s+1)^{1-\alpha}}} = \frac{1}{s^{\alpha}}\left(1-\frac{\lambda(s+1)^{\alpha-1}}{s^{\alpha}}+\frac{\lambda^{2}(s+1)^{2\alpha-2}}{s^{2\alpha}}-\frac{\lambda^{3}(s+1)^{3\alpha-3}}{s^{3\alpha}}+\dots\right)$$
$$= \frac{1}{s^{\alpha}}-\frac{\lambda(s+1)^{\alpha-1}}{s^{2\alpha}}+\frac{\lambda^{2}(s+1)^{2\alpha-2}}{s^{3\alpha}}-\frac{\lambda^{3}(s+1)^{3\alpha-3}}{s^{4\alpha}}+\dots .$$

Now, we take this equality and plug into (4.9), we obtain

$$R_{\alpha-1}\left(z(t)\right) = F(s)\left(\frac{1}{s^{\alpha}}-\frac{\lambda(s+1)^{\alpha-1}}{s^{2\alpha}}+\frac{\lambda^{2}(s+1)^{2\alpha-2}}{s^{3\alpha}}-\frac{\lambda^{3}(s+1)^{3\alpha-3}}{s^{4\alpha}}+\dots\right)$$
$$+z(\alpha-1)\left(\frac{1}{s^{\alpha}}-\frac{\lambda(s+1)^{\alpha-1}}{s^{2\alpha}}+\frac{\lambda^{2}(s+1)^{2\alpha-2}}{s^{3\alpha}}-\frac{\lambda^{3}(s+1)^{3\alpha-3}}{s^{4\alpha}}+\dots\right),$$

where $F(s) = R_0\left(f(t+\alpha-1)\right)(s)$.

Before going further, we need to rewrite the notations $\frac{1}{s^{\alpha}}, \frac{(s+1)^{\alpha-1}}{s^{2\alpha}}, \frac{(s+1)^{2\alpha-2}}{s^{3\alpha}}, \dots$ by using the R-transform.

To do that we need to use Lemma 2.3.1 (i), first we have

$$\frac{1}{s^{\alpha}} = \frac{1}{\Gamma(\alpha)}R_{\alpha-1}\left(t^{(\alpha-1)}\right). \tag{4.10}$$

We write the second term as

$$\frac{1}{s^{2\alpha}} = \frac{1}{\Gamma(2\alpha)}R_{2\alpha-1}\left(t^{(2\alpha-1)}\right). \tag{4.11}$$

Therefore, by the definition of the R-transform we have

$$R_{2\alpha-1}\left(t^{(2\alpha-1)}\right)(s) = \sum_{t=2\alpha-1}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}t^{(2\alpha-1)}.$$

Let $t \to t + \alpha - 1$, then we have

$$R_{2\alpha-1}\left(t^{(2\alpha-1)}\right)(s) = \sum_{t=\alpha}^{\infty}\left(\frac{1}{s+1}\right)^{t+\alpha-1+1}(t+\alpha-1)^{(2\alpha-1)}$$

$$= \frac{1}{(s+1)^{\alpha-1}}\sum_{t=\alpha-1}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}(t+\alpha-1)^{(2\alpha-1)}$$

$$= (s+1)^{1-\alpha}R_{\alpha-1}\left((t+\alpha-1)^{(2\alpha-1)}\right)(s).$$

Therefore, (4.11) becomes

$$\frac{(s+1)^{\alpha-1}}{s^{2\alpha}} = \frac{1}{\Gamma(2\alpha)}R_{\alpha-1}\left((t+\alpha-1)^{(2\alpha-1)}\right)(s). \tag{4.12}$$

Similarly, we can write

$$\frac{1}{s^{3\alpha}} = \frac{1}{\Gamma(3\alpha)}R_{3\alpha-1}\left(t^{(3\alpha-1)}\right)(s). \tag{4.13}$$

Now, we have

$$R_{3\alpha-1}\left(t^{(3\alpha-1)}\right)(s) = \sum_{t=3\alpha-1}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}t^{(3\alpha-1)}.$$

Let $t \to t + 2\alpha - 2$, then we obtain

$$R_{3\alpha-1}\left(t^{(3\alpha-1)}\right)(s) = \sum_{t=\alpha+1}^{\infty}\left(\frac{1}{s+1}\right)^{t+2\alpha-2+1}(t+2\alpha-2)^{(3\alpha-1)}$$

$$= \frac{1}{(s+1)^{2\alpha-2}}\sum_{t=\alpha-1}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}(t+2\alpha-2)^{(3\alpha-1)},$$

since for $t = \alpha$ and $t = \alpha - 1$, the summand is zero. By the definition of the R-transform we have

$$R_{3\alpha-1}\left(t^{(3\alpha-1)}\right)(s) = (s+1)^{2-2\alpha}R_{\alpha-1}\left((t+2\alpha-2)^{(3\alpha-1)}\right)(s).$$

Therefore, (4.13) becomes

$$\frac{(s+1)^{2\alpha-2}}{s^{3\alpha}} = \frac{1}{\Gamma(3\alpha)} R_{\alpha-1}\left((t+2\alpha-2)^{(3\alpha-1)}\right)(s). \tag{4.14}$$

Now, to find $F(s)\left(\frac{1}{s^{\alpha}} - \frac{\lambda(s+1)^{\alpha-1}}{s^{2\alpha}} + \frac{\lambda^2(s+1)^{2\alpha-2}}{s^{3\alpha}} - \frac{\lambda^3(s+1)^{3\alpha-3}}{s^{4\alpha}} + \ldots\right)$, we need to use convolution product. By the definitions of the R-transform and the falling factorial, we see that $R_{\alpha-2}(f(t))(s) = R_{\alpha-1}(f(t))(s)$. Therefore, we rewrite the equations (4.10), (4.12), and (4.14) we obtain

$$\begin{aligned} R_{\alpha-1}(z(t)) = F(s) R_{\alpha-2}\left(\frac{1}{\Gamma(\alpha)} t^{(\alpha-1)} - \frac{\lambda}{\Gamma(2\alpha)}(t+\alpha-1)^{(2\alpha-1)} + \frac{\lambda^2}{\Gamma(3\alpha)}(t+2\alpha-2)^{(3\alpha-1)} - \ldots\right) \\ + z(\alpha-1) R_{\alpha-1}\left(\frac{1}{\Gamma(\alpha)} t^{(\alpha-1)} - \frac{\lambda}{\Gamma(2\alpha)}(t+\alpha-1)^{(2\alpha-1)} + \frac{\lambda^2}{\Gamma(3\alpha)}(t+2\alpha-2)^{(3\alpha-1)} - \ldots\right). \end{aligned}$$

Since there is a pattern among these terms, we rewrite them by using the summation operator. Therefore, we have

$$R_{\alpha-1}(z(t)) = F(s) R_{\alpha-2}(E(t,-\lambda,\alpha))(s) + z(\alpha-1) R_{\alpha-1}(E(t,-\lambda,\alpha))(s), \tag{4.15}$$

where $E(t,-\lambda,\alpha) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma((n+1)\alpha)} (t+n(\alpha-1))^{((n+1)\alpha-1)}$, $t \in \mathbb{N}_{\nu-1}$.

Now, choosing $\nu = \alpha - 1$ in Lemma 2.3.2, equation (4.15) becomes

$$R_{\alpha-1}(z(t)) = R_{\alpha-1}\left[f(t+\alpha-1) * E(t,-\lambda,\alpha) + z(\alpha-1) E(t,-\lambda,\alpha)\right](s). \tag{4.16}$$

Finally, we apply the inverse R-transform to equation (4.16), then we obtain the general solution of Theorem 4.2.1. □

4.2.1. Generalized Discrete Maxwell Model with Single Fractional Order Element. By choosing a discrete time domain, we convert the derivative operator to a difference operator in equation (4.3). Therefore, we would obtain the discrete Maxwell model, which is

$$\Delta\sigma(t) + \frac{1}{\tau}\sigma(t) = E\Delta\epsilon(t). \tag{4.17}$$

If we replace the spring in the Maxwell model with a generalized fractional order element, we can establish the appropriate constitutive relationship in the following manner.

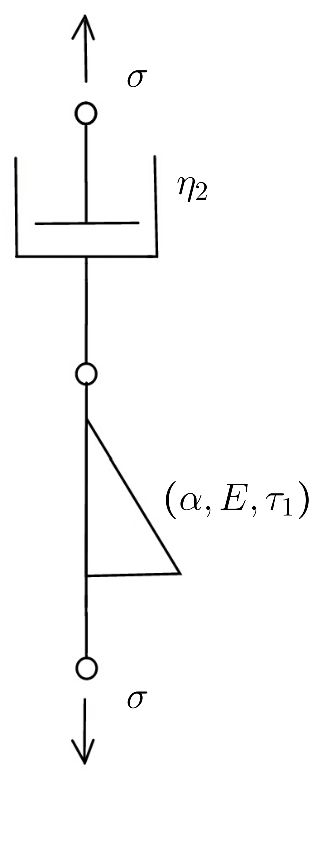

FIGURE 4.2.2. The fractional Maxwell model.

Thus, the stresses σ_1 and σ_2 for the individual elements are given by

$$\sigma_1(t) = E_1\tau_1^\alpha \frac{d^\alpha \epsilon_1(t)}{dt^\alpha} \quad and \quad \sigma_2(t) = \eta_2 \frac{d\epsilon_2(t)}{dt}.$$

Our purpose is to write these above equations in a discrete domain instead of a continuos domain.

We replace the spring in the discrete Maxwell model with a generalized fractional element (4.6), then we have

$$\sigma_1(t) = E_1\tau_1^\alpha \Delta^\alpha \epsilon_1(t) \quad and \quad \sigma_2(t) = \eta_2 \Delta\epsilon_2(t).$$

For $0 < \alpha < 1$, we have

$$\Delta\epsilon_1(t) = \Delta^{1-\alpha}\left[\Delta^\alpha \epsilon_1(t)\right] = \Delta^{1-\alpha}\left[\frac{\sigma_1(t)}{E_1\tau_1^\alpha}\right].$$

Since $\epsilon_1(t) + \epsilon_2(t) = \epsilon(t)$ and $\sigma(t) = \sigma_1(t) = \sigma_2(t)$, we have

$$\Delta\epsilon_1(t) + \Delta\epsilon_2(t) = \Delta\epsilon(t).$$

We can rewrite the strain relationship as the following fractional order difference equation

$$\frac{1}{E_1\tau_1^\alpha}\Delta^{1-\alpha}\sigma(t) + \frac{1}{\eta_2}\sigma(t-\alpha) = \Delta\epsilon(t).$$

If we let $\tau^{\alpha-1} = \dfrac{E_1\tau_1^\alpha}{\eta_2}$ and $E = E_1\dfrac{\tau_1^\alpha}{\tau^\alpha}$, we obtain the discrete fractional order equation of the Maxwell model for $t = 0, 1, 2, \dots,$

$$\Delta^{1-\alpha}\sigma(t) + \tau^{\alpha-1}\sigma(t-\alpha) = E\tau^\alpha \Delta\epsilon(t) \tag{4.18}$$

where $\sigma(t)$ is defined on $\{-\alpha, 1-\alpha, 2-\alpha, \dots\}$.

To solve (4.18), we will apply the R- transform to the equation. Then, we obtain

$$\frac{1}{\tau^{\alpha-1}} R_0\left(\Delta^{1-\alpha}\sigma(t)\right) + R_0\left(\sigma(t-\alpha)\right) = E\tau R_0\left(\Delta\epsilon(t)\right). \tag{4.19}$$

Now, we choose $\nu = 1-\alpha$ in Lemma 2.3.3 and apply it to (4.19). We get

$$\frac{1}{\tau^{\alpha-1}} s^{1-\alpha} R_{-\alpha}\left(\sigma(t)\right) - \frac{1}{\tau^{\alpha-1}}\sigma(-\alpha) + R_0\left(\sigma(t-\alpha)\right) = E\tau R_0\left(\Delta\epsilon(t)\right).$$

If we apply the equality in [**11**], which is

$$R_0\left(\Delta\epsilon(t)\right) = -\epsilon(0) + sR_0\left(\epsilon(t)\right),$$

then we obtain

$$\frac{1}{\tau^{\alpha-1}} s^{1-\alpha} R_{-\alpha}\left(\sigma(t)\right) - \frac{1}{\tau^{\alpha-1}}\sigma(-\alpha) + R_0\left(\sigma(t-\alpha)\right) = E\tau\left(-\epsilon(0) + sR_0\left(\epsilon(t)\right)\right).$$

Assume that the initial conditions are zero, then we have

$$\frac{1}{\tau^{\alpha-1}} s^{1-\alpha} R_{-\alpha}\left(\sigma(t)\right) + R_0\left(\sigma(t-\alpha)\right) = E\tau s R_0\left(\epsilon(t)\right). \tag{4.20}$$

Now, let $I(t) = t-\alpha$ and $(\sigma \circ I)(t) = \sigma(t-\alpha)$, then by definition of the discrete R-transform we obtain

$$\begin{aligned} R_0\left((\sigma \circ I)(t)\right)(s) &= \sum_{t=0}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}(\sigma \circ I)(t) \\ &= \sum_{t=0}^{\infty}\left(\frac{1}{s+1}\right)^{t+1}\sigma(t-\alpha). \end{aligned}$$

Choose $u = t - \alpha$, then we have

$$R_0(\sigma(u))(s) = \sum_{u=-\alpha}^{\infty} \left(\frac{1}{s+1}\right)^{u+\alpha+1} \sigma(u)$$

Hence, we write

$$R_0(\sigma(t-\alpha))(s) = \left(\frac{1}{s+1}\right)^{\alpha} R_{-\alpha}(\sigma(t)).$$

Take this above equality and plug into (4.20), we get

$$\frac{1}{\tau^{\alpha-1}} s^{1-\alpha} R_{-\alpha}(\sigma(t)) + \left(\frac{1}{s+1}\right)^{\alpha} R_{-\alpha}(\sigma(t)) = E\tau s R_0(\epsilon(t)).$$

Therefore, we have

$$R_{-\alpha}(\sigma(t))\left[(\tau s)^{1-\alpha} + (s+1)^{-\alpha}\right] = E\tau s R_0(\epsilon(t)). \tag{4.21}$$

Now, we solve (4.21) for $R_{-\alpha}(\sigma(t))$. Then we have

$$\begin{aligned} R_{-\alpha}(\sigma(t)) &= \frac{E\tau s R_0(\epsilon(t))}{(\tau s)^{1-\alpha} + (s+1)^{-\alpha}} = \frac{E\tau \tau^{\alpha-1}}{\left[\frac{s^{1-\alpha}}{\tau^{\alpha-1}} + \frac{1}{(s+1)^{\alpha}}\right]\tau^{\alpha-1}} = \frac{E\tau^{\alpha}}{s^{1-\alpha} + \frac{(s+1)^{-\alpha}}{\tau^{1-\alpha}}} \\ &= \frac{E\tau^{\alpha}}{s^{1-\alpha} + \lambda(s+1)^{-\alpha}} = \frac{E\tau^{\alpha}}{s^{1-\alpha}}\left(\frac{1}{1 + \frac{\lambda(s+1)^{-\alpha}}{s^{1-\alpha}}}\right) \end{aligned}$$

where $R_0(\epsilon(t)) = \dfrac{1}{s}$ and $\lambda = \dfrac{1}{\tau^{1-\alpha}}$.

In fact, we have the following series expansion

$$\frac{1}{1 + \frac{\lambda(s+1)^{-\alpha}}{s^{1-\alpha}}} = 1 - \frac{\lambda(s+1)^{-\alpha}}{s^{1-\alpha}} + \frac{\lambda^2(s+1)^{-2\alpha}}{s^{2-2\alpha}} - \frac{\lambda^3(s+1)^{-3\alpha}}{s^{3-3\alpha}} + \frac{\lambda^4(s+1)^{-4\alpha}}{s^{4-4\alpha}} - \dots.$$

Therefore, we obtain

$$R_{-\alpha}(\sigma(t)) = E\tau^{\alpha}\left(\frac{1}{s^{1-\alpha}} - \frac{\lambda(s+1)^{-\alpha}}{s^{2-2\alpha}} + \frac{\lambda^2(s+1)^{-2\alpha}}{s^{3-3\alpha}} - \frac{\lambda^3(s+1)^{-3\alpha}}{s^{4-4\alpha}} + \frac{\lambda^4(s+1)^{-4\alpha}}{s^{5-5\alpha}} - \dots\right).$$

Now, applying the inverse R-transform we have

$$\sigma(t) = E\tau^{\alpha}\left[R_{\alpha}^{-1}\left(\frac{1}{s^{1-\alpha}}\right) - \lambda R_{\alpha}^{-1}\left(\frac{(s+1)^{-\alpha}}{s^{2-2\alpha}}\right) + \lambda^2 R_{\alpha}^{-1}\left(\frac{(s+1)^{-2\alpha}}{s^{3-3\alpha}}\right) - \dots\right]. \tag{4.22}$$

The equation (4.22) is a special case of Theorem 4.2.1. Therefore, applying the same process, we find the solution of the discrete fractional Maxwell model as

$$\sigma(t) = E\tau^{\alpha}\sum_{n=0}^{\infty}\frac{\lambda^n}{\Gamma[(n+1)(1-\alpha)]}(t+n(1-\alpha-1))^{((n+1)(1-\alpha)-1)},$$

which is known as the Mittag-Leffler function in [**2**].

4.2.2. Generalized Discrete Kelvin-Voigt Model with Single Fractional Order Element.

The discrete Kelvin-Voigt model is

$$\sigma(t) = E\epsilon(t) + \eta\Delta\epsilon(t). \tag{4.23}$$

Adding a spring in parallel with the dash-pot unit allows for the relaxation plateau to be observed experimentally. This is the Kelvin-Voigt Fractional model given by [**9**], which is

$$\sigma(t) = E_0\epsilon(t) + E_1\tau^{\alpha}\frac{d^{\alpha}\epsilon(t)}{dt^{\alpha}}, \tag{4.24}$$

and consisting of elastic and viscoelastic terms. E_0 and E_1 may have different values; however τ can be modified so that

$$\sigma(t) = E_0\epsilon(t) + E_0\tau'^{\alpha}\frac{d^{\alpha}\epsilon(t)}{dt^{\alpha}}. \tag{4.25}$$

This form is convenient when expressing the time-domain behavior in response to a step stimulus.

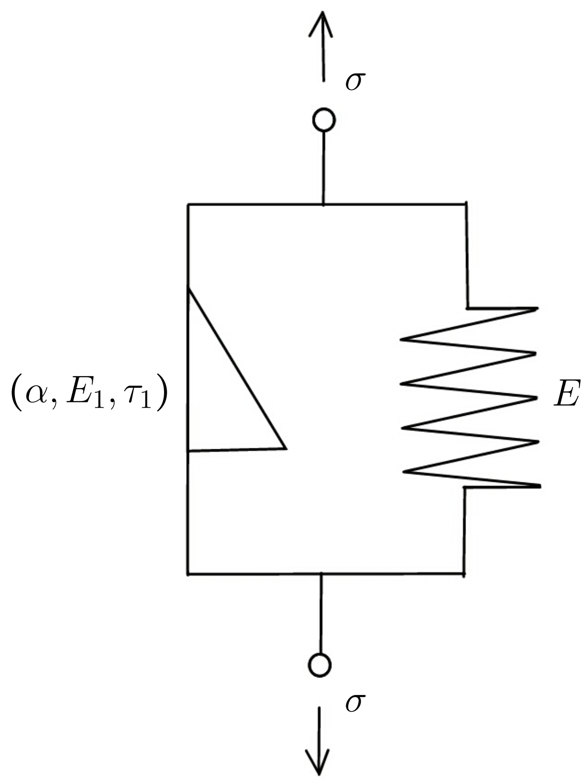

FIGURE 4.2.3. The Kelvin-Voigt fractional model.

Similarly, we rewrite the Kelvin-Voigt model by using a discrete time domain. Therefore, the equation (4.25) becomes

$$\sigma(t) = E_0\epsilon(t+\alpha-1) + E_0\tau'^{\alpha}\Delta^{\alpha}\epsilon(t) \tag{4.26}$$

which can be solved by using similar methods that we used before.

4.2.3. The Four-Parameter Model. In addition to the discrete fractional Maxwell and Kelvin-Voigt models, we give the following model based on four parameters, a continuous version of which was given in [**15**], as the most adequate for representing the viscoelastic behavior of certain materials from a rheological point of view:

$$b\Delta^{\alpha}\sigma(t) + \sigma(t+\alpha-1) = E_1\Delta^{\alpha}\epsilon(t) + E_0\epsilon(t+\alpha-1), \tag{4.27}$$

where $0 < \alpha \le 1$, with $b \ge 0$, $E_0 \ge 0$, $E_1 > 0$, $b \le \frac{E_1}{E_0}$. For a known tension $\sigma(t)$, we find that (4.27) can be represented as follows:

$$\Delta^{\alpha}z(t) + \lambda z(t+\alpha-1) = f(t+\alpha-1) \tag{4.28}$$

with $z(t+\alpha-1) = \dfrac{b}{E_1}\sigma(t+\alpha-1) - \epsilon(t+\alpha-1)$, $\lambda = \dfrac{E_0}{E_1}$, $A = \dfrac{bE_0 - E_1}{E_1^2}$, and $f(t+\alpha-1) = A\sigma(t+\alpha-1)$, where $t = 0, 1, 2, \ldots$, and $z(t)$ is defined on $\mathbb{N}_{\alpha-1}$.

By Theorem 4.2.1, the solution of the equation (4.28) is

$$z(t) = z(\alpha-1)E(t,-\lambda,\alpha) + \left[f(t+\alpha-1) * E(t,-\lambda,\alpha)\right],$$

where $E(t,-\lambda,\alpha) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma((n+1)\alpha)} (t+n(\alpha-1))^{((n+1)\alpha-1)}$.

Therefore, the solution of the equation (4.27) is

$$\epsilon(t) = \frac{b}{E_1}\sigma(t) - [\frac{b}{E_1}\sigma(\alpha-1) - \epsilon(\alpha-1)]E(t,-\lambda,\alpha) - \left[\frac{bE_0 - E_1}{E_1^2}\sigma(t+\alpha-1) * E(t,-\lambda,\alpha)\right].$$

Chapter 5

CONCLUSION AND FUTURE WORK

The fields of fractional calculus and discrete fractional calculus have attracted the interest of researchers from several areas. The theory of discrete fractional calculus and the theory of fractional calculus are parallel to each other in many directions. In these fields, there are still many open questions waiting to be answered. In this thesis, we closed some of the gaps in the analysis of discrete fractional operators and we constructed discrete fractional rheological constitutive equations. In the second chapter, we presented fundamental definitions and formulas in discrete fractional calculus for the convenience of the reader. In the third chapter, we introduced two new monotonicity concepts for a nonnegative or nonpositive valued function defined on a discrete domain. We gave examples to illustrate connections between these new monotonicity concepts and the traditional ones. We then proved some monotonicity criteria based on the sign of the fractional difference operator of a function f, $\Delta^{\nu} f$ with $0 < \nu < 1$. In the fourth chapter, we carried previous results about fractional rheological models to the discrete fractional case. We started this chapter by giving a brief introduction to Maxwell and Kelvin-Voigt models, and then we constructed and solved discrete fractional rheological constitutive equations.

For future work, we would like to see applications of monotonicity results on discrete fractional operators. Moreover, because discrete fractional rheological models are relatively new and undeveloped, some experimental data will be analyzed in terms of the proposed model containing discrete fractional derivatives. We also will pay some attention to the relationship between fractional rheological models and discrete fractional rheological models of the theory of linear viscoelasticity.

APPENDIX

An Illustration of Discrete Fractional Derivatives

The following Mathematica codes were used to compute and plot the graph in Figure 2.2.1.

- $g[0] := 0.53, g[1] := 0.56, g[2] := 0.65, g[3] := 0.79, g[4] := 0.92, g[5] := 0.99,$ $g[6] := 1.05, g[7] := 1.05, g[8] := 1.05, g[9] := 1.05, g[10] := 0.98, g[11] := 0.92,$ $g[12] := 0.88, g[13] := 0.84, g[14] := 0.77,$
- $y[t_, alp_] := 1/Gamma[1-alp]Sum[(Gamma[t+1-s-alp]*g[s])/Gamma[t+1-s], s, 0, t]-1/Gamma[1-alp]Sum[(Gamma[t-s-alp]*g[s])/Gamma[t-s], s, 0, t-1];$
- $DiscretePlot[y[t,0], y[t,0.2], y[t,0.4], y[t,0.6], y[t,0.8], g[t+1]-g[t], t, 0, 14,$ $PlotLegends->"Expressions"].$

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