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Retiling a Colored Hexagonal Plane

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A ***tiling*** is a collection of closed subsets $\mathcal{T}=\{\mathcal{T}_1, \mathcal{T}_2, \ldots\}$ of the plane such that $\mathcal{T}$ covers the plane and the interiors of the sets $\mathcal{T}_1, \mathcal{T}_2, \ldots$ are pairwise disjoint. The sets $\mathcal{T}_1, \mathcal{T}_2, \ldots$ are called ***tiles***[1]. Tilings have been used for centuries in mosaics and architecture; they form a bridge between art and mathematics. However, due to their intricacy and complexity, the mathematical study of tilings proves extremely difficult unless we impose severe restrictions on their properties.

A ***monohedral tiling*** is a tiling in which all the tiles are congruent to a single tile, called the ***prototile*** of the tiling. A ***regular tiling*** is a monohedral tiling in which the prototile is a regular polygon[2]. The only shapes that yield a regular tiling are the triangle, square, and hexagon[3] (Figure 1).

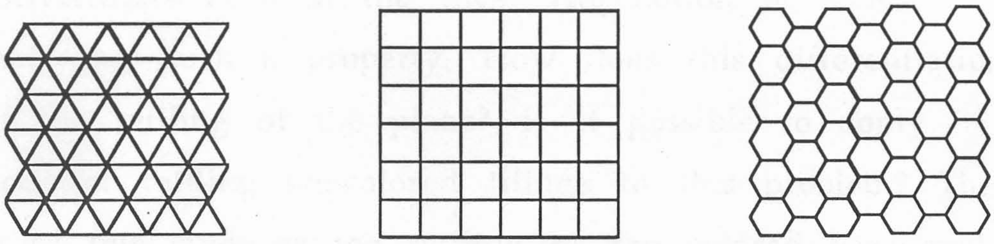


Figure 1. The three regular tilings.

An open question of interest is how to generate a regular tiling using a subset of the tiling. Given a regular tiling $\mathcal{T}$, we define an ***animal*** to be a finite collection of tiles of $\mathcal{T}$[4]. A tiling $\mathcal{T}$, or a closed subset $\tau \subseteq \mathcal{T}$, is ***retiled*** by an animal X if it can be partitioned by rigid motions of X. The question becomes, given an animal X of n tiles, is it possible to retile the tiling with X? For a square tiling, this is always possible for $n \leq 4$, and is generally not possible for $n \geq 6$[5]. Likewise every animal X will retile a hexagonal tiling for $n \leq 3$, but not every animal will retile for $n \geq 6$ (Figure 2).

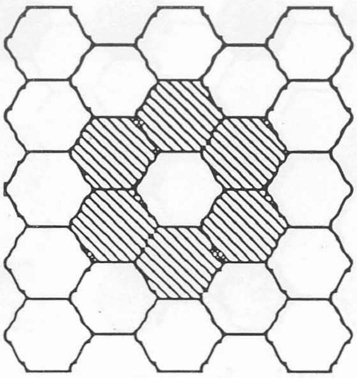


Figure 2. A 6-celled animal that does not retile the plane.

In the tilings discussed thus far, all the tiles were "equal" in the sense that there was no distinguishing property to differentiate between the tiles. The notion of "color" presents just such a property. How does this differentiation affect the retiling of the plane? Is it possible to apply methods of retiling noncolored tilings to this problem? The focus of this study is the retiling of the colored hexagonal tiling H (Figure 3). For the purpose of simplification, the research has been limited to methods of retiling by translations only.

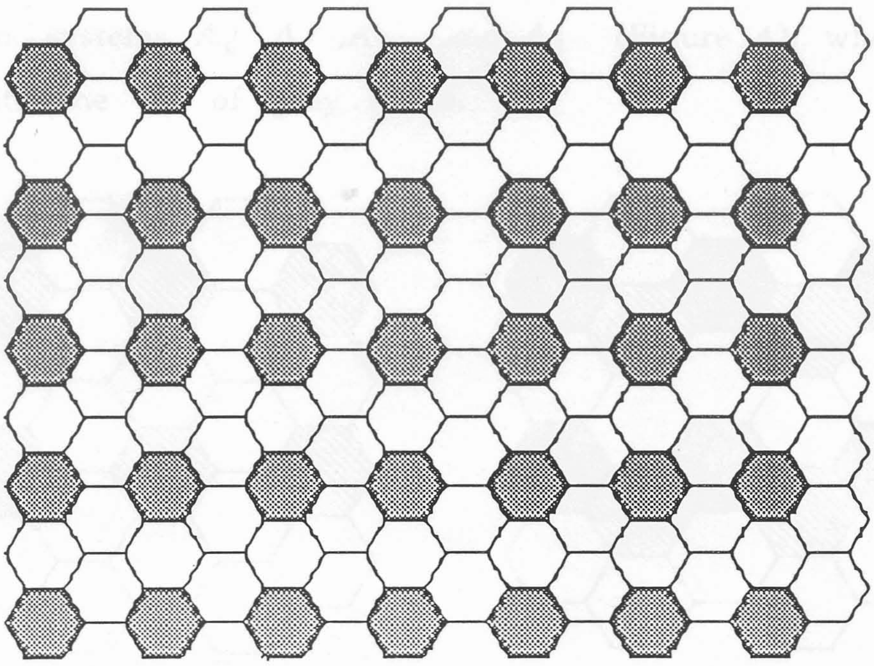


Figure 3. The colored hexagonal tiling H.

1. Definitions

Let X be an animal in a tiling, and let T be a set of translations of the tiling. Then T is a ***slide*** of X if X retiles the plane under the translations T. The ***focus*** of X, denoted F(X), is the set of all slides of X.

A ***polyhex*** P is an animal in a hexagonal tiling. The corresponding term for a square tiling is a ***block***. Any animal which retiles its tiling using translations only is a ***generator*** of the tiling.

In the colored hexagonal tiling H the ratio of gray tiles to white tiles is 1:3; any generator of H must exhibit this same ratio. Note also that the coloring of H divides it into four axis systems A_0 ,A_1 ,A_2 , and A_3 (Figure 4), where A_0 represents the set of gray hexes.

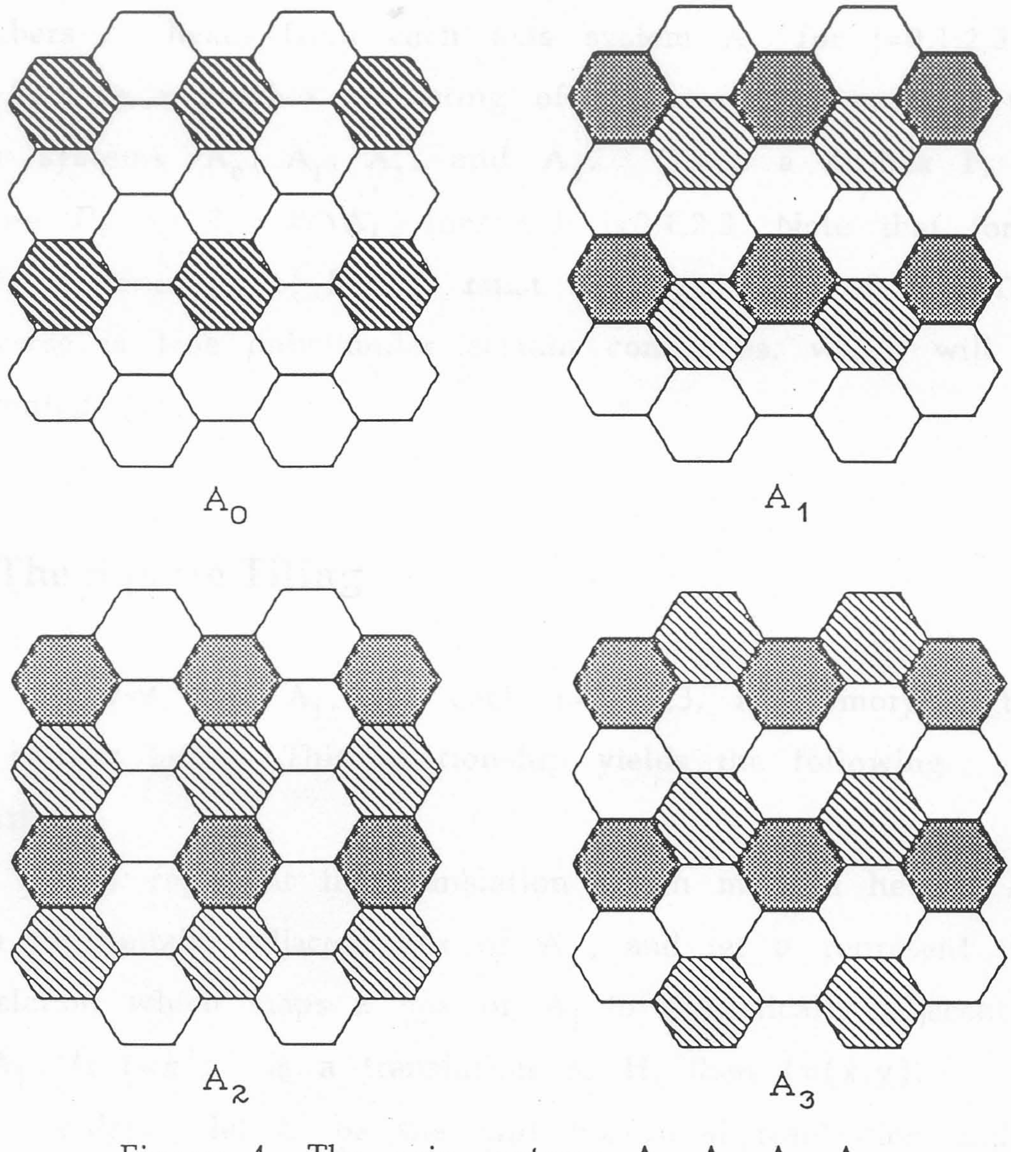


Figure 4. The axis systems A_0 ,A_1 ,A_2 ,A_3 .

For a polyhex P to be a generator it is necessary that P contains at least one gray hex, and that each translation of P maps a gray hex to another gray hex. Translating P in this fashion also implies that any hex of A_j is mapped to another hex of A_j . Thus for P to be a generator of the tiling, it is not sufficient for P to exhibit a ratio of three white hexes to each gray hex; more specifically, it must contain equal numbers of hexes from each axis system A_j for j=0,1,2,3. A ***4n-hex*** is a polyhex consisting of n hexes from each of the axis systems A_0, A_1, A_2, and A_3. Given a 4n-hex P, we define P_j by $P_j = P \cap A_j$ for each j=0,1,2,3. Note that for P to be a generator of H, P_j must be a generator of A_j . The converse is true only under certain conditions, which will be presented later.

2. The Square Tiling

Observe that A_j, for each j=0,1,2,3, is isomorphic to the integer lattice. This relationship yields the following notation.

Let h represent the translation which maps a hex of A_j to a horizontally adjacent hex of A_j , and let v represent the translation which maps a hex of A_j to a vertically adjacent hex of A_j . If $t=h^x v^y$ is a translation of H, then $t=(x,y)$.

Similarly, let h' be the unit horizontal translation and v' the unit vertical translation of the square tiling. If $t'=(h')^x(v')^y$ is a translation of the square tiling, then $t'=(x,y)$.

Let p and q be animals in a square or hexagonal tiling such that $t(p)=q$ for some translation $t=(x,y)$. Then $p+(x,y) = q$, or equivalently, $q-p = (x,y)$.

The equivalence relation correspondence, denoted $\cong$, is defined on the set of translations of the plane as follows. By definition, $h \cong h'$ and $v \cong v'$. Let t and t' be translations of a hexagonal and square tiling, respectively. Then $t \cong t'$ if and only if $t=(x,y)$ and $t'=(x,y)$.

The problem of retiling A_j for a given j is now reduced to the problem of retiling the square tiling, as presented in the following theorem. Recall that F(X) denotes the focus of X.

<u>Theorem 2.1</u> Let P be a 4n-hex. For each j=0,1,2,3, let B_j represent an n-block in Z^2 that is an isomorphic image of P_j. Then:

1) $F(P_j) \cong F(B_j)$ for each j=0,1,2,3.

2) $F(P) = \cap F(P_j) \cong \cap F(B_j)$.

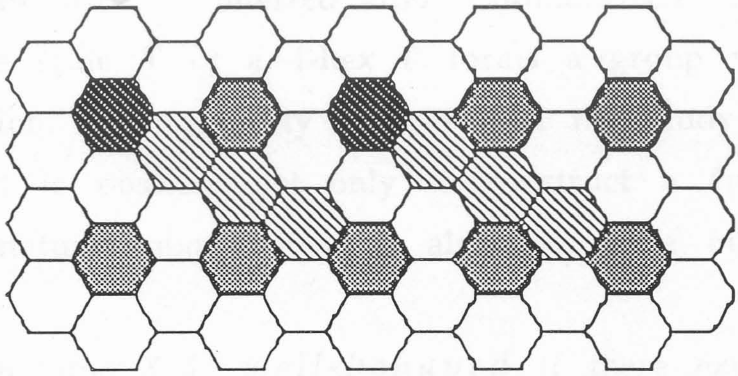
<u>Proof.</u> Observe that for each j=0,1,2,3, A_j is isomorphic to Z^2. Thus T is a slide of P_j in A_j if and only if T is a slide of X_j in Z^2. So $F(P_j)=F(B_j)$ for each j=0,1,2,3.

Clearly if T is a slide of P, then T must be a slide of P_j in A_j for each j=0,1,2,3. Furthermore, for P to retile the plane by translations, it is necessary that P_j retile A_j for each j. Hence T is a slide of P if and only if T is a slide of P_j in A_j for each j. Thus $F(P) = \cap F(P_j) = \cap F(B_j)$. □

This relationship is particularly helpful when considering uniform 4n-hexes. A 4n-hex P is ***uniform*** if there exists a translation of A_j onto A_k that maps P_j onto P_k for each

$0 \leq j,k \leq 3$. Each polyhex $P_j \subseteq P$ is thus isomorphic to a fixed block B in the square tiling.

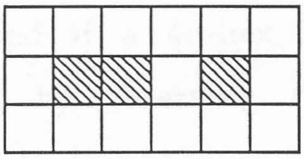
Figure 5. A uniform 8-hex.



Corollary 2.2 Let P be a uniform 4n-hex. Then P is a generator of H if $n \leq 2$.

This result naturally follows from Theorem 2.1 upon observing that all 1-blocks and 2-blocks are generators (that is, they retile by translations only) of the square tiling. For $n \geq 3$, there exists an n-block that will not retile the square tiling using translations only (Figure 6).

Figure 6. A 3-block that does not retile the square tiling using translations only.



3. Well-Behaved Generators

It is clear that any 4-hex retiles the colored hexagonal tiling H; this can be directly inferred from Corollary 2.2. It is easily seen that the slide T of a 4-hex P forms a group with respect to composition. This property is useful in the study of larger generators. It is possible not only to construct a 4n-hex generator for any natural number n, but also to derive other generators from it.

An animal generator X is *well-behaved* if there exists a slide T of X that forms a group with respect to composition. The slide T is then called a *slide group*. The *core* of a generator X, denoted C(X), is the set of all slide groups T such that X retiles under T. In other words, $C(X) = \{T \in F(X) : (T, \circ) \text{ is a group}\}$.

It is important to note that any group of translations acting on the plane is isomorphic to $\mathfrak{Z} \times \mathfrak{Z}$. Thus for any well-behaved 4n-hex generator with slide group T, $T \approx \mathfrak{Z} \times \mathfrak{Z}$. However, not all 4n-hex generators are well-behaved; for example, the 8-hex in Figure 5 is a non well-behaved generator of H. It may be determined if a 4n-hex P is a well-behaved generator of the tiling by observing the behavior of its components P_j.

<u>Theorem 3.1</u> A 4n-hex P is a well-behaved generator if and only if

1) P_j is a well-behaved generator of A_j for all $j=0,1,2,3$; and

2) $\cap C(P_j) \neq \emptyset$.

Furthermore, $C(P) = \cap C(P_j)$.

__Proof.__ Suppose P is well-behaved. Then clearly $\cap C(P_j) = C(P) \neq \emptyset$. Furthermore, since P retiles the plane under a slide group T, then P_j must retile A_j under T for each j=0,1,2,3. So by definition, P_j is a well-behaved generator of A_j .

Suppose P_j is well-behaved for each j, and $\cap C(P_j) \neq \emptyset$. Let $T \in \cap C(P_j)$; then P retiles the plane under T. Thus $\cap C(P_j) \subseteq C(P)$.

Likewise, if P retiles the plane under a slide group T, then P_j retiles A_j under T for each j=0,1,2,3. By definition $T \in \cap C(P_j)$. Hence $C(P) \subseteq \cap C(P_j)$. □

Combining the results of Theorems 2.1 and 3.1 leads to the following conclusion.

__Corollary 3.2__ A 4n-hex P is a well-behaved generator if and only if

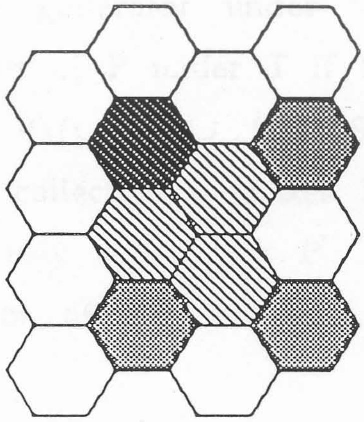
1) B_j is a well-behaved generator of the square tiling for all j=0,1,2,3; and

2) $\cap C(B_j) \neq \emptyset$.

It follows that $C(B) = \cap C(B_j)$.

Thus a 4n-hex generator of the tiling may be constructed for any natural number n. A ***fundamental 4-hex F*** is pictured in Figure 7. A *fundamental 4n-hex* F^{4n} consists of n horizontally adjacent fundamental 4-hexes. This provides a well-behaved 4n-hex generator for any positive integer n.

Figure 7. A fundamental 4-hex.



<u>Theorem 3.3</u> F^{4n} is a well-behaved generator of H for any natural number n.

<u>Proof.</u> Since F^{4n} is uniform for any n, it suffices to observe the behavior of the gray hexes F_0^{4n}. This corresponds to a block B of n horizontally adjacent squares in a square tiling. It is easily seen that the block B retiles the tiling under the slide group $T = \langle (n,0),(0,1) \rangle$. B is a well-behaved generator of the square tiling; thus from Corollary 3.2 it follows that F^{4n} is a well-behaved generator of H. □

The fundamental 4n-hex F^{4n} presents a well-behaved 4n-hex generator of H for any natural number n. However, it may also be used to construct other well-behaved generators.

4. Derivation and Equivalency of Generators

Let P be a well-behaved 4n-hex generator under the slide group T. Then Q is a ***derivation*** of P under T if there exist polyhexes S_j in P such that $Q = P \backslash (\cup S_j) \cup (\cup t_j(S_j))$ for some translations $t_j \in T$. That is, a collection of hexes $P' \subseteq P$ is replaced by hexes that were previously retiled by P'. This produces another well-behaved generator of the tiling.

<u>Theorem 4.1</u> Let P be a well-behaved 4n-hex generator under the slide group T, and let Q be a derivation of P under T. Then Q retiles the plane by the translations T.

<u>Proof.</u> Label the hexes of P as $p_1, p_2, \ldots, p_{4n}$, where $p_{nj+1}, \ldots, p_{nj+n} \in A_j$. Then label the hexes of Q as $q_1, \ldots, q_{4n}$, by $q_{nj+k} = t_{nj+k}(p_{nj+k})$, j=0,1,2,3, where $t_{nj+k} \in T$ is the translation which maps p_{nj+k} to q_{nj+k}. Since Q is a derivation of P, this is a one-to-one correspondence.

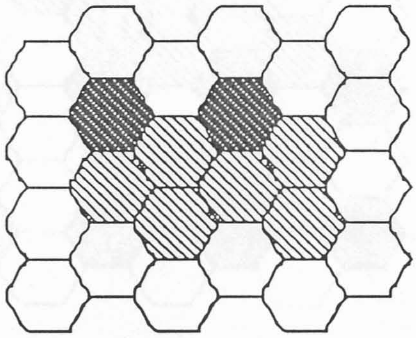
Let $R \in A_j$ for any j=0,1,2,3. Since P retiles the plane by T, there exists a translation $u \in T$ such that $u(p_{nj+k}) = R$ for some k, $1 \leq k \leq n$. Since $q_{nj+k} = t_{nj+k}(p_{nj+k})$, R may also be covered by $u \circ t_{nj+k}^{-1}(q_{nj+k})$. T is a group; therefore $u \circ t_{nj+k}^{-1} \in T$. This holds true for all hexes R in A_j for any j. Thus Q retiles the plane by the translations T. □

This definition of derivation defines an equivalence relation on the set of well-behaved 4n-hexes. Given well-behaved 4n-hexes P and Q, Q is ***equivalent to P mod T,*** denoted Q = P mod T, if Q is derived from P under T. Note that Q = P mod T if and only if $T \in C(P) \cap C(Q)$. If C(P) = C(Q), then P and Q are ***equivalent,*** denoted P≡Q.

It is possible that a well-behaved generator P may retile the plane under two or more nondisjoint slide groups (Figure 8). This case presents another interesting property.

Figure 8. (a) The fundamental 8-hex F^8 retiles under the two retiling groups illustrated in (b) and (c).

(a)



(b)

1	2	1	2	1	2
1	2	1	2	1	2
1	2	1	2	1	2

(c)

2	1	2	1	2	1
1	2	1	2	1	2
2	1	2	1	2	1

<u>Theorem 4.2</u> Let P be a well-behaved 4n-hex generator with core C(P). If Q is derived from P using only translations $t_j \in \cap C(P)$, then $C(P) \subseteq C(Q)$.

<u>Proof.</u> Let $C(P)=\{T_1, \ldots, T_m\}$ be the core of a well-behaved polyhex P, and let Q be derived from P using the collection of translations $T \subseteq \cap C(P)$. Then for each $T_k \in C(P)$, Q is derived from P under T_k . From Theorem 4.1 it follows that Q retiles the plane under T_k . Thus $T_k \in C(Q)$ for all $T_k \in C(P)$. Therefore $C(P) \subseteq C(Q)$. □

Note that this does not imply that $C(Q) \subseteq C(P)$! A counterexample is presented in Figure 9.

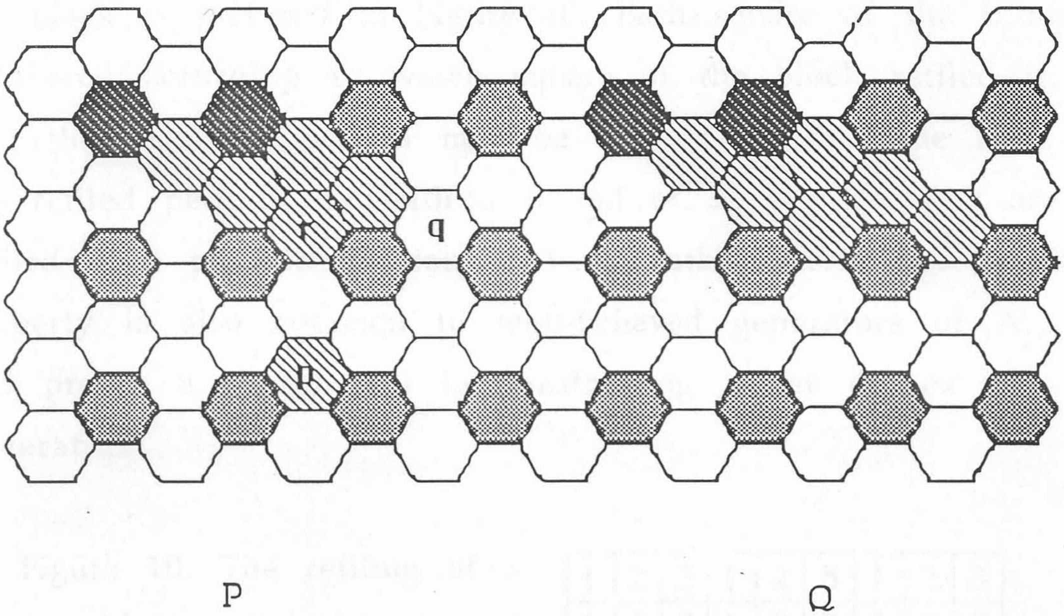


Figure 9. First note that P is a well-behaved generator of H, since it retiles H under the group $T = \langle (2,0),(1,1) \rangle$.

Q may be derived from P by translating the hex p to the hex q. All the slides of P contain this translation. The hex q must be retiled by the hex p, since retiling q by r leads to overlap. Thus Q is derived from P under the translation $t \in \cap C(P)$; hence $C(P) \subseteq C(Q)$.

However, C(P) is clearly not equal to C(Q). Q retiles the plane under the additional group $T'=\langle (2,0),(0,1) \rangle$, whereas P does not.

5. The Periodic Strip

An example of a method for retiling the square tiling by a 3-block is pictured in Figure 10. Each square of the tiling is numbered according to which square of the block retiled it, so that the resulting pattern may be seen. Note that the rows are retiled periodically with a period of 3. The columns are retiled in a periodic fashion as well, with a period of 1. This property is also common to well-behaved generators of A_j, and proves a useful tool in constructing larger 4n-hex generators.

Figure 10. The retiling of a square tiling by a horizontal connected 3-block.

1	2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2	3

An ***axis l of A_j*** is a line which passes through the centers of at least two hexes of A_j. A hex is said to ***lie on l*** if l passes through its center. A ***strip*** is the set of all hexes of A_j that lie on an axis l. Let l be an axis of A_j for some j=0,1,2,3, let p and q be distinct hexes of A_j that lie on l, and let t be the translation such that $t(\mathrm{p}) = \mathrm{q}$. Then p and q are ***l-adjacent*** if the strip l can be retiled by the group of translations $\langle (t(\mathrm{p}) \rangle$.

Let P be a well-behaved 4n-hex generator with slide group T, and let l be a strip of A_j for some j=0,1,2,3. Then there exist a subset R of P_j and a subgroup S of T such that l is retiled by R under the group S. Furthermore, the number of hexes p in R is a divisor of n.

<u>Theorem 4.1</u> Let 1 be a strip of A_j for some j=0,1,2,3, and let R be a generator of 1 under the group of translations S. Let p be the number of hexes in R. Then $p \mid n$.

<u>Proof</u> Recall that for any slide group T, $T \approx \mathfrak{Z} \times \mathfrak{Z}$; it follows that $S \approx \mathfrak{Z}$. S must therefore by cyclic, which causes l to be retiled by R in a periodic fashion. Define u to be the translation which maps a hex of l to an l-adjacent hex. Then $S = \langle u^p \rangle$.

Observe that A_j may be retiled by retiling each strip parallel to l. Any strip parallel to l is also generated by a subset of P_j under a group of translations. Since the slide T is a group, the generators of the strips parallel to l are pairwise disjoint; furthermore, they partition P_j.

Let Y be a generator of a strip m parallel to l such that $R \neq Y$, and let q be the number of hexes in Y. Then $Z = \langle u^q \rangle$ is the slide group for Y. If $p < q$ then the translation $u^p(Y)$ leads to overlap. A similar contradiction is reached if $q < p$.

Therefore p = q. Since this is true for any strip *m* parallel to *l*, it follows that $p \mid n$. □

An analogous proof shows that the same result holds for well-behaved generators of the square tiling.

6. 8-Hexes

Having discussed some general properties of 4n-hex generators, the study turns to considering generators for specific values of n. Corollary 3.2 implies that any 4-hex is a well-behaved generator of the tiling. From Theorem 2.1 it follows that any uniform 8-hex is a generator of the tiling. It is further possible to broaden the discussion to include non uniform 8-hex generators, as well as to characterize those which are well-behaved.

The characterization of well-behaved 8-hexes requires a determination of the slide groups for 8-hexes. Given an 8-hex generator P it is necessary that the two hexes of P_j retile A_j for each j=0,1,2,3. Furthermore, given any well-behaved 2-block in the square tiling, it is possible to construct a corresponding uniform 8-hex. Hence the slide groups for 8-hexes in H are determined by the slide groups for 2-blocks in the square tiling.

<u>Theorem 6.1</u> There are exactly three slide groups for 2-blocks:

$T_1 = \langle (0,2), (1,0) \rangle$;

$T_2 = \langle (0,1), (2,0) \rangle$; and

$T_3 = \langle (1,1), (2,0) \rangle$.

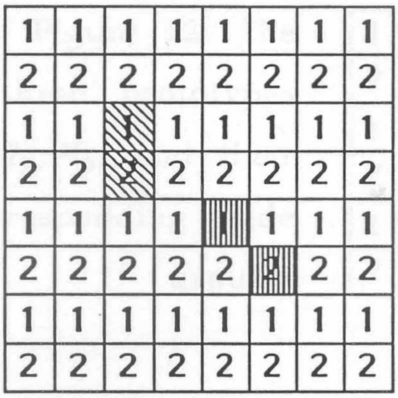
Proof. Label the two squares of the block X_1 and X_2, and let Y be a square horizontally adjacent to X_1.

Case I: X_1 retiles Y. Then if the translations form a group, the horizontal strip l containing X_1 and Y must be retiled entirely by X_1. Likewise, each horizontal strip adjacent to l must be retiled completely by X_2. Thus each vertical strip is retiled in an alternating fashion, producing the pattern in Figure 11(a). This pattern corresponds to the slide group T_1.

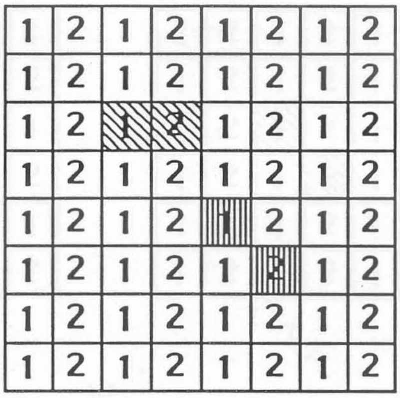
Case II: X_2 retiles Y. Then in order for the slide to form a group, the horizontal strip l containing X_1 and Y must be retiled in an alternating pattern. Let Z be a square vertically adjacent to X_1.

Subcase 1: X_1 retiles Z. Then the vertical strip containing X_1 and Z must be retiled entirely by X_1. Each vertical strip must thus be retiled in a solid pattern (Figure 11(b)). This pattern is produced by the slide group T_2.

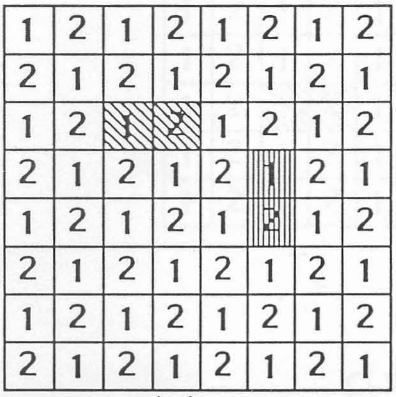
Subcase 2: X_2 retiles Z. Then the vertical strip containing X_1 and Z must also be retiled in an alternating pattern. This produces the checkerboard pattern shown in Figure 11(c). This pattern corresponds to the slide group T_3. □



(a) T_1



(b) T_2



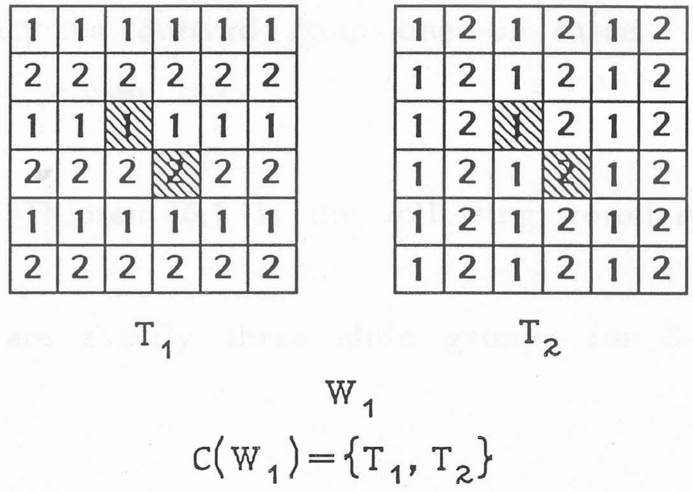
(c) T_3

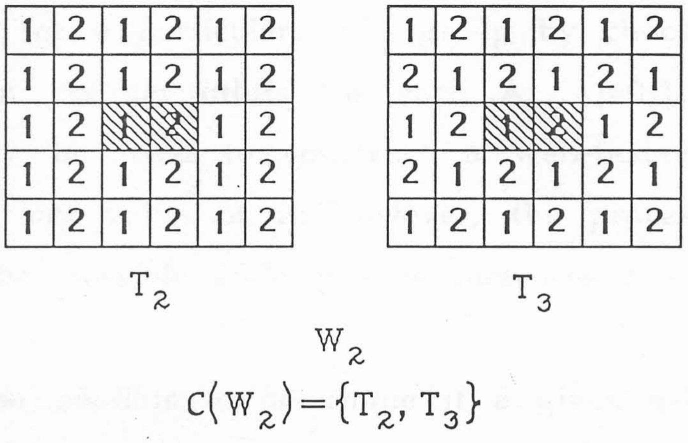
Figure 11. Patterns and prototypes for the three slide groups (a) T_1, (b) T_2, and (c) T_3.

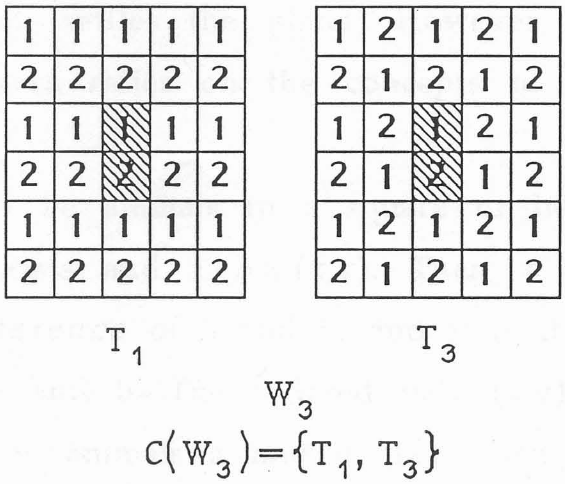
Thus the pattern produced by the slide group divides the tiling into two sets of tiles. Two connected prototypes of each slide group are presented in Figure 11. However, a well-behaved 2-block generator of the tiling may be constructed for any slide group by choosing any one square from each of the orbits of X_1 and X_2. Such a generator is easily seen to be a derivation of one of the prototypes.

Observe in Figure 11 that there are three connected prototypes for well-behaved 2-blocks. Each prototype retiles the plane under exactly two slide groups (Figure 12). Any

Figure 12. The three prototypes W_1, W_2, W_3, and their corresponding slide groups.







well-behaved 2-block may be derived from one of these prototypes; this will be proven later.

A direct result of Theorem 6.1 is the following corollary.

Corollary 6.2. There are exactly three slide groups for 8-hexes: T_1, T_2, and T_3.

As before, it is possible to construct a well-behaved 8-hex generator for a particular slide group by choosing any pair of hexes from distinct orbits for each A_j, j=0,1,2,3. This procedure may be used to construct a well-behaved 4n-hex generator for any value of n; however, the process of determining the possible slide groups increases in complexity as n increases.

It is also possible to determine if a given 8-hex is a generator of the tiling. From Corollary 2.2 it is known that any uniform 8-hex retiles the plane. However, the study of non uniform 8-hexes relies on the concepts of difference and parity.

Let a and b be animals in a square or hexagonal tiling such that a - b exists and a - b = (x,y). Then x is the ***horizontal difference*** of a and b, and y is the ***vertical difference*** of a and b. The ordered pair (x,y) is the ***difference*** of the animals a and b. Let $Z=\{z_1, z_2\}$ be a 2-block in the square tiling or a 2-hex in A_j for some j=0,1,2,3 such that $z_2 - z_1 = (x,y)$. Then the ***parity*** of Z, denoted $||Z||$, is given by (x mod 2, y mod 2). The parity of Z then determines whether it is well-behaved.

<u>Theorem 6.3</u> Let $B=\{b_1, b_2\}$ be a 2-block in the square tiling. Then B is a well-behaved generator of the tiling if and only if $||B|| \neq (0,0)$.

<u>Proof.</u> Suppose $||B|| \neq (0,0)$. Then there exist integers x and y such that $b_1 - b_2 = (2x,2y) + (0,1)$; hence $b_1 = b_2 + (2x,2y) + (0,1)$.

Note that $||W_1|| = (1,1)$, $||W_2|| = (1,0)$, and $||W_3|| = (0,1)$, as shown in Figure 12. Let $W \in \{W_1, W_2, W_3\}$ such that $||W|| = ||B||$. Label the squares of W as w_1 and w_2, and position W_1 so that w_1 coincides with b_1. Now $w_2 - b_2 = (2x,2y) \in \cap T_j \subseteq \cap C(W)$; thus w_2 retiles b_2. Therefore B is a derivation of W under any one of the two slide groups in C(W). From Theorems 4.1 and 4.2 it follows that B is well-behaved, with $C(B) \supseteq C(W)$. Furthermore, Figure 11 illustrates that there is no 2-block that retiles under all three slide groups; thus $C(B) = C(W)$.

Now let B be a well-behaved generator of the tiling. Suppose $||B|| = (0,0)$; then $b_1 - b_2 = (2x,2y)$ for some integers x and y. Consider the square $b = b_1 + (2,0)$. From Theorem 6.1 it follows that b_1 must retile b. Thus b_1 must retile $b_1 + (2k,0)$ for all integers k; in particular, b_1 retiles $b_1 + (-2x,0)$. Similarly, b_2 must retile $b_2 + (0,2y)$. But

$$b_1 - b_2 = (2x,2y)$$
$$b_1 - b_2 = (2x,0)+(0,2y)$$

$$\begin{aligned} C(P) &= \cap C(P_j) \\ &= C(W) \cap C(X) \cap C(Y) \cap C(Z) \\ &= C(W) \cap C(X) \cap C(Y) \\ &= \emptyset \end{aligned}$$

However, this contradicts the assumption that P is well-behaved. Thus $|S| \leq 2$.

Suppose (1) and (2) are true. Since $\|P_j\| \neq (0,0)$ for all $j=0,1,2,3$, P_j is well-behaved for all j. If $|S|=1$, then P is uniform and therefore well-behaved.

Suppose $|S|=2$. Let W, X and Y be as before. As illustrated in Figure 12, the intersection of any two of the cores of W, X and Y is non empty; thus $C(P) = \cap C(P_j) \neq \emptyset$. Hence P is well-behaved. □

7. Areas for Further Study

This study has identified some general properties of generators of the tiling, and provided a detailed description of 4-hexes and 8-hexes. Yet there is still much to explore. The complete classification of 8-hexes requires an investigation of non well-behaved 8-hexes. It has been shown that uniform 8-hexes retile the plane (Figure 5), but there is not yet a detailed characterization of non well-behaved, non uniform 8-hexes.

This research may also be expanded to consider larger 4n-hexes. An algorithm similar to the one presented in

Theorem 6.1 may be used to determine the slide groups for 4n-hexes for $n > 2$; however, the process of characterization grows increasingly difficult with higher values of n. A systematic method of identifying and classifying higher-order generators also requires an understanding of non well-behaved 4n-hex generators for $n > 2$.

A third avenue for exploration is to consider retiling methods using rotations and reflections as well as translations (Figure 13). This poses additional problems, since rotations and reflections do not preserve the axes systems as translations do. Nevertheless, it will also yield a diverse new class of generators.

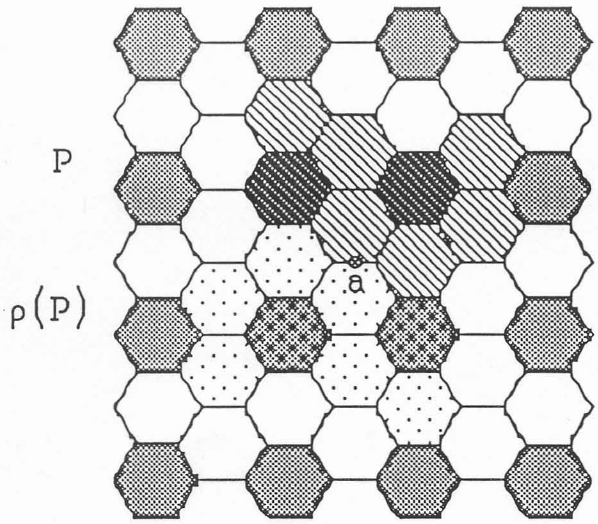


Figure 13. A polyhex P which requires a rotation to retile the colored hexagonal tiling. Let ρ be a rotation of the plane through 180 degrees about the point a. The polyhex formed by $P \cup \rho(P)$ is a well-behaved 16-hex generator that retiles the

plane under the slide groups $S_1 = \langle (2,0),(0,2) \rangle$, $S_2 = \langle (2,0),(1,2) \rangle$, $S_3 = \langle (2,1),(0,2) \rangle$, and $S_4 = \langle (2,1),(1,2) \rangle$.

Notes

1. Branko Grunbaum and G.C. Shephard, Tilings and Patterns, W.H. Freeman and Company, NY, 1987, p. 16.

2. Grunbaum and Shephard, p. 20.

3. Grunbaum and Shephard, p. 58.

4. Don Coppersmith, "Each Four-Celled Animal Tiles the Plane," *Journal of Combinatorial Theory*, Series A **40**, 1985, p. 444. Coppersmith defines this term in the context of a square tiling only. This capacity is also served by the more common term "block"; thus we have altered the definition to serve as a general term for a regular tiling.

5. Coppersmith, p. 444.