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Minimizing Travel Time Through Multiple Media With Various Borders

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MINIMIZING TRAVEL TIME THROUGH MULTIPLE MEDIA WITH VARIOUS BORDERS

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MINIMIZING TRAVEL TIME THROUGH MULTIPLE MEDIA WITH VARIOUS BORDERS

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I dedicate this thesis to my parents, Mark and Wanda Miick, who never let me settle for less than my abilities and aspirations dictate. I also dedicate this work to Zach Robinson for his unending and invaluable support throughout graduate school. All of the little (and big) things he does means more than he can know.

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CONTENTS

MINIMIZING TRAVEL TIME THROUGH MULTIPLE MEDIA WITH VARIOUS BORDERS

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This thesis consists of two main chapters along with an introduction and conclusion. In the introduction, we address the inspiration for the thesis, which originates in a common calculus problem wherein travel time is minimized across two media separated by a single, straight boundary line. We then discuss the correlation of this problem with physics via Snells Law.

The first core chapter takes this idea and develops it to include the concept of two media with a circular border. To make the problem easier to discuss, we talk about it in terms of running and swimming speeds. We first address the case where the starting and ending points for the passage are both on the boundary. We find the possible optimal paths, and also determine the conditions under which we travel along each path. Next we move the starting point to a location outside the boundary. While we are not able to determine the exact optimal path, we do arrive at some conclusions about what does not constitute the optimal path.

In the second chapter, we alter this problem to address a rectangular enclosed boundary, which we refer to as a swimming pool. The variations in this scenario prove complex enough that we focus on the case where both starting and ending points are on the boundary. We start by considering starting and ending points on adjacent sides of the rectangle. We identify three possibilities for the fastest path, and are able to identify the conditions that will make each path

optimal. We then address the case where the points are on opposite sides of the pool. We identify the possible paths for a minimum time and once again ascertain the conditions that make each path optimal.

We conclude by briefly designating some other scenarios that we began to investigate, but were not able to explore in depth. They promise insightful results, and we hope to be able to address them in the future.

INTRODUCTION

We have all likely seen the problem in a calculus textbook. A generic person wants to travel from point A to point B. One piece of the journey is along a straight stretch (a road, a beach, etc.). The other part of the journey typically goes in a straight line, but may proceed at any angle, and is at a different speed. The task is to find the path that will take the least amount of time.

This problem was in particular made famous in [**6**] where the author explores the problem with his dog at the beach. The straight line is the coastline, and the other portion of the path is through the water to a thrown tennis ball. He finds that most of the time his dog "chose a path that agreed remarkably closely with the optimal path."

While we do not include a trip with a beloved canine in our exploration, this thesis expands on this idea. We look at how the problem is changed when the shape of the border, locations of starting and ending points, and rates of travel are changed.

This type of optimization problem has extensions and applications in various other areas of study. It is closely related to Snell's Law and the Least Time Principle in physics. If we consider our swim rate to be zero, this becomes an obstacle problem and extends into graph theory with visibility graphs. While we use the concepts of running and swimming as the means of traveling through different media in order to make the speeds, paths, and travel times easier to identify and discuss, we can generalize all of our results so that running is the same as traveling through any medium at some speed r and swimming is the same as traveling through a second medium at some speed s.

Our work will focus on closed boundaries instead of the standard straight-line border. We will begin with a circular border, which can be thought of as a circular pond, and then move to a rectangular shape, such as a swimming pool.

In optimizing our travel time, we will find that it simplifies things to consider different cases. One standard set of cases that will appear in each situation is based on the ratio of our run speed to our swim speed: is running faster than swimming or vice versa? In some of our scenarios, one of these will prove to be trivial while in others, the results are quite complex.

Another set of cases that we will consider is based on the location of our points. There are three possibilities for the location of each point: on the boundary, inside the boundary, and outside the boundary. We will refer to these as On, In, and Out respectively. As we have a starting point and an ending point, and know that traveling in one direction would take the same amount of time as traveling in the opposite direction (meaning that order does not matter and we can switch our starting and ending points without altering the results), we have six possible combinations among these point locations. We will focus on the On to On case and the Out to On case.

Although the scenarios may seem similar in many ways, we will find that small changes provide interesting outcomes.

Snell's Law

Snell's Law as summarized in [**9**], "gives the relationship between angles of incidence and refraction for a wave impinging on an interface between two media with different indices of refraction."

To understand Snell's Law, we must understand the concept of refraction. Refraction occurs when a wave encounters a boundary between media at an angle, such as a light wave passing from air into a pane of glass. In [**2**], Henderson explains that when the light wave passes from air into glass, it causes a decrease in the speed and wavelength of the light wave due to the fact that glass is more optically dense than air. In particular, when the wave approaches the boundary at an angle, this causes the light to bend, which is called refraction.

When light passes a border between media in which it travels at different speeds, it will bend according to the angle at which it hits the boundary and the speed at which it travels in each medium. When light travels from a medium in which it travels faster to one in which it travels slower, it will bend toward the normal line at this point. Similarly, if the light is traveling from a slower medium to a faster medium, it will bend away from the normal.

Henderson provides an insightful analogy to explain why this happens. Consider a tractor that is traveling over asphalt toward a rectangular patch of grass with asphalt on the other side as well as depicted in Figure 0.0.1. When the wheels of the tractor enter the grassy area, they sink into the ground and move slower. However, since the tractor is traveling at an angle, the wheel closer to the grass will slow down before the other wheel. So when the tractor encounters the boundary, for

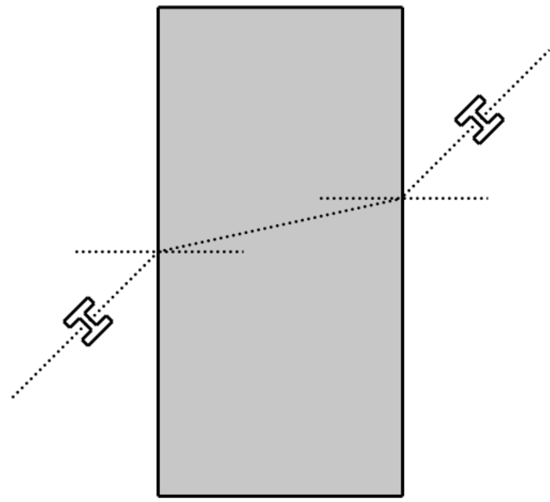

FIGURE 0.0.1. Refraction Tractor Analogy

a brief period of time, one wheel is turning slower than the other causing the tractor to turn toward the wheel that entered first. In particular, it will turn toward the normal line until both wheels are on the grass. Once this happens, the wheels will turn at the same speed again and the tractor will continue on in a straight line. Similarly, when the tractor reaches the other side of the grassy area, the wheel closer to the border (the same wheel as before) will reach the border first and begin to spin faster than the other, causing the tractor to turn away from the wheel that exited the grass first, and hence away from the normal line. The same thing would happen when a beam of light passes through a rectangular piece of glass.

Snell's Law is also based on the Least Time Principle, otherwise known as Fermat's Principle. It states that a beam of light traveling between two points will always travel at a minimum time, although it is pointed out in [**8**] that the original statement of the principle was not general and is more accurately stated that the path will be a minimum, maximum, or saddle point.

So although we know how a beam of light will refract when passing between media, can we predict by how much it will refract? That is, can we find the angle at which it will depart from the boundary? Snell's Law answers this question. In determining Snell's Law, we actually use the angles with the normal line rather than the boundary (although these will be complementary angles so we can determine the angle in question given the angle with the boundary).

Theorem 0.0.1 (Snell's Law). *A beam of light is passing from medium 1 with a refractive index of* n_1 *to medium 2 with a refractive index of* n_2*. It approaches the boundary at an angle of* θ_1 *(known as the angle of incidence) with the normal line and departs at an angle of* θ_2 *(known as the angle of refraction) with the normal line. Then*

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

Proof. We are told in [**7**] that the refractive index of a medium is equal to the velocity of light in empty space divided by its velocity in the medium. So if we let v be the velocity of light in a vacuum, and r be light's speed in the medium with refractive index n_1, then $n_1 = \frac{v}{r}$. Similarly, if we let s be the speed of light in the medium with refractive index n_2, and then $n_2 = \frac{v}{s}$. Furthermore, let A be our starting point with distance a_1 to the normal line and distance a_2 to the boundary, and B be our ending point with distance b_1 to the normal line and b_2 to the boundary. Then we can illustrate the setup of this problem as in Figure 0.0.2.

Now, we can set up a time function, T, to represent the time it will take to travel a path from A to B, depending on where we hit the boundary. Let d be the horizontal distance from A to B so that $d = a_1 + b_1$. Since A and B are stationary, d

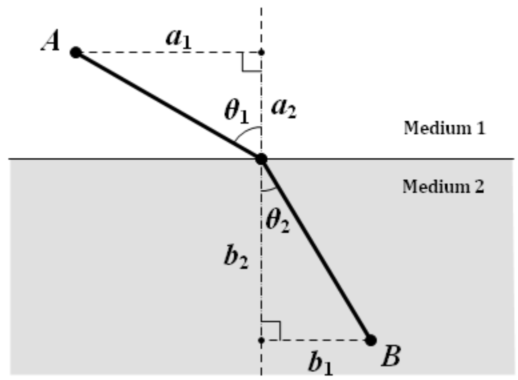

FIGURE 0.0.2. Snell's Law

will be fixed while a_1 and b_1 will change, so we will have that $b_1 = d - a_1$. Then we can write T as a function of a_1 (with a_2 and b_2 as constants):

$$T(a_1) = \frac{\sqrt{a_1^2 + a_2^2}}{r} + \frac{\sqrt{(d - a_1)^2 + b_2^2}}{s}.$$

The Least Time Principle tells us that the light will travel along an optimal path, or at least one that provides a critical point. So we can find where the light is hitting the boundary by setting the derivative equal to zero. So our critical point is where

$$T'(a_1) = \frac{2a_1}{2r\sqrt{a_1^2 + a_2^2}} + \frac{2(d - a_1)(-1)}{2s\sqrt{(d - a_1)^2 + b_2^2}} = 0$$

$$\frac{a_1}{r\sqrt{a_1^2 + a_2^2}} - \frac{(d - a_1)}{s\sqrt{(d - a_1)^2 + b_2^2}} = 0$$

$$\frac{a_1}{r\sqrt{a_1^2 + a_2^2}} = \frac{(d - a_1)}{s\sqrt{(d - a_1)^2 + b_2^2}}$$

$$\frac{a_1}{r\sqrt{a_1^2 + a_2^2}} = \frac{b_1}{s\sqrt{b_1^2 + b_2^2}}.$$

But $\frac{a_1}{\sqrt{a_1^2 + a_2^2}} = \sin\theta_1$ and $\frac{b_1}{\sqrt{b_1^2 + b_2^2}} = \sin\theta_2$, so this gives

$$\frac{\sin\theta_1}{r} = \frac{\sin\theta_2}{s},$$

and multiplying by v gives

$$\frac{v\sin\theta_1}{r} = \frac{v\sin\theta_2}{s}$$

$$n_1\sin\theta_1 = n_2\sin\theta_2,$$

which is Snell's Law. □

CHAPTER 1

CIRCLES

The first adaptation that we will make is to consider a circular pond instead of a straight shoreline. Suppose we are trying to travel from point A to point B. We can run at a rate of r units, and swim at a rate of s units. How can we make the journey in the shortest amount of time?

1.1. On to On

We will first consider the case where both the starting and ending points are on the edge of the pond. We then have the option of either swimming across the pond, running around the edge of the pond, or running part of the way around the pond and swimming the other part.

One of our cases will prove to be trivial. We know that the shortest distance between two points is a straight line, a simple idea that is actually quite complex to prove. This can be done using Calculus of Variations. Blochle gives a nice proof of this in [**1**]. If our two points are both on the circular boundary, then the shortest distance between them will be the chord connecting these points; in other words, the all-swimming path. Then if we have $s \geq r$, the all-swim path will not only be the shortest path, but also the one that can be traversed at the fastest speed. Thus the optimal path will always be the all-swim path in this case.

Then let us consider the more interesting case where $r > s$. This scenario was previously explored by students Isaac Forshee and Stephen King under the guidance of Dr. Tom Richmond. Although the shortest distance is still described by the all-swim path, if our run rate is fast enough, it may actually be optimal to run around the pond, or possibly run part of the way and swim part of the way. This leads to our first result.

THEOREM 1.1.1. *Suppose we are trying to travel from a point, A, on the edge of a circular pond to another point, B, also on the edge of the pond, with a running rate of r and a swimming rate of s. The fastest path from A to B is obtained either by running the entire way or by swimming the entire way.*

PROOF. Without loss of generality, we can scale our circular pond to be represented by the unit circle, and position it on the coordinate plane so that our starting point A is located at the point $(1, 0)$, and our ending point is on the half of the circle lying above the x-axis. We will define this ending point to be $B = (\cos\alpha, \sin\alpha)$.

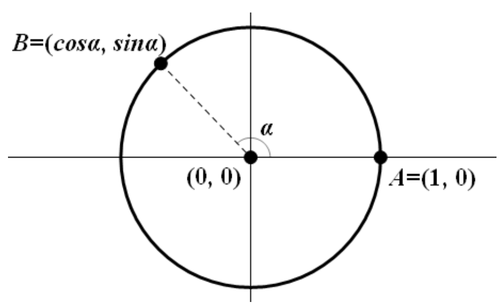

FIGURE 1.1.1. Circle On to On

The sum of the length of two chords subtended by two angles of a circle is longer than the length of a single chord subtended by the sum of those angles. This can easily be seen in Figure 1.1.2 where one of the chords is rotated about the center until the chords share an endpoint, and then follows from the fact that the sum of the lengths of two sides of a triangle is greater than the length of the third side.

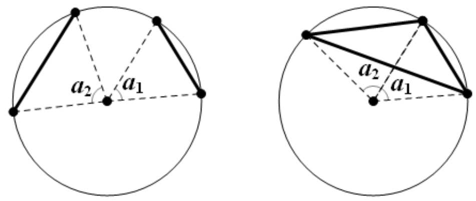

FIGURE 1.1.2. Chords

Since traversing a path along a chord of a circle will always involve swimming, the speed will be the same so a longer distance implies a longer travel time. Therefore, we only need to consider paths that include one chord of swimming. Furthermore, two chords subtended by the equal angles will have the same length and thus the same travel time, as will two arcs subtended by equal angles. So without loss of generality, we can assume that all of the swimming will be done at the end of the path, and all of the running will be done at the beginning of the path. Thus, we will run along the edge of the circle to a point $C = (\cos\theta, \sin\theta)$ where $0 \le \theta \le \alpha \le \pi$, and then swim along the straight-line path from C to B, as shown by the bold path in Figure 1.1.3.

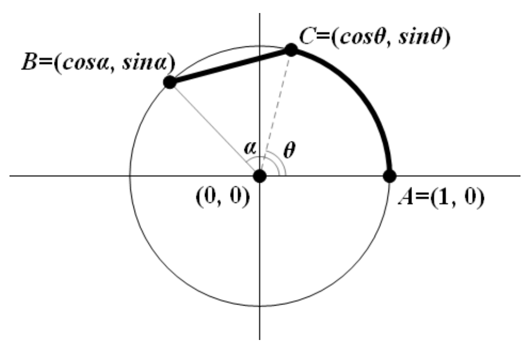

FIGURE 1.1.3. Run-Swim Path

Now as $\text{Time} = \dfrac{\text{Distance}}{\text{Rate}}$, the time it takes to travel the path can be given as a function of θ by

$$\begin{aligned}
T(\theta) &= \frac{\theta}{r} + \frac{\sqrt{(\cos\alpha - \cos\theta)^2 + (\sin\alpha - \sin\theta)^2}}{s} \\
&= \frac{\theta}{r} + \frac{\sqrt{\cos^2\alpha - 2\cos\alpha\cos\theta + \cos^2\theta + \sin^2\alpha - 2\sin\alpha\sin\theta + \sin^2\theta}}{s} \\
&= \frac{\theta}{r} + \frac{\sqrt{(\cos^2\alpha + \sin^2\alpha) + (\cos^2\theta + \sin^2\theta) - 2(\cos\alpha\cos\theta + \sin\alpha\sin\theta)}}{s} \\
&= \frac{\theta}{r} + \frac{\sqrt{2 - 2\cos(\alpha - \theta)}}{s} \qquad \text{(Pythagorean and Sum-Difference Trig Identities)} \\
&= \frac{\theta}{r} + \frac{\sqrt{4\left(\dfrac{1 - \cos\left(2\dfrac{\alpha - \theta}{2}\right)}{2}\right)}}{s} \\
&= \frac{\theta}{r} + \frac{2\sqrt{\sin^2\left(\dfrac{\alpha - \theta}{2}\right)}}{s} \qquad \text{(Half Angle Formula).}
\end{aligned}$$

Furthermore, we can note that $0 \le \theta \le \alpha \le \pi$ implies that $0 \le \alpha - \theta \le \pi$, and hence $0 \le \dfrac{\alpha - \theta}{2} \le \dfrac{\pi}{2}$. Then it must be true that $\sin\left(\dfrac{\alpha - \theta}{2}\right) \ge 0$. So we can simplify the time function to

$$T(\theta) = \frac{\theta}{r} + \frac{2\sin\left(\dfrac{\alpha - \theta}{2}\right)}{s}. \tag{1.1.1}$$

We can also arrive at this function by finding a formula for the length of a chord of a circle. The chord and the radii to the endpoints of the chord form an isosceles triangle. Let the chord be subtended by an angle γ. If we rotate our circle so that the x-axis bisects γ, we divide the isosceles triangle into two congruent right triangles, meaning that the x-axis also perpendicularly bisects the chord. This can be seen in the following figure (Figure 1.1.4). Then since the radius of the circle is 1, we must have that the length of the opposite edge of the top triangle is $\sin\left(\frac{\gamma}{2}\right)$ and thus the length of the entire chord is double this amount.

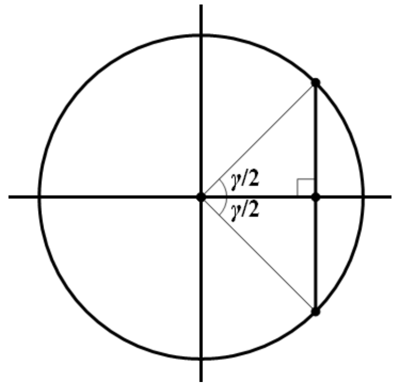

FIGURE 1.1.4. Chord Length

In our problem, the chord is subtended by the angle $\alpha - \theta$, so the time it takes to travel our path will be the length of the arc divided by the run speed plus the length of the chord divided by the swim speed. This is the same as the formula given in Equation 1.1.1.

To minimize the time it takes to travel from A to B, we begin by finding the derivative of this time function, $T(\theta)$:

$$T'(\theta) = \frac{1}{r} + \frac{2\cos\left(\frac{\alpha-\theta}{2}\right)}{s}\left(\frac{-1}{2}\right) = \frac{1}{r} - \frac{\cos\left(\frac{\alpha-\theta}{2}\right)}{s}.$$

However, in this case, the second derivative actually gives us more insight into the existence of a minimum:

$$T''(\theta) = -\frac{-\sin\left(\frac{\alpha-\theta}{2}\right)}{s}\left(\frac{-1}{2}\right) = \frac{-\sin\left(\frac{\alpha-\theta}{2}\right)}{2s}.$$

We have already established that $\sin\left(\frac{\alpha-\theta}{2}\right)$ must be positive, and obviously $2s$ is positive, so $T''(\theta)$ must be negative on this interval, meaning that $T(\theta)$ is strictly concave down. Hence there can be no local minimum (as this would require $T(\theta)$ to be concave up), and the absolute minimum for $\theta \in [0, \alpha]$ must be at one of the endpoints. That is, the minimum travel time comes from swimming the entire way from A to B ($\theta = 0$) or from running the entire way ($\theta = \alpha$). $\square$

If we compare the travel times for the all-running path and the all-swimming path, we can find a formula to determine the run rate to swim rate ratio that will cause the all-running path to be optimal.

Corollary 1.1.2. *The all-running path is optimal whenever* $\frac{r}{s} > \frac{\alpha}{2\sin\left(\frac{\alpha}{2}\right)}$, *and the all-swimming path is optimal whenever* $\frac{r}{s} < \frac{\alpha}{2\sin\left(\frac{\alpha}{2}\right)}$. *They are equally advantageous when equality holds.*

Proof. As found in Equation 1.1.1, the time it takes to travel from A to B is given by the formula $T(\theta) = \frac{\theta}{r} + \frac{2\sin\left(\frac{\alpha-\theta}{2}\right)}{s}$. Theorem 1.1.1 states that the optimal path will be provided by either running around the circle without swimming, or by swimming the straight-line distance between the points. Thus, we only need to consider times given by the corresponding values of θ. The all-swim time will be given by $T(0)$, and the all-run time will be given by $T(\alpha)$. Thus the all-swim time is

$$T(0) = \frac{0}{r} + \frac{2\sin\left(\frac{\alpha-0}{2}\right)}{s} = \frac{2\sin\left(\frac{\alpha}{2}\right)}{s},$$

and the all-run time is

$$T(\alpha) = \frac{\alpha}{r} + \frac{2\sin\left(\dfrac{\alpha-\alpha}{2}\right)}{s} = \frac{\alpha}{r} + \frac{2\sin 0}{s} = \frac{\alpha}{r}.$$

Then we can easily see that the all-running path will be optimal whenever $T(\alpha) < T(0)$. That is, when

$$\frac{\alpha}{r} < \frac{2\sin\left(\dfrac{\alpha}{2}\right)}{s} \iff s\alpha < 2r\sin\left(\frac{\alpha}{2}\right) \iff \frac{\alpha}{2\sin\left(\dfrac{\alpha}{2}\right)} < \frac{r}{s}.$$

The dual argument then follows that the all-swimming path will be optimal whenever $\frac{r}{s} < \frac{\alpha}{2\sin\left(\frac{\alpha}{2}\right)}$, and similarly, the paths would take equal amounts of time when $\frac{r}{s} = \frac{\alpha}{2\sin\left(\frac{\alpha}{2}\right)}$. □

Note that we cannot algebraically solve the inequalities found in Corollary 1.1.2 for the angle α. However, we can use this formula to find an interesting result when our starting and ending points are diametrically opposite. In this case, with our strategic positioning of a circle, a diametrically opposite ending point would be given when $\alpha = \pi$. Then if we substitute this into the formula found in Corollary 1.1.2, we see that the optimal path will be all-running whenever

$$\frac{r}{s} > \frac{\pi}{2\sin\left(\dfrac{\pi}{2}\right)} = \frac{\pi}{2}, \tag{1.1.2}$$

and consequently, all-swimming will be optimal whenever

$$\frac{r}{s} < \frac{\pi}{2}.$$

So if we are traveling from one point on the edge of a circular pond to another point on the edge of the pond, we know that we will always run all the way around or swim straight between the points. In addition, given the angle between the starting and ending points, we can determine the running and swimming speeds that would cause us to either run or swim.

1.2. Out to On

Having fully analyzed the On to On case, next we will consider the same situation of a round pond, but with one difference: our starting point is outside the circle instead of on the edge. One of the main difficulties with this scenario comes from the fact that there are so many cases to consider based on ending point placement and run/swim speeds.

Before we begin, we take a moment to prove a fairly obvious result. If we change the location on the boundary so that the distance traveled at the slower speed is increased as well as the total distance, this new path will never be optimal.

Lemma 1.2.1. *Suppose we are traveling through two media with speeds x and y where $x < y$. Let d_x be the distance traveled at speed x and d_y be the distance traveled at speed y. If a new path has distances traveled at speeds x and y that are d'_x and d'_y respectively so that $d'_x > d_x$ and $d'_x + d'_y > d_x + d_y$, this new path will never be optimal.*

Proof. The travel times for the distances traveled at speed x will be $\frac{d_x}{x}$ and $\frac{d'_x}{x}$ for the original and new paths respectively. Similarly, the travel times for the portions of these paths traveled at speed y will be $\frac{d_y}{y}$ and $\frac{d'_y}{y}$ respectively. Then the original path will be faster (implying that the new path cannot be optimal) if and only if

$$\frac{d'_x}{x} + \frac{d'_y}{y} > \frac{d_x}{x} + \frac{d_y}{y}$$
$$\frac{d'_y - d_y}{y} > \frac{d_x - d'_x}{x}. \qquad (1.2.1)$$

If $d'_y \geq d_y$, then $\frac{d'_y - d_y}{y} \geq 0 > \frac{d_x - d'_x}{x}$, giving the desired result. So assume $d'_y < d_y$. Then,

$$d'_x + d'_y > d_x + d_y \iff d'_x - d_x > d_y - d'_y \iff \frac{d'_x - d_x}{x} > \frac{d_y - d'_y}{x},$$

so $x < y$ implies that $\frac{d_y - d'_y}{x} > \frac{d_y - d'_y}{y}$ and hence $\frac{d'_x - d_x}{x} > \frac{d_y - d'_y}{y}$. Multiplying through by -1 gives the inequality in Equation 1.2.1. Thus the original path is faster and the new path cannot be optimal. □

First we will address the case where the ending point, B is "visible" from the starting point, A; that is, the straight-line path from A to B would not cross the water. In this scenario, it is trivial to consider $r \geq s$. The straight-line path from A to B would have the shortest distance, and be traveled at the fastest speed, making this route optimal. So suppose $s > r$. If B falls on the line connecting the center of the circle to A, then B is the point on the circle that is closest to A. Thus if we were to run to any other point on the circle before swimming to B, it would increase the total distance traveled as well as the distance traveled at the slower speed, so by Lemma 1.2.1, this would not be optimal. So suppose B does not fall on the line connecting A and the center of the circle.

THEOREM 1.2.2. *Suppose we are traveling from a point A outside a circular pond to a point B on the edge of the pond with running speed r and swimming speed s, and where the line connecting A and B can be traversed without any swimming. Let α be the angle between the line connecting the center of the pond to B and the line connecting A to B. If $\frac{r}{s} < \sin\alpha$, running straight to B is not the optimal solution.*

PROOF. If B is on the line connecting A and the center of the circle, then $\alpha = 0$ and we can never have $\frac{r}{s} < \sin\alpha$. So consider the case where B is not on the line connecting A and the center of the circle. Position the pond on the coordinate plane so that the center of the circle is at the origin and B is located at the point $(1, 0)$. Use the symmetry of the circle to position A in the first quadrant at some point (a_1, a_2) as shown in Figure 1.2.1. We will compare running straight to B with

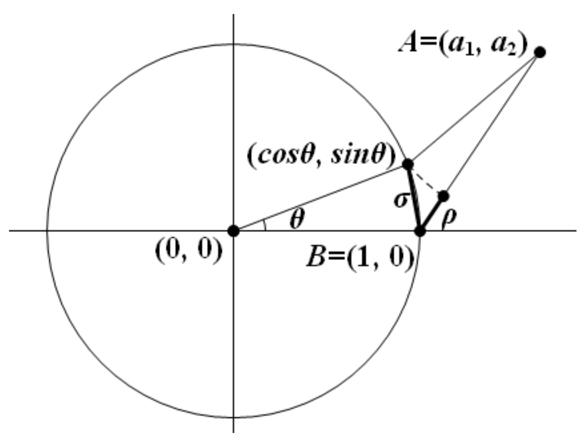

FIGURE 1.2.1. Circle Out to On Visible Ending Point

running to some other point $(\cos\theta, \sin\theta)$ on the circle and then swimming to B. Let $a = \sqrt{(a_1 - 1)^2 + a_2^2}$, the distance from B to A. Also, let σ be the swim distance, and ρ be the decrease in run distance caused by running to the new point instead of B.

Then the new path will be faster if and only if

$$\frac{a-\rho}{r}+\frac{\sigma}{s}<\frac{a}{r} \iff \frac{a}{r}-\frac{\rho}{r}+\frac{\sigma}{s}<\frac{a}{r} \iff -\frac{\rho}{r}+\frac{\sigma}{s}<0$$
$$\iff \frac{\sigma}{s}<\frac{\rho}{r} \iff \sigma r<\rho s \iff \frac{r}{s}<\frac{\rho}{\sigma}.$$

Now, $\sigma=\sqrt{(\cos\theta-1)^2+\sin^2\theta}$ and $\rho=\sqrt{(a_1-1)^2+a_2^2}-\sqrt{(a_1-\cos\theta)^2+(a_2-\sin\theta)^2}$. Then

$$\lim_{\theta\to 0}\frac{\rho}{\sigma}=\lim_{\theta\to 0}\frac{\sqrt{(a_1-1)^2+a_2^2}-\sqrt{(a_1-\cos\theta)^2+(a_2-\sin\theta)^2}}{\sqrt{(\cos\theta-1)^2+\sin^2\theta}},$$

which is a limit of the form $\frac{0}{0}$ so we can apply L'Hopital's Rule to get

$$\lim_{\theta\to 0}\frac{\rho}{\sigma}=\lim_{\theta\to 0}\frac{\dfrac{-(2(a_1-\cos\theta)(\sin\theta)+2(a_2-\sin\theta)(-\cos\theta))}{2\sqrt{(a_1-\cos\theta)^2+(a_2-\sin\theta)^2}}}{\dfrac{2(\cos\theta-1)(-\sin\theta)+2\sin\theta\cos\theta}{2\sqrt{(\cos\theta-1)^2+\sin^2\theta}}}$$

$$=\lim_{\theta\to 0}\frac{(-a_1\sin\theta+\cos\theta\sin\theta+a_2\cos\theta-\sin\theta\cos\theta)\sqrt{(\cos\theta-1)^2+\sin^2\theta}}{\sqrt{(a_1-\cos\theta)^2+(a_2-\sin\theta)^2}(-\cos\theta\sin\theta+\sin\theta+\sin\theta\cos\theta)}$$

$$=\lim_{\theta\to 0}\frac{a_2\cos\theta-a_1\sin\theta}{\sqrt{(a_1-\cos\theta)^2+(a_2-\sin\theta)^2}}\sqrt{\frac{(\cos\theta-1)^2+\sin^2\theta}{\sin^2\theta}}$$

$$=\lim_{\theta\to 0}\left(\frac{a_2\cos\theta-a_1\sin\theta}{\sqrt{(a_1-\cos\theta)^2+(a_2-\sin\theta)^2}}\right)\sqrt{\left(\lim_{\theta\to 0}\frac{\cos\theta-1}{\sin\theta}\right)^2+1}$$

$$=\lim_{\theta\to 0}\left(\frac{a_2\cos\theta-a_1\sin\theta}{\sqrt{(a_1-\cos\theta)^2+(a_2-\sin\theta)^2}}\right)\sqrt{\left(\lim_{\theta\to 0}\frac{-\sin\theta}{\cos\theta}\right)^2+1}$$

$$=\left(\frac{a_2(1)-a_1(0)}{\sqrt{(a_1-1)^2+(a_2-0)^2}}\right)\sqrt{(0)^2+1}$$

$$=\frac{a_2}{\sqrt{(a_1-1)^2+a_2^2}}.$$

Now we note that $\sqrt{(a_1-1)^2+a_2^2}$ is the distance between A and B and a_2 is the length of the perpendicular segment from A to the x-axis, which connects B and the center of the circle. So if we let α be the angle between the x-axis and the line connecting A to B, $\lim_{\theta\to 0}\frac{\rho}{\sigma}=\sin\alpha$. This can be seen in Figure 1.2.2.

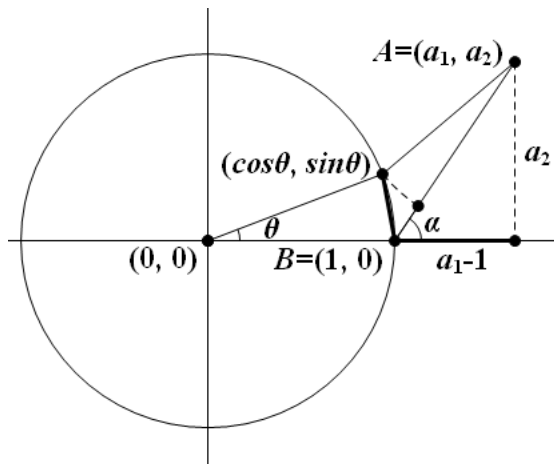

FIGURE 1.2.2. Angle to A

Suppose $\frac{r}{s}<\sin\alpha=\lim_{\theta\to 0}\frac{\rho}{\sigma}$, so $\sin\alpha-\frac{r}{s}>0$. Then there exists $\varepsilon>0$ such that $\sin\alpha-\frac{r}{s}>\varepsilon$. Furthermore, $\lim_{\theta\to 0}\frac{\rho}{\sigma}=\sin\alpha$ implies that for the given $\varepsilon>0$, there exists $\delta>0$ such that $|\frac{\rho}{\sigma}-\sin\alpha|<\varepsilon<\sin\alpha-\frac{r}{s}$ whenever $0<|\theta|=\theta<\delta$. But $|\frac{\rho}{\sigma}-\sin\alpha|<\sin\alpha-\frac{r}{s}$ means that $-\sin\alpha+\frac{r}{s}<\frac{\rho}{\sigma}-\sin\alpha<\sin\alpha-\frac{r}{s}$ which implies that $-\sin\alpha+\frac{r}{s}+\sin\alpha<\frac{\rho}{\sigma}-\sin\alpha+\sin\alpha$ so $\frac{r}{s}<\frac{\rho}{\sigma}$ whenever $\theta<\delta$. Thus there exist angles θ such that the corresponding $\frac{\rho}{\sigma}$ ratios are greater than $\frac{r}{s}$ and hence it is faster to run to one of these points and then swim to B than it is to run straight to B. $\square$

As an example of this, consider $A=(3,2)$, $r=0.5$, $s=1$, and $\theta=0.1$. Then the all-running path would have a time of $\frac{2\sqrt{2}}{0.5}\approx 5.65685$ and the path with swimming would have a time of

$\frac{\sqrt{(3-\cos 0.1)^2+(2-\sin 0.1)^2}}{0.5}+\sqrt{(1-\cos 0.1)^2+\sin^2 0.1}\approx 5.62468$. Hence the all-running path takes longer and is not optimal.

Next we will address the more interesting case where the ending point, B is not "visible" from the starting point, A. That is, it is beyond the tangential point from A to the circle as depicted in Figure 1.2.3.

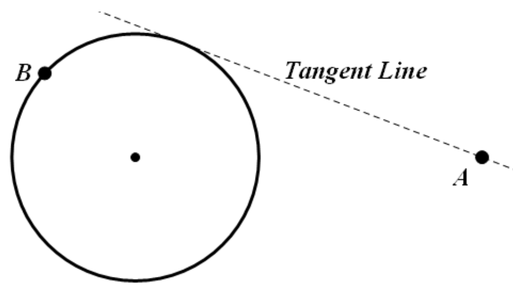

FIGURE 1.2.3. Circles Out to On General Case

We start with a very significant result. As already noted, the shortest distance between two points is a straight line, but is the shortest time also found by taking this path? When different speeds are involved, it is not always the case. We will address this in the next two theorems.

THEOREM 1.2.3. *When traveling from a point A outside a circular pond to a point B on the edge of the pond, where the line from A to B does not contain a diameter of the circle, the straight-line path from A to B is optimal if and only if the swim rate s equals the run rate r.*

PROOF. ($\Leftarrow$) Suppose $r = s$. Then this is the equivalent of traveling the entire path at one speed, and the shortest path will have the shortest time. The shortest distance between two points is a straight line, so this will be the optimal path.

(⇒) Suppose the points are not diametrically opposite. Position the circle on the coordinate grid so that the line connecting A and B lies on the x-axis and the other point where the circle intersects the line is at the origin. Let point B have coordinates $(-b, 0)$ and point A have coordinates $(a, 0)$ as seen in Figure 1.2.4. Furthermore, let the smaller portion of the circle be above the x-axis. If the line connecting the points does not contain a diameter, the portion of the curve near the intersection can be given by a differentiable function (it would never contain a point with a vertical tangent line). Call this function $f(x)$.

First consider the case with $s > r$. Moving up from the origin (which would be done by decreasing the x-coordinate) would increase the run distance as well as the overall distance, and from Lemma 1.2.1 would not be optimal. So instead, suppose we move down (by increasing the x-coordinate) to a point $(x, f(x))$. Then we have increased the swimming distance by a distance of σ and decreased the running distance by a distance of ρ.

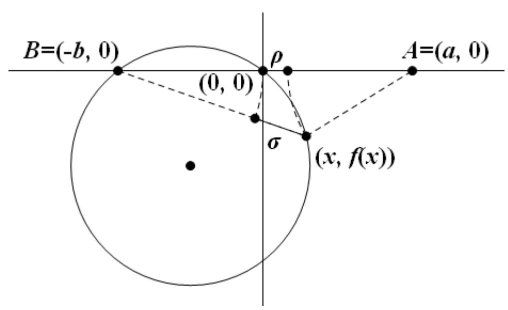

Figure 1.2.4. Straight-Line Path for Out to On Case (Swimming Faster)

With this setup, the original path can be traveled at a time of $\frac{a}{r} + \frac{b}{s}$, and the new path can be traveled at a time of $\frac{a-\rho}{r} + \frac{b+\sigma}{s}$. Then the new path is faster if and only if

$$\frac{a-\rho}{r}+\frac{b+\sigma}{s}<\frac{a}{r}+\frac{b}{s} \iff \frac{a}{r}-\frac{\rho}{r}+\frac{b}{s}+\frac{\sigma}{s}<\frac{a}{r}+\frac{b}{s} \iff -\frac{\rho}{r}+\frac{\sigma}{s}<0$$
$$\iff \frac{\sigma}{s}<\frac{\rho}{r} \iff \sigma r<\rho s \iff \frac{r}{s}<\frac{\rho}{\sigma}. \tag{1.2.2}$$

Now, we will show that $\lim_{x\to 0}\frac{\rho}{\sigma}=1$.

First note that $\rho = a-\sqrt{(f(x))^2+(x-a)^2}$, and $\sigma=\sqrt{(f(x))^2+(x+b)^2}-b$.

Then we need to find

$$\lim_{x\to 0}\frac{\rho}{\sigma}=\lim_{x\to 0}\frac{a-\sqrt{(f(x))^2+(x-a)^2}}{\sqrt{(f(x))^2+(x+b)^2}-b}. \tag{1.2.3}$$

We note that as $x\to 0$, $f(x)\to 0$ as well, so we have a limit of form $\frac{0}{0}$, and therefore we can apply L'Hopital's Rule, to obtain the equivalent problem

$$\lim_{x\to 0}\frac{-\dfrac{2f(x)f'(x)+2(x-a)}{2\sqrt{(f(x))^2+(x-a)^2}}}{\dfrac{2f(x)f'(x)+2(x+b)}{2\sqrt{(f(x))^2+(x+b)^2}}}.$$

Since the line connecting the points does not contain a diameter of the circle, the derivative is defined at $x=0$, and thus $f'(0)$ is a constant (as opposed to being undefined). Then we can simplify this limit:

$$\begin{aligned}\lim_{x\to 0}\frac{\rho}{\sigma}&=\lim_{x\to 0}-\frac{\sqrt{(f(x))^2+(x+b)^2}(f(x)f'(x)+(x-a))}{(f(x)f'(x)+(x+b))\sqrt{(f(x))^2+(x-a)^2}} \qquad (1.2.4)\\ &=-\frac{\sqrt{0^2+(0+b)^2}(0\cdot f'(0)+(0-a))}{(0\cdot f'(0)+(0+b))\sqrt{0^2+(0-a)^2}}\\ &=\frac{-a\sqrt{b^2}}{-b\sqrt{a^2}}=\frac{ab}{ab}=1.\end{aligned}$$

Now since $s > r$, we know that $\frac{r}{s} < 1$ so $1 - \frac{r}{s} > 0$, and there exists some $\varepsilon > 0$ such that $\varepsilon < 1 - \frac{r}{s}$. Furthermore, from Equation 1.2.2, we know that a path other than the straight-line path is optimal if its $\frac{\rho}{\sigma}$ ratio is greater than $\frac{r}{s}$. But $\lim_{x \to 0} \frac{\rho}{\sigma} = 1$ implies that for the above $\varepsilon > 0$, there exists $\delta > 0$ such that $|\frac{\rho}{\sigma} - 1| < \varepsilon$ whenever $0 < |x| < \delta$. If $\frac{\rho}{\sigma} \geq 1$ then we already have $\frac{r}{s} < \frac{\rho}{\sigma}$, so assume $\frac{\rho}{\sigma} < 1$. Then there exist points $(x, f(x))$ where the corresponding $\frac{\rho}{\sigma}$ ratios satisfy
$-(\frac{\rho}{\sigma} - 1) = -\frac{\rho}{\sigma} + 1 < \varepsilon < 1 - \frac{r}{s}$ and hence $-\frac{\rho}{\sigma} < -\frac{r}{s}$ so $\frac{\rho}{\sigma} > \frac{r}{s}$. Hence it is faster to travel from A to one of these points and then swim to B rather than take the straight-line path.

If we instead consider the case where $r > s$, we can position the circle and points the same as before. However, in this case, moving down by increasing x will increase total distance while increasing the distance traveled at the slower speed as well, which is not optimal by Lemma 1.2.1. So instead, we move up from the origin by decreasing x, increasing the run distance by ρ and decreasing the swim distance by σ as shown in Figure 1.2.5.

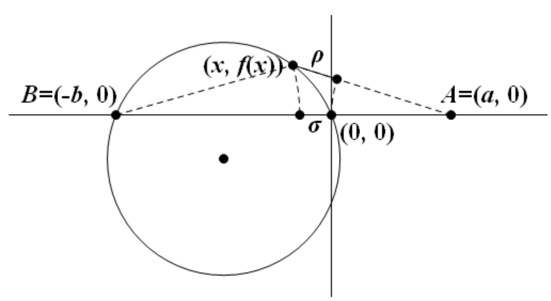

FIGURE 1.2.5. Straight-Line Path for Out to On Case (Running Faster)

This time, the new path can be traveled at a time of $\frac{a+\rho}{r} + \frac{b-\sigma}{s}$, and will be faster if and only if

$$\frac{a+\rho}{r} + \frac{b-\sigma}{s} < \frac{a}{r} + \frac{b}{s} \iff \frac{a}{r} + \frac{\rho}{r} + \frac{b}{s} - \frac{\sigma}{s} < \frac{a}{r} + \frac{b}{s} \iff \frac{\rho}{r} - \frac{\sigma}{s} < 0$$
$$\iff \frac{\rho}{r} < \frac{\sigma}{s} \iff \rho s < \sigma r \iff \frac{\rho}{\sigma} < \frac{r}{s}. \tag{1.2.5}$$

In this case, $\rho = \sqrt{(f(x))^2 + (x-a)^2} - a$ and $\sigma = b - \sqrt{(f(x))^2 + (x+b)^2}$ and thus

$$\frac{\rho}{\sigma} = \frac{\sqrt{(f(x))^2 + (x-a)^2} - a}{b - \sqrt{(f(x))^2 + (x+b)^2}} = \frac{a - \sqrt{(f(x))^2 + (x-a)^2}}{\sqrt{(f(x))^2 + (x+b)^2} - b}.$$

Then $\lim_{x\to 0} \frac{\rho}{\sigma}$ is the same as in Equation 1.2.3, and we know this limit to be 1.

Now since $r > s$, we know that $\frac{r}{s} > 1$, so there exists $\varepsilon > 0$ such that $\frac{r}{s} - 1 > \varepsilon$. But $\lim_{x\to 0} \frac{\rho}{\sigma} = 1$, implies that for $\varepsilon > 0$, there exists $\delta > 0$ such that $|\frac{\rho}{\sigma} - 1| < \varepsilon < \frac{r}{s} - 1$ whenever $0 < |x| < \delta$. If $\frac{\rho}{\sigma} \leq 1$, then we already have that $\frac{\rho}{\sigma} < \frac{r}{s}$, so assume $\frac{\rho}{\sigma} > 1$. Then there exist points $(x, f(x))$ where the corresponding $\frac{\rho}{\sigma}$ ratios satisfy $\frac{\rho}{\sigma} - 1 < \frac{r}{s} - 1$ and hence $\frac{\rho}{\sigma} < \frac{r}{s}$. Equation 1.2.5 tells us that a path other than the straight-line path is optimal if its $\frac{\rho}{\sigma}$ ratio is less than $\frac{r}{s}$. Thus it is faster to travel from A to one of these points and then swim to B rather than take the straight-line path.

Therefore, if $r \neq s$ and the points are not diametrically opposite, the straight line path is never optimal. □

This theorem provides the meaningful result that minimizing distance does not necessarily minimize travel time unless the traveling speed is constant over the entire journey. In fact, if the line connecting the starting and ending points does not intersect the border between the media perpendicularly, minimizing distance will never be optimal.

This proof is supported by the relationship between this problem and Fermat's Principle. If we aim a beam of light originating at A through a circular (or more accurately, cylindrical) piece of glass directly toward B on the other side of the circle, the light wave will encounter the curved boundary of the glass at an angle causing it to refract away from the straight-line path, unable to reach B.

Since the previous proof excluded the case where the points were diametrically opposite, we will now address this case. It is very interesting because it leads to some specific results about the possibly optimal paths from A to B as well as when we might take each one.

THEOREM 1.2.4. *When traveling from a point A outside a circular pond to a point B on the edge of the pond with a running speed of r and swimming speed of s, if the line connecting the points contains a diameter of the circle, then the straight-line path from A to B is optimal if and only if $s \geq r$.*

PROOF. ($\Leftarrow$) Suppose $s > r$ and the points are diametrically opposite. If we run to any point other than the point on the straight-line path between A and B before swimming, it will increase the distance at the running speed while increasing the total distance traveled. Then from Lemma 1.2.1, this will never be optimal. Hence the straight-line path will be optimal. If $s = r$ then the entire path will be traveled at one speed, and the path with the shortest distance, the straight-line path, will have the fastest time.

($\Rightarrow$) Suppose $r > s$ and the points are diametrically opposite. Position the pond on the coordinate plane, and scale it so that the center of the circle is located at the origin and B is located at $(-1, 0)$. Then the diameter of the circle will be along the x-axis, and A will be also be located on the x-axis at some point $(a, 0)$, where $a > 1$. Since this setup will be symmetric with respect to the x-axis, we only need to consider paths involving the northern half of the circle. Suppose we run from A to some visible point $C = (\cos\theta, \sin\theta)$ on the edge of the pond, and then continue on to point B.

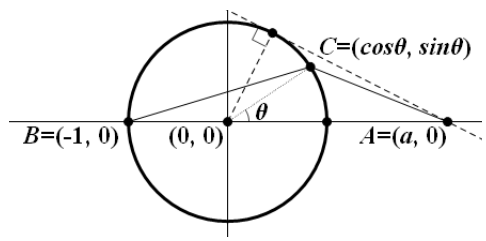

FIGURE 1.2.6. A and B Diametrically Opposite

If the optimal path from A to B passes through a point C, then it must contain the optimal path from C to B. Theorem 1.1.1 tells us that we will travel from C to B by either running the entire way around the edge of the circle, or by swimming straight from C to B. If the optimal path from C to B involves running around the circle, then we must pass the tangential point to the circle from A. However, it would then be faster to run straight from A to this tangential point instead of going there by way of point C. So, if we run to a point C on the circle that is not the tangential point, it will only be optimal if we then swim from C to B. We will construct a time function, $T(\theta)$, to calculate the time it takes to traverse such a path.

First we will find the domain for θ. We know that the tangent line to the circle passing through A will form a right angle with the radius of the circle. We can find the coordinates of this tangential point by using the similar right triangles shown in Figure 1.2.7. It is clear from the larger triangle that $\cos\gamma = \frac{1}{a}$, and hence

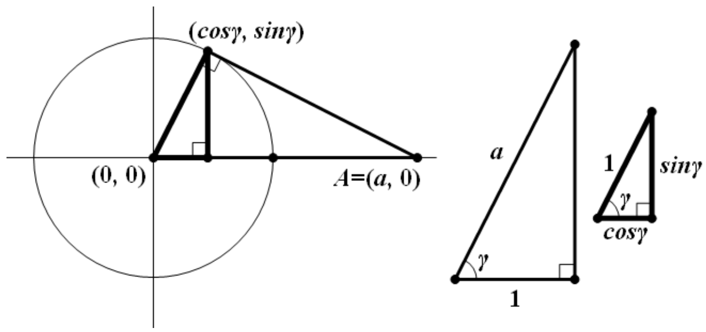

FIGURE 1.2.7. Circles Out to On Tangential Point

$\gamma = \cos^{-1}\frac{1}{a}$. Hence θ cannot exceed $\cos^{-1}\frac{1}{a}$. We can also find $\sin\gamma$ by using the Pythagorean Theorem to obtain $\sin\gamma = \sqrt{1-\frac{1}{a^2}} = \frac{\sqrt{a^2-1}}{a}$. We note that any line tangent to the circle at $(\cos\theta, \sin\theta)$ for $\theta \geq \frac{\pi}{2}$ would have a slope greater than or equal to zero, with a y-intercept greater than or equal to one, so it could never pass through a point on the positive x-axis, specifically A. Therefore, we have that $\theta \in \left[0, \cos^{-1}\frac{1}{a}\right] \subset \left[0, \frac{\pi}{2}\right)$.

Now, returning to Figure 1.2.6, we can construct a time function, $T(\theta)$, for a path that consists of running to point C and then swimming to B:

$$\begin{aligned} T(\theta) &= \frac{\sqrt{(\cos\theta - a)^2 + (\sin\theta)^2}}{r} + \frac{\sqrt{(\cos\theta+1)^2+(\sin\theta)^2}}{s} \\ &= \frac{\sqrt{\cos^2\theta - 2a\cos\theta + a^2 + \sin^2\theta}}{r} + \frac{\sqrt{\cos^2\theta + 2\cos\theta + 1 + \sin^2\theta}}{s} \\ &= \frac{\sqrt{a^2 - 2a\cos\theta + 1}}{r} + \frac{\sqrt{2+2\cos\theta}}{s}. \end{aligned} \tag{1.2.6}$$

Then we can find critical points by setting the derivative equal to zero. The derivative will be

$$\begin{aligned} T'(\theta) &= \frac{2a\sin\theta}{2r\sqrt{a^2-2a\cos\theta+1}} + \frac{-2\sin\theta}{2s\sqrt{2+2\cos\theta}} \\ &= \frac{a\sin\theta}{r\sqrt{a^2-2a\cos\theta+1}} - \frac{\sin\theta}{s\sqrt{2+2\cos\theta}}. \end{aligned} \tag{1.2.7}$$

So $T'(\theta) = 0$ if and only if

$$\frac{a\sin\theta}{r\sqrt{a^2-2a\cos\theta+1}} - \frac{\sin\theta}{s\sqrt{2+2\cos\theta}} = 0.$$

Since both denominators are nonzero, this will be true if $\sin\theta = 0$ and hence $\theta = 0$, or if $\sin\theta \neq 0$ and

$$\begin{aligned} \frac{a\sin\theta}{r\sqrt{a^2-2a\cos\theta+1}} &= \frac{\sin\theta}{s\sqrt{2+2\cos\theta}} \\ \frac{a}{r\sqrt{a^2-2a\cos\theta+1}} &= \frac{1}{s\sqrt{2+2\cos\theta}} \\ \frac{a^2}{r^2(a^2-2a\cos\theta+1)} &= \frac{1}{s^2(2+2\cos\theta)} \\ 2a^2s^2 + 2a^2s^2\cos\theta &= a^2r^2 - 2ar^2\cos\theta + r^2 \\ 2a^2s^2\cos\theta + 2ar^2\cos\theta &= a^2r^2 + r^2 - 2a^2s^2 \\ \cos\theta(2a^2s^2 + 2ar^2) &= a^2r^2 + r^2 - 2a^2s^2 \\ \cos\theta &= \frac{a^2r^2 + r^2 - 2a^2s^2}{2a^2s^2 + 2ar^2} \\ \theta &= \cos^{-1}\frac{a^2r^2 + r^2 - 2a^2s^2}{2a^2s^2 + 2ar^2}. \end{aligned}$$

Next we find the second derivative to determine what type of extrema these critical points represent.

$$T'(\theta) = \frac{a}{r}\sin\theta(a^2 - 2a\cos\theta + 1)^{-\frac{1}{2}} - \frac{1}{s}\sin\theta(2 + 2\cos\theta)^{-\frac{1}{2}},$$

so

$$\begin{aligned} T''(\theta) = \frac{a}{r}\sin\theta\left(-\frac{1}{2}\right)(a^2 - 2a\cos\theta + 1)^{-\frac{3}{2}}(2a\sin\theta) + \frac{a}{r}\cos\theta(a^2 - 2a\cos\theta + 1)^{-\frac{1}{2}} \\ - \frac{1}{s}\sin\theta\left(-\frac{1}{2}\right)(2 + 2\cos\theta)^{-\frac{3}{2}}(-2\sin\theta) - \frac{1}{s}\cos\theta(2 + 2\cos\theta)^{-\frac{1}{2}} \\ = -\frac{a^2\sin^2\theta}{r(a^2 - 2a\cos\theta + 1)^{\frac{3}{2}}} + \frac{a\cos\theta}{r\sqrt{a^2 - 2a\cos\theta + 1}} - \frac{\sin^2\theta}{s(2 + 2\cos\theta)^{\frac{3}{2}}} - \frac{\cos\theta}{s\sqrt{2 + 2\cos\theta}}. \end{aligned}$$

Evaluating this for $\theta = 0$ gives

$$T''(0) = \frac{a}{r\sqrt{a^2 - 2a + 1}} - \frac{1}{s\sqrt{2 + 2}} = \frac{a}{r(a-1)} - \frac{1}{2s}.$$

If $T''(0) \leq 0$ then we must have

$$\frac{a}{r(a-1)} \leq \frac{1}{2s} \iff \frac{2a}{a-1} \leq \frac{r}{s},$$

but

$$\frac{2a}{a-1} > \frac{2a}{a} = 2 > \frac{\pi}{2} \approx 1.57,$$

and from Equation 1.1.2, we know that if $\frac{r}{s} > \frac{\pi}{2}$, it will be faster to run the entire way rather than swimming. So if the optimal path will include any swimming we must have $\frac{r}{s} \leq \frac{\pi}{2} < \frac{2a}{a-1}$ which means $T''(0) > 0$ so $T'(\theta)$ is increasing at 0 and hence $T'(\theta)$ is negative immediately before 0 and positive immediately after 0. Thus

$T(\theta)$ is decreasing immediately before 0 and increasing immediately after 0 so $T(\theta)$ is concave up and thus has a local minimum at $\theta = 0$. Furthermore, since $T(\theta)$ is continuous, and must be increasing to the right of zero, it cannot have a local minimum at the next critical point, $\theta = \cos^{-1} \dfrac{a^2r^2 + r^2 - 2a^2s^2}{2a^2s^2 + 2ar^2}$. Thus the minimum value of $T(\theta)$ on its domain must be at one of the endpoints. That is, either we travel the straight-line path by running directly to the pond and then swimming across the diameter, or we run to the tangential point and continue by either running around the edge of the pond, or swimming directly from the tangential point to B.

Finally, we compare the paths that include a swimming portion. The straight-line path will be faster than the tangential path if and only if

$$T(0) < T(\cos^{-1} \frac{1}{a})$$

$$\frac{a-1}{r} + \frac{2}{s} < \frac{\sqrt{a^2 - 2a\left(\frac{1}{a}\right) + 1}}{r} + \frac{\sqrt{2 + \frac{2}{a}}}{s}$$

$$\frac{a-1}{r} - \frac{\sqrt{a^2-1}}{r} < \frac{\sqrt{2 + \frac{2}{a}}}{s} - \frac{2}{s}$$

$$\frac{a-1-\sqrt{a^2-1}}{r} < \frac{\sqrt{2 + \frac{2}{a}} - 2}{s}$$

$$\frac{a-1-\sqrt{a^2-1}}{\sqrt{2 + \frac{2}{a}} - 2} < \frac{r}{s}.$$

However, graphical evidence indicates that $\dfrac{a-1-\sqrt{a^2-1}}{\sqrt{2 + \frac{2}{a}} - 2}$ is decreasing for $a > 1$ as shown in Figure 1.2.8. Also,

```
Plot[(a - 1 - Sqrt[a^2 - 1])/Sqrt[(2 (a+1))/a - 2], {a, 1, 100}, PlotRange → {1.5, 2}]
```

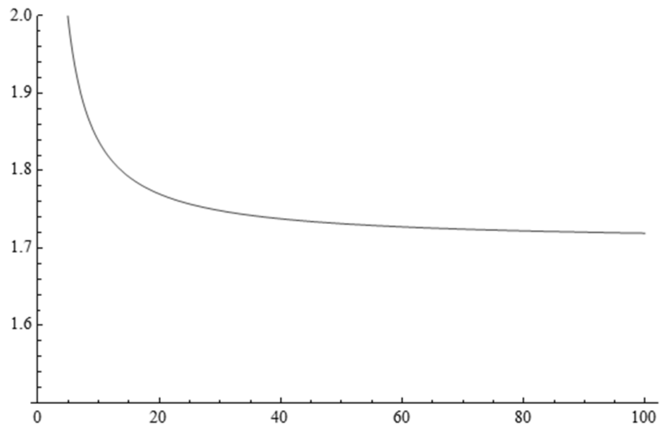

FIGURE 1.2.8. Graph of $\dfrac{a-1-\sqrt{a^2-1}}{\sqrt{2+\dfrac{2}{a}}-2}$ for $a > 1$

$$\lim_{a\to\infty}(a-\sqrt{a^2-1}) = \lim_{a\to\infty}\frac{a^2-(a^2-1)}{a+\sqrt{a^2-1}} = \lim_{a\to\infty}\frac{1}{a+\sqrt{a^2-1}} = 0 \text{ so}$$

$$\lim_{a\to\infty}\frac{a-1-\sqrt{a^2-1}}{\sqrt{2+\dfrac{2}{a}}-2} = \frac{\lim\limits_{a\to\infty}(a-\sqrt{a^2-1})-1}{\sqrt{\lim\limits_{a\to\infty}\left(2+\dfrac{2}{a}\right)}-2} = \frac{-1}{\sqrt{2}-2} = \frac{1}{2-\sqrt{2}}.$$

Thus if $\dfrac{r}{s} > \dfrac{a-1-\sqrt{a^2-1}}{\sqrt{2+\dfrac{2}{a}}-2}$, it must also be true that $\dfrac{r}{s} > \dfrac{1}{2-\sqrt{2}} \approx 1.7 > \dfrac{\pi}{2} \approx 1.57$.

But if $\dfrac{r}{s} > \dfrac{\pi}{2}$, it is faster to run the entire way rather than swim. Hence, it is never optimal to take the straight-line path from A to B if $r > s$. □

Now that we know that when A and B are diametrically opposite, traveling straight from A to B is only optimal if $s \geq r$, it is only a short step further to determine the conditions that make each of the possible paths optimal.

Corollary 1.2.5. *Suppose we are traveling from a point A outside a circular pond to a point B on the edge of the pond with running speed r and swimming speed s, where $r > s$. If the line from A to B contains a diameter of the circle, the optimal path will consist of running to the tangential point on the pond and then either running the rest of the way around the pond, or swimming straight to B. If $\frac{r}{s} > \frac{\pi - \cos^{-1}\frac{1}{a}}{\sqrt{2+\frac{2}{a}}}$, the optimal choice is to run around the edge of the pond. If $\frac{r}{s} < \frac{\pi - \cos^{-1}\frac{1}{a}}{\sqrt{2+\frac{2}{a}}}$, the optimal choice is to swim to B.*

Proof. Theorem 1.2.4 tells us that the straight-line path from A to B will never be the optimal path. Also, in proving this theorem, we found that if the optimal path involves swimming, it must be from one of the endpoints of the domain of $\theta \in \left[0, \cos^{-1}\frac{1}{a}\right]$ as depicted in Figure 1.2.6. Since the minimum time is not from the straight-line path ($\theta = 0$), it must be at the other endpoint, which represents the tangential point on the circle from A. If we were to run the entire distance from A to B, we would want to minimize the distance, which we would do by running to the tangential point and then running the rest of the way around the edge. Hence either path we take must begin by running to this tangential point. Since this part of the journey is the same in both cases, the only part that we need to compare to determine the optimal path is the journey from the tangential point to B. But this is just an On to On case.

Corollary 1.1.2 tells us that when traveling from a point on a circle to another point on the circle, we will run around the edge if $\frac{r}{s} > \frac{\alpha}{2\sin\left(\frac{\alpha}{2}\right)} = \frac{\alpha}{\sqrt{2 - 2\cos\alpha}}$, and swim straight to B if $\frac{r}{s} < \frac{\alpha}{2\sin\left(\frac{\alpha}{2}\right)} = \frac{\alpha}{\sqrt{2-2\cos\alpha}}$, where α is the angle between the radii to the two points. In the proof of Theorem 1.2.4, we found that the angle to the tangential point is $\gamma = \cos^{-1}\frac{1}{a}$, so the angle between the tangential point and B is $\alpha = \pi - \gamma$, and $\cos\alpha = \cos(\pi-\gamma) = -\cos\gamma = -\frac{1}{a}$. Then the optimal path is to run the entire way if $\frac{r}{s} > \frac{\pi - \cos^{-1}\frac{1}{a}}{\sqrt{2+\frac{2}{a}}}$, and run to the tangential point and then swim straight to B if $\frac{r}{s} < \frac{\pi - \cos^{-1}\frac{1}{a}}{\sqrt{2+\frac{2}{a}}}$. □

Now that the diametrically opposite case is fully exhausted, we return to the non-diametrically opposite case. We know that for the right $\frac{r}{s}$ ratio when $r > s$, we might minimize the swim distance by running the entire way. So the question arises as to whether we might want to minimize the run time for the right ratio if $r < s$. Since A is outside the circle, we must run part of the way in order to get to the pond, so the minimum run distance would be found by running perpendicularly to the pond. We know that we will never do this if the points are diametrically opposite, but what if they are not? This leads us to our next theorem.

THEOREM 1.2.6. *When traveling from a point A outside a circular pond to a point B on the edge of the pond, where the line connecting A and B does not contain a diameter of the pond (which may or may not be between the points), it will never be optimal to minimize the run distance by running perpendicular to the circle and then swimming.*

Proof. We want to compare the path that consists of running directly to the pond and then swimming to B with a path in which we run to another point on the circle and then swim. Let the run speed be r and the swim speed be s.

First, assume $r > s$. If we compare this perpendicular path to the straight-line path, the perpendicular route increases total distance while increasing the slower, swim distance. Then by Lemma 1.2.1, the perpendicular route will not be optimal. Also, Theorem 1.2.3 tells us that the straight-line path will be optimal if $r = s$.

Suppose $s > r$. Position the pond on the xy-plane so that the point where the line connecting A to the center of the pond intersects the circle is at the origin, and A and B are both above the x-axis as shown in Figure 1.2.9. (Since A and B are not diametrically opposite, the angle between them is less than π, and this is possible.) Then we want to compare the path in which we run from A to the origin and then swim to B with a path which involves running to another point $(x, f(x))$ on the circle and then swimming. This point must be above the x-axis or else we are increasing total distance while increasing the slower run distance which is not optimal. Let ρ represent the increase in run distance, and σ represent the decrease in swim distance that is caused by moving from the origin to this new point. Also, let a represent the distance from the origin to A, and b represent the distance from the origin to B.

Now, the new path will be optimal if and only if $\frac{a+\rho}{r} + \frac{b-\sigma}{s} < \frac{a}{r} + \frac{b}{s}$. Note that this is the same inequality as in Equation 1.2.5, so the new path will be optimal if and only if $\frac{\rho}{\sigma} < \frac{r}{s}$. We will find the limit of $\frac{\rho}{\sigma}$ as x approaches zero. Note that $\sigma = \sqrt{b_1^2 + b_2^2} - \sqrt{(x-b_1)^2 + (f(x)-b_2)^2}$ and $\rho = \sqrt{(x-a_1)^2 + (f(x)-a_2)^2} - \sqrt{a_1^2 + a_2^2}$,

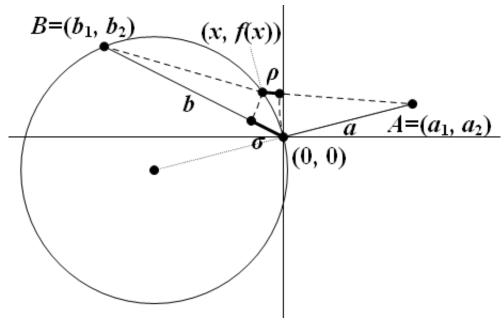

FIGURE 1.2.9. Circles Out to On Perpendicular Path Comparison

and as x approaches zero, $f(x)$ approaches zero as well. Then

$$\lim_{x\to 0}\frac{\rho}{\sigma}=\lim_{x\to 0}\frac{\sqrt{(x-a_1)^2+(f(x)-a_2)^2}-\sqrt{a_1^2+a_2^2}}{\sqrt{b_1^2+b_2^2}-\sqrt{(x-b_1)^2+(f(x)-b_2)^2}}.$$

This limit is of the form $\frac{0}{0}$, so we can apply L'Hopital's Rule. Then

$$\begin{aligned}\lim_{x\to 0}\frac{\rho}{\sigma}&=\lim_{x\to 0}\frac{\dfrac{2(x-a_1)+2(f(x)-a_2)f'(x)}{2\sqrt{(x-a_1)^2+(f(x)-a_2)^2}}}{-\dfrac{2(x-b_1)+2(f(x)-b_2)f'(x)}{2\sqrt{(x-b_1)^2+(f(x)-b_2)^2}}}\\&=\lim_{x\to 0}-\frac{(x-a_1+f(x)f'(x)-a_2f'(x))\sqrt{(x-b_1)^2+(f(x)-b_2)^2}}{\sqrt{(x-a_1)^2+(f(x)-a_2)^2}(x-b_1+f(x)f'(x)-b_2f'(x))}.\end{aligned}\tag{1.2.8}$$

Now, since A lies on a line through the origin, we can write $a_2=k(a_1)$, where k is the slope of this line. That is, $k=\frac{a_2}{a_1}\neq 0$. Furthermore, since this line will be perpendicular to the tangent line at the origin, we know that $f'(0)=-\frac{1}{k}$. Hence

$$\begin{aligned}\lim_{x\to 0}\frac{\rho}{\sigma}&=\lim_{x\to 0}-\frac{\left(x-a_1+f(x)\left(-\frac{1}{k}\right)-ka_1\left(-\frac{1}{k}\right)\right)\sqrt{(x-b_1)^2+(f(x)-b_2)^2}}{\sqrt{(x-a_1)^2+(f(x)-ka_1)^2}\left(x-b_1+f(x)\left(-\frac{1}{k}\right)-b_2\left(-\frac{1}{k}\right)\right)}\\&=\frac{(0-a_1+0+a_1)\sqrt{(0-b_1)^2+(0-b_2)^2}}{\sqrt{(0-a_1)^2+(0-ka_1)^2}\left(0-b_1+0+\frac{b_2}{k}\right)}\end{aligned}$$

$$= \frac{0}{\sqrt{a_1^2 + (ka_1)^2}\left(\frac{b_2}{k} - b_1\right)} = 0.$$

So as long as $\frac{b_2}{k} - b_1 \neq 0$, we will have $\lim_{x \to 0} \frac{\rho}{\sigma} = 0$. But if $\frac{b_2}{k} - b_1 = 0$, this means that $b_2 = kb_1$ and B is on the same line through the origin as A. But our hypothesis was that the line connecting A and B does not contain a diameter of the circle, so B cannot be on this line and therefore this limit is zero.

Both of our speeds are positive, so $\frac{r}{s} > 0$, which means that there exists $\varepsilon > 0$ such that $\frac{r}{s} > \varepsilon$. Also, $\lim_{x \to 0} \frac{\rho}{\sigma} = 0$ means that for $\varepsilon > 0$, there exists $\delta > 0$ such that $|\frac{\rho}{\sigma} - 0| = \frac{\rho}{\sigma} < \varepsilon < \frac{r}{s}$ whenever $0 < |x| < \delta$. Thus there exist points $(x, f(x))$ such that the corresponding $\frac{\rho}{\sigma}$ ratios are less than $\frac{r}{s}$ and therefore the paths through these points are faster than the path to the perpendicular point on the circle. Thus the perpendicular path is never optimal. □

This result is interesting, because in general, it shows that minimizing the distance traveled at the slower speed would not necessarily increase our overall speed, as instinct might suggest. In fact, in the given scenario, this will never happen.

Once again, this result relates to the correspondence between this scenario and the Least Time Principle. If we were to aim a beam of light at a circular, or rather cylindrical, piece of glass (or other material) along the normal line, it would not bend or refract since the angle of incidence would be 0, leading to the equation $0 = n_2 \sin\theta_2$ so either $n_2 = 0$ or $\sin\theta_2 = 0$. A piece of glass would not have refractive index of zero, and in fact, materials with such a property have only recently been created, the first of which was developed at Columbia Engineering School in 2011 according to [**3**]. Hence we would not have $n_2 = 0$ (and can generalize this for most materials other than glass as well), so we must have $\sin\theta_2 = 0$ and thus θ_2 is either 0 or π. So the light would continue along the normal, and thus could not bend to reach B.

While we have not obtained an exact equation to identify the point on the pond to which running from A and then swimming to B will provide the optimal path, we have managed to narrow it down through the last few theorems. We begin by positioning the pond on the xy-plane so that the center is at the origin, and the radius is one. Let A be on the x-axis at $(a, 0)$, and B be on the top half of the circle at (b_1, b_2). In the proof of Theorem 1.2.4, we found that the tangential point to the circle from A will be located at the point $\left(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\right)$. This is shown in Figure 1.2.10.

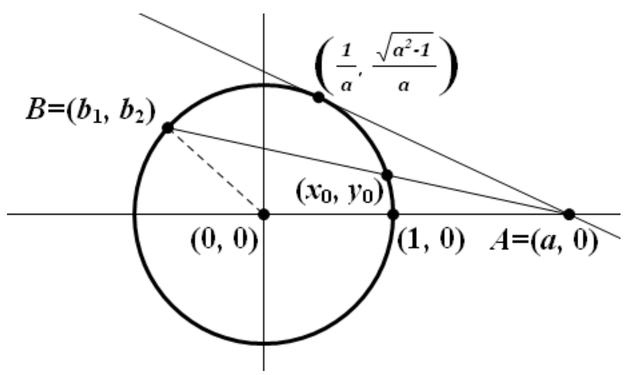

Figure 1.2.10. General Circle Out to On Case

We can also find the exact point where the straight-line path from A to B intersects the circle. We begin by finding the equation of this line, which we can do by using the two points on the line, $(a, 0)$ and (b_1, b_2). Since B is on the northern half of the circle, $b_2 = \sqrt{1-b_1^2}$, and the slope of the line will be $\dfrac{b_2}{b_1 - a} = \dfrac{\sqrt{1-b_1^2}}{b_1 - a}$.

Then the equation of the line will be given by

$$y = \frac{\sqrt{1 - b_1^2}}{b_1 - a}(x - a).$$

We want to find where this line intersects the top half of the unit circle, so we are trying to find where $\sqrt{1 - x^2} = \frac{\sqrt{1 - b_1^2}}{b_1 - a}(x - a)$. Both sides will be positive, so we can find the intersection points by squaring both sides and moving all terms to one side of the equation.

$$1 - x^2 = \frac{(1 - b_1^2)(x - a)^2}{(b_1 - a)^2}$$

$$(1 - x^2)(b_1^2 - 2ab_1 + a^2) = (1 - b_1^2)(x^2 - 2ax + a^2)$$

$$b_1^2 - 2ab_1 + a^2 - x^2b_1^2 + 2x^2ab_1 - x^2a^2 = x^2 - 2xa + a^2 - x^2b_1^2 + 2xab_1^2 - a^2b_1^2$$

$$0 = x^2 - x^2b_1^2 + x^2b_1^2 - 2x^2ab_1 + x^2a^2 - 2xa + 2xab_1^2 + a^2 - a^2b_1^2 - b_1^2 + 2ab_1 - a^2$$

$$0 = x^2 - 2x^2ab_1 + x^2a^2 - 2xa + 2xab_1^2 - a^2b_1^2 - b_1^2 + 2ab_1$$

$$0 = x^2(1 - 2ab_1 + a^2) + x(-2a + 2ab_1^2) + (2ab_1 - a^2b_1^2 - b_1^2) \tag{1.2.9}$$

This equation can be very difficult to solve further. However, we already know that the line and circle intersect at point B; we just need to find the other point. But this means that b_1 must be an x-intercept of the function in Equation 1.2.9, and hence $(x - b_1)$ must be a factor. Then we can find the other factor using long division.

$$
\begin{array}{r|l}
 & x\,(1-2ab_1+a^2) \;+\; (b_1+a^2b_1-2a) \\
\hline
x - b_1 & x^2(1-2ab_1+a^2) \;+\; x(-2a+2ab_1^2) \;+\; (2ab_1-a^2b_1^2-b_1^2) \\
 & -x^2(1-2ab_1+a^2) \;+\; x(b_1-2ab_1^2+a^2b_1) \\
\hline
 & x(b_1+a^2b_1-2a) \;+\; (2ab_1-a^2b_1^2-b_1^2) \\
 & -\; x(b_1+a^2b_1-2a) \;+\; (b_1^2+a^2b_1^2-2ab_1) \\
\hline
 & 0.
\end{array}
$$

By setting this quotient equal to zero and solving, we can get that $x_0 = \dfrac{2a-b_1-a^2b_1}{1-2ab_1+a^2}$, and y_0 can easily be found by substituting this value into the equation for the line:

$$
\begin{aligned}
y_0 &= \frac{\sqrt{1-b_1^2}}{b_1-a}\left(\frac{2a-b_1-a^2b_1}{1-2ab_1+a^2}-a\right)\\
&= \frac{\sqrt{1-b_1^2}}{b_1-a}\left(\frac{2a-b_1-a^2b_1-a+2a^2b_1-a^3}{1-2ab_1+a^2}\right)\\
&= \frac{\sqrt{1-b_1^2}}{b_1-a}\left(\frac{a-b_1+a^2b_1-a^3}{1-2ab_1+a^2}\right)\\
&= \frac{\sqrt{1-b_1^2}}{b_1-a}\left(\frac{-(b_1-a)+a^2(b_1-a)}{1-2ab_1+a^2}\right)\\
&= \frac{\sqrt{1-b_1^2}}{b_1-a}\left(\frac{(b_1-a)(a^2-1)}{1-2ab_1+a^2}\right)\\
&= \frac{(a^2-1)\sqrt{1-b^2}}{1-2ab_1+a^2}.
\end{aligned}
$$

Hence the line connecting A and B intersects the circle at the point $(x_0, y_0) = \left(\dfrac{2a-b_1-a^2b_1}{1-2ab_1+a^2}, \dfrac{(a^2-1)\sqrt{1-b^2}}{1-2ab_1+a^2}\right).$

If $s > r$, running to a point above the straight-line path would increase total distance while increasing the distance traveled at the slower speed. This would not be optimal via Lemma 1.2.1, so we must instead run to a point below the straight-line path. Thus we would run to some point $(x, \sqrt{1-x^2})$ where $x \in \left(\frac{2a - b_1 - a^2 b_1}{1 - 2ab_1 + a^2}, 1\right)$. Theorems 1.2.3 and 1.2.6 tell us that the endpoints are not included.

If $r > s$, we know that we will run to a point above the straight-line path for the optimal time since running to a point below this path would increase the total distance while increasing the distance traveled at the slower speed, a course that would not be optimal according to Lemma 1.2.1. If we run to a point other than the tangential point, then we will continue to B by swimming; but if we run to the tangential, we may either continue running along the edge of the circle, or swim to B. Thus, if $r > s$, we would run to some point $(x, \sqrt{1-x^2})$ where $x \in \left[\frac{1}{a}, \frac{2a - b_1 - a^2 b_1}{1 - 2ab_1 + a^2}\right)$.

Proposition 1.2.7. *If* $\frac{r}{s} > \frac{\pi - \cos^{-1}\frac{1}{a}}{\sqrt{2 + \frac{2}{a}}}$*, the optimal path will be the all-running path, in which we run to the tangential point on the circle, and then continue running around the edge to the ending point,* B*.*

Proof. Suppose $\frac{r}{s} > \frac{\pi - \cos^{-1}\frac{1}{a}}{\sqrt{2 + \frac{2}{a}}}$. Corollary 1.2.5 tells us that if we are running from A to the point diametrically opposite from A, the optimal path will be the all-running path. This case is the one in which the ending point is the farthest from A that is possible in the Out to On case. Then for any other ending point B on the circle that is beyond the tangential point, the all running path from A to the diametrically opposite point must pass through B. But if a path is optimal, it must be optimal for all portions of the journey along that path. Since part of this journey to the diametrically opposite point involves traveling from A to B, that portion of the path must be optimal as well. Hence if $\frac{r}{s} > \frac{\pi - \cos^{-1}\frac{1}{a}}{\sqrt{2 + \frac{2}{a}}}$, the optimal path from A to B is the all-running path. □

We should note that this claim is only true in one direction. There may be values for $\frac{r}{s}$ that are less than $\frac{\pi - \cos^{-1}\frac{1}{a}}{\sqrt{2 + \frac{2}{a}}}$ which would also make it optimal to take the all-running path since the angle subtending the arc from the tangential point to B will be smaller than the one subtending the arc to the diametrically opposite point.

We now have several interesting results about what may or may not constitute the optimal route from a point outside a circular pond to a point on the edge of the pond.

CHAPTER 2

RECTANGLES

We now transition to a similar problem that, in a way, combines the ideas of a straight shoreline and a circular pond by considering an enclosed body of water with straight edges. Specifically, we will consider a rectangular pool. As with the case of a circular pond, we begin with the case where the starting and ending points are both on the edge of the pool. We discover that the optimal path is easier to determine in the On to On case when considering the pond due to the uniformity of the circle. The corners in this scenario create more cases to consider.

We can begin by ruling out one of our standard cases. Once again, the shortest distance between two points is a straight line, so if our swim speed is greater than or equal to our run speed, this minimum distance will also be traveled at the fastest speed, taking the shortest amount of time. So we will always swim straight from A to B when considering a faster (or equal) swim speed. Then we can focus on the more interesting case where the run speed is greater than the swim speed.

First we will look at the case where the points are on adjacent sides of the rectangle, which we will refer to as the 2-sided case, and then move on to the 3-sided case where the points are on opposite sides of the rectangle.

2.1. The 2-Sided Case

Suppose we are traveling from a point A on the edge of a rectangular pool to another point B on an adjacent edge of the pool. What is important in determining the optimal route is the ratio of the run speed to the swim speed, so for simplicity, let the swim speed be 1 unit. This way we will only have to be concerned with one speed variable instead of two.

We begin by determining if we would run the entire way, swim the entire way, or use some combination of running and swimming.

THEOREM 2.1.1. *Suppose we are traveling from a point A on one side of a rectangular pool to a point B on an adjacent side of the pool with swim speed $s = 1$ and run speed $r > 1$. Furthermore, suppose the distance from the corner between the sides to A is less than (or equal to) the distance to B. The optimal path will either be to swim directly from A to B, run around the edge from A to B, or to swim from A to some point on the adjacent side and then run to B. We do not need to consider cases which would have a running portion followed by a swimming portion and then another running portion.*

PROOF. Position the pool on the xy-plane so that the corner between the sides containing the starting and ending points is at the origin, A is at the point $(a, 0)$, and B is at the point $(0, b)$ with $b \geq a$ as shown in Figure 2.1.1. Any path from A to B will involve running along the x-axis to a point $(x, 0)$ where $0 \leq x \leq a$, swimming to a point $(0, y)$ where $0 \leq y \leq b$, and then running the rest of the way to B.

The run time will be $\frac{a-x}{r} + \frac{b-y}{r}$ and the swim time will be $\sqrt{x^2 + y^2}$. We can then create a function $T(x, y)$ to give the time to travel from A to B,

$$T(x,y) = \frac{a-x+b-y}{r} + \sqrt{x^2+y^2}. \tag{2.1.1}$$

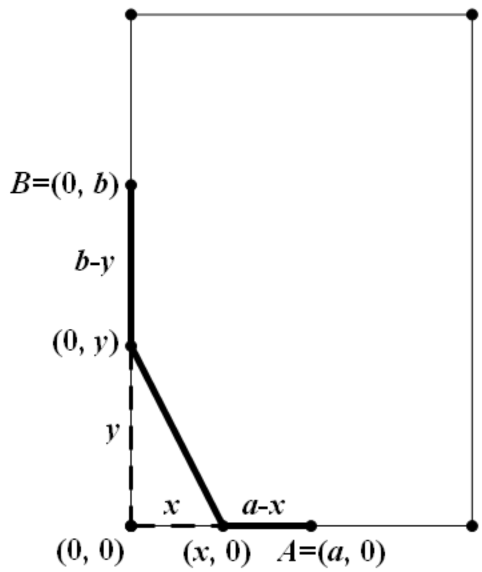

FIGURE 2.1.1. 2-Sided Rectangle Path Possibilities

We want to minimize $T(x,y)$ for $(x,y) \in [0,a] \times [0,b]$. This minimum could occur at an interior critical point or on the boundary of this rectangular domain. First we explore the possibility of an interior critical point.

For any x and y that provide a local minimum travel time, the partial derivatives with respect to each variable must both be equal to zero. Now,

$$\frac{\partial T}{\partial x}(x,y) = -\frac{1}{r} + \frac{2x}{2\sqrt{x^2+y^2}}$$

so $\frac{\partial T}{\partial x}(x,y) = 0$ if and only if $\frac{x}{\sqrt{x^2+y^2}} = \frac{1}{r}$. Similarly,

$$\frac{\partial T}{\partial y}(x,y) = -\frac{1}{r} + \frac{2y}{2\sqrt{x^2+y^2}}$$

so $\frac{\partial T}{\partial y}(x, y) = 0$ if and only if $\frac{y}{\sqrt{x^2 + y^2}} = \frac{1}{r}$. There can only be a local minimum point if both of these derivatives are equal to zero and hence

$$\frac{y}{\sqrt{x^2 + y^2}} = \frac{1}{r} = \frac{x}{\sqrt{x^2 + y^2}},$$

which only happens if $x = y$. So if we cut off a corner of the rectangle, it must form an isosceles triangle which means we would swim away from the edge at a 45° angle.

Since $x = y$, we can adjust our time function from Equation 2.1.1 so that it is a function of only one variable instead of two:

$$T(x) = \frac{a + b - 2x}{r} + x\sqrt{2}.$$

Then the only critical point(s) will be where

$$T'(x) = \frac{-2}{r} + \sqrt{2} = 0 \quad \Longleftrightarrow \quad \sqrt{2} = \frac{2}{r} \quad \Longleftrightarrow \quad r = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Thus we will only be able to cut off a corner if the run rate is $\sqrt{2}$, and even then, the value of x doesn't matter! In fact, if $r = \sqrt{2}$, $T(x)$ simplifies to

$$T(x) = \frac{a + b - 2x}{\sqrt{2}} + x\sqrt{2} = \frac{a + b - 2x}{\sqrt{2}} + \frac{2x}{\sqrt{2}} = \frac{a + b}{\sqrt{2}},$$

which is just a constant function. So if $r = \sqrt{2}$, cutting off any corner with equal sides will give the same time, and we can just consider this the same as the all-running path corresponding to $x = y = 0$. If $r \neq \sqrt{2}$, there will not be an interior critical point, and the minimum must happen on the boundary. Thus we do not need to consider any run-swim-run paths.

Now we will consider the behavior of T on the boundary. Since we have two variables, these boundary points are like the edges of a rectangle themselves. That is, the boundary would be the outside edges of the domain, $[0,a] \times [0,b] \subset \mathbb{R}^2$. So we would have $x \in \{0,a\}$ and $y \in [0,b]$, or $y \in \{0,b\}$ and $x \in [0,a]$. We will need to consider the cases where $x = y = 0$ and where $x = a$ and $y = b$ as they are the all-running path around the edge of the pool from A to B and the all-swimming path directly between the two points respectively. The cases where $x = 0$ and $y \in (0,b]$ and where $y = 0$ and $x \in (0,a]$ are trivial because they would be the same distance as the all-running path but with one leg of the journey done by swimming along the edge of the pool. Since swimming is slower, these could never be minimum paths. This leaves the cases where $x = a$ and $y \in [0,b]$ or where $y = b$ and $x \in [0,a]$. Henceforth, we will refer to these paths as the swim-run path and run-swim path respectively.

Next, we will find the point to which we want to swim on the y-axis in the case of the swim-run option, and verify that it is a minimum. (The run-swim option will follow similarly.) An updated picture will help us revise our time function again. See Figure 2.1.2. In the above paragraph, we discerned that this swim-run path will occur when $x = a$ and $y \in [0,b]$, so we only need to consider one variable, y, giving

$$T(y) = \sqrt{a^2 + y^2} + \frac{b - y}{r}. \tag{2.1.2}$$

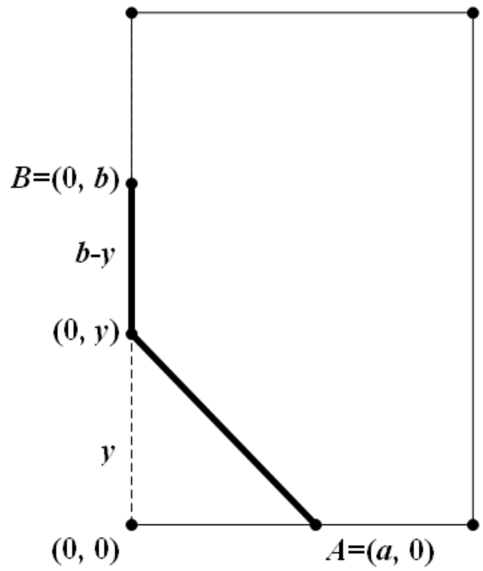

FIGURE 2.1.2. Swim-Run 2-Sided Case

Then for a minimum to occur, we must have $T'(y) = 0$, so

$$\begin{aligned} T'(y) &= \frac{2y}{2\sqrt{a^2+y^2}} - \frac{1}{r} = 0 \\ \frac{y}{\sqrt{a^2+y^2}} &= \frac{1}{r} \\ \frac{y^2}{a^2+y^2} &= \frac{1}{r^2} \\ r^2y^2 &= a^2+y^2 \\ y^2(r^2-1) &= a^2 \\ y^2 &= \frac{a^2}{r^2-1} \\ y &= \frac{a}{\sqrt{r^2-1}}. \end{aligned} \tag{2.1.3}$$

We can take the second derivative to see that this is in fact a local minimum.

$$T'(y) = y(a^2+y^2)^{-\frac{1}{2}} - \frac{1}{r}$$

so

$$
\begin{aligned}
T''(y) &= y\left(-\frac{1}{2}\right)(a^2+y^2)^{-\frac{3}{2}}(2y)+(a^2+y^2)^{-\frac{1}{2}} \\
&= \frac{-y^2}{(a^2+y^2)^{\frac{3}{2}}}+\frac{1}{(a^2+y^2)^{\frac{1}{2}}} \\
&= \frac{-y^2}{(a^2+y^2)^{\frac{3}{2}}}+\frac{a^2+y^2}{(a^2+y^2)^{\frac{3}{2}}} \\
&= \frac{a^2}{(a^2+y^2)^{\frac{3}{2}}}. \qquad (2.1.4)
\end{aligned}
$$

Since this must always be positive, the derivative is always increasing and must be negative to the left of the critical point and positive to the right of the critical point. Thus our time function is decreasing to the left of the critical point and increasing to the right of the critical point, making it concave up over its domain. Therefore our critical point from Equation 2.1.3 must be a local minimum. Furthermore, it will be an absolute minimum if it falls within our domain (if $0 \le \frac{a}{\sqrt{r^2-1}} \le b$), but not if it falls outside the domain. However, we should note that this function, $T(y)$, does not represent the all-run path for any $y \in [0,b]$, so we will still need to compare the times for these two paths at a later stage.

Now, we just found that the critical point of the time function given in Equation 2.1.2 occurs when $y = \frac{a}{\sqrt{r^2-1}}$. If we substitute this value back into the time function, we find that the travel time corresponding to this swim-run path will be given by

$$
T\left(\frac{a}{\sqrt{r^2-1}}\right) = \sqrt{a^2+\left(\frac{a}{\sqrt{r^2-1}}\right)^2} + \frac{b-\left(\frac{a}{\sqrt{r^2-1}}\right)}{r}
$$

$$= \sqrt{\frac{a^2r^2 - a^2}{r^2 - 1} + \frac{a^2}{r^2 - 1}} + \frac{b}{r} - \frac{a}{r\sqrt{r^2 - 1}}$$

$$= \frac{ar}{\sqrt{r^2 - 1}} + \frac{b}{r} - \frac{a}{r\sqrt{r^2 - 1}}$$

$$= \frac{ar^2 - a}{r\sqrt{r^2 - 1}} + \frac{b}{r}$$

$$= \frac{a(r^2 - 1)}{r\sqrt{r^2 - 1}} + \frac{b}{r}$$

$$= \frac{a\sqrt{r^2 - 1} + b}{r}. \tag{2.1.5}$$

Similarly, if we let $a = b$, $b = a$, and $y = x$, we can use the same equations to determine the time function and local minimum for the case where $y = b$ and $x \in [0, a]$. The time function will be

$$T(x) = \sqrt{b^2 + y^2} + \frac{a - x}{r}. \tag{2.1.6}$$

Then a local minimum will occur only if $T'(x) = 0$, which only happens where $x = \frac{b}{\sqrt{r^2 - 1}}$. Also, the second derivative will be $\frac{b^2}{(b^2 + y^2)^{\frac{3}{2}}}$ which is always positive, verifying that the function is concave up and this critical point does provide a local minimum. Furthermore, the time it takes to travel this run-swim path will be $T\left(\frac{b}{\sqrt{r^2 - 1}}\right) = \frac{b\sqrt{r^2 - 1} + a}{r}$.

Now, the run-swim path will be faster than the swim-run path if and only if

$$\frac{b\sqrt{r^2 - 1} + a}{r} < \frac{a\sqrt{r^2 - 1} + b}{r}$$

$$b(\sqrt{r^2 - 1} - 1) < a(\sqrt{r^2 - 1} - 1).$$

However, since $b \geq a$, this can be true if and only if

$$\sqrt{r^2-1}-1<0 \quad \iff \quad \sqrt{r^2-1}<1.$$

But if $\sqrt{r^2-1}<1$, then

$$\frac{b}{\sqrt{r^2-1}} > b \geq a,$$

and the critical point is not in the domain of $T(x)$. Thus for $b \geq a$, we only need to consider swim-run paths in which we swim from A to $\left(0, \frac{a}{\sqrt{r^2-1}}\right)$ and then run the rest of the way to B. □

Now that we know which paths could be optimal, we can begin to compare the times for each path to find out when each one is optimal.

THEOREM 2.1.2. *When traveling between two points A and B on adjacent sides of a rectangular pool, where the distance from the corner between the sides to A and B are one unit and $b \geq 1$ units respectively with a swim speed of $s = 1$ and run speed of $r > s = 1$, the following are true:*

(1) *The all-running path will be optimal to the all-swim path if $r > \sqrt{2}$ or if $r \leq \sqrt{2}$ and $b < \frac{1-r\sqrt{2-r^2}}{r^2-1}$ or $b > \frac{1+r\sqrt{2-r^2}}{r^2-1}$. If equality holds for b, they are equally optimal. Otherwise, the all-swim path is preferable.*

(2) *The all-running path will be optimal to the swim-run path if and only if $r > \sqrt{2}$ or $b < \frac{1}{\sqrt{r^2-1}}$. Note that if $r = \sqrt{2}$, they will take equal amounts of time.*

(3) *The all-swimming path will be optimal to the swim-run path if and only if $r < \sqrt{2}$ and $b < \dfrac{1}{\sqrt{r^2 - 1}}$. Note that if equality holds for b, they will take equal amounts of time.*

PROOF. Position the pool on the xy-plane as in the proof of Theorem 2.1.1, depicted in Figure 2.1.1. Furthermore, scale the figure so that A is located at $(1, 0)$.

(1) The all-run path will be optimal to the all-swim path if and only if the all-run time is less than the all-swim time so

$$\begin{aligned} \frac{a+b}{r} &< \sqrt{a^2+b^2} \\ \frac{1+b}{r} &< \sqrt{1+b^2} \\ \frac{1+2b+b^2}{r^2} &< 1+b^2 \\ 1+2b+b^2 &< r^2+r^2b^2 \\ 0 &< r^2b^2 - b^2 - 2b + r^2 - 1 \\ 0 &< (r^2-1)b^2 - 2b + (r^2-1). \end{aligned}$$

We notice that the right-hand side of this inequality is just a parabola with b as the variable, so we can use the quadratic formula to find where equality holds:

$$\begin{aligned} b &= \frac{2 \pm \sqrt{4 - 4(r^2-1)^2}}{2(r^2-1)} \\ &= \frac{1 \pm \sqrt{1 - r^4 + 2r^2 - 1}}{r^2-1} \\ &= \frac{1 \pm r\sqrt{2-r^2}}{r^2-1}. \end{aligned}$$

Since this problem only applies to $r > s = 1$, $r^2 > 1$ and $r^2 - 1 > 0$. Thus the leading coefficient is positive and the parabola opens upward so it will be greater than zero when $b < \dfrac{1 - r\sqrt{2 - r^2}}{r^2 - 1}$ or $b > \dfrac{1 + r\sqrt{2 - r^2}}{r^2 - 1}$. However, this will only be true if the discriminant is zero or more. If the discriminant is negative, then the parabola is completely above the x-axis and the all-run path is always optimal to the all-swim path. This will happen when

$$2 - r^2 < 0 \iff 2 < r^2 \iff r > \sqrt{2}.$$

Thus the all-run path will be optimal if and only if $r > \sqrt{2}$ or $r \leq \sqrt{2}$ with $b < \dfrac{1 - r\sqrt{2 - r^2}}{r^2 - 1}$ or $b > \dfrac{1 + r\sqrt{2 - r^2}}{r^2 - 1}$. Note that if $r = \sqrt{2}$, then this parabola will never be below the x-axis, and will intersect the x-axis when $b = 1$. This implies that the paths will have equal time when equality holds, which will happen if and only if $r \leq \sqrt{2}$ and $b = \dfrac{1 \pm r\sqrt{2 - r^2}}{r^2 - 1}$.

(2) First we note that the swim-run path is only minimal if the critical point, $\dfrac{1}{\sqrt{r^2 - 1}}$ falls within the domain of the function. So if the critical point is not in the domain, i.e. $b < \dfrac{1}{\sqrt{r^2 - 1}}$, the all-run path must be optimal to the swim-run path. If $b \geq \dfrac{1}{\sqrt{r^2 - 1}}$, the critical point will fall within the domain, and the all-run path will be optimal to the swim-run path if and only if

$$\frac{1 + b}{r} < \frac{a\sqrt{r^2 - 1} + b}{r}$$

$$1 + b < \sqrt{r^2 - 1} + b$$

$$1 < \sqrt{r^2 - 1}$$

$$1 < r^2 - 1$$

$$r > \sqrt{2}.$$

(3) In the proof of Theorem 2.1.1, Equation 2.1.2 applies for $y \in [0, b]$. If the critical point, $y = \dfrac{a}{\sqrt{r^2-1}} = \dfrac{1}{\sqrt{r^2-1}}$ falls inside the domain, this path will be the minimum. If not, the minimum will fall on an endpoint. Now, the critical point is a positive number since $r > 1$ implies $\sqrt{r^2-1} > 0$. So if the critical point falls outside the domain, it must be above the upper limit. In the proof of Theorem 2.1.1, we determined that $T(y)$ is decreasing for $y \leq \dfrac{1}{\sqrt{r^2-1}}$, so it is decreasing over the entire domain. Then the minimum would be at the upper endpoint of the domain where $y = b$, which is the all-swim path.

Suppose $r > \sqrt{2}$. Then

$$r^2 > 2 \iff \sqrt{r^2-1} > 1 \iff \frac{1}{\sqrt{r^2-1}} < 1,$$

and since we are labeling the rectangle so that $b \geq a = 1$, the critical point will always be in the domain and the swim-run path will be optimal to the all-swim path.

If $r = \sqrt{2}$ then $\dfrac{1}{\sqrt{r^2-1}} = 1 \leq b$, and the critical point will always be in the domain. If $b = 1$, this would make the critical point equal to b so the swim-run path is the same as the all-swim path, and thus would take the same time to travel.

Suppose $r < \sqrt{2}$. Then we will have $\dfrac{1}{\sqrt{r^2 - 1}} > 1$ so this critical point may or may not be in the domain of T. If $b \geq \dfrac{1}{\sqrt{r^2 - 1}}$, the critical point is in the domain and the swim-run path is faster. However, if $b = \dfrac{1}{\sqrt{r^2 - 1}}$, the critical point *is* the upper endpoint of the domain and thus the swim-run path is the same as the all-swim path. If $b < \dfrac{1}{\sqrt{r^2 - 1}}$, then the critical point is outside the domain and the optimal path is the all-swim path. □

Now that we know how to determine the faster of any two paths, we can put all three parts of the previous theorem together to find out when each path is optimal *overall*. One more piece of information will allow us to pinpoint these conditions exactly.

LEMMA 2.1.3. *Suppose* $1 < r < \sqrt{2}$. *Then*

$$\frac{1 - r\sqrt{2 - r^2}}{r^2 - 1} < 1 < \frac{1}{\sqrt{r^2 - 1}} < \frac{1 + r\sqrt{2 - r^2}}{r^2 - 1}. \tag{2.1.7}$$

PROOF. First we will prove the middle inequality, which follows directly from the fact that $r < \sqrt{2}$. This implies that $r^2 < 2$ and then $\sqrt{r^2 - 1} < 1$ so $\dfrac{1}{\sqrt{r^2 - 1}} > 1$.

Next we will prove the right inequality. $\dfrac{1}{\sqrt{r^2 - 1}} = \dfrac{\sqrt{r^2 - 1}}{r^2 - 1}$, so to prove that $\dfrac{1}{\sqrt{r^2 - 1}} < \dfrac{1 + r\sqrt{2 - r^2}}{r^2 - 1}$, it is sufficient to prove that

$$\frac{\sqrt{r^2 - 1}}{r^2 - 1} < \frac{1 + r\sqrt{2 - r^2}}{r^2 - 1},$$

and $r > 1$ implies that $r^2 - 1 > 0$ so we can simplify this even further to

$$\sqrt{r^2 - 1} < 1 + r\sqrt{2 - r^2}.$$

Now, $r < \sqrt{2}$ tells us that $-r^2 > -2$, so $2 - r^2 > 0$ and $r\sqrt{2 - r^2} > 0$ (and we already mentioned that $\sqrt{r^2 - 1} < 1$). Thus

$$\sqrt{r^2 - 1} < 1 < 1 + r\sqrt{2 - r^2}, \tag{2.1.8}$$

which proves the right inequality.

Now we will prove the left inequality. We start by rationalizing the numerator of the left side to get

$$\begin{aligned} \frac{1 - r\sqrt{2 - r^2}}{r^2 - 1} &= \frac{1 - r\sqrt{2 - r^2}}{r^2 - 1}\left(\frac{1 + r\sqrt{2 - r^2}}{1 + r\sqrt{2 - r^2}}\right) \\ &= \frac{1 - r^2(2 - r^2)}{(r^2 - 1)(1 + r\sqrt{2 - r^2})} \\ &= \frac{1 - 2r^2 + r^4}{(r^2 - 1)(1 + r\sqrt{2 - r^2})} \\ &= \frac{(r^2 - 1)^2}{(r^2 - 1)(1 + r\sqrt{2 - r^2})} \\ &= \frac{r^2 - 1}{1 + r\sqrt{2 - r^2}}. \end{aligned} \tag{2.1.9}$$

Equation 2.1.8 tells us that $1 < 1 + r\sqrt{2 - r^2}$, so $\dfrac{r^2 - 1}{1 + r\sqrt{2 - r^2}} < \dfrac{r^2 - 1}{1}$. But $r < \sqrt{2}$ implies that $r^2 - 1 < 1$ so $\dfrac{1 - r\sqrt{2 - r^2}}{r^2 - 1} < 1$, and the left side of the inequality is also proven. Thus the inequality in Equation 2.1.7 is true. □

Now we can put all of this information together to know exactly when each path is optimal.

THEOREM 2.1.4. *Suppose we are traveling between two points A and B on adjacent sides of a rectangular pool where the distance from the corner between the sides to A and B are one unit and $b \geq 1$ units respectively with a swim speed of $s = 1$ and run speed of $r > s = 1$. Excluding cases where travel times and/or paths are equal, the following are true:*

(1) *The all-running path where we run around the edge of the pool will be optimal if and only if $r > \sqrt{2}$.*

(2) *The all-swimming path where we swim directly from A to B will be optimal if and only if $r < \sqrt{2}$ and $b < \dfrac{1}{\sqrt{r^2 - 1}}$.*

(3) *The swim-run path where we swim from A to the point on the adjacent edge that is $\dfrac{1}{\sqrt{r^2 - 1}}$ units from the corner between the sides and then run the rest of the way to B will be optimal if and only if $r < \sqrt{2}$ and $b > \dfrac{1}{\sqrt{r^2 - 1}}$.*

PROOF. (1) ($\Leftarrow$) Part (1) of Theorem 2.1.2 tells us that if $r > \sqrt{2}$, the all-running path is optimal to the all-swimming path, and part (2) of that theorem tells us that under the same condition, the all-running path is optimal to the swim-run path. Hence the all-run path is optimal overall if $r > \sqrt{2}$.

($\Rightarrow$) If $r < \sqrt{2}$, part (1) of Theorem 2.1.2 tells us that the all-swim path is optimal to the all-run path if $\dfrac{1 - r\sqrt{2 - r^2}}{r^2 - 1} < b < \dfrac{1 + r\sqrt{2 - r^2}}{r^2 - 1}$. But

Lemma 2.1.3 tells us that

$$\frac{1 - r\sqrt{2 - r^2}}{r^2 - 1} < 1 < \frac{1}{\sqrt{r^2 - 1}} < \frac{1 + r\sqrt{2 - r^2}}{r^2 - 1},$$

so $b \geq a = 1$ implies that we cannot have $b < \frac{1 - r\sqrt{2 - r^2}}{r^2 - 1}$. Also, if $b > \frac{1 + r\sqrt{2 - r^2}}{r^2 - 1}$, then $b > \frac{1}{\sqrt{r^2 - 1}}$ and the swim-run path is optimal to the all-run path via part (2) of Theorem 2.1.2. Thus either the all-swim path or the swim-run path is optimal to the all-run path if $r < \sqrt{2}$.

(2) Suppose $r < \sqrt{2}$ and $b < \frac{1}{\sqrt{r^2 - 1}}$. As already established in the proof of part (1), Lemma 2.1.3 tells us that we cannot have $b < \frac{1 - r\sqrt{2 - r^2}}{r^2 - 1}$. Also, this lemma indicates that if $b < \frac{1}{\sqrt{r^2 - 1}}$, then $b < \frac{1 + r\sqrt{2 - r^2}}{r^2 - 1}$ so part (1) of Theorem 2.1.2 indicates that the all-swim path is optimal to the all-run path. Furthermore, part (3) indicates that this route is optimal to the swim-run path as well, making it optimal overall. Similarly, if $r > \sqrt{2}$, part (1) indicates that all-running is optimal to the all-swim path, and if $b > \frac{1}{\sqrt{r^2 - 1}}$, part (3) implies the swim-run path is optimal to the all-swim path.

(3) Suppose $r < \sqrt{2}$ and $b > \frac{1}{\sqrt{r^2 - 1}}$. Then part (2) of Theorem 2.1.2 implies that the swim-run path is preferable to the all-run path, and part (3) tells us that the swim-run path is faster than the all-swim path. Similarly, if $r > \sqrt{2}$ or $b < \frac{1}{\sqrt{r^2 - 1}}$, part (2) of the theorem indicates that the all-run path is optimal to the swim-run path so it is not optimal.

Thus if $r > \sqrt{2}$, the all-run path is optimal. If $r < \sqrt{2}$, we consider the value of b. If $b > \dfrac{1}{\sqrt{r^2 - 1}}$, the swim-run path is optimal, and if $b < \dfrac{1}{\sqrt{r^2 - 1}}$, the all-swim path is optimal. □

2.2. The 3-Sided Case

As in the previous case we will consider traveling between points on the edge of a rectangular pool. However, this time, suppose the points are on opposite edges of the pool instead of adjacent edges. As already mentioned, if $s > r$, this is trivial because the fastest time will come from traveling the shortest possible distance at the fastest possible time, which can be done by swimming directly between the points. So we will focus on the case where $r > s$. Also, as in the previous case, the optimal path will depend on the run speed to swim speed ratio rather than the actual speeds, so we can simplify this ratio by letting $s = 1$.

We will need to consider paths that travel directly across the pool as well as paths that touch a third side. Suppose we are traveling from a point A that is a units from the third side to a point C that is c units from the third side. Traveling from A to C will be the same as traveling from C to A, so without loss of generality, suppose $c > a$. (If not, we can simply relabel the points.) Also suppose that the length of the third side is b. In general, we can position the pool on the xy-plane so that the bottom corner of the pool below A is at the origin. Then A will be located at $(0, a)$ and C will be located at (b, c) as shown in Figure 2.2.1.

First we consider paths that do not touch a third side.

Theorem 2.2.1. *Suppose we are traveling from a point A on the edge of a rectangular pool to a point C on the opposite edge of the pool with running speed r*

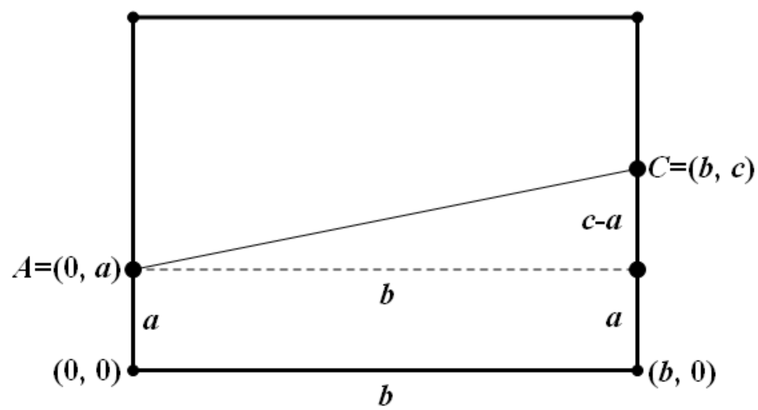

FIGURE 2.2.1. General 3-Sided Rectangular Pool

and swimming speed $s = 1$ *with* $r > 1$. *If we do not consider paths that involve touching a third side of the pool, then the optimal path will be to swim to a point on the opposite side that has a vertical distance of* $\frac{b}{\sqrt{r^2 - 1}}$ *units from* A *and then run to* C, *assuming this point falls below* C. *Otherwise, the fastest path will be to swim directly from* A *to* C.

PROOF. First we note that if we were to run from A to some point on the same side before swimming to a point on the other side and running the rest of the way to C, it would have the same time as a route with a parallel swim path beginning at A. So we only need to consider swim-run paths. To make things simpler, we will shift the problem down so that A is at the origin as shown in Figure 2.2.2.

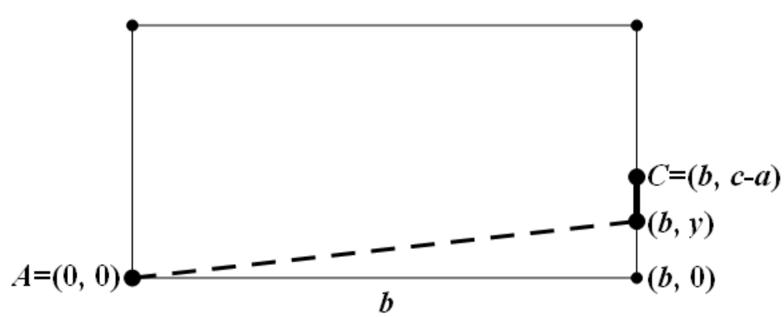

FIGURE 2.2.2. Rectangular Pool Between Opposite Sides

Now, if the path does not touch the third side of the pool, we must swim from A to some point on the opposite side of the pool. We would never swim to a point below $(b,0)$ since that would increase the swim distance and the total distance so it would not be optimal via Lemma 1.2.1. Similarly, we would never swim to a point above C as this would result in the same problem. So we must swim to a point between $(b,0)$ and C. Call this point (b,y). Then the time that it takes to travel from A to C by way of (b,y) will be given by the function

$$T(y) = \frac{c-a-y}{r} + \sqrt{b^2+y^2}$$

for $y \in [0, c-a]$.

From both the figure and the time function, it is clear that this is just like the 2-sided case and Equation 2.1.2, except with $a = b$ and $b = c-a$. Also, we do not know that $c - a \geq b$. Then if we likewise replace a with b and b with $c-a$ in Equation 2.1.3, we get that the critical point will occur at $y = \dfrac{b}{\sqrt{r^2-1}}$. Furthermore, Equation 2.1.4 tells us that $T''(x) = \dfrac{b^2}{(b^2+y^2)^{\frac{3}{2}}}$, which is always positive so this point must be a local minimum. Thus if the point falls in the domain of y, it will provide the minimum time.

Also, $\dfrac{b}{\sqrt{r^2-1}}$ is positive so the point will never fall below the domain (below 0), and thus if it is not in the domain, it must fall above $c-a$. Since $y = \dfrac{b}{\sqrt{r^2-1}}$ always gives a local minimum, T must be decreasing to the left of $\dfrac{b}{\sqrt{r^2-1}}$ meaning that the absolute minimum on the domain would be at the upper endpoint where $y = c-a$, which would correspond to the all-swim path directly from A to C.

Finally, the equivalent of Equation 2.1.5 tells us that the time it would take to travel this optimal swim-run path is

$$T\left(\frac{b}{\sqrt{r^2-1}}\right) = \frac{b\sqrt{r^2-1}+(c-a)}{r}. \tag{2.2.1}$$

This gives the optimal path if it does not involve touching the bottom edge. If we shift this back up to its original location where the bottom left corner is at the origin instead of A, then the critical point would be located at $y = \dfrac{b}{\sqrt{r^2-1}} + a$, and the time function would be given by

$$T(y) = \frac{c-y}{r} + \sqrt{b^2+(y-a)^2}.$$

The time that it takes to travel this shifted path would be the same as traveling the path before it was shifted. □

Now we will consider routes that do include travel to a third side. Without loss of generality, suppose it is the bottom. If the top provides the optimal path, we can simply reflect the rectangle across the line of symmetry so that it is on the bottom. We still need to determine the appropriate side of the rectangle to use, but we will address this at a later time. We know that the optimal path will involve some combination of running and/or swimming. First we will identify these possible paths.

Theorem 2.2.2. *Suppose we are traveling from a point* $A = (0, a)$ *on the edge of a rectangular pool with the bottom left corner at the origin to a point* $C = (b, c)$ *on the opposite edge of the pool, with swimming speed* $s = 1$ *and running speed* $r > 1$*. If* *we only consider paths that involve touching a third side of the pool, then the optimal path will either be to run the entire way around the edge of the pool from* A *to* C*, or to swim to a point on the bottom edge that is* $\frac{a}{\sqrt{r^2-1}}$ *from the edge containing* A*, run to a point that is* $\frac{c}{\sqrt{r^2-1}}$ *from the edge containing* C*, and then swim to* C*.*

Proof. We begin by noting that any path that arrives at a point B on the bottom edge by swimming and leaves B by swimming as well would not be optimal. If we swam to B from some point A' on the edge containing A to B and then from B to some point C' on the side containing C, it would be faster to just swim directly from A' to C' rather than going by way of B since the path would be traveled at only one speed and thus faster if traveled by way of the shortest distance.

We also note that if the optimal path involves touching some point B on the bottom edge of the pool, the all running path will be optimal if and only if $r > \sqrt{2}$. The optimal path from A to C can only be optimal if for any two points along the path, the overall route contains the optimal path between the points. So if B is on the optimal path from A to C, it must include the optimal way to get from A to B. Since these points are on adjacent sides of the rectangle, we can use the results of Theorem 2.1.4 to evaluate whether or not this portion of the path is optimal, and Theorem 2.1.4 says that the all-running path is optimal if and only if $r > \sqrt{2}$. Similarly, the all-running path from B to C would only be optimal under the same conditions. Thus the all-running path from A to C would be optimal if and only if $r > \sqrt{2}$.

Now, suppose $r < \sqrt{2}$, and the optimal path touches a point B on the bottom. We would never run to some point A' then swim to a point on the adjacent edge. If the point was before B, Theorem 2.1.1 tells us that we would never take a run-swim-run path that cuts off a corner unless it is the all-swim path or swim-run path. So the path from A to B through A' could only be optimal if the point that we swim to on the adjacent edge is B. But then we must be able to continue optimally from B to C, which must be done by either swimming or running and swimming. If we leave B by swimming that would not be optimal because we arrived at B by swimming, so we must run to some point B' and then swim to C. But then the path from A to B' must be optimal, which it cannot be since it involves a run-swim-run path between points on adjacent sides. A similar analysis would indicate that we will never swim to a point C' on the side containing C and then run the rest of the way.

Thus if we are traveling through any point B on the bottom edge, we must leave A by swimming and arrive at C by swimming. The proof of Theorem 2.1.1 indicates that a path involving swimming consists of either swimming directly between the points or swimming to the point at $\frac{a}{\sqrt{r^2-1}}$ units from the corner between the sides and then running. Suppose we swim to some point other than $\left(\frac{a}{\sqrt{r^2-1}}, 0\right)$. If we leave this point by swimming, it will not be optimal, so suppose we run to some other point B and then swim to C. But since we did not swim to $\frac{a}{\sqrt{r^2-1}}$ (or directly to B), this cannot be the optimal path from A to B, and thus the total path is not optimal.

Suppose we swim from A to the point $\left(\frac{a}{\sqrt{r^2-1}}, 0\right)$. We cannot leave this point by swimming or it will not be optimal, so we must run to some other point B and then swim to C. But then the path from $\left(\frac{a}{\sqrt{r^2-1}}, 0\right)$ to C must be optimal, and since this must be a run-swim path, the only way it can be optimal is if B is $\frac{c}{\sqrt{r^2-1}}$ units from the edge containing C.

Thus we know that if $r < \sqrt{2}$ and the optimal path touches a point on the bottom edge of the rectangle, then the optimal route will be a swim-run-swim path. $\square$

This result leads directly to some conditions indicating when the all-running path and swim-run-swim paths are optimal.

Corollary 2.2.3. *The optimal path from A to C can only be the all-running path if $r \geq \sqrt{2}$, and can only be the swim-run-swim path if $r < \sqrt{2}$ and $b > \frac{a+c}{\sqrt{r^2-1}}$. Then if $r < \sqrt{2}$ and $b \leq \frac{a+c}{\sqrt{r^2-1}}$, the optimal path will not touch the bottom edge.*

Proof. In the proof Theorem 2.2.2, we determined that if a path touches any point B on the bottom edge of the pool, it must be optimal from A to B and from B to C. The only way for the all-running path to be optimal is if $r > \sqrt{2}$. Otherwise a swim-run path or all-swimming path from A to B would be optimal (and if $r = \sqrt{2}$, the all-running path will have the same time as the swim-run path). Note that $r \geq \sqrt{2}$ does not necessarily imply that the all-running path is optimal, just that it could be.

Theorem 2.2.2 also states that if the optimal path touches the bottom edge and is not the all-running route, we will swim to a point that is $\frac{a}{\sqrt{r^2-1}}$ units from the side containing A, then run to a point that is $\frac{c}{\sqrt{r^2-1}}$ units from the edge containing C. Therefore, this path can only be taken if the points $\left(\frac{a}{\sqrt{r^2-1}}, 0\right)$ and $\left(b - \frac{c}{\sqrt{r^2-1}}, 0\right)$ exist on the third side, $\left(b - \frac{c}{\sqrt{r^2-1}}, 0\right)$ is to the right of

$\left(\frac{a}{\sqrt{r^2-1}}, 0\right)$, and $\frac{a}{\sqrt{r^2-1}} \neq b - \frac{c}{\sqrt{r^2-1}}$. Thus we must have that

$$b > \frac{a}{\sqrt{r^2-1}} + \frac{c}{\sqrt{r^2-1}} = \frac{a+c}{\sqrt{r^2-1}}.$$

Once again, these conditions are necessary to make a swim-run-swim path optimal, but not sufficient.

Now, suppose $r < \sqrt{2}$ and $b \leq \frac{a+c}{\sqrt{r^2-1}}$. Conditions are not met for the all-run path or the swim-run-swim path to be optimal. But these are the only paths that can touch the bottom edge. Thus the optimal path must be one that does not go through any point on the bottom edge. □

Now that we have identified all of the possible paths, we can take a moment to identify the times for each path. We have already found the time for the paths that do not touch the bottom edge, and the time for the all-run path is obviously $\frac{a+b+c}{r}$, so let us focus on the swim-run-swim path. Figure 2.2.3 depicts this route.

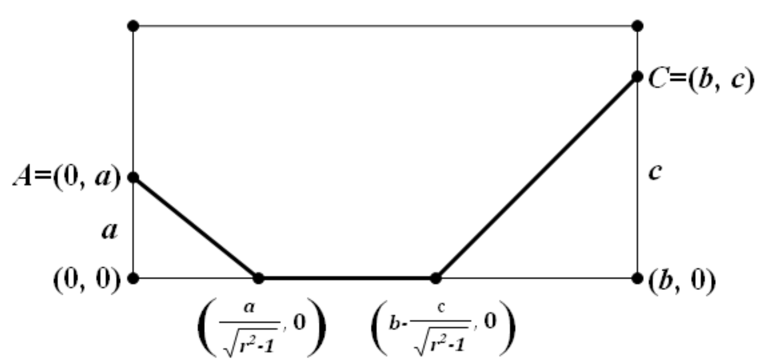

FIGURE 2.2.3. 3-Sided Rectangle Swim-Run-Swim Path

We will be able to simplify some parts of this formula using calculations from previous theorems and formulas. The time it will take to travel this path is

$$
\begin{aligned}
T &= \sqrt{a^2 + \left(\frac{a}{\sqrt{r^2-1}}\right)^2} + \sqrt{c^2 + \left(\frac{c}{\sqrt{r^2-1}}\right)^2} + \frac{b - \frac{c}{\sqrt{r^2-1}} - \frac{a}{\sqrt{r^2-1}}}{r} \\
&= \frac{ar}{\sqrt{r^2-1}} + \frac{cr}{\sqrt{r^2-1}} + \left(\frac{b}{r} - \frac{a+c}{r\sqrt{r^2-1}}\right) \\
&= \frac{r^2(a+c) + b\sqrt{r^2-1} - (c+a)}{r\sqrt{r^2-1}} \\
&= \frac{(r^2-1)(a+c) + b\sqrt{r^2-1}}{r\sqrt{r^2-1}} \\
&= \frac{(r^2-1)(a+c)\sqrt{r^2-1} + b(r^2-1)}{r(r^2-1)} \\
&= \frac{b + (a+c)\sqrt{r^2-1}}{r}. && (2.2.2)
\end{aligned}
$$

Before getting into the details needed to determine conditions for a minimum path for the general 3-sided case, we take a moment to address the simple case where $a = c$.

Theorem 2.2.4. *Suppose we are traveling from a point $A = (0, a)$ on the edge of a rectangular pool (with the bottom left corner at the origin) to a point $C = (b, a)$ on the opposite edge of the pool with swimming speed $s = 1$ and running speed $r > 1$ as depicted in Figure 2.2.4. The following are true:*

(1) *The all-running path is optimal if and only if $r > \sqrt{2}$ and $b > \dfrac{2a}{r-1}$.*

(2) *The swim-run-swim path is optimal if and only if $r < \sqrt{2}$ and*
$$b > \frac{2a\sqrt{r^2-1}}{r-1} = 2a\sqrt{1 + \frac{2}{r-1}}.$$

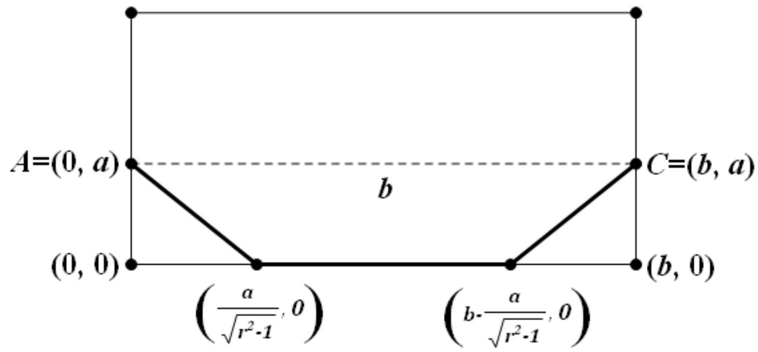

FIGURE 2.2.4. 3-Sided Rectangle with Equal Sides

(3) *The all-swimming path is optimal if and only if* $r > \sqrt{2}$ *and* $b < \dfrac{2a}{r-1}$, *or* $r < \sqrt{2}$ *and* $b < \dfrac{2a\sqrt{r^2-1}}{r-1} = 2a\sqrt{1+\dfrac{2}{r-1}}$.

PROOF. First we will note that the fastest path that does not touch the bottom edge will be the direct path from A to C as the domain of y in the time function for the swim-run path would be $[0, c-a] = [0, 0]$. Thus the travel time for this path will simply be b. Also, since $c = a$, the travel time for the all-run path will be $\dfrac{2a+b}{r}$, and using Equation 2.2.2, we get that the swim-run-swim time will be $\dfrac{b+2a\sqrt{r^2-1}}{r}$.

(1) Corollary 2.2.3 tells us that a necessary condition for the all-run path to be optimal is that $r > \sqrt{2}$ because otherwise the swim-run-swim path will be faster. Then the all-running path will be preferable to the all-swimming path if and only if the running time is less than the swimming time, or

$$\frac{2a+b}{r} < b$$

$$2a + b < br$$

$$2a < b(r-1)$$

$$b > \frac{2a}{r-1}. \tag{2.2.3}$$

Thus the all-run time is optimal overall if and only if both of these conditions hold.

(2) Corollary 2.2.3 states that for the swim-run-swim path to be optimal, we must have $r < \sqrt{2}$ or else the all-running path will be faster. Then we just need to compare this path with the all-swimming path. The swim-run-swim path will be optimal if and only if

$$\begin{aligned}\frac{b + 2a\sqrt{r^2 - 1}}{r} &< b \\ 2a\sqrt{r^2 - 1} &< b(r - 1) \\ b &> \frac{2a\sqrt{r^2 - 1}}{r - 1}.\end{aligned} \tag{2.2.4}$$

We will note that this result backs up what we already found in Corollary 2.2.3 since $r > 1$ implies that $r^2 > r$ so $\frac{2a\sqrt{r^2 - 1}}{r - 1} > \frac{2a\sqrt{r^2 - 1}}{r^2 - 1}$. Then if $r < \sqrt{2}$ and $b > \frac{2a\sqrt{r^2 - 1}}{r - 1} > \frac{2a\sqrt{r^2 - 1}}{r^2 - 1}$, the necessary condition from the corollary is met.

(3) Corollary 2.2.3 tells us that if $r > \sqrt{2}$, the swim-run path cannot be optimal and thus we only need to compare the all-swimming time to the all-running time. Furthermore, if we reverse the inequality in Equation 2.2.3, we find that the all-swimming time will be faster if and only if $b < \frac{2a}{r - 1}$. Similarly, if $r < \sqrt{2}$, we only need to compare this path to the swim-run-swim path, so reversing the inequality in Equation 2.2.4 implies that the all swimming path will be optimal if and only if $b < \frac{2a\sqrt{r^2 - 1}}{r - 1}$. Thus we must have either $r > \sqrt{2}$ and $b < \frac{2a}{r - 1}$, or $r < \sqrt{2}$ and $b < \frac{2a\sqrt{r^2 - 1}}{r - 1} = 2a\sqrt{1 + \frac{2}{r - 1}}$. □

It is interesting to note that in the comparison between the all-running path and the all-swimming path, we found that the all-running path will be optimal if $b > \frac{2a}{r-1}$. However, this was only the case for $r > \sqrt{2}$, so $\frac{2a}{r-1} < \frac{2a}{\sqrt{2}-1}$. Thus if $b > \frac{2a}{\sqrt{2}-1} = 2a(\sqrt{2}+1)$, then the all-running path will be faster than the all-swimming path regardless of the running rate. Furthermore, in the comparison between the all-swimming path and the swim-run-swim path, the all-swimming path will be optimal if $b < \frac{2a\sqrt{r^2-1}}{r-1}$. But this was only true for $1 < r < \sqrt{2}$, so

$$\frac{2a\sqrt{r^2-1}}{r-1} = \frac{2a(r^2-1)}{(r-1)\sqrt{r^2-1}} = \frac{2a(r+1)(r-1)}{(r-1)\sqrt{r^2-1}} = \frac{2a(r+1)}{\sqrt{r^2-1}} > 2a(r+1) > 4a.$$

Thus if $b < 4a$, the all-swimming path will be optimal to the swim-run-swim path regardless of the running rate.

Since we have addressed the case where $a = c$, we will be assuming that $c > a$ in all future scenarios. To simplify the problem by removing one of the variables from our calculations, we will scale the pool so that when it is placed on the xy-plane, the vertical distance from A to C (specifically $c - a$) is equal to one, or so that $c = a + 1$.

In the proof of Theorem 2.2.2, we determined that if the optimal path involves travel through a point B on the bottom edge of the pool, it will be the all-running path if $r > \sqrt{2}$, and the swim-run-swim path if $r < \sqrt{2}$. So in determining the optimal overall path, we can consider these two separate cases, and then only must compare the appropriate path with the ones that do not involve touching a point on the bottom edge as found in Theorem 2.2.1.

Case 1: $r > \sqrt{2}$

First we consider the case where $r > \sqrt{2}$ so that if the optimal path involves touching a point on the bottom edge, it must be the all-running path.

THEOREM 2.2.5. *Suppose we are traveling from a point $A = (0, a)$ on the edge of a rectangular pool (with the bottom left corner at the origin) to a point $C = (b, a+1)$ on the opposite edge of the pool with swimming speed $s = 1$ and running speed $r > \sqrt{2}$. The all-running path in which we run around the edge of the pool from A to C will be optimal to the all-swim path in which we swim directly from A to C if and only if $r > \sqrt{1 + (2a+1)^2}$ or $b < \dfrac{2a+1-r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$ or $b > \dfrac{2a+1+r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$.*

PROOF. The all-run path will have a travel time equal to $\dfrac{2a+1+b}{r}$, and the all-swim path will have a travel time equal to $\sqrt{1+b^2}$. Then the all-run path will be optimal to the all-swim path if and only if

$$\frac{2a+1+b}{r} < \sqrt{1+b^2}$$

$$\frac{(2a+1)^2 + 2b(2a+1) + b^2}{r^2} < 1 + b^2$$

$$(2a+1)^2 + 2b(2a+1) + b^2 < r^2 + b^2 r^2$$

$$0 < (r^2-1)b^2 - 2b(2a+1) + (r^2 - (2a+1)^2). \qquad (2.2.5)$$

The right side of this inequality is a parabola in terms of b which opens upward since $r > 1$ implies $r^2 - 1 > 0$. Then it will cross the x-axis whenever

$$\begin{aligned} b &= \frac{2(2a+1) \pm \sqrt{4(2a+1)^2 - 4(r^2-1)(r^2-(2a+1)^2)}}{2(r^2-1)} \\ &= \frac{2(2a+1) \pm 2\sqrt{(2a+1)^2 - r^4 + r^2(2a+1)^2 + r^2 - (2a+1)^2}}{2(r^2-1)} \\ &= \frac{(2a+1) \pm r\sqrt{1+(2a+1)^2 - r^2}}{r^2-1}. \end{aligned}$$

Now, if the discriminant is negative, there will not be any x-intercepts and the parabola is above the x-axis over all real numbers, making the inequality in Equation 2.2.5 always true and the all-run path consequently faster. This will happen if and only if

$$\begin{aligned} 0 &> 1 + (2a+1)^2 - r^2 \\ r^2 &> 1 + (2a+1)^2 \\ r &> \sqrt{1+(2a+1)^2}. \end{aligned}$$

Otherwise we will still have the all-run path optimal if and only if
$b < \dfrac{(2a+1) - r\sqrt{1+(2a+1)^2 - r^2}}{r^2-1}$ or $b > \dfrac{(2a+1) + r\sqrt{1+(2a+1)^2 - r^2}}{r^2-1}$. □

Theorem 2.2.6. *Suppose we are traveling from a point $A = (0, a)$ on the edge of a rectangular pool (with the bottom left corner at the origin) to a point $C = (b, a+1)$ on the opposite edge of the pool with swimming speed $s = 1$ and running speed $r > \sqrt{2}$. The all-running path, in which we run around the edge of the pool from*

A to C will be optimal to the swim-run path in which we swim from A to the point $(b, \frac{b}{\sqrt{r^2-1}}+a)$ *and then run the rest of the way to C if and only if* $b > \frac{2a}{\sqrt{r^2-1}-1}$.

PROOF. The all-run path will have a travel time equal to $\frac{2a+1+b}{r}$, and Equation 2.2.1 indicates that with $c = a+1$, the travel time for the swim-run path will be $\frac{b\sqrt{r^2-1}+1}{r}$. Then the all-run path will be faster than the swim-run path if and only if

$$\frac{2a+1+b}{r} < \frac{b\sqrt{r^2-1}+1}{r}$$

$$2a+1+b < b\sqrt{r^2-1}+1$$

$$2a < b(\sqrt{r^2-1}-1)$$

$$b > \frac{2a}{\sqrt{r^2-1}-1}.$$

□

Now that we have the pair-wise comparisons between the possible paths, we can combine Theorem 2.2.5 and Theorem 2.2.6 with Theorem 2.2.1 to get the overall results for this case.

THEOREM 2.2.7. *Suppose we are traveling from a point* $A = (0, a)$ *on the edge of a rectangular pool (with the bottom left corner at the origin) to a point* $C = (b, a+1)$ *on the opposite edge of the pool with swimming speed* $s = 1$ *and running speed* $r > \sqrt{2}$*. The path in which we run around the edge of the pool from A to C will be referred to as the all-run path, the option in which we swim directly from A to C will be called the all-swim path, and the route in which we swim from A to the point* $\left(b, \frac{b}{\sqrt{r^2-1}}+a\right)$ *and then run the rest of the way to C will be referred to as the swim-run path. The following are true:*

(1) *The all-run path is optimal if and only if* $b > \frac{2a}{\sqrt{r^2-1}-1}$ *and either* $r > \sqrt{1+(2a+1)^2}$ *or* $b > \frac{2a+1+r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$.

(2) *The swim-run path is optimal if and only if* $b < \frac{2a}{\sqrt{r^2-1}-1}$ *and* $b < \sqrt{r^2-1}$.

(3) *The all-swim path is optimal if and only if* $r < \sqrt{1+(2a+1)^2}$ *and* $\sqrt{r^2-1} < b < \frac{2a+1+r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$.

PROOF. The three possible paths are depicted in Figure 2.2.5.

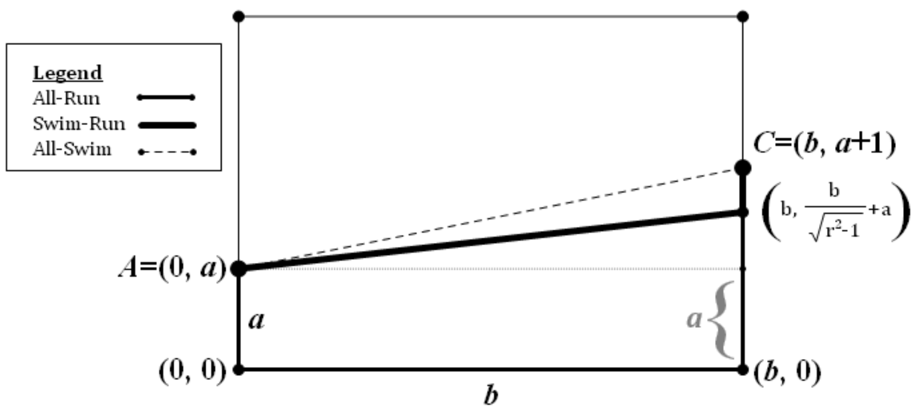

FIGURE 2.2.5. 3-Sided Rectangle Possible Paths for $r > \sqrt{2}$

(1) For the all-run path to be optimal, it must simultaneously be faster than both the swim-run path and the all-swim path. The first condition that we must have $b > \frac{2a}{\sqrt{r^2-1}-1}$ comes directly from Theorem 2.2.6, and will determine whether or not the route in question is faster than the swim-run path.

For the all-run path to be faster than the all-swim path, we have several options provided by Theorem 2.2.5. We can either have $r > \sqrt{1+(2a+1)^2}$ or $b < \frac{(2a+1)-r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$ or $b > \frac{(2a+1)+r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$.
However, having $b < \frac{(2a+1)-r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$ is only necessary if

$r < \sqrt{1+(2a+1)^2}$. We will show that if the first condition is satisfied to make the all-run path optimal to the swim-run path, then we will never have $b < \dfrac{(2a+1) - r\sqrt{1+(2a+1)^2 - r^2}}{r^2-1}$.

Suppose $r < \sqrt{1+(2a+1)^2}$. This implies that $r^2 - (2a+1)^2 < 1$, which leads to the following:

$$\begin{aligned}
\frac{(2a+1) - r\sqrt{1+(2a+1)^2 - r^2}}{r^2-1} &= \frac{(2a+1)^2 - r^2(1+(2a+1)^2 - r^2)}{(r^2-1)(2a+1+r\sqrt{1+(2a+1)^2-r^2})} \\
&= \frac{(2a+1)^2 - r^2 - r^2(2a+1)^2 + r^4}{(r^2-1)(2a+1+r\sqrt{1+(2a+1)^2-r^2})} \\
&= \frac{-(2a+1)^2(r^2-1) + r^2(r^2-1)}{(r^2-1)(2a+1+r\sqrt{1+(2a+1)^2-r^2})} \\
&= \frac{r^2 - (2a+1)^2}{2a+1+r\sqrt{1+(2a+1)^2-r^2}} \\
&< \frac{1}{(2a+1+r\sqrt{1+(2a+1)^2-r^2})} \\
&< 1. \qquad (2.2.6)
\end{aligned}$$

The last line is due to the fact that $a \geq 0$ and $r\sqrt{1+(2a+1)^2-r^2} > 0$ so $2a+1+r\sqrt{1+(2a+1)^2-r^2} > 1$.

Now suppose $b < 1$. Then $2ab < 2a$ and $2ab + b < 2a + b$. But $r < \sqrt{1+(2a+1)^2}$ implies that $r^2 < 1 + (2a+1)^2$ so $\sqrt{r^2-1} < 2a+1$ so

$$b\sqrt{r^2-1} < b(2a+1) = 2ab + b < 2a + b.$$

It follows that

$$b\sqrt{r^2-1} - b < 2a$$

$$b(\sqrt{r^2-1} - 1) < 2a$$

$$b < \frac{2a}{\sqrt{r^2-1}-1}.$$

So if $r < \sqrt{1+(2a+1)^2}$ and $b < \dfrac{(2a+1) - r\sqrt{1+(2a+1)^2 - r^2}}{r^2-1}$, then $b < 1$ by Equation 2.2.6. But if $b < 1$, then $b < \dfrac{2a}{\sqrt{r^2-1}-1}$ so swim-run is optimal to all-run by Theorem 2.2.6 and we do not need to consider this option.

Thus the all-run path is optimal if and only if $b > \dfrac{2a}{\sqrt{r^2-1}-1}$ and either $r > \sqrt{1+(2a+1)^2}$ or $b > \dfrac{2a+1+r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$.

(2) The swim-run path will be optimal if and only if it is faster than the all-run path and the all-swim path simultaneously. Theorem 2.2.1 tells us that the swim-run path will be optimal to the all-swim path if and only if $\dfrac{b}{\sqrt{r^2-1}} + a < c$. With $c = a+1$, this translates to $\dfrac{b}{\sqrt{r^2-1}} < 1$. Thus the swim-run path will be faster than the all-swim path if and only if $b < \sqrt{r^2-1}$. Also, Theorem 2.2.6 states that this path will be faster than the all-run path if and only if $b < \dfrac{2a}{\sqrt{r^2-1}-1}$. So it will be optimal overall if and only if both of these conditions hold.

(3) The all-swim path will be optimal if and only if it is faster than both the all-run path and the swim-run path. From Theorem 2.2.5, this path will be faster than the all-run path if and only if $r < \sqrt{1+(2a+1)^2}$ and $\dfrac{(2a+1) - r\sqrt{1+(2a+1)^2-r^2}}{r^2-1} < b < \dfrac{(2a+1) + r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$.

Furthermore, from Theorem 2.2.1, it will be faster than the swim-run path if and only if $\dfrac{b}{\sqrt{r^2-1}} + a > c = a+1$, or $b > \sqrt{r^2-1}$. However, $r > \sqrt{2}$ implies that $\sqrt{r^2-1} > 1$ so if $b > \sqrt{r^2-1}$, then $b > \dfrac{(2a+1) - r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$ based on Equation 2.2.6. So we only need the requirement that $\sqrt{r^2-1} < b < \dfrac{(2a+1) + r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$ along with $r < \sqrt{1+(2a+1)^2}$. □

We should note that in the condition for the all-swim path to be optimal, it is not necessarily true that $\sqrt{r^2-1} < \dfrac{(2a+1) + r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$. However, if this is the case, it simply means that the necessary inequality is impossible to satisfy and the all-swim path will never be optimal. For example, consider $r = 3$ and $a = 2$. Then $\sqrt{r^2-1} \approx 2.82843$ and $\dfrac{(2a+1) + r\sqrt{1+(2a+1)^2-r^2}}{r^2-1} \approx 2.17116$. Since it is impossible to have $2.82843 < b < 2.17116$, the all-swimming path can never be optimal.

Using these results, we can get a generalization about the optimal paths as b gets smaller or larger.

Corollary 2.2.8. *Under the hypotheses of Theorem 2.2.7, for a fixed run rate $r > \sqrt{2}$ and distance $a > 0$, the all-run path will always be optimal for b sufficiently large.*

Proof. Suppose r and a are fixed. Then $\dfrac{2a}{\sqrt{r^2-1}-1}$ and $\dfrac{(2a+1) + r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$ will be constants, so we can find some number $M \geq \max\left\{\dfrac{2a}{\sqrt{r^2-1}-1}, \dfrac{(2a+1) + r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}\right\}$ such that the all-run path will be optimal whenever $b > M$ according to Theorem 2.2.7. □

Unfortunately, it is not possible to simplify these results any further as we cannot find a definite order to the other boundary numbers for b: $\sqrt{r^2-1}$, $\dfrac{2a}{\sqrt{r^2-1}-1}$, and $\dfrac{(2a+1)+r\sqrt{1+(2a+1)^2-r^2}}{r^2-1}$. For each pair, there are cases which will provide different orders.

Case 2: $r<\sqrt{2}$

In this case, if the optimal path passes through a point on the bottom edge of the rectangle, it will be by way of the swim-run-swim path as opposed to the all-run path by Corollary 2.2.3.

THEOREM 2.2.9. *Suppose we are traveling from a point $A=(0,a)$ on the edge of a rectangular pool (with the bottom left corner at the origin) to a point $C=(b,a+1)$ on the opposite edge of the pool with swimming speed $s=1$ and running speed $1<r<\sqrt{2}$. The all-swim path in which we swim directly from A to C will be optimal to the swim-run-swim path in which we swim to the point $\left(\dfrac{a}{\sqrt{r^2-1}},0\right)$, run to the point $\left(b-\dfrac{c}{\sqrt{r^2-1}},0\right)$, and then swim to C if and only if* $\dfrac{2a+1-2r\sqrt{a^2+a}}{\sqrt{r^2-1}}<b<\dfrac{2a+1+2r\sqrt{a^2+a}}{\sqrt{r^2-1}}$.

PROOF. The all-swim path will be faster than the swim-run-swim path if and only if the travel time for the path is less. Equation 2.2.2 tells us that the swim-run-swim path can be traversed at a time of $\dfrac{b+(a+c)\sqrt{r^2-1}}{r}$, so this will be the case if and only if

$$\sqrt{1+b^2}<\frac{b+(a+c)\sqrt{r^2-1}}{r}$$

$$r^2+r^2b^2<b^2+2b(2a+1)\sqrt{r^2-1}+(2a+1)^2(r^2-1)$$

$$b^2(r^2-1) - 2b(2a+1)\sqrt{r^2-1} - (2a+1)^2(r^2-1) + r^2 < 0.$$

We notice that this is a parabola with a variable of b, so the inequality will be true if b is between the x-intercepts of the parabola since the leading coefficient is positive and the parabola opens up. First we simplify the discriminant for the quadratic formula:

$$\begin{aligned}
&4(2a+1)^2(r^2-1) + 4(r^2-1)((2a+1)^2(r^2-1) - r^2) \\
&= 4(r^2-1)[(2a+1)^2 + (2a+1)^2(r^2-1) - r^2] \\
&= 4(r^2-1)[(2a+1)^2 + (2a+1)^2r^2 - (2a+1)^2 - r^2] \\
&= 4(r^2-1)[(2a+1)^2r^2 - r^2] \\
&= 4r^2(r^2-1)[4a^2+4a+1-1] \\
&= 16r^2(r^2-1)(a^2+a).
\end{aligned}$$

Now we apply the full quadratic formula to get that

$$\begin{aligned}
b &= \frac{2(2a+1)\sqrt{r^2-1} \pm \sqrt{16r^2(r^2-1)(a^2+a)}}{2(r^2-1)} \\
&= \frac{2(2a+1)\sqrt{r^2-1} \pm 4r\sqrt{r^2-1}\sqrt{a^2+a}}{2(r^2-1)} \\
&= \frac{2a+1 \pm 2r\sqrt{a^2+a}}{\sqrt{r^2-1}}.
\end{aligned}$$

Thus the all-swim path will be preferable if and only if

$$\frac{2a+1-2r\sqrt{a^2+a}}{\sqrt{r^2-1}} < b < \frac{2a+1+2r\sqrt{a^2+a}}{\sqrt{r^2-1}}. \qquad \square$$

It is interesting to note that for $c = a + 1$, these boundary points are equal to $\frac{a + c \pm 2r\sqrt{ac}}{\sqrt{r^2 - 1}}$. Furthermore, the lower endpoint will be negative if and only if $a + c < 2r\sqrt{ac}$ or $r > \frac{a + c}{2\sqrt{ac}}$. However, you might notice that this is the ratio of the arithmetic mean of the two numbers to their geometric mean. And the arithmetic mean of two positive numbers is greater than their geometric mean, proofs of which were collected by Muirhead in [**5**]. Hence the ratio is always greater than one, so for any given a and c, we can find some r such that $r < \frac{a + c}{2\sqrt{ac}}$, and the lower boundary point will be positive. Therefore, it is possible to have $b < \frac{2a + 1 - 2r\sqrt{a^2 + a}}{\sqrt{r^2 - 1}}$, making the swim-run-swim case optimal.

Now we will compare the swim-run case with the swim-run-swim case.

Theorem 2.2.10. *Suppose we are traveling from a point* $A = (0, a)$ *on the edge of a rectangular pool (with the bottom left corner at the origin) to a point* $C = (b, a + 1)$ *on the opposite edge of the pool with swimming speed* $s = 1$ *and running speed* $1 < r < \sqrt{2}$. *The swim-run path in which we swim from* A *the point* $\left(b, \frac{b}{\sqrt{r^2 - 1}} + a\right)$ *and then run the rest of the way to* C *will be optimal to the swim-run-swim path in which we swim to the point* $\left(\frac{a}{\sqrt{r^2 - 1}}, 0\right)$ *run to the point* $\left(b - \frac{c}{\sqrt{r^2 - 1}}, 0\right)$, *and then swim to* C *if and only if* $b < \frac{(2a + 1)\sqrt{r^2 - 1} - 1}{\sqrt{r^2 - 1} - 1}$.

Proof. The swim-run path will be faster than the swim-run-swim path if and only if the time for the swim-run path given in Equation 2.2.1 is less than the time for the swim-run-swim path given in Equation 2.2.2, or

$$\frac{1 + b\sqrt{r^2 - 1}}{r} < \frac{b + (2a + 1)\sqrt{r^2 - 1}}{r}$$

$$b(\sqrt{r^2-1}-1) < (2a+1)\sqrt{r^2-1}-1$$
$$b < \frac{(2a+1)\sqrt{r^2-1}-1}{\sqrt{r^2-1}-1}$$
$$b < \frac{1-(2a+1)\sqrt{r^2-1}}{1-\sqrt{r^2-1}}. \qquad \square$$

From this result, we can see that we have to have $b < 1$ if the the swim-run path is to be preferable to the swim-run-swim path. This is due to the fact that $a > 0$ (if $a = 0$, this would fall into the 2-sided case) and thus $2a + 1 > 1$ and $(2a+1)\sqrt{r^2-1} > \sqrt{r^2-1}$. If the numerator is negative then the swim-run case would never be optimal as this would require b to be negative, providing a contradiction, so the numerator must be positive and

$$b < \frac{1-(2a+1)\sqrt{r^2-1}}{1-\sqrt{r^2-1}} < \frac{1-\sqrt{r^2-1}}{1-\sqrt{r^2-1}} = 1.$$

These results easily fit together to summarize the optimal paths for this case, and follow directly from the given theorems.

All that is left for the 3-sided case is to determine which 3 sides we will use. In all of the calculations thus far, we assumed that paths touching a third side would use the bottom. However, this may not be the case. It may be that a path traveling to the top would be optimal. However, this problem would be exactly the same as the scenario already analyzed only with a 180° rotation. However, we do need to determine whether to perform this rotation or not. This will actually be a very simple matter.

As before, we allow the distance from A to the bottom to be a and the distance from C to the bottom to be c. Now, call the distance from A to the top a' and the distance from C to the top c'. Then we will perform the rotation if and only if $a' + c' < a + c$. Since the all-swim and swim-run times are based solely on the length of b and the difference between a and c, they would not be affected by a rotation. The only times that would change would be times for paths that touch a point on a third side, the all-run path or the swim-run-swim path. Then $a' + c' < a + c$ if and only if the run time around the top, $\frac{a' + b + c'}{r}$ is less than the run time around the bottom, $\frac{a + b + c}{r}$. Also, the swim-run-swim time using the top, $\frac{1}{r}\left(b + (a' + c')\sqrt{r^2 - 1}\right)$, is less than the swim-run-swim time using the bottom, $\frac{1}{r}\left(b + (a + c)\sqrt{r^2 - 1}\right)$ if and only if $a' + c' < a + c$.

CONCLUSION AND FUTURE PROJECTS

Many results were found, but there is also a lot of room for further investigation. We obtained nice results for the On to On case in each scenario, but these are only two of the many boundary shapes that could be considered. Furthermore, there were several combinations of point locations that were not yet explored.

The next project that we would like to accomplish is to expand the investigation of the circular Out to On case to a general Out to In case. In many parts, the fact that the ending point was on a circular boundary did not greatly affect the result, which instead depended on whether the function for this border was differentiable. We believe that this will extend to a case with a general boundary shape that is strictly increasing or strictly decreasing. Preliminary investigation suggests that we can find a point on the boundary that provides a locally minimum time by finding where $\frac{r}{s} = \frac{\sin\theta_1}{\sin\theta_2}$ where θ_1 and θ_2 are the angles of incidence and refraction respectively. This would obviously be an extension of Snell's Law from a straight boundary line to a curved one, where we would apply Snell's Law to the family of tangent lines.

We also began some study of the Out to In case with circles that pointed to some interesting conclusions in the future. We ran into some trouble when the derivative could not be solved to obtain an exact determination of the path that would involve running and swimming. However, using Monte Carlo graphs, we did find some interesting visualizations about the areas that would require us to swim the entire way.

Furthermore, we would like to extend these specific problems to related ones in other fields of study. One interesting option would be to take one of the speeds to be zero, giving obstacles instead of passable areas. In this case, visibility graphs are used to determine the optimal path, some of which was addressed in [**4**]. Once a weighted graph is created, there are algorithms to find the minimum path, such as Dijkstra's algorithm as addressed in [**10**].

We were able to obtain many fascinating conclusions, but much room for further exploration remains.

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