Simple Robust Controllers for Delayed Systems

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Abstract: A way for treating general delayed systems with uncertain delays in both the numerator and denominator is shown. The proposed procedure is demonstrated by an example. A simple controller is derived via algebraic theory and the structured singular value, which treats uncertain time delay in both the numerator and denominator of an anisochronic system. The overall performance is verified by simulations and compared with standard tool for robust control design.

Keywords: Robust control, linear fractional transformation, time delay, algebraic synthesis, structured singular value.

1. INTRODUCTION

The task of controller design for time delay systems is a frequent problem and many papers have been published on this topic during last decades. The control theory dealing with this type of plant is very complex and this paper is not summarizing the common knowledge in this field. The aim of this paper is an application of robust control design to general anisochronic plant with time delay in both the numerator and denominator, and the algebraic approach together with the structured singular value is used as a tool.

The algebraic theory [e.g. Kučera (1993), Prokop and Corriou (1997), Vidyasagar (1985)] is well known and its importance is growing due to the simplicity of controller derivation and the fact that some crucial properties of the resulting feedback loop can be easily influenced by the choice of the controller structure, which is not hard to do within the scope of this approach. The structured singular value denoted μ [see Doyle (1982, 1985)] provides a measure of robust stability and performance that can take into account many aspects of controller design including sensor noise, dynamic perturbations as well as parametric uncertainties in case they can be treated via linear fractional transformation (LFT). However, standard tools for μ synthesis are not able to design controller with a predefined structure. The algebraic approach provides methodology for synthesis of very simple controllers (PI, PID), yet with an excellent functionality compared with the *D*-*K* iteration, which is a reference method for this type of controller design. Due to the multimodality of the cost function an algorithm for global optimization is employed for tuning nominal closed-loop pole placement, where the peak of the μ function in frequency domain gives the measure of controller stability and performance. Knowing this, we can simulate behavior of the resulting feedback system for the worst-case perturbation causing the highest value of μ. But this is not the only issue that can be addressed by the structured singular value. In this case, it is also possible to design controllers that have some specific properties such as stability and performance for the whole range of time delays. This means that in real world resulting feedback loop characteristics will not degrade if time delay varies from 0 to a value defined as the worst possible case.

Many procedures has been developed for control of time delay systems including LFT approaches using multiplicative uncertainty or internal model control (IMC) dealing with design in the ring of meromorphic functions [e.g. Zítek and Hlava (2001) and Zítek and Kučera (2003)]. Methods handling time delay systems via multiplicative uncertainty are well known. However, techniques for systems with time delay in both the numerator and denominator use mainly IMC design, which deals robustness in a less easy way.

In this paper, a general scheme for treating anisochronic delayed systems via LFT will be shown alongside with an example of application to such a system with time delay in both the numerator and denominator. The controller design is performed using algebraic μ-synthesis [see Dlapa et al. (2009)] as well as a comparison study with a standard tool - *D*-*K* iteration. The overall performance is verified by simulation of step response for different values of time delays and for simple and two degrees of freedom feedback loops [1DOF, 2DOF, see Prokop and Corriou (1997)].

2. MODELING OF DELAYED SYSTEMS VIA LFT

Consider general delayed system with uncertain time delays in both the numerator and denominator:

$$P(s) \equiv \frac{(b_0 + b_1 s + \ldots + b_n s^n) e^{-\tau_b s}}{s^n + a_0 e^{-\tau_0 s} + a_1 s e^{-\tau_1 s} + \ldots + a_{n-1} s^{n-1} e^{-\tau_{n-1} s}}$$

$$\tau_b \in [0, T_{db}],\ \tau_i \in [0, T_{di}]\ i = 0, 1, \ldots, n-1 \qquad (1)$$

This family of plants has uncertain retarded quasi-polynomial in denominator. The highest *s*-power represents a delayless term and τ_b, τ_i are non-negative delay parameters. The delays vary in the intervals of zero to a predefined value representing the upper bound for each time delay.

This plant can be (with some conservatism) expressed via LFT in Fig. 1. Perturbations δ_{delb}, $\delta_{deli} \in \mathbf{C}$ satisfy conditions

$$|\delta_{delb}| < 1\,,\ |\delta_{deli}| < 1 \tag{2}$$

And for weights W_{delb} and W_{deli} the following inequalities must be held for all $\omega \in \mathbf{R}$:

$$|W_{delb}(j\omega)| > |1 - e^{j\omega T_{db}}| \tag{3}$$

$$|W_{deli}(j\omega)| > |1 - e^{j\omega T_{di}}|,\ i = 0,1,\ldots,n-1 \tag{4}$$

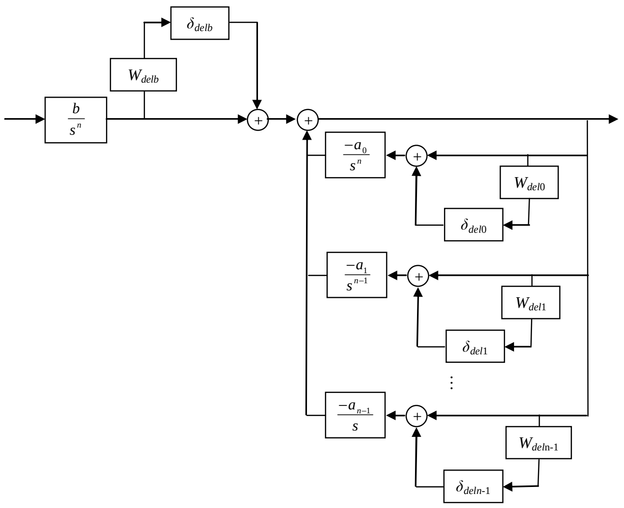

Fig. 1. LFT interconnection of general delayed system.

3. ALGEBRAIC μ-SYNTHESIS

The plant (1) can be treated by the interconnection in Fig. 2 with sensitivity function as performance indicator.

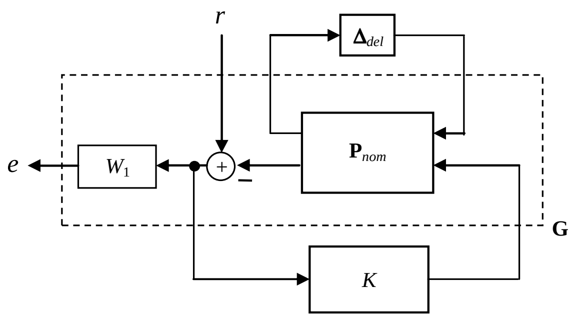

Fig. 2. Closed-loop interconnection.

Perturbation matrix has the form:

$$\Delta_{del} \equiv \begin{bmatrix} \delta_{delb} & 0 \\ 0 & \Delta_{dela} \end{bmatrix},\ \Delta_{dela} \equiv \begin{bmatrix} \delta_{del0} & 0 & \cdots & 0 \\ 0 & \delta_{del1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \delta_{deln-1} \end{bmatrix},$$

$$|\delta_{delb}| < 1\,,\ |\delta_{deli}| < 1\,,\ \delta_{delb},\ \delta_{deli} \in \mathbf{C},\ i = 0,\ldots,n-1 \tag{5}$$

The interconnection in Fig. 2 can be transformed to a simplified scheme in Fig. 3.

For stability and performance the following theorem holds:

Theorem 1 [Doyle (1982)]: For Δ_{del} defined by (5) is the loop in Fig. 2 well posed, internally stable and $\|\mathbf{F}_L[\mathbf{F}_U(\mathbf{G},\Delta_{del}),K]\|_\infty \le 1$ iff

$$\sup_{\substack{\omega \in R \\ K \text{ stabilizing } \mathbf{G}}} \mu_\Delta[\mathbf{F}_L(\mathbf{G},K)(j\omega)] \le 1 \tag{6}$$

with $\Delta \equiv \begin{bmatrix} \delta_P & 0 \\ 0 & \Delta_{del} \end{bmatrix}$, $|\delta_P| < 1$, $\delta_P \in \mathbf{C}$.

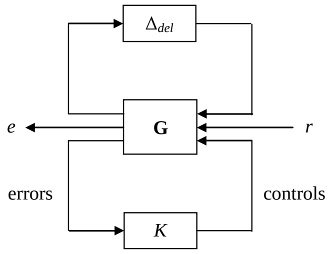

Fig. 3. Closed-loop interconnection.

Define sensitivity function as transfer function from reference *r* to error *e* in Fig. 4:

$$S \equiv \frac{1}{1+PK} \tag{7}$$

Now, as a consequence of Theorem 1, a sufficient condition for robust stability and performance of the feedback loop in Fig. 4 can be formed for sensitivity function *S* and the family of plants (1).

Corollary 1: For the set of plants (1) feedback loop in Fig. 4 is internally stable and $\|SW_1\|_\infty \le 1$ if conditions (3), (4) and (6) hold.

Proof: For Δ_{del} defined by (5) the numerator and denominator of $\mathbf{F}_U(\mathbf{P}_{nom}, \Delta_{del})$ is

$$b\,(1 + \delta_{delb} W_{delb}) \tag{8}$$

$$s^n + \sum_{i=0}^{n-1} a_i s^i (1 + \delta_{deli} W_{deli}) \tag{9}$$

Elements $(1+\delta_{delb}W_{delb})$ and $(1+\delta_{deli}W_{deli})$, $i = 1, \ldots, n-1$ fully cover frequency properties of time delays in the numerator and denominator of the set of plants (1) if conditions (3) and (4) hold. This is apparent, since for each element $e^{-\tau_b s}$, $a_i e^{-\tau_i s}$, $i = 0, \ldots, n-1$ and $s = j\tilde{\omega}$, $\tilde{\omega} \in \mathbf{R}$ $\tilde{\delta}_{delb}$ and $\tilde{\delta}_{deli}$ exist such that $e^{-\tau_b j\tilde{\omega}} = 1 + \tilde{\delta}_{delb} W_{delb}(j\tilde{\omega})$, $e^{-\tau_i j\tilde{\omega}} = 1 + \tilde{\delta}_{deli} W_{deli}(j\tilde{\omega})$, $|\tilde{\delta}_{delb}|$, $|\tilde{\delta}_{deli}| < 1$, $\tilde{\delta}_{delb}$, $\tilde{\delta}_{deli} \in \mathbf{C}$, $i = 0, \ldots, n-1$. The proof then follows from Theorem 1.

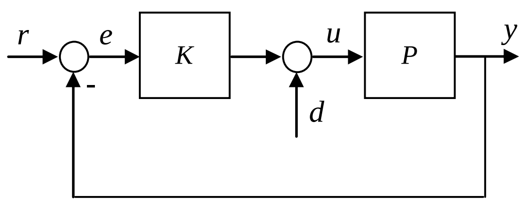

Fig. 4. Feedback loop.

The algebraic synthesis can be applied to the nominal plant

$$P_0(s) \equiv \frac{b_0 + b_1 s + \ldots + b_n s^n}{s^n + a_0 + a_1 s + \ldots + a_{n-1} s^{n-1}} \tag{10}$$

which can be transformed to:

$$P_0(s) = \frac{\dfrac{b_0 + b_1 s + \ldots + b_n s^n}{(\alpha_1 + s)(\alpha_2 + s)\cdot \ldots \cdot(\alpha_n + s)}}{\dfrac{s^n + a_0 + a_1 s + \ldots + a_{n-1} s^{n-1}}{(\alpha_1 + s)(\alpha_2 + s)\cdot \ldots \cdot(\alpha_n + s)}} = \frac{B}{A},\ A, B \in \mathbf{R}_{\mathrm{PS}} \tag{11}$$

Then the controller is obtained as a solution to the Diophantine equation:

$$AD_K + BN_K = 1, \qquad D_K, N_K \in \mathbf{R}_{\mathrm{PS}} \tag{12}$$

Equation (12) is often called the Bezout identity, and all feedback controllers N_K/D_K are given by

$$K = \frac{N_K}{D_K} = \frac{N_{K_0} - AT}{D_{K_0} + BT}, \quad N_{K_0}, D_{K_0} \in \mathbf{R}_{\mathrm{PS}} \tag{13}$$

where N_{K_0}, $D_{K_0} \in \mathbf{R}_{\mathrm{PS}}$ are particular solutions of (12) and T is an arbitrary element of $\mathbf{R}_{\mathrm{PS}}$.

The controller (13) derived as a solution to (12) safeguards that the nominal feedback loop in Fig. 5 is BIBO stable, which is important for appropriate theorems related to the structured singular value. If the nominal feedback system has a pole in the right half plane then these theorems cannot be used. However, this is not the case if the BIBO stability is held.

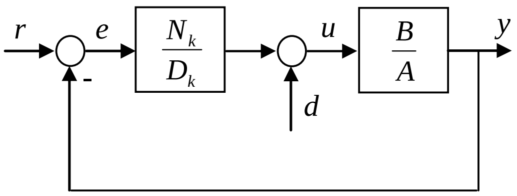

Fig. 5. Nominal feedback loop.

The aim of synthesis is to design a controller which satisfies condition:

$$\sup_{\substack{\omega \\ K \text{ stabilizing } \mathbf{G}}} \mu_{\mathbf{\Delta}}[\mathbf{F}_L(\mathbf{G}, K)(\omega, \alpha_1, \ldots, \alpha_{n+n_1+n_2}, t_1, \ldots, t_{n_2})] \le 1,\ \omega \in (-\infty, +\infty) \tag{14}$$

where $n + n_1 + n_2$ is the order of the nominal feedback system, n_1 is the order of particular solution K_0, t_i are arbitrary parameters in $T = \dfrac{t_0 + t_1 s + \ldots + t_{n_2} s^{n_2}}{(\alpha_{n_1+1} + s)\cdot \ldots \cdot(\alpha_{n_1+n_2} + s)}$ and $\mu_{\mathbf{\Delta}}$ denotes the structured singular value of LFT on generalized plant $\mathbf{G}$ and controller K with

$$\mathbf{\Delta} \equiv \begin{bmatrix} \mathbf{\Delta}_{del} & 0 \\ 0 & \delta_P \end{bmatrix},\ \delta_P < 1,\ \delta_P \in \mathbf{C} \tag{15}$$

where $\mathbf{\Delta}_{del}$ denotes the perturbation matrix (5) and δ_P is a complex number corresponding with the robust performance condition.

Tuning parameters are positive and constrained to real axis since parameters of transfer function have to be real and due to the fact that non-real poles cause oscillation of nominal feedback loop.

A crucial problem of the cost function in (14) is the fact that many local extremes are present. Hence, in most cases, local optimization does not yield a suitable or even stabilizing solution. This can be overcome via evolutionary optimization, which solves the task very efficiently.

4. EXAMPLE – UNSTABLE DELAYED SYSTEM

Consider the set of anisochronic systems with time delay in the numerator and denominator:

$$P(s) \equiv \frac{3e^{-\tau_1 s}}{5s - e^{-\tau_2 s}},\ \tau_1 \in [0, 4],\ \tau_2 \in [0, 0.8] \tag{16}$$

This set of plants is treated via LFT using the scheme in Fig. 6. Weights W_{del1} and W_{del2} can be obtained from the inequalities:

$$\left|W_{deli}\right| > \left|1 - e^{j\omega T_{di}}\right|,\ i = 1, 2;\ T_{d1} = 4, T_{d2} = 0.8 \tag{17}$$

It follows from Fig. 7 and 8 that

$$W_{del1} = \frac{2s}{2s+1} 2.5,\ W_{del2} = \frac{0.4s}{0.4s+1} 2.5 \tag{18}$$

satisfy (17) with very small conservatism.

Now, it is easy to create an open-loop interconnection with weighted sensitivity function as performance indicator. Recall closed-loop interconnection depicted in Fig. 2 with the open-loop in dashed rectangle denoted $\mathbf{G}$.

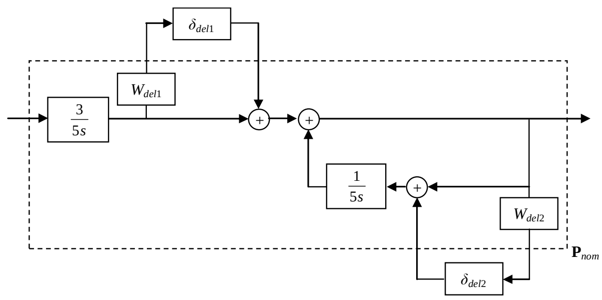

Fig. 6. LFT model of plant.

The perturbation matrix has the form:

$$\mathbf{\Delta}_{del} \equiv \begin{bmatrix} \delta_{del1} & 0 \\ 0 & \delta_{del2} \end{bmatrix},$$

$$\left|\delta_{del1}\right| < 1,\ \left|\delta_{del2}\right| < 1,\ \delta_{del1}, \delta_{del2} \in \mathbf{C} \tag{19}$$

and performance weight is a 3rd order transfer function:

$$W_1 = \frac{0.004}{10s^3 + 100s^2 + s + 1\cdot 10^{-5}} \tag{20}$$

The weight W_1 has a small factor for s^0 in denominator so that the DGKF formulae Doyle et al. (1989) can be used.

The plant for which the controller is derived is a nominal system:

$$P_0(s) \equiv \frac{3}{5s-1} \tag{21}$$

Plant P_0 is unstable due to positive feedback in the interconnection in Fig. 6. If the nominal plant is stable then negative feedback in Fig. 6 must be used and W_{del2} must be chosen so that $|W_{del2}| \geq 2$. The interconnection for stable plant has, however, a drawback in higher conservatism. Moreover, it can be proved that no stabilizing controller exists (for these particular weights) since robust stability conditions for transfer functions between the outputs and inputs from weights W_{del1}, W_{del2} and perturbations δ_{del1}, δ_{del2} yield inequalities for gain of the controller that cannot be satisfied at frequencies close to 1 rad/s.

Nominal plant P_0 can be transformed to:

$$P_0(s) = \frac{\dfrac{3}{\alpha_1 + s}}{\dfrac{5s-1}{\alpha_1 + s}} = \frac{B}{A}, \qquad A, B \in \mathbf{R}_{\mathrm{PS}} \tag{22}$$

Then the controller is obtained as a solution to Diophantine equation (12) with all controllers N_K/D_K given by (13) implying BIBO stable nominal feedback loop.

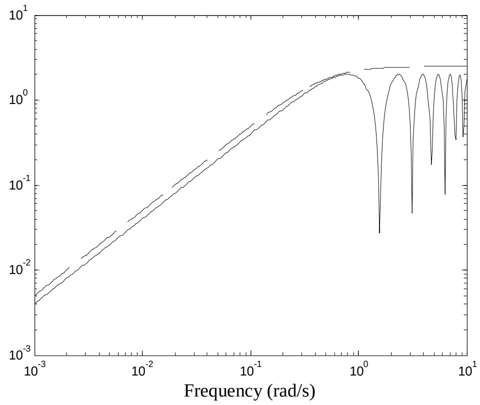

Fig. 7. Bode plot W_{del1} (dashed) and the right side of (17). (solid).

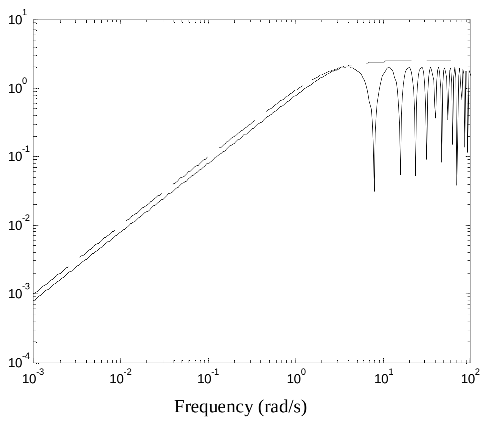

Fig. 8. Bode plot W_{del2} (dashed) and the right side of (17). (solid).

For plant (21) the controller is a 4th order transfer function derived from (13) as

$$K = \frac{N_K}{D_K} = \frac{N_{K_0} - AT}{D_{K_0} + BT} = \frac{\dfrac{n_{K_0 1}s + n_{K_0 0}}{(\alpha_2 + s)} - A\dfrac{t_2 s^2 + t_1 s}{(\alpha_3 + s)(\alpha_4 + s)}}{\dfrac{d_{K_0 1}s}{(\alpha_2 + s)} + B\dfrac{t_2 s^2 + t_1 s}{(\alpha_3 + s)(\alpha_4 + s)}} \tag{23}$$

The denominator of (23) is divisible by s so that the asymptotic tracking for the stepwise reference signal can be achieved.

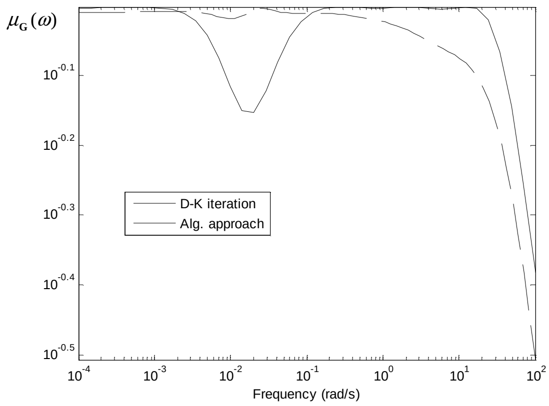

Fig. 9. Mu-plot for D-K iteration (dashed) and algebraic approach (solid).

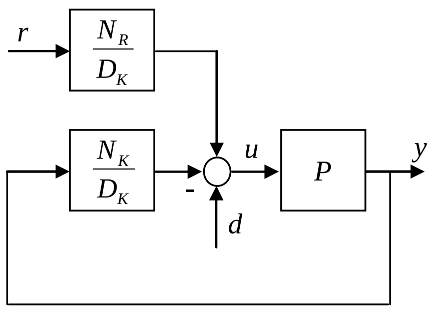

Fig. 10. 2DOF interconnection for simulation.

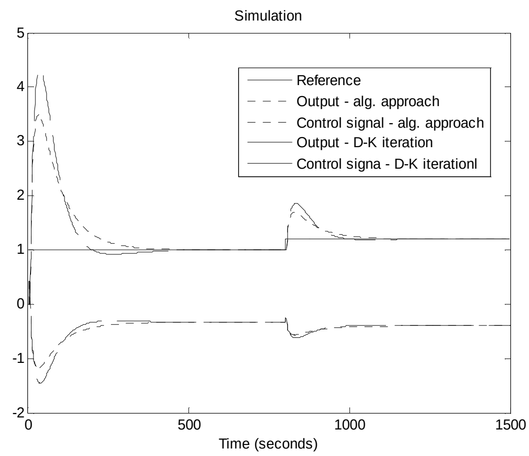

Fig. 11. Simulation for 1DOF structure (τ_1 = 4, τ_2 = 0.8).

The aim of synthesis is to design a controller which satisfies condition (14). Evolutionary optimization by Differential

Migration (DM, see Dlapa, 2009) gave the poles and arbitrary parameters as follows:

$$\alpha_1 = 0.023,\ \alpha_2 = 31.973,\ \alpha_3 = 23.264,\ \alpha_4 = 1.771 \qquad (24)$$

$$t_1 = 24.50,\ t_2 = 44.89 \qquad (25)$$

and controller

$$K_A(s) = \frac{n_K}{d_K} = \frac{29.16s^4 + 522.7s^3 + 1003s^2 + 389s + 1.159}{s^4 + 39.76s^3 + 538.6s^2 + 862.1s} \qquad (26)$$

The *D-K* iteration, which is a reference method, yields the controller

$$K_{D-K}(s) = \frac{21.94s^4 + 210.3^3 + 105.1s^2 + 1.203s + 0.003}{s^4 + 35.26s^3 + 248.3s^2 + 2.19s + 2\cdot 10^{-5}} \qquad (27)$$

Both controllers satisfy condition (14) (see Fig. 9).

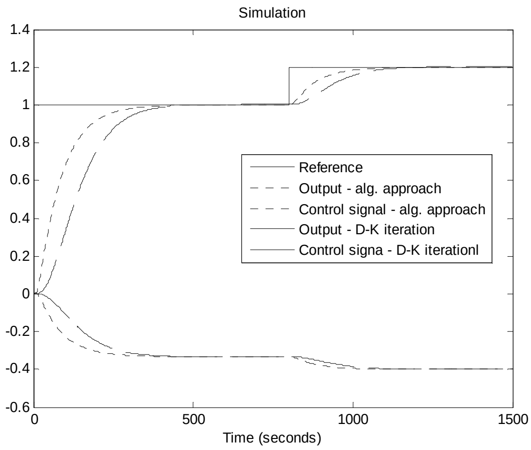

Fig. 12. Simulation for with 2DOF structure (τ_1 = 4, τ_2 = 0.8).

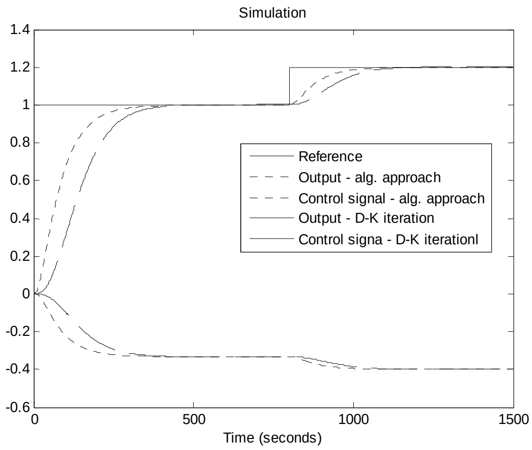

Fig. 13. Simulation for 2DOF structure (τ_1 = 2, τ_2 = 0.4).

Simulations have been performed for 1DOF and 2DOF feedback loop with real-plant *P*, i.e. with transport delays present in the simulation model. Two-degree-of-freedom controller for the *D-K* iteration has been obtained by putting n_R equal to the parameter with zero exponent of *s*, i.e., $n_R = 0.003$. The interconnection of 2DOF system is in Fig. 10. For details on 2DOF controllers in $\mathbf{R}_{\mathrm{PS}}$ see Prokop and Corriou (1997).

Simulation for both controllers with 1DOF structure and stepwise reference signal is in Fig. 11. Simulation for 2DOF structure and the same reference signal is in Fig. 12. It is apparent that the *D-K* iteration has a non-zero steady state error for both 1DOF and 2DOF interconnection, which is not the case of the algebraic approach. Set point tracking is similar for both procedures.

The same simulations but with lower time delays are depicted in Fig. 13. It can be observed that the properties of feedback loop do not degrade if the time delays vary in the intervals of 0 to 4 s and 0 to 0.8 s for τ_1 and τ_2, respectively. For the 2DOF structure there is no overshoot present, which is not true for 1DOF feedback loop.

Remark: It is apparent that for some values of time delays no stabilizing controller exists. There is, however, no general rule for examining the existence of the robust controller except Corollary 1 and the analysis of robust stability conditions, which is not straightforward due to the fact that a controller must be connected for the applicability of Corollary 1, and stability conditions are not easy to analyze so that explicit formulae for time delays and dynamics can be derived (in the general case).

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