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Continued Radicals and Cantor Sets

Thomas Tyler Clark
Western Kentucky University, thomas.clark973@topper.wku.edu

CONTINUED RADICALS AND CANTOR SETS

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Thomas Tyler Clark

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CONTINUED RADICALS AND CANTOR SETS

Date Recommended 4/24/12

Tom Richmond

Dr. Tom Richmond, Director of Thesis

Dr. Dominic Lanphier

John S. Spraker

Dr. John Spraker

Kinchel C Doerner 18-May-2012

Dean, Graduate Studies and Research Date

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CONTINUED RADICALS AND CANTOR SETS

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Directed by: Tom Richmond, Dominic Lanphier, and John Spraker

Department of Mathematics and Computer Science Western Kentucky University

We examine the formation of sets homeomorphic to the ternary Cantor set by continued radicals. We determine properties of bridges and gaps and calculate the thickness of the Cantor set. From this we apply information from continued fractions to continued radicals to obtain new results. We also consider the measure of several Cantor sets.

Chapter 1

Introduction

We define an Iterated Function System to be a finite set of contraction mappings from a compact metric space onto itself. IFS's are often associated with fractals and their related image compression techniques. Commonly used examples of IFS's include infinite products, series, continued fractions, and continued radicals.

Mathematicians have been studying continued fractions for quite some time with the phrase first being used in Wallis' 1653 work titled, *Arithmetica infinitorum.* Continued fractions are of the form,

$$x = a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}},$$

where $a_i, b_{i+1} \in \mathbb{R}$ for $i \in \mathbb{N} \cup \{0\}$ Note that the most commonly studied continued fractions have the property that $a_i, b_{i+1} \in \mathbb{Z}^+ \cup \{0\}$ for $i \in \mathbb{N} \cup \{0\}$.

Continued radicals, although related to continued fractions, have received notably less attention from researchers. Continued radicals are composed of a sequence of nested radicals. A nested radical is of the form,

$$\sqrt{x_0 + \sqrt{x_1 + \sqrt{x_2 + \ldots + \sqrt{x_n}}}},$$

where $x_i \in \mathbb{R}^+ \cup \{0\}$ for $i \in \mathbb{N} \cup \{0\}$, whereas a continued radical is of the form,

$$\lim_{n \to \infty} \sqrt{x_0 + \sqrt{x_1 + \sqrt{x_2 + \ldots + \sqrt{x_n}}}}.$$

Similarly, the most commonly studied continued radicals have $x_i \in \mathbb{Z}^+$ for $i \in \mathbb{N} \cup \{0\}$. However, it is necessary that $x_i > 0$ for all $i \in \mathbb{N} \cup \{0\}$ to make the partial expressions be defined in $\mathbb{R}$.

For ease of notation, we often represent nested radicals by $\sqrt{x_0, x_1, x_2, \ldots, x_n}$ and continued radicals by $\sqrt{x_0, x_1, x_2, \ldots}$.

The following theorem, proven by Sizer[**23**], demonstrates the significance of continued radicals.

THEOREM 1.1. *Any positive real number can be represented as a continued radical* $\sqrt{a_0 + \sqrt{a_1 + \ldots}}$ *where* $a_0 \in \mathbb{N} \cup \{0\}$ *and for* $i \geq 1$, $a_i \in \{0, 1, 2\}$.

It is important to note that the above representation only yields positive real numbers; however, multiplying by (-1) will give the negative real numbers, hence, allowing us to represent any real number in this manner. This theorem demonstrates the relevance of continued radicals. Prior to utilizing Theorem 1.1 to construct real numbers, we must analyze the convergence of continued radicals. It is necessary to consider the nested radicals, sometimes referred to as partial expressions, that compose the continued radical and verify their convergence.

Laugwitz [**16**] and Sizer [**23**] have extensively considered the conditions for convergence. Since our work entails studying continued radicals of the form $\sqrt{a_1, a_2, \ldots}$ such that $a_i \in A$ where A is a specified finite set of nonnegative integers, we need only consider the convergence results noted by Johnson and Richmond [**15**].

THEOREM 1.2. *If the sequence* $(a_i)_{i=1}^{\infty}$ *of nonnegative numbers is bounded above, then* $\sqrt{a_1, a_2, a_3, \ldots}$ *converges.*

Consider the sequence $(a_i)_{i=1}^{\infty} = (n, n, \ldots)$. We have that $(a_i)_{i=1}^{\infty}$ is bounded above. Thus, by Theorem 1.2, we have that $\phi_n = \sqrt{n, n, n, \ldots}$ converges for any nonnegative real number.

PROPOSITION 1.3. *We have that* $\phi_n = \dfrac{1 + \sqrt{4n + 1}}{2}$ *for* $n \in \mathbb{R}^+$.

Proof. Note that

$$\begin{aligned}\phi_n &= \sqrt{n, n, \ldots} \\ \implies \phi_n^2 &= n + \sqrt{n, n, \ldots} \\ &= n + \phi_n \\ \implies \phi_n^2 &- \phi_n - n = 0.\end{aligned}$$

By the quadratic equation, we get that $\phi_n = \sqrt{n, n, \ldots} = \frac{1+\sqrt{4n+1}}{2}$. $\square$

The following example demonstrates the use of Proposition 1.3 in constructing the golden ratio.

Example 1.4. Consider $\sqrt{1, 1, \cdots} = \phi_1$ which converges by Theorem 1.2. Then we have that by Proposition 1.3, $\phi_1 = \sqrt{1, 1, \ldots} = \frac{1+\sqrt{4(1)+1}}{2} = \frac{1+\sqrt{5}}{2}$.

Although Theorem 1.2 is sufficient for our study, it is important to note that the converse statement is not true. The following example was introduced by Ramanujan in 1911. As noted by Borwein and de Barra [**8**], Ramanujan did not consider the convergence of the continued radical but only to what they converged. We will, however, consider the convergence.

Example 1.5 (Ramanujan). The continued radical

$$\begin{aligned}&\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\cdots}}} \\ &= \sqrt{1 + \sqrt{2^2 + \sqrt{2^4 3^2 + \sqrt{2^8 3^4 4^2 + \cdots}}}} \\ &= \sqrt{1!^2 + \sqrt{2!^2 1!^2 + \sqrt{3!^2 2!^2 1!^2 + \sqrt{4!^2 3!^2 2!^2 1!^2 + \cdots}}}}\end{aligned}$$

converges to 3.

Lemma 1.6. $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}}$ *converges.*

Since $1!^2, 2!^2 1!^2, 3!^2 2!^2 1!^2, \ldots$ is not bounded above, Theorem 1.2 does not apply. Therefore, we must first prove that the continued radical actually converges.

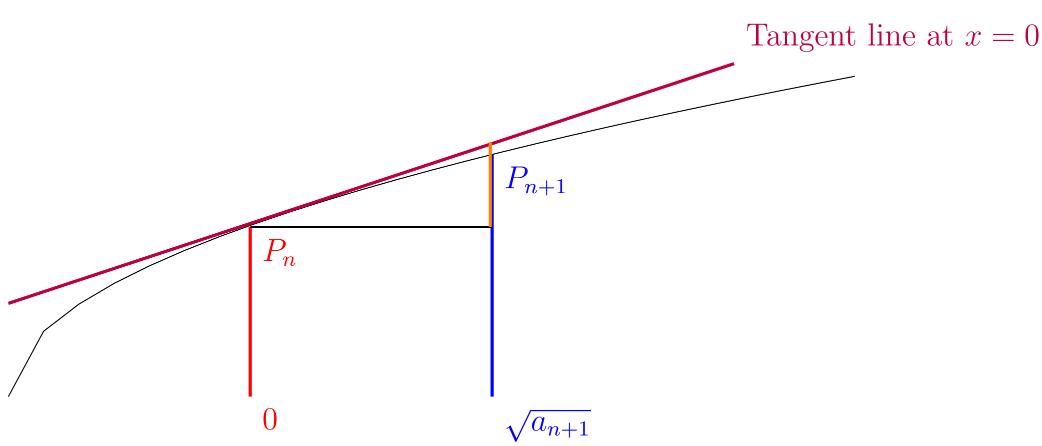
Proof 1 of Lemma 1.6. Let $a_n = (1!2! \cdots n!)^2$. We want to show that the sequence $P_1 = \sqrt{1}$, $P_2 = \sqrt{1, (1!2!)^2}$, $P_3 = \sqrt{1, (1!2!)^2, (1!2!3!)^2}$, $\ldots$ of partial expressions converges. It is obvious that $P_1, P_2, \ldots$ increases. Since an increasing sequence that is bounded above converges, it suffices to show that the sequence P_1, P_2 of partial expressions is bounded above.

Let $f(x) = \sqrt{a_1, a_2, \ldots, a_{n-1}, a_n + x}$. Then we have the following

$$\begin{aligned} f(0) &= \sqrt{a_1, a_2, \ldots, a_{n-1}, a_n} = P_n, \\ f(\sqrt{a_{n+1}}) &= \sqrt{a_1, a_2, \ldots, a_{n-1}, a_n + \sqrt{a_{n+1}}} = P_{n+1}, \text{ and} \\ f'(x) &= \frac{1}{2^n \sqrt{a_1, \ldots, a_n + x}\sqrt{a_2, \ldots, a_n + x}, \ldots \sqrt{a_n + x}}. \end{aligned}$$

Since $2^n, \sqrt{a_1, \ldots, a_n + x}, \sqrt{a_2, \ldots, a_n + x}, \ldots, \sqrt{a_n + x} > 0$ and $\sqrt{a_1, \ldots, a_n + x}, \sqrt{a_2, \ldots, a_n + x}, \ldots, \sqrt{a_n + x}$ are increasing in x, we have that $f'(x)$ is positive and decreasing which implies that $f''(x) < 0$; thus, f is concave down and increasing. From this, we can construct the following figure.

Figure 1.1. Fluctuation in Function Versus Fluctuation in Tangent Line



We have that the function change will be less than the tangent line change. Furthermore, we have that f lies below the tangent line for all x since f is concave down. Thus, we get the following

$$
\begin{aligned}
P_{n+1} &< P_n + \text{ (fluctuation of tangent line)} \\
&= P_n + df|_{x=0,\Delta x=\sqrt{a_{n+1}}} \\
&= P_n + [f'(0)][\Delta x] \\
&= P_n + \frac{\sqrt{a_{n+1}}}{2^n\sqrt{a_1,\ldots,a_n}\sqrt{a_2,\ldots,a_n}\cdots\sqrt{a_{n-1},a_n}\sqrt{a_n}}.
\end{aligned}
$$

Letting

$$
\varepsilon_n = \frac{\sqrt{a_{n+1}}}{\sqrt{a_1,\ldots,a_n}\sqrt{a_2,\ldots,a_n}\cdots\sqrt{a_{n-1},a_n}\sqrt{a_n}},
$$

we obtain the following construction.

$$
\begin{aligned}
&P_2 < P_1 + \frac{\varepsilon_1}{2} \\
&P_3 < P_2 + \frac{\varepsilon_2}{2^2} < P_1 + \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2^2} \\
&\quad\vdots \\
&P_j < P_1 + \sum_{i=1}^{j}\frac{\varepsilon_i}{2^i}
\end{aligned}
$$

If $\displaystyle\sum_{i=0}^{\infty}\frac{\varepsilon_i}{2^i}$ converges to some L, then $P_j < P_1 + L$ for all j. This indicates that the partial expressions are bounded.

We have that $\displaystyle\sum_{i=0}^{\infty}\frac{1}{2^i}$ converges since it is a geometric series with $r = \frac{1}{2}$. Now if the sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ is bounded by M, then $\displaystyle\sum_{i=1}^{\infty}\frac{\varepsilon_i}{2^i}$ would converge by the comparison test. For $n \geq 5$,

$$
\begin{aligned}
\varepsilon_n &= \frac{\sqrt{a_{n+1}}}{\sqrt{a_1,\ldots,a_n}\sqrt{a_2,\ldots,a_n}\cdots\sqrt{a_{n-1},a_n}\sqrt{a_n}} \\
&= \frac{\sqrt{((n+1)!n!(n-1)!\cdots 2!1!)^2}}{\sqrt{a_1,\ldots,a_n}\sqrt{a_2,\ldots,a_n}\cdots\sqrt{a_{n-2},a_{n-1},a_n}\sqrt{a_{n-1},a_n}\sqrt{(n!(n-1)!\cdots 2!1!)^2}} \\
&= \frac{(n+1)!}{\sqrt{a_1,\ldots,a_n}\sqrt{a_2,\ldots,a_n}\cdots\sqrt{a_{n-2},a_{n-1},a_n}\sqrt{[(n-1)!(n-2)!\cdots 2!1!)^2+\sqrt{a_n}}} \\
&< \frac{(n+1)!}{\sqrt{a_1,\ldots,a_n}\sqrt{a_2,\ldots,a_n}\cdots\sqrt{a_{n-2},a_{n-1},a_n}[(n-1)!(n-2)!\cdots 2!1!]} \\
&= \frac{(n+1)n}{\sqrt{a_1,\ldots,a_n}\sqrt{a_2,\ldots,a_n}\cdots\sqrt{a_{n-2},a_{n-1},a_n}(n-2)!\cdots 2!1!} \\
&< \frac{(n+1)n}{(n-2)(n-3)(n-4)} \\
&= \left(\frac{n+1}{n-2}\right)\left(\frac{n}{n-3}\right)\left(\frac{1}{n-4}\right) \\
&= \left(\frac{(n-2)+3}{n-2}\right)\left(\frac{(n-3)+3}{n-3}\right)\left(\frac{1}{n-4}\right) \\
&= \left(1+\frac{3}{n-2}\right)\left(1+\frac{3}{n-3}\right)\left(\frac{1}{n-4}\right) \le 5.
\end{aligned}
\tag{1.1}
$$

To obtain the second inequality, note that $1 < \phi_1 \le \sqrt{a_1, a_2, \ldots}$; therefore, $\sqrt{a_1,\ldots,a_n}\sqrt{a_2,\ldots,a_n}\cdots\sqrt{a_{n-2},a_{n-1},a_n} > 1$.

Thus, $\{\varepsilon_n\}_{n=1}^{\infty}$ is bounded by $\max\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, 5\}$. So, $\{\varepsilon_n\}_{n=1}^{\infty}$ is bounded and $\sum_{i=1}^{\infty} \frac{\varepsilon_i}{2^i}$ converges to some L; therefore, the partial expressions P_j are increasing and bounded above by $P_1 + L$ and $\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}$ converges. □

The above proof demonstrates a unique use of differentials. We may also utilize the following theorem by Sizer [**23**] (see also [**16**]) to show that these continued radicals converges.

THEOREM 1.7. *Suppose a_i is real for all $i \ge 0$ and that $a_i \ge 0$ for $i \ge 0$. Then $\sqrt{a_0, a_1, \ldots}$ converges if and only if the set $S = \{\sqrt[2^i]{a_i} \mid i \ge 1\}$ is bounded.*

Proof 2 of Lemma 1.6. We first need to consider the convergence properties of the right hand side. We will use Theorem 1.7 to accomplish this.

Let $a_n = n!^2(n-1)!^2(n-2)!^2 \ldots 3!^2 2!^2 1!^2$. For $n \geq 4$, we have the following construction.

$$
\begin{aligned}
&(n-2)!(n-3)!\cdots 2!1! \geq 2 > 1 + \frac{2}{n-1} \\
&\Longrightarrow (n-1)!(n-2)!\cdots 2!1! > (n-1) + 2 = n+1 \\
&\Longrightarrow n!(n-1)!(n-2)!\cdots 2!1! > (n+1)! \\
&\Longrightarrow [n!(n-1)!(n-2!)\cdots 2!1!]^2 > (n+1)!n!(n-1)!\cdots 2!1! \\
&\Longrightarrow [n!(n-1)!(n-2!)\cdots 2!1!]^{\frac{1}{2^{n-1}}} > [(n+1)!n!(n-1)!\cdots 2!1!]^{\frac{1}{2^n}} \\
&\Longrightarrow \sqrt[2^n]{a_n} > \sqrt[2^{n+1}]{a_{n+1}}
\end{aligned}
$$

Thus we have that the sequence $\{\sqrt[2^n]{a_n}\} = \{\sqrt[2^n]{n!^2(n-1)!^2(n-2)!^2 \ldots 3!^2 2!^2 1!^2}\}$ is decreasing for $n \geq 4$ and therefore is bounded.

Hence, $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\cdots}}}$ converges. □

Thus, we have a sequence of nonnegative numbers that is not bounded above, yet the continued radical formed by the sequence converges. This example illustrates that the converse of Theorem 1.2 is not necessarily true.

Now that we have proven convergence, we can consider to what Ramanujan's continued radical converges. The proof below is related to a proof given by Sury [**25**].

Let $f(x) = \sqrt{1 + x\sqrt{1 + (x+1)\sqrt{1 + \cdots}}}$. By squaring both sides, we obtain,

$$f^2(x) = 1 + x\sqrt{1 + (x+1)\sqrt{1 + (x+2)\sqrt{1 + \cdots}}}$$
$$= 1 + x \cdot f(x+1)$$
$$\implies x \cdot f(x+1) - f^2(x) + 1 = 0.$$

This is a first order, non-linear difference equation with variable coefficients. Commonly used solution methods such as Ricatti Equations do not seem to work nicely for this example. However, by inspection, it seems that a solution is a linear function $f(x) = mx + b$. From the definition, $f(0) = 1$. This gives that $f(0) = 0 + b = 1$; hence, $b = 1$. Thus we have $f(x) = mx + 1$. Substituting this into the difference equation, we get

$$x \cdot (m(x+1) + 1) - (mx+1)^2 + 1$$
$$= mx^2 + mx + x - m^2x^2 - 2mx - 1 + 1$$
$$= mx^2 - mx + x - m^2x^2 = 0$$

Furthermore, the above holds for all $x \in \mathbb{R}$, so we can choose an x to solve for m. Take $x = 1$ to get

$$m(1)^2 - m(1) + (1) - m^2(1)^2$$
$$= m - m + 1 - m^2$$
$$= 1 - m^2 = 0$$
$$\implies 1 = m^2 \implies m = \pm 1.$$

Taking $m = -1$ would yield negative solutions which is not possible; hence, $m = 1$ and $f(x) = x + 1$. Thus we have that $x + 1$ is a particular solution to the difference equation. Hence, we get $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \ldots}}} = f(2) = 2 + 1 = 3$.

We also need to consider some relations between continued radicals. First, we have the following definition.

Definition 1.8. *We say that* $\langle x_1, x_2, \ldots, x_n \rangle \leq \langle y_1, y_2, \ldots, y_n \rangle$ *coordinate-wise if* $x_i \leq y_i$ *for all* $i \in \{1, 2, \ldots, n\}$.

This definition gives rise to the following Proposition by Johnson and Richmond [**15**].

Proposition 1.9. *If* $a_i \geq b_i \geq 0$ *for all* $i \in \mathbb{N}$, *then* $\sqrt{a_1, a_2, \ldots, a_k} \geq \sqrt{b_1, b_2, \ldots, b_k}$ *for all* $k \in \mathbb{N}$.

The converse of Proposition 1.9 does not hold.

Our main goals for studying continued radicals involve Cantor sets. Bartle and Sherbert [**7**] give the following definition.

Definition 1.10. *We define the Middle-Thirds Cantor set,* $\mathcal{C}_3$, *to be the intersection of the sets* C_n *for* $n \in \mathbb{N}$, *obtained by successive removal of open middle thirds, starting with the unit interval* $[0, 1]$.

Note that the Middle-thirds Cantor set is $\mathcal{C}_3 = \bigcap_{n \in \mathbb{Z}^+} C_n$ where $I = [0, 1]$, $C_0 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and $C_n = [C_{n-1}] \setminus \left[\bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) \right]$.

Johnson and Richmond [**15**] gives the following theorem regarding Cantor sets formed by continued radicals.

Theorem 1.11. *If* m_1 *and* m_2 *are natural numbers with* $m_1 < m_2$, *then the set* $D = \{\sqrt{a_1, a_2, \ldots} \mid a_i \in \{m_1, m_2\} \, \forall i \in \mathbb{N}\}$ *is homeomorphic to the Middle-Thirds Cantor set,* $\mathcal{C}_3$.

It is important to note that this theorem does not take into consideration of allowing $m_1 = 0$. The Cantor sets from Theorem 1.11 will be the focus of this thesis. A more extensive discussion of Theorem 1.11 is given in Chapter 2.

Now, we need to consider a key component making continued radicals IFS's.

Definition 1.12. *Let $\langle M, \rho \rangle$ be a metric space. If $T : M \to M$, we say that T is a <u>contraction</u> on M if there exists $\alpha \in (0,1)$ such that $\rho(T(x), T(y)) \leq \alpha\rho(x,y)$, for every $x, y \in M$.*

Proposition 1.13. *The continued radical function $h(x) = \sqrt{w_1, w_2, \ldots, w_n, x}$ for $w_i \in \mathbb{N}$ is a contraction on $\langle [\frac{1}{4} + \varepsilon, \infty), |y - x| \rangle$.*

Proof. Let $f(x) = \sqrt{x}$. Assume $x > y \geq \frac{1}{4} + \varepsilon$ for some $\varepsilon > 0$. We have

$$\begin{aligned} f(x) - f(y) &= \left(\sqrt{x} - \sqrt{y}\right) \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \frac{x - y}{\sqrt{x} + \sqrt{y}} \\ &= \frac{1}{\sqrt{x} + \sqrt{y}}(x - y) \text{ for } x > y \geq \frac{1}{4} + \varepsilon \\ &\leq \frac{1}{2\sqrt{y}}(x - y) \text{ for } y \geq \frac{1}{4} + \varepsilon \\ &< \frac{1}{2\sqrt{\frac{1}{4} + \varepsilon}}(x - y) = k(x - y) \\ &\text{where } k = \frac{1}{2\sqrt{\frac{1}{4} + \varepsilon}} < 1 \end{aligned}$$

Thus, $f(x) = \sqrt{x}$ is a contraction on $[\frac{1}{4} + \varepsilon, \infty)$. On $[a, \infty)$, for $a \geq 1$, this proof shows $\sqrt{x}$ is a contraction with contraction constant $k = \dfrac{1}{2\sqrt{a}}$.

Now, let $g(x) = \sqrt{w + \sqrt{x}} = f(w + \sqrt{x})$ for $w, x > y \geq a \geq 1$ and $x \in [a, \infty)$ and $(w + \sqrt{x}) \in [w, \infty)$, we have

$$
\begin{aligned}
g(x) - g(y) &= f(w + \sqrt{x}) - f(w + \sqrt{y}) \\
&\leq \frac{1}{2\sqrt{w}}(\sqrt{x} - \sqrt{y}) \\
&= \frac{1}{2\sqrt{w}}[f(x) - f(y)] \\
&= \frac{1}{4\sqrt{wa}}(x - y).
\end{aligned}
$$

Iterating $h(x) = \sqrt{w_1, w_2, \ldots, w_n, x}$, we get $h(x) - h(y) \leq \dfrac{1}{2^{n+1}\sqrt{w_1 w_2 \cdots w_n a}}$. Hence, h is a contraction with $k = \dfrac{1}{2^{n+1}\sqrt{w_1 w_2 \cdots w_n a}}$ whenever $x \in [a, \infty)$ and $w_i \geq a$. □

In [**4**] and [**6**], Astels studies Cantor sets derived from partial fractions. To understand these results more clearly, we must define bridges and gaps of a Cantor set.

Definition 1.14. *The bridges on a given level n of a Cantor set are the remaining closed intervals on that level. We denote these bridges by $B_{n,i}$ for $1 \leq i \leq 2^n$.*

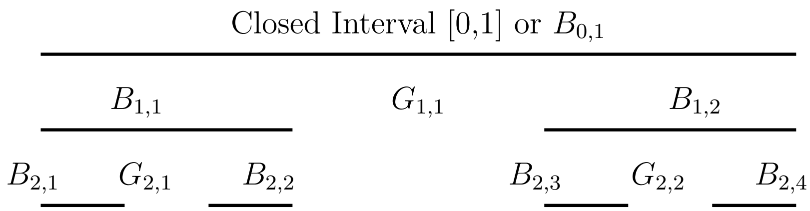
Definition 1.15. *The gaps on a given level n of a Cantor set are the open intervals removed from that level. We denote these gaps by $G_{n,k}$ for $1 \leq k \leq 2^{n-1}$.*

It is important to note that Definition 1.15 tells that the number of gaps applies only to a particular level n instead of the entire Cantor set.

Figure 1 illustrates gaps and bridges. Cantor sets can be generated through many different methods. However, Astels' results follow from those generated by ordered derivations. Astels [**4**] gives the following definition.

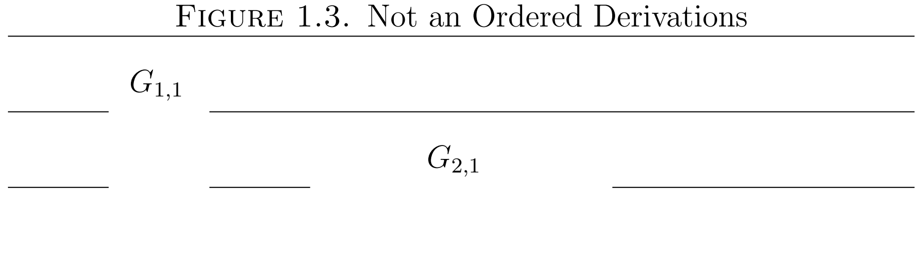
Definition 1.16. *A derivation $\mathcal{D}$ is ordered if for any bridges A and E of $\mathcal{D}$ with $A = B_{n,i} \cup G_{n,k} \cup B_{n,i+1}$, $E = B_{m,i} \cup G_{m,k} \cup B_{m,i+1}$ where $m \geq n$ and $E \subseteq A$, we have $\mid G_{n,k} \mid \geq \mid G_{m,k} \mid$.*

Figure 1.2. Gaps and Bridges



The construction given in Definition 1.10, which is the natural derivation, is an ordered derivation. Consider Figure 1.3. We have that A is the union of $G_{1,1}$ and its adjacent bridges and E is the union of $G_{2,1}$ and its adjacent bridges. Furthermore, $|G_{1,1}| < |G_{2,1}|$. Notice that this construction would be the natural derivation of the Cantor set; however, we replace the second level with two levels that do not have decreasing gaps.

Figure 1.3. Not an Ordered Derivations



Chapter 2

Properties of Bridges and Gaps

In this chapter, we discuss various properties of the generated Cantor sets. Gap and bridge length are particularly important for examining the thickness of the Cantor sets in Chapter 3. One of the main tools used in examining these properties is the Generalized Mean Value Theorem [**7**] also known as the Cauchy Mean Value Theorem.

Theorem 2.1 (Generalized Mean-Value Theorem(GMVT)). *Suppose $f(x)$ and $g(x)$ are continuous on the closed interval $[\alpha, \beta]$ and differentiable on (α, β), and assume that $g'(x) \neq 0$ for all $x \in (\alpha, \beta)$. Then there exists $c \in (\alpha, \beta)$ such that*

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(c)}{g'(c)}.$$

Before we analyze the components of the generated Cantor sets, we must understand the construction.

Consider the set of real numbers representable as a continued radical $\sqrt{m_1, m_2, \ldots}$ such that $m_i \in \{a, b\}$, $a, b \in \mathbb{N}$, and $a < b$. The elements of this set are called representable. Then the smallest number representable is $\sqrt{a, a, \ldots} = \phi_a$ and the largest is $\sqrt{b, b, \ldots} = \phi_b$. This result follows directly from Proposition 1.9. Thus, we have that the numbers representable are in the closed interval $[\phi_a, \phi_b]$. This result is mentioned by Johnson and Richmond [**15**].

Now, if we consider the representable numbers whose first term is a, we notice that the largest is $\sqrt{a, b, b, \ldots} = \sqrt{a + \phi_b}$. Analogously, we have that the smallest representable number whose first term is b is $\sqrt{b, a, a, \ldots} = \sqrt{b + \phi_a}$. Furthermore, Johnson and Richmond [**15**] tell us that for $a, b \in \mathbb{N}$ and $a < b$, then $\sqrt{a + \phi_b} < \sqrt{b + \phi_a}$. Thus, we have a jump between the largest representable number with first term a and the smallest representable number with first term b. This produces a gap in $[\phi_a, \phi_b]$. If we remove this gap, we obtain two bridges, namely, $[\phi_1, \sqrt{a+\phi_b}]$ and $[\sqrt{b+\phi_a}, \phi_b]$.

Now, if we fix the first term, we have that the second term may be a or b. Therefore, we can calculate the largest and smallest representable number similar to the above construction. This will leave us with two addition gaps, one for the case that the second term is a and another for the case that the second term is b. We can inductively continue this process.

Suppose we fix the first n terms of the continued radical. That is, we fix the values of $\overrightarrow{w} = \langle m_1, m_2, \ldots, m_n \rangle$ such that $m_i \in \{a, b\}$. We have that $\overrightarrow{w}$ has length n and leads to the gap on level $n+1$. When discussing gaps and bridges on level $n+1$, we refer to $\overrightarrow{w}$ as the direction vector. The sequence of a's and b's in $\overrightarrow{w}$ determine the left endpoint of a bridge on level n. Notice that since $a < b$, when reading the coordinates of $\overrightarrow{w}$, we move to the left bridge on the level below for each a and the right bridge for each b. Thus we have that $\overrightarrow{w}$ directs us to the left endpoint of a bridge on level n. Furthermore, the jump from a to b in coordinate $n+1$ will produce a gap on level $n+1$ in the bridge on level n whose left endpoint was determined by $\overrightarrow{w}$. Thus, all the gaps and bridges on level $n+1$ are determined by all possible choices of direction vectors $\overrightarrow{w}$ of length n.

We begin by considering a gap and its two adjacent bridges. Proposition 2.2 shows that the left bridge is larger than the right bridge.

Proposition 2.2. *Given a gap, $G_{n,k}$ and two adjacent bridges, $B_{n,i}$ and $B_{n,i+1}$, we have that $\mid B_{n,i} \mid > \mid B_{n,i+1} \mid$.*

Proof. Suppose that $\overrightarrow{w} = \sqrt{w_1, w_2, \ldots, w_n}$ navigates us to a certain level on a generated Cantor set. Then $\mid B_{n,i} \mid = \sqrt{\overrightarrow{w}, a+\phi_b} - \sqrt{\overrightarrow{w}, a+\phi_a}$ and

$\mid B_{n,i+1} \mid = \sqrt{\overrightarrow{w}, b+\phi_b} - \sqrt{\overrightarrow{w}, b+\phi_a}$. We can then define
$h(x) = \sqrt{\overrightarrow{w}, x+\phi_b} - \sqrt{\overrightarrow{w}, x+\phi_a}$. Note that $\mid B_{n,i} \mid = h(a)$ and $\mid B_{n,i+1} \mid = h(b)$.

We now need to consider

$$\begin{aligned} h'(x) = &\frac{1}{2^{k+1}}\Big(\frac{1}{\sqrt{w_1,\ldots,w_k,x+\phi_b}\sqrt{w_2,\ldots,w_k,x+\phi_b}\cdots\sqrt{w_k,x+\phi_b}\sqrt{x+\phi_b}} \\ &- \frac{1}{\sqrt{w_1,\ldots,w_k,x+\phi_a}\sqrt{w_2,\ldots,w_k,x+\phi_a}\cdots\sqrt{w_k,x+\phi_a}\sqrt{x+\phi_a}}\Big). \end{aligned}$$

We want to show that $h(x)$ is decreasing, so we can consider each portion of the denominator separately. We define new functions,
$g_i(y) = \sqrt{w_i, w_{i+1}, \ldots, w_k, x+y}$. Since the composition of increasing functions is increasing, each $g_i(y)$ is increasing as a function of y.

Therefore, we get that, $g_i(x+\phi_b) > g_i(x+\phi_a) > 0$. Taking reciprocals, we get $0 < \dfrac{1}{g_i(x+\phi_b)} < \dfrac{1}{g_i(x+\phi_a)}$. Hence, we get that,

$$\begin{aligned} &\prod_{i=1}^{k}\frac{1}{g_i(x+\phi_b)} < \prod_{i=1}^{k}\frac{1}{g_i(x+\phi_a)} \\ &\implies \prod_{i=1}^{k}\frac{1}{g_i(x+\phi_b)} - \prod_{i=1}^{k}\frac{1}{g_i(x+\phi_a)} < 0 \\ &\implies h'(x) = \frac{1}{2^{k+1}}\prod_{i=1}^{k}\frac{1}{g_i(x+\phi_b)} - \frac{1}{2^{k+1}}\prod_{i=1}^{k}\frac{1}{g_i(x+\phi_a)} < 0 \end{aligned}$$

Hence, $h(x)$ is decreasing. Ergo, $h(a) > h(b)$ and $\mid B_{n,i} \mid > \mid B_{n,i+1} \mid$. $\square$

We consider an analogous idea for gaps. Suppose we choose a bridge $B_{n,i}$ on level n of the Cantor set. Then proceeding through the derivation of the Cantor set to level $n+2$, we have two gaps below $B_{n,i}$. Proposition 2.3 compares the lengths of these two gaps.

Proposition 2.3. *Of the two gaps that are 2 levels below any bridge, the largest gap is on the left.*

Proof. The length of a gap two levels below the bridge whose left endpoint is determined by $\overrightarrow{w}$ is to be,

$h(x) = \sqrt{\overrightarrow{w}, x, b+\phi_a} - \sqrt{\overrightarrow{w}, x, a+\phi_b}$, where $x \in \{a, b\}$.

Then $h'(x) = \left[2^{n+1}\sqrt{w_1, w_2, \ldots, w_n, x, b+\phi_a}\sqrt{w_2, \ldots, w_n, x, b+\phi_a} \cdots \sqrt{x, b+\phi_a}\right]^{-1}$ $- \left[2^{n+1}\sqrt{w_1, w_2, \ldots, w_n, x, a+\phi_b}\sqrt{w_2, \ldots, w_n, x, a+\phi_b} \cdots \sqrt{x, a+\phi_b}\right]^{-1}$.

We also have that

$$\begin{aligned} &\sqrt{b+\phi_a} > \sqrt{a+\phi_b}; \\ &\text{thus, } \frac{1}{\sqrt{b+\phi_a}} < \frac{1}{\sqrt{a+\phi_b}}. \end{aligned} \tag{2.1}$$

Let $A = \left[2^{n+1}\sqrt{w_1, w_2, \ldots, w_n, x, b+\phi_a}\sqrt{w_2, \ldots, w_n, x, b+\phi_a} \cdots \sqrt{x, b+\phi_a}\right]^{-1}$ and $B = \left[2^{n+1}\sqrt{w_1, w_2, \ldots, w_n, x, a+\phi_b}\sqrt{w_2, \ldots, w_n, x, a+\phi_b} \cdots \sqrt{x, a+\phi_b}\right]^{-1}$. Then, each component of B is greater than the corresponding component of A. Thus, equation (2.1) implies that $h'(x) < 0$; hence, h is decreasing.

Therefore, of the two gaps that are two levels below any bridge, the largest gap is on the left. $\square$

Now that we have determined the larger of two bridges below a certain bridge, we need to consider the largest gap on a given level n. Proposition 2.4 gives this result.

Proposition 2.4. *The largest gap on a given level n is the left-most gap $G_{n,1}$.*

Proof. Note that $\mid G_{n,1} \mid = \sqrt{a_1, \ldots, a_k, b+\phi_a} - \sqrt{a_1, \ldots, a_k, a+\phi_b}$, where $a_1 = a_2 = \cdots = a_k = a$, is the length of the left-most gap and $\sqrt{w_1, \ldots, w_k, b+\phi_a} - \sqrt{w_1, \ldots, w_k, a+\phi_b}$ represents the length of $G_{n,i}$ for $1 < i \in \mathbb{N}$ (i.e. a gap to the right of $G_{n,1}$). We define $f(x) = \sqrt{a_1, a_2, \ldots, a_k, x}$ and

$g(x) = \sqrt{w_1, \ldots, w_k, x}$. Furthermore,

$$f'(x) = \frac{1}{2^{k+1}} \frac{1}{\sqrt{a_1, \ldots, a_k, x}\sqrt{a_2, \ldots, a_k, x} \cdots \sqrt{a_k, x}\sqrt{x}} \text{ and}$$

$$g'(x) = \frac{1}{2^{k+1}} \frac{1}{\sqrt{w_1, \ldots, w_k, x}\sqrt{w_2, \ldots, w_k, x} \ldots \sqrt{w_k, x}\sqrt{x}}.$$

Using the Generalized Mean-Value Theorem, with $\alpha = a + \phi_b$ and $\beta = b + \phi_a$, there exists $c \in (\alpha, \beta)$ such that,

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(c)}{g'(c)} = \frac{2^{k+1}\sqrt{w_1, \ldots, w_k, c}\sqrt{w_2, \ldots, w_k, c} \cdots \sqrt{w_k, c}\sqrt{c}}{2^{k+1}\sqrt{a, \ldots, a, c}\sqrt{a, \ldots, a, c} \cdots \sqrt{a, c}\sqrt{c}}$$
$$= \frac{\sqrt{w_1, \ldots, w_k, c}\sqrt{w_2, \ldots, w_k, c} \cdots \sqrt{w_k, c}}{\sqrt{a, \ldots, a, c}\sqrt{a, \ldots, a, c} \cdots \sqrt{a, c}}.$$

Furthermore, since we have that $\overrightarrow{w} > \overrightarrow{a}$ in the coordinate-wise order and $w_i > a$ for some $i \leq k$, we have that $\frac{f'(c)}{g'(c)} > 1$.

Hence, we have that

$$\begin{aligned} f(\beta) - f(\alpha) &= \sqrt{a_1, \ldots, a_k, b + \phi_a} - \sqrt{a_1, \ldots, a_k, a + \phi_b} \\ &> \sqrt{w_1, \ldots, w_k, b + \phi_a} - \sqrt{w_1, \ldots w_k, a + \phi_b} \\ &= g(\beta) - g(\alpha). \end{aligned}$$

Therefore, the left-most gap is larger than every gap to its right. □

Corollary 2.5 gives the analogous property for bridges. The proof is similar to that of Proposition 2.4.

Corollary 2.5. *The longest bridge on a given level n is the left-most bridge $B_{n,1}$.*

Now that we have located the position of the longest gap and bridge, we need to consider the position of the shortest gap and bridge.

Proposition 2.6. *The shortest gap on a given level n is the right-most gap $G_{n,2^{n-1}}$.*

PROOF. Note that $\mid G_{n,2^{n-1}} \mid = \sqrt{b_1, \ldots, b_k, b+\phi_a} - \sqrt{b_1, \ldots, b_k, a+\phi_b}$, where $b_1 = b_2 = \cdots = b_k = b$, is the length of the right-most gap and $\sqrt{w_1, \ldots, w_k, b+\phi_a} - \sqrt{w_1, \ldots, w_k, a+\phi_b}$ represents the length of $G_{n,m}$ where $2^{n-1} > m \in \mathbb{N}$ (i.e. a gap to the left of $G_{n,2^{n-1}}$). We define $f(x) = \sqrt{b_1, b_2, \ldots, b_k, x}$ and $g(x) = \sqrt{w_1, \ldots, w_k, x}$. Furthermore,
$f'(x) = \dfrac{1}{2^{k+1}} \dfrac{1}{\sqrt{b_1, \ldots, b_k, x}\sqrt{b_2, \ldots, b_k, x} \cdots \sqrt{b_k, x}\sqrt{x}}$ and
$g'(x) = \dfrac{1}{2^{k+1}} \dfrac{1}{\sqrt{w_1, \ldots, w_k, x}\sqrt{w_2, \ldots, w_k, x} \ \ldots \ \sqrt{w_k, x}\sqrt{x}}$.

Using the Generalized Mean-Value Theorem, with $\alpha = a + \phi_b$ and $\beta = b + \phi_a$, there exists $c \in (\alpha, \beta)$ such that

$$\begin{aligned}\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(c)}{g'(c)} &= \frac{2^{k+1}\sqrt{w_1, \ldots, w_k, c}\sqrt{w_2, \ldots, w_k, c} \cdots \sqrt{w_k, c}\sqrt{c}}{2^{k+1}\sqrt{b_1, \ldots, b_k, c}\sqrt{b_2, \cdots, b_k, c} \cdots \sqrt{b_k, c}\sqrt{c}} \\ &= \frac{\sqrt{w_1, \cdots, w_k, c}\sqrt{w_2, \cdots, w_k, c} \cdots \sqrt{w_k, c}}{\sqrt{b_1, \ldots, b_k, c}\sqrt{b_2, \ldots, b_k, c} \cdots \sqrt{b_k, c}}.\end{aligned}$$

Furthermore, since we have that $\overrightarrow{w} < \overrightarrow{b}$, in the coordinate-wise order, and $w_i < b$ for some $i \leq k$, we have that $\frac{f'(c)}{g'(c)} < 1$.

Hence, we have that

$$\begin{aligned}f(\beta) - f(\alpha) &= \sqrt{b_1, \ldots, b_k, b+\phi_a} - \sqrt{b_1, \ldots, b_k, a+\phi_b} \\ &< \sqrt{w_1, \ldots, w_k, b, \phi_a} - \sqrt{w_1, \ldots, w_k, a, \phi_b} \\ &= g(\beta) - g(\alpha).\end{aligned}$$

Therefore, the right-most gap is smaller than every gap to its left. □

Example 2.8 below illustrates this fact on level 4 of the Cantor set generated by continued radicals whose entries are from $\{1, 2\}$.

COROLLARY 2.7. *The shortest bridge on a given level n is the right-most bridge $B_{n,2^n}$.*

TABLE 2.1. Gap Lengths of Cantor set generated by $M = \{1, 2\}$

Gap Location	Gap Length
1	0.0154802834320609
2	0.0126629571021913
3	0.0132093368773177
4	0.0109449447186351

Corollaries 2.5 and 2.7 say that the longest bridge on level n is the left-most bridge and the shortest bridge is the right-most bridge. However, it is important to note that the bridges are not strictly decreasing in length from left to right. Example 2.8 demonstrates this.

EXAMPLE 2.8. Suppose we consider continued radicals whose entries come from $M = \{1, 2\}$. The Table 2.1 indicates the lengths of the gaps on the 4th level of the generated Cantor set from left to right.

Although we notice that $G_{4,1}$ (the left-most gap) is the largest gap and $G_{4,4}$ (the right-most gap) is the smallest gap, it is important to notice that $| G_{4,2} |<| G_{4,3} |$. This shows that the gap lengths do not decrease as we move from left to right.

Chapter 3

Thickness

In this chapter, we discuss the thickness of Cantor sets. Newhouse [**18**] defined the thickness of a Cantor set to be a nonnegative quantity that gives conditions on when two Cantor sets intersect. Astels [**6**] gives the following definition.

Definition 3.1. *We define the* <u>*thickness of a level* n</u> *to be,*

$$\tau(n) = \inf_{w} \left(\frac{min(\mid B_{n,i} \mid, \mid B_{n,i+1} \mid)}{\mid G_{n,k} \mid} \right),$$

where w *ranges through all location vectors of a fixed length determined by the level* n *and* $B_{n,i}$ *and* $B_{n,i+1}$ *are adjacent to* $G_{n,k}$.

Definition 3.1 only gives us the thickness of a certain level of the Cantor set. However, we also need to consider the thickness over the entire (ordered) derivation. Similarly, we have the following definition.

Definition 3.2. *We define the* <u>*thickness of a derivation* $\mathcal{D}$</u> *to be,*

$$\tau(\mathcal{D}) = \inf_{w} \left(\frac{min(\mid B_{n,i} \mid, \mid B_{n,i+1} \mid)}{\mid G_{n,k} \mid} \right),$$

where w *is a direction vector of any length and* $B_{n,i}$ *and* $B_{n,i+1}$ *are adjacent to* $G_{n,k}$.

Astels [**4**], [**6**] gives information regarding the thickness of Cantor sets generated by continued fractions. Here, we consider the analogous construction for continued radicals.

Proposition 3.3. $\tau(n)$ *is formed by the left-most gap and the right bridge adjacent to the gap.*

PROOF. To prove this, we need to show that for $\overrightarrow{w} > \overrightarrow{a}$,

$$
\begin{aligned}
0 &< \frac{\sqrt{a_1, a_2, \ldots, a_k, b+\phi_b} - \sqrt{a_1, a_2, \ldots, a_k, b+\phi_a}}{\sqrt{a_1, a_2, \ldots, a_k, b+\phi_a} - \sqrt{a_1, a_2, \ldots, a_k, a+\phi_b}} \\
&< \frac{\sqrt{w_1, w_2, \ldots, w_k, b+\phi_b} - \sqrt{w_1, w_2, \ldots, w_k, b+\phi_a}}{\sqrt{w_1, w_2, \ldots, w_k, b+\phi_a} - \sqrt{w_1, w_2, \ldots, w_k, a+\phi_b}}.
\end{aligned}
$$

Define $r(x) = \sqrt{a_1, a_2, \ldots, a_k, x}$ and $s(x) = \sqrt{w_1, w_2, \ldots, w_k, x}$. Then the above construction is analogous to showing that

$$
\frac{\frac{r(b+\phi_b)-r(b+\phi_a)}{s(b+\phi_b)-s(b+\phi_a)}}{\frac{r(b+\phi_a)-r(a+\phi_b)}{s(b+\phi_a)-s(a+\phi_b)}} > 1.
$$

By the *Generalized Mean Value Theorem* we have that

$$
\frac{\frac{r(b+\phi_b)-r(b+\phi_a)}{s(b+\phi_b)-s(b+\phi_a)}}{\frac{r(b+\phi_a)-r(a+\phi_b)}{s(b+\phi_a)-s(a+\phi_b)}} = \frac{\frac{r'(c_1)}{s'(c_1)}}{\frac{r'(c_2)}{s'(c_2)}}
$$

for some $c_1 \in (b+\phi_a, b+\phi_b)$ and some $c_2 \in (a+\phi_b, b+\phi_a)$. This says that $c_2 < c_1$. Since we want that

$$
\frac{\frac{r'(c_1)}{s'(c_1)}}{\frac{r'(c_2)}{s'(c_2)}} < 1 \text{ or equivalently } \frac{r'(c_1)}{s'(c_1)} < \frac{r'(c_2)}{s'(c_2)},
$$

it will suffice to show that $\frac{r'(x)}{s'(x)}$ is a decreasing function.

Note that

$$
r'(x) = \frac{1}{2^{n+1}\sqrt{a_1, a_2, \ldots, a_n, x}\sqrt{a_2, \ldots, a_n, x} \cdots \sqrt{a_n, x}\sqrt{x}} \text{ and}
$$

$$
s'(x) = \frac{1}{2^{n+1}\sqrt{w_1, w_2, \ldots, w_n, x}\sqrt{w_2, \ldots, w_n, x} \cdots \sqrt{w_n, x}\sqrt{x}}.
$$

Now we consider

$$
\frac{r'(x)}{s'(x)} = \left(\frac{\sqrt{w_1, \ldots, w_n, x}}{\sqrt{a_1, \ldots, a_n, x}}\right)\left(\frac{\sqrt{w_2, \ldots, w_n, x}}{\sqrt{a_2, \ldots, a_n, x}}\right) \cdots \left(\frac{\sqrt{w_n, x}}{\sqrt{a_n, x}}\right)
$$

where $a_i \leq w_i$ for all i and $a_j < w_j$ for some $1 \leq j \leq n$. Let us consider the j^{th} factor of this expression. We have $\dfrac{\sqrt{w_j, \ldots, w_n, x}}{\sqrt{a_j, \ldots, a_n, x}}$ and need to determine if it is decreasing or not. Let

$$\begin{aligned} A = &(a_j + \sqrt{a_{j+1}, \ldots, a_n, x})(\sqrt{a_{j+1}, \ldots, a_n, x} \cdots \sqrt{a_n, x}) \\ &\qquad - (w_j + \sqrt{w_{j+1}, \ldots, w_n, x})(\sqrt{w_{j+1}, \ldots, w_n, x} \cdots \sqrt{w_n, x}) \text{ and} \end{aligned}$$

$$B = \left[2^{n-j+2}\sqrt{w_j, \ldots, w_n, x} \cdots \sqrt{w_n, x}\sqrt{x}\sqrt{a_j, \ldots, a_n, x} \cdots \sqrt{a_n, x}(\sqrt{a_j, \ldots, a_n, x})^2\right].$$

Thus, $\dfrac{d}{dx}\left(\frac{\sqrt{w_j,\ldots,w_n,x}}{\sqrt{a_j,\ldots,a_n,x}}\right) = \dfrac{A}{B}$.

We have that the denominator is a positive value and the numerator is a non-postive value. Hence, $\dfrac{d}{dx}\left(\frac{\sqrt{w_j,\ldots,w_n,x}}{\sqrt{a_j,\ldots,a_n,x}}\right) \leq 0$. Since we know that $a_j < w_j$ for some j, we know that the derivative of at least one of these factors is non-zero which implies it is decreasing.

Therefore, $\dfrac{r'(x)}{s'(x)}$ is the product of all non-increasing functions and at least one decreasing function and is decreasing. This completes the proof that $\tau(n)$ is formed by the left-most gap and the right bridge adjacent to the gap. $\square$

Thus, we have that

$$\tau(n) = \left(\frac{\text{length of right adjacent bridge}}{\text{length of left-most gap}}\right) = \left(\frac{|B_{n,2}|}{|G_{n,1}|}\right).$$

We have a similar result for $\tau(\mathcal{D})$ with a direction vector w of any length.

Our first conjecture was that the length of the largest gap on a given level is always smaller than the smallest gap on the level above. However, as demonstrated in Example 3.4, this is not the case.

Example 3.4. Suppose we consider continued radicals whose entries come from $M = \{1, 3\}$. Table 3.1 indicates the lengths of the smallest and largest gaps on the first five levels of the generated Cantor set.

TABLE 3.1. Gap Lengths of Cantor set generated by $M = \{1, 3\}$

Level	Smallest Gap Length	Largest Gap Length
1	0.0742850921310610	0.0960336969295997
2	0.0162383151756316	0.0290808163357608
3	0.00353095745036294	0.00893019445346545
4	0.000766916858749292	0.00275422498885553
5	0.000166531540288216	0.000850591283299251

We see that the largest gap on level 5 is larger than the smallest gap on level 4. This is a contradiction to our initial conjecture.

Although our initial thoughts were not correct, this does not contradict any results. In fact, it would be a much stronger statement than we need to prove that the derivation of our Cantor set is an ordered derivation. We do, however, have the following result.

PROPOSITION 3.5. *For levels k and $k+1$, we have that $\tau(k) > \tau(k+1)$.*

PROOF. To prove this, we need to show that

$$\frac{\sqrt{\overrightarrow{a}, b+\phi_b} - \sqrt{\overrightarrow{a}, b+\phi_a}}{\sqrt{\overrightarrow{a}, b+\phi_a} - \sqrt{\overrightarrow{a}, a+\phi_b}} > \frac{\sqrt{a, \overrightarrow{a}, b+\phi_b} - \sqrt{a, \overrightarrow{a}, b+\phi_a}}{\sqrt{a, \overrightarrow{a}, b+\phi_a} - \sqrt{a, \overrightarrow{a}, a+\phi_b}} > 0.$$

Define $\overrightarrow{a} = \langle a_1, a_2, \ldots a_n \rangle$ where $a_i = a$ for $i \in \{1, 2, \ldots, n\}$, $r(x) = \sqrt{\overrightarrow{a}, x}$, and $s(x) = \sqrt{a, \overrightarrow{a}, x}$. Then the above construction is analogous to showing that

$$\frac{\frac{r(b+\phi_b)-r(b+\phi_a)}{s(b+\phi_b)-s(b+\phi_a)}}{\frac{r(b+\phi_a)-r(a+\phi_b)}{s(b+\phi_a)-s(a+\phi_b)}} > 1.$$

By the *GMVT* we have that

$$\frac{\frac{r(b+\phi_b)-r(b+\phi_a)}{s(b+\phi_b)-s(b+\phi_a)}}{\frac{r(b+\phi_a)-r(a+\phi_b)}{s(b+\phi_a)-s(a+\phi_b)}} = \frac{\frac{r'(c_1)}{s'(c_1)}}{\frac{r'(c_2)}{s'(c_2)}}$$

for some $c_1 \in (b+\phi_a, b+\phi_b)$ and some $c_2 \in (a+\phi_b, b+\phi_a)$. This says that $c_2 < c_1$. Since we want that

$$\frac{\frac{r'(c_1)}{s'(c_1)}}{\frac{r'(c_2)}{s'(c_2)}} > 1 \text{ or equivalently } \frac{r'(c_1)}{s'(c_1)} > \frac{r'(c_2)}{s'(c_2)},$$

it will suffice to show that $\frac{r'(x)}{s'(x)}$ is an increasing function.

Note that

$$r'(x) = \frac{1}{2^{n+1}\sqrt{a_1, a_2, \ldots, a_n, x}\sqrt{a_2, \ldots, a_n, x}\cdots\sqrt{a_n, x}\sqrt{x}} \text{ and}$$

$$s'(x) = \frac{1}{2^{n+2}\sqrt{a_1, a_2, \ldots, a_n, a_{n+1}, x}\sqrt{a_2, \ldots, a_n, a_{n+1}, x}\cdots\sqrt{a_n, a_{n+1}, x}\sqrt{a_{n+1}, x}\sqrt{x}}.$$

Now we consider

$$\frac{r'(x)}{s'(x)} = 2\left(\frac{\sqrt{a_1, \ldots, a_n, a_{n+1}, x}}{\sqrt{a_1, \ldots, a_n, x}}\right)\left(\frac{\sqrt{a_2, \ldots, a_n, a_{n+1}, x}}{\sqrt{a_2, \ldots, a_n, x}}\right)\cdots\left(\frac{\sqrt{a_n, a_{n+1}, x}}{\sqrt{a_n, x}}\right)(\sqrt{a_{n+1}, x}).$$

Simplifying this, we get $\dfrac{r'(x)}{s'(x)} = 2\sqrt{a_1, a_2, \ldots, a_n, a_{n+1}, x}$. This is an increasing function in x.

Thus, the thickness on a given level is larger than the thickness on all levels below. □

We calculate the thickness of the middle-thirds Cantor set and the middle-p% Cantor set in Propositions 3.6 and 3.7, respectively.

PROPOSITION 3.6. *The thickness of the middle-third Cantor set ($\mathcal{C}_3$) is 1.*

PROOF. To construct the middle-third Cantor set, we begin with the closed interval $[0, 1]$. We first remove the open interval $(\frac{1}{3}, \frac{2}{3})$ which is the middle third of $[0, 1]$. From here we have two bridges and a gap. We remove the middle third from each of the two bridges. We repeat this process infinitely to obtain the middle-third Cantor set as shown in Figure 3.1 below.

Now, on a given level, we have that the gap, $G_{n,k}$ is the same length as its adjacent bridges $B_{n,i}$ and $B_{n,i+1}$. Thus we have that

$$\begin{aligned}\tau(\mathcal{C}_3) &= \inf_w \left(\frac{min(\mid B_{n,i} \mid, \mid B_{n,i+1} \mid)}{\mid G_{n,k} \mid} \right) \\ &= \inf_w \left(\frac{B_{n,i}}{B_{n,i}} \right) = \inf_w(1) = 1.\end{aligned} \tag{3.1}$$

Thus, $\tau(\mathcal{C}_3) = 1$. □

FIGURE 3.1. Middle Thirds Cantor Set

PROPOSITION 3.7. *Suppose we have the "middle-p%" Cantor set ($\mathcal{C}_p$). Then* $\tau(\mathcal{C}_p) = \frac{1}{2}\left(\frac{1-p}{p}\right)$.

PROOF. To construct the middle-p% Cantor set, we begin with the closed interval $[a, b]$. We first remove the middle-p% of the interval and have two bridges and a gap. Each of the bridges have length $\frac{1-p}{2}$%. Then we have that

$$\begin{aligned}\tau(\mathcal{C}_p) &= \left(\frac{min(\mid B_{n,i} \mid, \mid B_{n,i+1} \mid)}{\mid G_{n,k} \mid} \right) \\ &= \left(\frac{\frac{1-p}{2}}{p} \right) = \frac{1-p}{2p} \\ &= \frac{1}{2}\left(\frac{1-p}{p}\right).\end{aligned} \tag{3.2}$$

Since we have constructed this Cantor set such that it is an ordered derivation, we have that $\tau(\mathcal{C}_p) = \frac{1}{2}\left(\frac{1-p}{p}\right)$. □

Notice that for $p > \frac{1}{3}$, we have $\tau < 1$ and $p < \frac{1}{3}$ gives $\tau > 1$.

By Proposition 3.5, we have that the thicknesses of the levels are monotone decreasing. Furthermore, by Definition 3.2 we have that the thickness of the Cantor set generated by continued radicals composed from $\{a, b\}$ is

$$\tau(a,b) = \lim_{n\to\infty} \frac{\sqrt{\overrightarrow{a}, b+\phi_b} - \sqrt{\overrightarrow{a}, b+\phi_a}}{\sqrt{\overrightarrow{a}, b+\phi_a} - \sqrt{\overrightarrow{a}, a+\phi_b}}.$$

Note that for continued fractions, there is a closed form solution to the values of thickness of Cantor sets generated by continued fractions composed from $\{a, b\}$. These calculations involve utilizing common recurrence relations for continued radicals as noted by Astels [**5**].

We have tried several methods to compute this for the analogous case using continued radicals; however, we have not been able to find an appropriate approach. We cannot use the generalized mean value theorem because we have components of three types in the thickness equation rather than just two types (i.e., we have $b+\phi_b$, $b+\phi_a$, and $a+\phi_b$).

Notice that as $n \to \infty$, the numerator and denominator are both approaching $\phi_a - \phi_a = 0$. Thus, we have an indeterminate form and can try applying L'Hôpital's Rule. We note that the limit is dependent upon n; hence, we need to consider the discrete analogue of L'Hôpital's Rule. This, however, ends in a more complicated expression than the original formula.

Thus, the obvious approaches do not work. We have, however, created a *Mathematica* program to approximate these results for us. Some of these values are shown in Table 3.2. First we need to give the following definition from Astels [**6**].

Definition 3.8. *The* <u>*normalized thickness*</u> *of a Cantor set is*

$$\gamma(\mathcal{D}) = \frac{\tau(\mathcal{D})}{\tau(\mathcal{D})+1}.$$

TABLE 3.2. Thickness of Cantor Sets generated by continued radicals generated by M

M	$\tau(\mathcal{D})$	$\gamma(\mathcal{D})$
{1,2}	0.556376528100861	0.357481957646694
{1,100}	0.0459757109111117	0.0439548551955034
{50,51}	0.0751391042757936	0.0698877977528376
{50,100}	0.0533173194238223	0.0506184778704556
{100,101}	0.0523015447772998	0.0497020507447496
{100,1000}	0.0159497775609435	0.0156993760058053

Although we cannot find a closed form solution for the thickness, we do have a few conjectures.

CONJECTURE 3.9. *We have that $\tau(1,2) > \tau(1,b)$ for $b > 2$.*

Table 3.3 on page 28 gives numerical approximations that are evidence for Conjecture 3.9. This conjecture suggests we should consider the properties of the function $\tau(a,x)$ for a fixed a. These results are hypothesized in Conjecture 3.10 below.

CONJECTURE 3.10. *For a fixed a, $\tau(a,x)$ is decreasing as a function of x.*

Evidence for Conjecture 3.10 is provided in Table 3.4 on page 29.

Conjecture 3.10 may give rise to the thought $\tau(x,b)$ could be decreasing as a function of x. However, Table 3.5 on page 30 shows that this is not the case. We have that $T(x,50)$ is increasing until $x = 11$ and decreasing afterwards.

Calculations of these thicknesses can be used to determine when the summation of points of Cantor sets create an interval as defined in Definition 3.11.

DEFINITION 3.11. *We define the summation of points of Cantor sets to be*

$$\mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_\alpha = \{a + b + \cdots + j : a \in \mathcal{C}_1, b \in \mathcal{C}_2, \ldots, j \in \mathcal{C}_\alpha\}.$$

We now consider the following definition by Astels [**6**].

TABLE 3.3. Thickness of Cantor sets generated by continued radicals composed from $\{1, b\}$

b	$\tau(\mathcal{D})$	$\gamma(\mathcal{D})$
2	0.556376528100861	0.357481957646694
3	0.437824182513184	0.304504672989925
4	0.367002837961805	0.268472623296821
5	0.319183276235179	0.241955217281178
6	0.284382621892108	0.221415812581741
7	0.257737462417361	0.204921511936197
8	0.236573198321508	0.191313541845017
9	0.219287896775729	0.179849154047712
10	0.204859058883445	0.170027404759931
11	0.192601000815585	0.161496594991847
12	0.182035388923834	0.154001640415830
13	0.172817577278206	0.147352478873372
14	0.164692458768075	0.141404245840378
15	0.157466841086656	0.136044364725665
16	0.150991524167825	0.131183871468552
17	0.145149306072251	0.126751424729146
18	0.139846738350957	0.122689071824934
19	0.135008324546636	0.118949193258616
20	0.130572352758641	0.115492257032502
21	0.126487846651723	0.112285140960628
22	0.122712297883725	0.109299860805866
23	0.119209954632136	0.106512593225923
24	0.115950512495125	0.103902916121140
25	0.112908100950810	0.101453211504479

DEFINITION 3.12. *Let $\mathcal{C}_j$ be the Cantor set generated by continued radicals from $\{j_1, j_2\}$. Then I_j is the interval $[\phi_{j_1}, \phi_{j_2}]$. We say a sequence of intervals $(I_1, \ldots, I_k)$ is* <u>*compatible*</u> *if $|I_{r+1}| \geq |G^j_{1,1}|$ and $|I_1| + \cdots + |I_r| \geq |G^{r+1}_{1,1}|$ for $r = 1, \ldots, k-1$ and $j = 1, \ldots, r$ where $G^j_{1,1}$ is gap $G_{1,1}$ of $\mathcal{C}_j$.*

We note that in Astels' definition, he considers the gap of maximal size in $\mathcal{C}_j$; however, in our case, gap $G^j_{1,1}$ is the gap of maximal size.

Theorem 3.13 by Astels [**6**] tells when the summation of points of a Cantor set create an interval.

TABLE 3.4. Thickness of Cantor sets generated by continued radicals composed from $\{4, b\}$

b	$\tau(n)$	b	$\tau(n)$
5	0.285816254186594	27	0.110923577381784
6	0.259770836834869	28	0.108567206087050
7	0.239200307246454	29	0.106338304997015
8	0.222448777882524	30	0.104226052792637
9	0.208482689881702	31	0.102220859481861
10	0.196619767662607	32	0.100314192868143
11	0.186389439624914	33	0.0984984339069825
12	0.177455479884038	34	0.0967667554459796
13	0.169570560825889	35	0.0951130200179106
14	0.162548254233971	36	0.0935316932546668
15	0.156245086779371	37	0.0920177701829647
16	0.150548648487237	38	0.0905667122015165
17	0.145369483265354	39	0.0891743929612277
18	0.140635419367533	40	0.0878370517025801
19	0.136287518356967	41	0.0865512528682555
20	0.132277124248259	42	0.0853138510197214
21	0.128563676867600	43	0.0841219602556500
22	0.125113066418082	44	0.0829729274666039
23	0.121896378023872	45	0.0818643088712489
24	0.118888921721093	46	0.0807938493697479
25	0.116069474375705	47	0.0797594643240645
26	0.113419680998674	48	0.0787592234358816

THEOREM 3.13. *Let k be a positive integer and for $j = 1, \cdots, k$ let $\mathcal{C}_j$ be a Cantor set derived from I_j. Put $S_\gamma = \gamma(\mathcal{C}_1) + \cdots + \gamma(\mathcal{C}_k)$ and assume that $(I_1, \cdots, I_k)$ is compatible. If $S_\gamma \geq 1$, then $\mathcal{C}_1 + \cdots + \mathcal{C}_k = I_1 + \cdots + I_k$.*

Otherwise, $\gamma(\mathcal{C}_1 + \cdots + \mathcal{C}_k) \geq S_\gamma$ and $dim_H(\mathcal{C}_1 + \cdots + \mathcal{C}_k) \geq \dfrac{log(2)}{log\left(1 + \frac{1}{S_\gamma}\right)}$.

Notice that in Theorem 3.13, $\mathcal{C}_1 + \cdots + \mathcal{C}_k = I_1 + \cdots + I_k$ implies that we get a solid interval as the sum of Cantor sets. The analogous question for when an interval is obtained as the sum of Cantor sets generated by partial fractions with small partial quotients was considered by Astels [**6**] and Hlavka [**13**]. We have that

Table 3.5. Thickness of Cantor sets generated by continued radicals composed from $\{a, 50\}$

a	$\tau(n)$	a	$\tau(n)$
1	0.0719453007816807	26	0.0773909563241597
2	0.0746331011003041	27	0.0773203095231361
3	0.0760266087893125	28	0.0772500423692610
4	0.0768541403989782	29	0.0771802622971120
5	0.0773768979382290	30	0.0771110545742078
6	0.0777162826204823	31	0.0770424863761889
7	0.0779373990911899	32	0.0769746100714446
8	0.0780785236296758	33	0.0769074658814049
9	0.0781636558225034	34	0.0768410840447848
10	0.0782085788450794	35	0.0767754865854873
11	0.0782240615400340	36	0.0767106887621612
12	0.0782176696188142	37	0.0766467002608114
13	0.0781948463590507	38	0.0765835261790597
14	0.0781595858088356	39	0.0765211678407435
15	0.0781148673055409	40	0.0764596234717979
16	0.0780629444189142	41	0.0763988887622992
17	0.0780055420312839	42	0.0762798211349541
18	0.0779439937378542	43	0.0762798211349541
19	0.0778793394868764	44	0.0762214707585727
20	0.0778123961414160	45	0.0761638957244926
21	0.0777438092381146	46	0.0761070847036113
22	0.0776740914620632	47	0.0760510257105330
23	0.0776036515908413	48	0.0759957062641625
24	0.0775328165046553	49	0.0759411135222135
25	0.0774618480879752		

partial fractions with small partial quotients are denoted by $F(n)$ and have the form

$$1 + \frac{1}{a + \frac{1}{b + \frac{1}{c + \dots}}},$$

where $\{a, b, c, \dots\}$ are from $\{1, 2, \dots, n\}$.

It is also important to recognize that if you have a very small interval that cannot span a large gap, then you cannot obtain a solid interval as the sum of Cantor sets generated by continued fractions with components from $\{a, b\}$. For example, consider the Cantor sets $\mathcal{C}_j$ with components from $\{j_1, j_2\}$ and $\mathcal{C}_k$ with components from $\{k_1, k_2\}$. If we do not have that $|I_k| = \phi_{k_2} - \phi_{k_1} > G^j_{1,1}$ and

$|I_j| = \phi_{j_2} - \phi_{j_1} > G^k_{1,1}$ then we would not be able to form an interval with $\mathcal{C}_j + \mathcal{C}_k$. If we have that either I_j or I_k is much smaller than the maximal gap of $\mathcal{C}_k$ or $\mathcal{C}_j$ respectively, then we cannot form an interval. Thus, the assumption of compatibility is very important in Theorem 3.13. We give results regarding the compatibility of intervals in Proposition 3.14 and Corollary 3.15.

PROPOSITION 3.14. *To determine compatibility of intervals for Theorem* 3.13, *it is sufficient to show that the shortest interval is larger than the longest gap. If this occurs, then the intervals are compatible. If this does not occurs, then the intervals may or may not be compatible.*

COROLLARY 3.15. *If* $\mathcal{C}_1 = \mathcal{C}_2 = \ldots = \mathcal{C}_k$, *then* $I_1, I_2, \ldots, I_k$ *are compatible.*

Table 3.6 on page 32 gives the number of Cantor sets generated by $\{a, b\}$ required to create an interval for $b = 1, 2, \ldots, 20$. To calculate these numbers, we must take the ceiling of $\dfrac{1}{\gamma(C_k)}$. Consider the following example.

EXAMPLE 3.16. Let $\mathcal{C}_{1,2}$ be the Cantor set generated by continued radicals composed from $\{1, 2\}$. Table 3.2 shows that $\gamma(\mathcal{C}_{1,2}) \approx 0.357481957646694$. Now, $\dfrac{1}{\gamma(\mathcal{C}_{1,2})} \approx 2.79734397$. Furthermore, $\lceil 2.79734397 \rceil = 3$. Thus, we need at least three copies of $\mathcal{C}_{1,2}$ to guarantee that their sum is an interval.

Similar calculations can be done to produce Table 3.6. This data leads to the following conjecture.

CONJECTURE 3.17. *The minimum number of Cantor sets generated by continued radicals composed from* $\{a, b\}$ *required to guarantee an interval is three.*

This conjecture suggests that we cannot take two Cantor sets and add their elements together to create an interval.

Table 3.6. Minimum Number of Cantor sets generated by continued radicals composed from $\{a, b\}$ required to guarantee an interval

a \ b	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	3	4	4	5	5	5	6	6	6	7	7	7	8	8	8	8	9	9	9
2		4	4	5	5	5	6	6	6	7	7	7	8	8	8	8	9	9	9
3			5	5	5	6	6	6	7	7	7	7	8	8	8	8	9	9	9
4				5	5	6	6	6	7	7	7	7	8	8	8	8	9	9	9
5					5	6	6	6	7	7	7	7	8	8	8	8	9	9	9
6						6	6	6	7	7	7	7	8	8	8	8	9	9	9
7							6	6	7	7	7	8	8	8	8	8	9	9	9
8								7	7	7	7	8	8	8	8	9	9	9	9
9									7	7	7	8	8	8	8	9	9	9	9
10										7	7	8	8	8	8	9	9	9	9
11											7	8	8	8	8	9	9	9	9
12												8	8	8	8	9	9	9	9
13													8	8	8	9	9	9	9
14														8	9	9	9	9	9
15															9	9	9	9	9
16																9	9	9	9
17																	9	9	9
18																		9	9
19																			9

Notice that in Table 3.6, we have that $\mathcal{C}_i = \mathcal{C}_j$ for all i, j. We should also consider adding the elements of Cantor sets $\mathcal{C}_1, \mathcal{C}_2, \ldots$ generated by continued radicals composed from $\{a, b\}$ whenever $\mathcal{C}_i \neq \mathcal{C}_j$ for some $i \neq j$. We consider this scenario in Table 3.7 on page 33.

We need to consider the compatibility of $\mathcal{C}_{1,4}$, $\mathcal{C}_{5,8}$, and $\mathcal{C}_{6,10}$. We have that the shortest interval is $[\phi_5, \phi_8]$ with length approximately 0.580993. Furthermore, the longest gap is $G_{1,1}^{1,4}$ with length approximately 0.136642601485087. Thus we have that the shortest interval is longer than the longest gap. Hence, $\mathcal{C}_{1,4}$, $\mathcal{C}_{5,8}$, and $\mathcal{C}_{6,10}$ are compatible.

TABLE 3.7. Minimum Number of Cantor sets generated by continued radicals composed from $\{1, 4\}$, $\{5, 8\}$, and $\{6, 10\}$ required to guarantee an interval

# of $\{1, 4\}$	# of $\{5, 8\}$	# of $\{6, 10\}$	Total # of Cantor sets
0	1	6	7
0	2	4	6
0	3	3	6
0	4	2	6
0	5	1	6
1	0	5	6
1	1	4	6
1	2	3	6
1	3	2	6
1	4	1	6
1	5	0	6
2	0	3	5
2	1	2	5
2	2	1	5
2	3	0	5
3	0	2	5
3	1	1	5
3	2	0	5

Chapter 4

Measure

Generally, the Cantor set is thought to have measure zero. In this chapter we examine the measure of the middle-thirds Cantor set, the ε-Cantor set, and the Cantor sets generated by continued radicals generated by the set $\{a, b\}$.

Proposition 4.1. *The middle-thirds Cantor set, $\mathcal{C}_3$ has measure zero.*

Proof. Recall the notation from Chapter 1. We have that I contains no gaps (i.e. removed intervals), C_0 contains $1 = 2^0$ gap, $C_1 \cup C_0^c$ contains $2 = 2^1$ gaps, $\ldots$ $C_m \left[\bigcup_{i=0}^{m-1} C_i^c \right]$ contains 2^m gaps, $\ldots$. Therefore, we have that on C_m, we remove 2^m intervals. Furthermore, we see that the length of the gaps removed will be 0, $\frac{1}{3}$, $\frac{1}{3^2}$, $\frac{1}{3^3}, \ldots$ for $I, C_0, C_1, C_2, \ldots$, respectively. Furthermore, we have that $\mu([0,1] \setminus \mathcal{C})$ is the sum of the lengths of the gaps; thus, we have

$$
\begin{aligned}
\mu([0,1] \setminus \mathcal{C}) &= \frac{2^0}{3^1} + \frac{2^1}{3^2} + \frac{2^2}{3^3} + \ldots \\
&= \sum_{m=0}^{\infty} \left(\frac{2^m}{3^{m+1}} \right) \\
&= \left(\frac{1}{3} \right) \sum_{m=0}^{\infty} \left(\frac{2}{3} \right)^m \\
&= \left(\frac{1}{3} \right) \frac{1}{1 - \frac{2}{3}} \\
&= \left(\frac{1}{3} \right) (3) = 1
\end{aligned}
$$

Thus we have that $\mu([0,1] \setminus \mathcal{C}_3) = 1 = \mu([0,1]) - \mu(\mathcal{C}_3) = 1 - \mu(\mathcal{C}_3)$. Thus, $\mu(\mathcal{C}_3) = 0$. □

It is important to note that the measure of Cantor sets is not always zero. Aliprantis and Burkinshaw [**1**] give an example of such a set.

Example 4.2 (The ε-Cantor set). To construct the ε-Cantor set, $\mathcal{C}_\varepsilon$, we begin with $0 < \varepsilon < 1$ and let $\delta = 1 - \varepsilon$. We begin with the closed interval $[0, 1] = A_0$. From the center of this interval, we remove an open interval with length $\frac{\delta}{2}$. Thus we have that $A_1 = [0, (\frac{1}{2} - \frac{\delta}{4})] \cup [(\frac{1}{2} + \frac{\delta}{4}), 1]$. Then $\mu(A_1) = 1 - \frac{\delta}{2}$.

Now, from the center of each of the two disjoint closed intervals of A_1, we remove an open interval of length $\frac{\delta}{2^3}$. The union of these four disjoint closed sets is A_2. Then we have $\mu(A_2) = 1 - [\mu(A_1) - (\frac{\delta}{8} + \frac{\delta}{8})] = 1 - (\frac{1}{2} + \frac{1}{4})\delta = 1 - \frac{3\delta}{4}$.

We have that A_n is the disjoint union of 2^n closed intervals of the same length that satisfy $\mu(A_n) = 1 - (\frac{1}{2} + \ldots + \frac{1}{2^n} + \frac{1}{2^{n+1}})\delta$. From the center of each of the 2^n disjoint closed intervals of A_n, we remove an open interval of length $\frac{\delta}{2^{2n+1}}$. Then we have that A_{n+1} is the union of these 2^{n+1} closed intervals. Thus we have that $\mu(A_{n+1}) = 1 - (\frac{1}{2} + \ldots + \frac{1}{2^n} + \frac{1}{2^{n+1}})\delta$.

We then have that $\mathcal{C}_\varepsilon = \bigcap\limits_{n=1}^{\infty} A_n$. Furthermore,

$$\mu(\mathcal{C}_\varepsilon) = \lim \mu(A_n) = 1 - \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right) \delta = 1 - \delta = \varepsilon > 0.$$

To see that the ε-Cantor set is in fact a Cantor set, we need to consider the following definition given by Cabrelli, Paulaskas, Lithuania, and Shonkwiler [**9**].

Definition 4.3. *A <u>general Cantor set</u> is a compact, perfect, totally disconnected subset of the real line.*

Note that Astels [**6**] gives a weaker definition of generalized Cantor sets. He states that a generalized Cantor set is any set $\mathcal{C}$ of real numbers of the form $\mathcal{C} = I \setminus \bigcup\limits_{i \geq 1} G_i$ where I is a finite closed interval and $\{G_i : i \geq 1\}$ is a countable collection of disjoint open intervals contained in I. This definition seems to not be equivalent to Definition 4.3. For our results, we will use the definition by Cabrelli, Paulaskas, Lithuania, and Shonkwiler [**9**].

PROPOSITION 4.4. *The ε-Cantor set, $\mathcal{C}_\varepsilon$, is a general Cantor set.*

PROOF. Note that the ε-Cantor set is closed and bounded; thus, by the Heine-Borel theorem, we have that it is a compact subset of $\mathbb{R}$.

Now, we need to show that $\mathcal{C}_\varepsilon$ is a perfect subset of $\mathbb{R}$. Note that a set A is said to be perfect if and only if every point in A is a limit point of A, or equivalently, if no point in A is isolated.

Now, consider the midpoints of the removed intervals on level n. We have that these are $M_n = \left\{ \frac{m}{2^n} \right\}$ where $m \in \{1, 2, \ldots, 2^n - 1\}$. Since $M = \bigcup_{n=1}^{\infty} M_n$ is a set of dyadic fractions, we have that M is dense on $[0, 1]$.

Furthermore, between any two midpoints of gaps, we have an interval. By the construction of $\mathcal{C}_\varepsilon$, the endpoints of intervals are never removed. Let $x \in \mathcal{C}_\varepsilon \setminus \{0, 1\} = D$ and $B(x, r)$ be an open-ball centered at x with radius r. We can find n such that $B(x, r)$ contains at least two midpoints of gaps. Therefore, $B(x, r)$ contains at least one interval; hence, it contains at least 2 distinct points $y, z \in D$. Thus, we have that x is not isolated and since 0 and 1 are limit points, every point in $\mathcal{C}_\varepsilon$ is a limit point of $\mathcal{C}_\varepsilon$. Thus, $\mathcal{C}_\varepsilon$ is a perfect subset of $\mathbb{R}$.

Now, since the midpoints of gaps are dense, any alleged interval will have points removed since midpoints of gaps are removed. Thus, $\mathcal{C}_\varepsilon$ is totally disconnected.

Therefore, $\mathcal{C}_\varepsilon$ is a compact, perfect, totally disconnected subset of the real line; thus, it is a general Cantor set. $\square$

The ε-Cantor set demonstrates the necessity of considering the measure of the Cantor sets generated by continued radicals generated with two values a, b.

PROPOSITION 4.5. *The measure of the Cantor sets generated by continued radicals generated with two values a, b is 0.*

Proof. Case 1: Suppose $a > 1$. Then the length of the left bridge is $\sqrt{\overrightarrow{a}, \phi_b} - \phi_a = |B_{n,1}|$. By Corollary 2.5, we have that the left-most bridge is the largest on a given level. Thus, each of the 2^n bridges on level n are smaller than or equal to the length of the left-most bridge.

Now $\sum_{i=1}^{2^n} |B_{n,i}| < 2^n |B_{n,1}| = 2^n(\sqrt{\overrightarrow{a}, \phi_b} - \phi_a)$.

We now need to consider $\lim_{n\to\infty} 2^n(\sqrt{\overrightarrow{a}, \phi_b} - \phi_a)$. We know that $f(x) = \sqrt{a_1, a_2, \ldots, a_n, x}$ is a contraction. Thus, by Proposition 1.13, we have that $(\sqrt{a_1, \ldots a_n, \phi_b} - \sqrt{a_1, \ldots, a_n, \phi_a}) \leq \frac{1}{2^{n+1}\sqrt{a^{n+1}}}(\phi_b - \phi_a)$. Therefore,

$$\begin{aligned} \lim_{n\to\infty} 2^n(\sqrt{\overrightarrow{a}, \phi_b} - \phi_a) &\leq \lim_{n\to\infty} 2^n \left(\frac{1}{2^{n+1}\sqrt{a^{n+1}}}\right)(\phi_b - \phi_a) \\ &= (\phi_b - \phi_a) \lim_{n\to\infty} \left(\frac{1}{2\sqrt{a^{n+1}}}\right) = 0. \end{aligned} \tag{4.1}$$

Thus, for $a > 1$, we have that the measure of the generated Cantor set is 0.

Case 2: Suppose $a = 1$. We have that the length of the bridges on level n is $\sqrt{w_1, w_2, \ldots, w_n, 1 + \phi_b} - \sqrt{w_1, w_2, \ldots w_n, 1 + \phi_1}$. Furthermore, $\overrightarrow{w}$ is a vector of length n generated by $\{1, b\}$. We have that

$$\begin{aligned} &\sqrt{w_1, w_2, \ldots, w_n, 1 + \phi_b} - \sqrt{w_1, w_2, \ldots w_n, 1 + \phi_1} \\ &\qquad < \frac{1}{2^{n+1}\sqrt{w_1 w_2 \cdots w_n 1}}(\phi_b - \phi_1) \\ &\qquad = \frac{1}{2^{n+1}\sqrt{b^k}}(\phi_b - \phi_1) \text{ if } \overrightarrow{w} \text{ has exactly } k\ b\text{'s.} \end{aligned}$$

Of the 2^n bridges, we need to consider how many actually have exactly k occurrences of b in their $\overrightarrow{w}$. We have that $\binom{n}{k}$ of these exist. Therefore, the sum of the lengths of the 2^n bridges on level n is less than

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^{n+1}\sqrt{b^k}}(\phi_b - \phi_1) = \frac{(\phi_b - \phi_1)}{2} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^n} \frac{1}{\sqrt{b^k}}.$$

Here we have that $\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^n} = 1$ since the sum of row n of Pascal's triangle is 2^n. Thus, $\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^n \sqrt{b^k}}$ is a weighted average of 1, $\frac{1}{\sqrt{b}}$, $\frac{1}{\sqrt{b^2}}$, ..., $\frac{1}{\sqrt{b^n}}$.

Now, let $\varepsilon > 0$ be given. We need to find the smallest j such that $\frac{1}{\sqrt{b^j}} < \frac{\varepsilon}{2}$. This gives us j terms that are greater than $\frac{\varepsilon}{2}$, namely 1, $\frac{1}{\sqrt{b}}, \frac{1}{\sqrt{b^2}}, \ldots, \frac{1}{\sqrt{b^{j-1}}}$. Furthermore, the remaining $n - j$ terms are all below $\frac{\varepsilon}{2}$. As $n \to \infty$, we have that their weights dominate the summation which makes the weighted average less than ε for all $\varepsilon > 0$.

Thus, for $a = 1$, we have that the measure is in fact zero. $\square$

For the case of $a = 1$, we have an alternate proof as shown below.

Proof 2 of Proposition 4.5. Suppose $a = 1$. We have that the length of the bridges on level n is $\sqrt{w_1, w_2, \ldots, w_n, 1 + \phi_b} - \sqrt{w_1, w_2, \ldots w_n, 1 + \phi_1}$. Furthermore, $\overrightarrow{w}$ is a vector of length n generated by $\{1, b\}$. We have that

$$\begin{aligned}
\sqrt{w_1, w_2, \ldots, w_n, 1 + \phi_b} - \sqrt{w_1, w_2, \ldots w_n, 1 + \phi_1} & \\
&< \frac{1}{2^{n+1}\sqrt{w_1 w_2 \cdots w_n (1)}} (\phi_b - \phi_a) \\
&= \frac{1}{2^{n+1}\sqrt{b^k}} (\phi_b - \phi_1) \text{ if } \overrightarrow{w} \text{ has exactly } k\ b\text{'s.}
\end{aligned}$$

Of the 2^n bridges, we need to consider how many actually have exactly k occurrences of b in their $\overrightarrow{w}$. We have that $\binom{n}{k}$ of these exist. Therefore, the sum of the lengths of the 2^n bridges on level n is less than

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^{n+1}\sqrt{b^k}} (\phi_b - \phi_1) \\
&< \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^{n}\sqrt{b^k}} (\phi_b - \phi_1) \text{ for } 2 \le b \in \mathbb{Z} \\
&< \sum_{k=0}^{2n} \binom{2n}{n} \frac{1}{2^{2n}\sqrt{2^k}} (\phi_b - \phi_1) \text{ (wlog use } 2n).
\end{aligned}$$

Romik [**21**] gives us Stirling's approximation which states that $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Thus we have that $(2n)! \sim 2\sqrt{\pi n}\left(\frac{2n}{e}\right)^{2n}$. Thus, we consider

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k=0}^{2n} \binom{2n}{n} \frac{1}{2^{2n}\sqrt{2^k}} (\phi_b - \phi_1) \\
&= \lim_{n\to\infty} \sum_{k=0}^{2n} \frac{(2n)!}{n!^2} \frac{1}{2^{2n}\sqrt{2^k}} (\phi_b - \phi_1) \\
&= \lim_{n\to\infty} \sum_{k=0}^{2n} \frac{2\sqrt{\pi n}\left(\frac{2n}{e}\right)^{2n}}{(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n)^2} \frac{1}{2^{2n}\sqrt{2^k}} (\phi_b - \phi_1) \\
&= \lim_{n\to\infty} \sum_{k=0}^{2n} \frac{1}{\sqrt{\pi n 2^k}} (\phi_b - \phi_1) \\
&= (\phi_b - \phi_1) \lim_{n\to\infty} \frac{1}{\sqrt{\pi n}} \sum_{k=0}^{2n} \frac{1}{\sqrt{2^k}} \\
&= (\phi_b - \phi_1) \left(\lim_{n\to\infty} \frac{1}{\sqrt{\pi n}}\right) \left(\lim_{n\to\infty} \sum_{k=0}^{2n} \frac{1}{\sqrt{2^k}}\right) = 0.
\end{aligned}$$

Note that $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\sqrt{2^k}}$ converges by the ratio test. Furthermore, $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{\pi n}} = 0$. Thus, the measure of the Cantor set generated by continued radicals generated by $\{1, b\}$ is zero. $\square$

We note here that there appears to be a relationship between this construction and the Gamma function. We have the common identity that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\displaystyle\binom{2n}{n} = \frac{\Gamma(2n+1)}{\Gamma^2(n+1)}$.

Chapter 5

Conclusion

We have been able to answer several questions regarding the lengths of gaps and bridges of Cantor sets generated by continued radicals. We have also studied the measure of these Cantor sets and gotten some results regarding their thickness.

The most important open question is the calculation of the thickness. Being able to find a closed form solution for the thickness of the Cantor sets generated by continued radicals would provide a nice result. This would allow us to obtain the exact thickness without using computational approximations. Since L'Hôpital's Rule gave us a more complicated expression, there may be a method of doing L'Hôpital's Rule in reverse to obtain a form that simplifies more easily. However, if this does not provide any insight, we may be able to use asymptotics to obtain a sharper approximation.

Even if the thickness cannot be calculated with a closed form solution, we give further consideration to Conjectures 3.10 and 3.17. Although numerical analysis provides evidence for these conjectures, there may be some other tools that can be used to prove them.

Furthermore, we have only considered continued radicals generated by $\{a, b\}$ where a, b are natural numbers. This does not consider continued radicals allowing zeros as their components. It would be interesting to see how allowing zero changes the results. Some preliminary numerical results indicate that if the thickness is unchanged, Conjecture 3.17 is false and the minimum number of Cantor sets required to guarantee an interval is two. However, as noted in the introduction, it has not been proven that continued radicals generated by $\{0, b\}$ is homeomorphic to the Middle-Thirds Cantor set. Therefore, it is necessary to consider this case.

This leads to the natural question of considering the use of all integers (including negative). With this, complex numbers arise creating complications with our constructions. However, Rhode [**20**] and Sizer and Wiredu [**24**] discuss continued radicals in the complex plane. Furthermore, Leighton and Thron [**17**] consider the analogous problem for continued fractions. Perhaps some of these results can be applied to continued radicals. Additionally, information may be obtained from the discussion by Wagon [**26**] regarding complex Cantor sets. Combining these two areas may produce some results regarding Cantor sets generated by complex continued radicals.

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