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An Orthogonal Scaling Vector Generating a Space of C^1 Cubic Splines Using Macroelements

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Abstract

The main result of this paper is the creation of an orthogonal scaling vector of four differentiable functions, two supported on $[-1, 1]$ and two supported on $[0, 1]$, that generates a space containing the classical spline space $\mathcal{S}_3^1(\mathbf{Z})$ of piecewise cubic polynomials on integer knots with one derivative at each knot. The author uses a macroelement approach to the construction, using differentiable fractal function elements defined on $[0, 1]$ to construct the scaling vector. An application of this new basis in an image compression example is provided.

AMS Subject Classification Numbers: 42C40, 65D15

Keywords: orthogonal bases, scaling vectors, spline spaces, macroelements, wavelets, multi-wavelets

1 Introduction

Orthogonal scaling vectors have numerous applications in signal and image processing, including image compression, denoising, and edge detection. Geronimo, Hardin, and Massopust were among the first to use more than one scaling function, in their construction of the GHM orthogonal scaling vector in [7]. The space generated by the GHM functions and their integer translates had approximation order 2, like the space generated by the single compactly-supported D4 orthogonal scaling function constructed by Daubechies in [3], but also contained the space of continuous piecewise linear polynomials on integer knots. Also, by using more than one function, they were able to build scaling functions and wavelets with symmetry properties. The only single compactly-supported scaling function with symmetry properties is the characteristic function on $[0, 1)$, known as the Haar basis. Other types of orthogonal scaling vectors have been constructed in [4], [8], [11], and [13] that have compactly-supported elements that lack continuous derivatives. Han and Jiang constructed a length-4 scaling vector in [9] that has approximation order 4 and two continuous derivatives.

Hardin and Kessler put the constructions of Geronimo and Hardin from [7] (with Massopust) and [4] (with Donovan) into a macroelement framework in [10]. The construction of scaling vectors from a single macroelement supported on $[0, 1]$ is important, since these types of scaling functions will be orthogonal interval-by-interval, and the intervals over which the basis is defined need not be of uniform length. This adaptability to an arbitrary partition of \mathbf{R} promises to give greater flexibility to the use of these types of bases in applications. While building bases on variable length intervals is outside the scope of this paper, we should keep in mind that most of the results here can be easily adapted to intervals of arbitrary length.

The main result in this paper is the creation of a orthogonal scaling vector of length 4 of compactly-supported, differentiable piecewise fractal interpolation functions, with symmetry properties, that generates a space including the classical spline space $\mathcal{S}_3^1(\mathbf{Z})$ of differentiable cubic polynomials on integer knots. This scaling vector has a symmetric/antisymmetric pair supported on $[-1, 1]$ and a symmetric/antisymmetric pair supported on $[0, 1]$. The construction will follow a macroelement approach; that is, we construct a macroelement with the desirable properties using fractal functions and then construct a scaling vector from the macroelement. Fractal functions have been used previously in [4], [6], [7], and [11] to construct orthogonal scaling vectors. We show that a shorter-length scaling vector can not be found with the same properties, and then show this new basis in use in an image compression example. We believe this to be the first scaling vector to be constructed with all of the above-mentioned properties.

1.1 Scaling Vectors

A vector $\Phi = (\phi_1, \phi_2, \dots, \phi_r)^T$ of functions defined on \mathbf{R}^k is said to be *refinable* if

$$\Phi = N^{\frac{k}{2}} \sum g_i \Phi(N \cdot -i) \quad (1)$$

for some integer dilation $N > 1$, $i \in \mathbf{Z}^k$, and for some sequence of $r \times r$ matrices g_i . (The normalization factor $N^{\frac{k}{2}}$ can be dropped, but is convenient for applications.) A *scaling vector* is a refinable vector Φ of square-integrable functions where the set of the components of Φ and their integer translates are linearly independent. An *orthogonal* scaling vector Φ is a scaling vector where the functions ϕ_1, \dots, ϕ_r are compactly supported and satisfy

$$\langle \phi_i, \phi_j(\cdot - n) \rangle = \delta_{i,j} \delta_{0,n}, \quad i, j \in \{1, \dots, r\}, \quad n \in \mathbf{Z}^k,$$

where the inner product is the standard $L^2(\mathbf{R}^k)$ integral inner product

$$\langle f, g \rangle = \int_{\mathbf{R}^k} f(x)g(x)dx$$

and δ is Kronecker's delta (1 if indices are equal, 0 otherwise.) A scaling vector Φ is said to *generate* a closed linear space denoted by

$$S(\Phi) = \text{clos}_{L^2} \text{span} \{ \phi_i(\cdot - j) : i = 1, \dots, r, j \in \mathbf{Z} \}.$$

Two scaling vectors Φ and $\tilde{\Phi}$ are *equivalent* if $S(\Phi) = S(\tilde{\Phi})$. The scaling vector $\tilde{\Phi}$ is said to *extend* Φ , or be an *extension* of Φ , if $S(\Phi) \subset S(\tilde{\Phi})$.

Scaling vectors are important because they provide a framework for analyzing functions in $L^2(\mathbf{R}^k)$. A *multiresolution analysis* (MRA) of $L^2(\mathbf{R}^k)$ of multiplicity r is a set of closed linear spaces (V_p) such that

1. $\dots \supset V_{-2} \supset V_{-1} \supset V_0 \supset V_1 \supset V_2 \dots$,
2. $\overline{\bigcup_{p \in \mathbf{Z}} V_p} = L^2(\mathbf{R}^k)$,
3. $\bigcap_{p \in \mathbf{Z}} V_p = \{0\}$,

4. $f \in V_0$ iff $f(N^{-j}\cdot) \in V_j$, and
5. there exists a set of functions ϕ_1, \dots, ϕ_r whose integer translates form a Riesz basis of V_0 .

From the above definitions, it is clear that scaling vectors can be used to generate MRA's, with $V_0 = S(\Phi)$. Jia and Shen proved in [15] that if the components of a scaling vector Φ are compactly-supported, then Φ will always generate an MRA. All the scaling vectors discussed in this paper will consist of compactly-supported functions, and therefore, will generate MRA's. A function vector $\Psi = (\psi_1, \dots, \psi_{r(N^k-1)})^T$, such that $\psi_i \in V_{-1}$ for $i = 1, \dots, r(N^k - 1)$ (see [14]) and such that $S(\Psi) = V_{-1} \ominus V_0$, is called a *multiwavelet*, and the individual ψ_i are called *wavelets*.

1.2 Macroelements on $[0, 1]$

We will use the notation $f^{(j)}(x)$ to denote the j^{th} derivative of $f(x)$, with the convention $f^{(0)}(x) = f(x)$. As a convenience, we will use the notation $f^{(j)}(0)$ and $f^{(j)}(1)$ to denote $\lim_{x \rightarrow 0^+} f^{(j)}(x)$ and $\lim_{x \rightarrow 1^-} f^{(j)}(x)$, respectively, although the notation may not always be mathematically rigorous.

A C^p macroelement defined on $[0, 1]$ is a vector of the form $(l_1, \dots, l_k, r_1, \dots, r_k, m_1, \dots, m_n)^T$ where the set of elements are linearly independent functions supported on $[0, 1]$ with p continuous derivatives such that

1. $l_i^{(j)}(0) = r_i^{(j)}(1) = 0$ for $i = 1, \dots, k$,
2. $m_i^{(j)}(0) = m_i^{(j)}(1) = 0$ for $i = 1, \dots, n$, and
3. $l_i^{(j)}(1) = r_i^{(j)}(0)$ for $i = 1, \dots, k$

for $j = 0, \dots, p$. A macroelement is *orthonormal* if $\langle l_i, r_j \rangle = 0$ for $i, j \in \{1, \dots, k\}$, $\langle l_i, m_j \rangle = \langle r_i, m_j \rangle = 0$ for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n\}$, and each element is normalized.

A macroelement Λ is *refinable* if there are $(2k+n) \times (2k+n)$ matrices p_0, \dots, p_{N-1} such that

$$\Lambda(x) = \sqrt{N} p_i \Lambda(Nx - i) \text{ for } x \in \left[\frac{i}{N}, \frac{i+1}{N} \right], \quad i = 0, \dots, N-1. \quad (2)$$

Because of the linear independence of the components of Λ , the matrix coefficients will be unique if they exist. Note that a refinable C^0 macroelement by necessity has $l_i(1) = r_i(0) \neq 0$ for some i , since if all elements were 0 at $x = 0, 1$, then each element would be 0 at N^j -adic points on $[0, 1]$ as $j \rightarrow \infty$. Hence, each element would be 0, a contradiction. Likewise, note that a refinable C^1 macroelement by necessity has $l'_i(1) = r'_i(0) \neq 0$ for some i , since if all elements had a zero derivative at $x = 0, 1$, then each element would be a constant function, also a contradiction.

Lemma 1. *A refinable C^p macroelement $\Lambda = (l_1, \dots, l_k, r_1, \dots, r_k, m_1, \dots, m_n)^T$ defined on $[0, 1]$ has an associated scaling vector Φ of length $k+n$ and support $[-1, 1]$. If the macroelement Λ is orthonormal, then the scaling vector Φ is equivalent to an orthogonal scaling vector.*

Proof. Let $l = (l_1, \dots, l_k)^T$, $r = (r_1, \dots, r_k)^T$, and $m = (m_1, \dots, m_n)^T$. Let a_i^l , a_i^r , and a_i^m be the $k \times k$, $k \times k$, and $k \times n$ matrices, respectively, such that

$$\begin{bmatrix} a_i^l & a_i^r & a_i^m \end{bmatrix} \Lambda(2x - i) = l(x) \text{ if } x \in \left[\frac{i}{N}, \frac{i+1}{N} \right], \quad i = 0, \dots, N-1.$$

Likewise, let b_i^l , b_i^r , and b_i^m be the $k \times k$, $k \times k$, and $k \times n$ matrices, respectively, such that

$$\begin{bmatrix} b_i^l & b_i^r & b_i^m \end{bmatrix} \Lambda(2x - i) = r(x) \text{ if } x \in \left[\frac{i}{N}, \frac{i+1}{N} \right], \quad i = 0, \dots, N-1$$

and let c_i^l , c_i^r , and c_i^m be the $n \times k$, $n \times k$, and $n \times n$ matrices, respectively, such that

$$\begin{bmatrix} c_i^l & c_i^r & c_i^m \end{bmatrix} \Lambda(2x - i) = m(x) \text{ if } x \in \left[\frac{i}{N}, \frac{i+1}{N} \right], \quad i = 0, \dots, N-1.$$

Then the matrix coefficients in (2) can be written in block-matrix form

$$p_i = \begin{bmatrix} a_i^l & a_i^r & a_i^m \\ b_i^l & b_i^r & b_i^m \\ c_i^l & c_i^r & c_i^m \end{bmatrix}.$$

Note that many of the block matrices are redundant: $a_{i-1}^l = a_i^r$, $b_{i-1}^l = b_i^r$, and $c_{i-1}^l = c_i^r$ for $i = 0, \dots, N-1$ since the macroelement components are continuous, and $a_{N-1}^l = b_0^r$ due to the endpoint conditions of the macroelement. Also, many of the block matrices are zero matrices: $a_0^r = b_{N-1}^l = 0_{k \times k}$ and $c_0^r = c_{N-1}^l = 0_{n \times k}$ due to the endpoint conditions of the macroelement.

Define

$$\phi_i(x) = \frac{1}{\sqrt{2}} \begin{cases} l_i(x+1) & \text{for } x \in [-1, 0] \\ r_i(x) & \text{for } x \in [0, 1] \end{cases}, \quad i = 1, \dots, k, \text{ and} \quad (3)$$

$$\phi_{k+i}(x) = m_i(x) \text{ for } x \in [0, 1], \quad i = 1, \dots, n. \quad (4)$$

Then the function vector $\Phi = (\phi_1, \dots, \phi_{k+n})^T$ satisfies (1), with

$$g_i = \begin{bmatrix} a_{N+i}^r & a_{N+i}^m \\ 0_{n \times k} & 0_{n \times n} \end{bmatrix} \quad i = -N, \dots, -1 \text{ and } g_i = \begin{bmatrix} b_i^r & b_i^m \\ c_i^r & c_i^m \end{bmatrix} \quad i = 0, \dots, N-1.$$

Hence, Φ is refinable, and supported completely in $[-1, 1]$.

If Λ is orthonormal, then by definition, Φ meets the criteria of an orthogonal scaling vector, except that possibly $\langle \phi_i, \phi_j \rangle \neq 0$ for $i, j \in \{1, \dots, k\}$, $i \neq j$, and for $i, j \in \{k+1, \dots, k+n\}$, $i \neq j$. However, we may replace $\{\phi_1, \dots, \phi_k\}$ with an orthonormal set $\{\tilde{\phi}_1, \dots, \tilde{\phi}_k\}$ and $\{\phi_{k+1}, \dots, \phi_{k+n}\}$ with an orthonormal set $\{\tilde{\phi}_{k+1}, \dots, \tilde{\phi}_{k+n}\}$, so that $\{\tilde{\phi}_1, \dots, \tilde{\phi}_k, \tilde{\phi}_{k+1}, \dots, \tilde{\phi}_{k+n}\}$ is an orthogonal scaling vector. \square

Let $\text{span } \Lambda$ refer to the span of the elements of Λ . Two macroelements Λ and $\tilde{\Lambda}$ are *equivalent* if $\text{span } \Lambda = \text{span } \tilde{\Lambda}$. The macroelement $\tilde{\Lambda}$ is said to *extend* Λ , or be an *extension* of Λ , if $\text{span } \Lambda \subset \text{span } \tilde{\Lambda}$. In this paper, we will extend macroelements for the purpose of extending scaling vectors, using the following lemma. We use the notation $\chi_{[a,b]}$ to be the characteristic function defined by

$$\chi_{[a,b]} = \begin{cases} 1 & \text{for } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2. Let Λ be a refinable C^p macroelement defined on $[0, 1]$, and let Φ be the associated scaling vector as defined in (3) and (4), for $p = 0, 1$. If $\tilde{\Lambda}$ is a C^p macroelement extension of Λ , then the associated scaling vector $\tilde{\Phi}$ as defined in (3) and (4) is an extension of Φ .

Proof. Let $\Lambda = \{l_1, \dots, l_k, r_1, \dots, r_k, m_1, \dots, m_n\}$ and $\tilde{\Lambda} = \{\tilde{l}_1, \dots, \tilde{l}_{k'}, \tilde{r}_1, \dots, \tilde{r}_{k'}, \tilde{m}_1, \dots, \tilde{m}_{n'}\}$, where $\tilde{\Lambda}$ is an extension of Λ , so $k \leq k'$ and $n \leq n'$. Consider a basis element $\phi \in \{\phi_i(\cdot - j) : \phi_i \in \Phi, i \in \{1, \dots, k+n\}, j \in \mathbf{Z}\}$. If $\text{supp } \phi \subset [j, j+1]$ for some $j \in \mathbf{Z}$, then $\phi(x+j) \in \text{span}\{m_1, \dots, m_n\} \subset \text{span}\{\tilde{m}_1, \dots, \tilde{m}_{n'}\}$. From the definition of $\tilde{\Phi}$ in (4), then $\phi \in S(\tilde{\Phi})$.

Otherwise, $\text{supp } \phi \subset [j, j+2]$ for some $j \in \mathbf{Z}$. Let $l = \phi\chi_{[j, j+1]}$ and $r = \phi\chi_{[j+1, j+2]}$. Then $l(x+j) \in \text{span}\{l_1, \dots, l_k, m_1, \dots, m_n\} \subset \text{span}\{\tilde{l}_1, \dots, \tilde{l}_{k'}, \tilde{m}_1, \dots, \tilde{m}_{n'}\}$ and $r(x+j+1) \in \text{span}\{r_1, \dots, r_k, m_1, \dots, m_n\} \subset \text{span}\{\tilde{r}_1, \dots, \tilde{r}_{k'}, \tilde{m}_1, \dots, \tilde{m}_{n'}\}$. From the match-up conditions in the definition of the macroelement and the definition of $\tilde{\Phi}$ in (3), then $\phi \in S(\tilde{\Phi})$. Thus, $S(\Phi) \subset S(\tilde{\Phi})$ and $\tilde{\Phi}$ is an extension of Φ . \square

In the following example, we show a simple way to extend a macroelement, and hence, a scaling vector.

Example 1. Let $l_1 = x\chi_{[0,1]}$ and $r_1 = (1-x)\chi_{[0,1]}$. Then $\Lambda = (l_1, r_1)^T$ is a C^0 macroelement. It is also refinable, since

$$\Lambda(x) = \sqrt{2} \begin{cases} \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \Lambda(2x) & \text{if } x \in [0, \frac{1}{2}] \\ \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{2\sqrt{2}} \end{bmatrix} \Lambda(2x-1) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}.$$

Therefore, we have the scaling vector $\Phi = (\phi_1)$ given in (3) and (4), with

$$\Phi(x) = \sqrt{2} \left[\frac{1}{2\sqrt{2}} \Phi(2x+1) + \frac{1}{\sqrt{2}} \Phi(2x) + \frac{1}{2\sqrt{2}} \Phi(2x-1) \right].$$

Then Φ is the linear B-spline, and generates $S(\Phi) = \mathcal{S}_1^0(\mathbf{Z})$, the spline space of continuous piecewise linear polynomials on integer knots.

Both Λ and Φ can be extended by the addition of the function $m_1(x) = \phi_2 = 4x(1-x)\chi_{[0,1]}$. Then $\tilde{\Lambda} = (l_1, r_1, m_1)^T$ is refinable, since

$$\tilde{\Lambda}(x) = \sqrt{2} \begin{cases} \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{4\sqrt{2}} \end{bmatrix} \tilde{\Lambda}(2x) & \text{if } x \in [0, \frac{1}{2}] \\ \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{4\sqrt{2}} \end{bmatrix} \tilde{\Lambda}(2x-1) & \text{if } x \in [\frac{1}{2}, 1] \end{cases},$$

and, by Lemma 1, we have the scaling vector $\tilde{\Phi} = (\phi_1, \phi_2)^T$, with

$$\tilde{\Phi}(x) = \sqrt{2} \left(\begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \tilde{\Phi}(2x+1) + \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{4\sqrt{2}} \end{bmatrix} \tilde{\Phi}(2x) + \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{4\sqrt{2}} \end{bmatrix} \tilde{\Phi}(2x-1) \right).$$

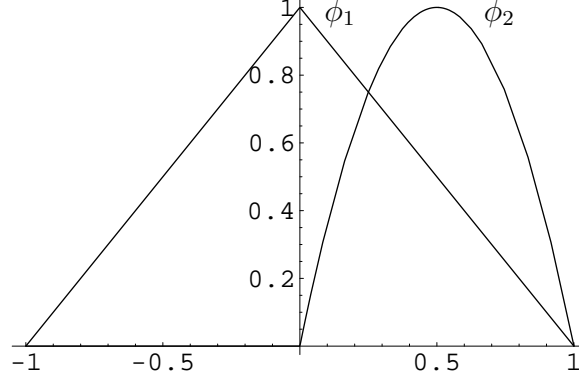


Figure 1: The scaling functions ϕ_1 and ϕ_2 from Example 1.

By Lemma 2, $\tilde{\Phi}$ is an extension of Φ . In fact, $S(\tilde{\Phi}) = \mathcal{S}_2^0(\mathbf{Z}) \supset \mathcal{S}_1^0(\mathbf{Z})$. Both functions are illustrated in Figure 1.

1.3 Fractal Interpolation Functions

Let $C_0([0, 1])$ denote the space of continuous functions defined over $[0, 1]$ that are 0 at $x = 0, 1$. Let $C_1([0, 1])$ denote the subspace of differentiable functions in $C_0([0, 1])$ whose derivatives are 0 at $x = 0, 1$. Let Λ be a refinable macroelement of length n , and let Π be a function vector of length k defined by

$$\Pi(x) = p_i \Lambda(Nx - i) \text{ for } x \in \left[\frac{i}{N}, \frac{i+1}{N} \right], i = 0, \dots, N-1,$$

for some $k \times n$ matrices p_i such that $\Pi(x) \in C_0([0, 1])^k$. Then a vector Γ of the form

$$\Gamma(x) = \Pi(x) + \sum_{i=0}^{N-1} s_i \Gamma(Nx - i) \in C_0([0, 1])^k, \quad (5)$$

where each s_i is a $k \times k$ matrix and $\max_i \|s_i\|_\infty < 1$, is a vector of *fractal interpolation functions* (FIF's). (See [1] and [2] for a more detailed introduction to FIF's.) By definition, the vector $\tilde{\Lambda} = (\Lambda^T, \Gamma^T)^T$ is a refinable C^0 macroelement that extends Λ .

Lemma 3. *Let Γ be a FIF satisfying (5). If $\Pi(x) \in C_1([0, 1])^k$ and $\max_{i=1, \dots, N-1} \|s_i\|_\infty < \frac{1}{N}$, then $\Gamma \in C_1([0, 1])^k$.*

Proof. Since Γ satisfies (5), then

$$\Gamma'(x) = \Pi'(x) + \sum_{i=0}^{N-1} N s_i \Gamma'(Nx - i).$$

Since $\Pi'(x) \in C_0([0, 1])^k$ and $\max_{i=1, \dots, N-1} \|N s_i\|_\infty = N \max_{i=1, \dots, N-1} \|s_i\|_\infty < N \frac{1}{N} = 1$, then $\Gamma'(x)$ is by definition a FIF, and $\Gamma'(x) \in C_0([0, 1])^k$. Therefore, $\Gamma(x) \in C_1([0, 1])^k$. \square

Consider a C^0 or C^1 macroelement $\Lambda = (l_1, \dots, l_k, r_1, \dots, r_k, m_1, \dots, m_n)^T$ defined on $[0, 1]$ that is not orthogonal. We can not simply apply the Gram-Schmidt process to the components of Λ to obtain an orthonormal macroelement, since the resulting functions will not satisfy the endpoint criteria. In fact, we can not apply the process to any subset of elements that includes a l_i and r_j and still have the same type of macroelement. (We could go from a C^1 macroelement to a C^0 macroelement, but the associated scaling vector would not be an extension of the original.) However, we can apply the Gram-Schmidt process to the set of functions $M = \{m_1, \dots, m_n\}$ to get $\tilde{M} = \{\tilde{m}_1, \dots, \tilde{m}_n\}$, and then subtract P_M , the orthogonal projection onto the space spanned by M , from each of the other elements, giving the equivalent macroelement

$$\tilde{\Lambda} = ((I - P_M)l_1, \dots, (I - P_M)l_k, (I - P_M)r_1, \dots, (I - P_M)r_k, \tilde{m}_1, \dots, \tilde{m}_n)^T.$$

If

$$\langle (I - P_M)l_i, (I - P_M)r_j \rangle = 0 \text{ for } i, j = 1, \dots, k, \quad (6)$$

(and each element is normalized), then $\tilde{\Lambda}$ is an orthonormal macroelement. This is the fractal function approach for extending a macroelement: add FIF's to the set M , hence the macroelement, so that (6) is satisfied. (See [5] for a broader discussion on constructing intertwined MRA's.)

Example 2. The scaling vector shown in this example was originally constructed by Geronimo, Hardin, and Massopust in [7], although not in the macroelement context, and is reconstructed by Hardin and Kessler in detail using macroelements in [10]. It is widely known as the GHM scaling vector.

Consider from Example 1 the C^0 macroelement Λ and the scaling vector $\Phi = (\phi_1)$ that generates $\mathcal{S}_1^0(\mathbf{Z})$. In order to extend Λ to an orthonormal C^0 macroelement, we construct a FIF satisfying

$$u(x) = \phi_1(2x - 1) + s_0 u(2x) + s_1 u(2x - 1), \quad \max_{i=0,1} |s_i| < 1,$$

such that $\langle (I - P_u)l_1, (I - P_u)r_1 \rangle = 0$. It was shown in [7] and [10] that the orthogonality condition is satisfied by $s_0 = s_1 = -\frac{1}{5}$. By letting

$$\check{l}_1 = \frac{(I - P_u)l_1}{\|(I - P_u)l_1\|}, \quad \check{r}_1 = \frac{(I - P_u)r_1}{\|(I - P_u)r_1\|}, \quad \text{and} \quad \check{m}_1 = \frac{u}{\|u\|},$$

we have the orthonormal C^0 macroelement $\check{\Lambda} = (\check{l}_1, \check{r}_1, \check{m}_1)^T$, equivalent to $(l_1, r_1, u)^T$ and an extension of Λ . The associated scaling vector $\check{\Phi} = (\check{\phi}_1, \check{\phi}_2)^T$ defined in (3) and (4) is the orthogonal GHM scaling vector, and is illustrated in Figure 2.

Example 3. The scaling vector shown in this example was originally constructed by Donovan, Geronimo, and Hardin in [4], although not in the macroelement context, and again by Hardin and Kessler in detail in [10] using a macroelement approach.

Consider from Example 1 the C^0 macroelement $\tilde{\Lambda}$ and the scaling vector $\tilde{\Phi} = (\phi_1, \phi_2)^T$ that generates $\mathcal{S}_2^0(\mathbf{Z})$. In order to extend $\tilde{\Lambda}$ to an orthonormal C^0 macroelement, we construct a FIF satisfying

$$u(x) = \phi_2(2x) - \phi_2(2x - 1) + su(2x) + su(2x - 1) \text{ for } |s| < 1,$$

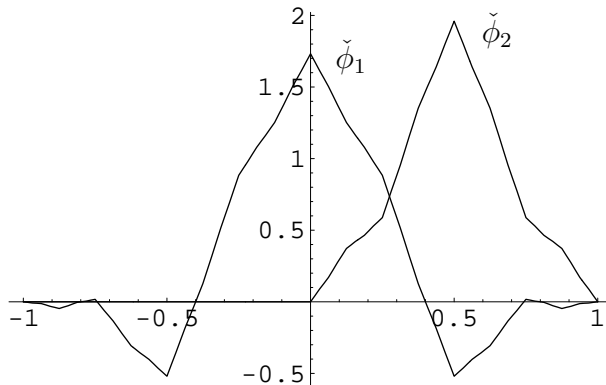


Figure 2: The orthogonal GHM scaling vector $\check{\Phi}$ from Example 2.

such that $\langle (I - P_M)l_1, (I - P_M)r_1 \rangle = 0$, where $M = \{\phi_2, u\}$. (Note that $\langle \phi_2, u \rangle = 0$, since ϕ_2 and u are symmetric and antisymmetric, respectively, about $x = \frac{1}{2}$.) It was shown in [4] and [10] that the orthogonality condition is satisfied by $s = \frac{2-\sqrt{10}}{6} \approx -0.1937$. By letting

$$\hat{l}_1 = \frac{(I - P_M)l_1}{\|(I - P_M)l_1\|}, \quad \hat{r}_1 = \frac{(I - P_M)r_1}{\|(I - P_M)r_1\|}, \quad \hat{m}_1 = \frac{m_1}{\|m_1\|}, \quad \text{and} \quad \hat{m}_2 = \frac{u}{\|u\|},$$

we have the orthonormal C^0 macroelement $\hat{\Lambda} = (\hat{l}_1, \hat{r}_1, \hat{m}_1, \hat{m}_2)^T$, equivalent to $(l_1, r_1, m_1, m_2)^T$ and an extension of $\tilde{\Lambda}$. The associated scaling vector $\hat{\Phi} = (\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3)^T$ defined in (3) and (4) is an orthogonal scaling vector, and is illustrated in Figure 3.

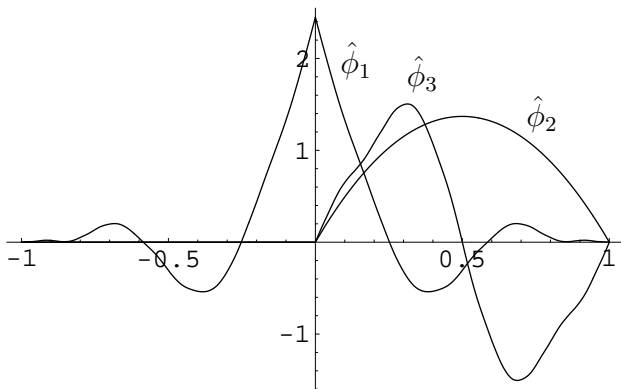


Figure 3: The orthogonal scaling vector $\hat{\Phi}$ from Example 3.

2 Main Results

In this section, we will construct a refinable C^1 macroelement on $[0, 1]$ which, in turn, provides a scaling vector Φ that generates $\mathcal{S}_3^1(\mathbf{Z})$. We will show that two additional functions are needed to extend that macroelement to an orthonormal macroelement that is refinable with dilation 2, and an explicit construction of that macroelement and the associated orthogonal scaling vector will be provided. Lastly, although general methods are available for finding a

multiwavelet associated with an orthogonal scaling vector (see [10], [16], and [17]), we will show an equivalent macroelement approach to constructing the wavelets.

2.1 Scaling Vector Generating $\mathcal{S}_3^1(\mathbf{Z})$

Consider the following basis for cubic polynomials defined on $[0, 1]$ and 0 elsewhere:

$$l_1(x) = (-2x^3 + 3x^2)\chi_{[0,1]}, \quad l_2(x) = (x^3 - x^2)\chi_{[0,1]},$$

$$r_1(x) = (2x^3 - 3x^2 + 1)\chi_{[0,1]}, \quad \text{and} \quad r_2(x) = (x^3 - 2x^2 + x)\chi_{[0,1]}.$$

One may verify that $\Lambda = (l_1, l_2, r_1, r_2)^T$ is a C^1 macroelement. It is also refinable, since

$$\Lambda = \sqrt{2} \begin{cases} \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{3}{4\sqrt{2}} & 0 & 0 \\ -\frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} & 0 & 0 \\ \frac{1}{2\sqrt{2}} & -\frac{1}{4\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \end{bmatrix} \Lambda(2x) \text{ if } x \in [0, \frac{1}{2}], \\ \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & \frac{3}{4\sqrt{2}} \\ 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{8\sqrt{2}} & -\frac{1}{4\sqrt{2}} \\ 0 & 0 & \frac{1}{2\sqrt{2}} & -\frac{3}{8\sqrt{2}} \\ 0 & 0 & \frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} \end{bmatrix} \Lambda(2x-1) \text{ if } x \in [\frac{1}{2}, 1]. \end{cases} \quad (7)$$

From direct computation, we know that

$$\langle l_1, r_1 \rangle = \frac{9}{70}, \quad \langle l_1, r_2 \rangle = \frac{13}{420}, \quad \langle l_2, r_1 \rangle = -\frac{13}{420}, \quad \text{and} \quad \langle l_2, r_2 \rangle = -\frac{1}{140},$$

so Λ is not an orthonormal macroelement. As in (3), we define

$$\phi_i(x) = \frac{1}{\sqrt{2}} \begin{cases} l_i(x+1) & \text{if } x \leq 0 \\ r_i(x) & \text{if } x \geq 0 \end{cases}, \quad i = 1, 2,$$

so that we have the scaling vector $\Phi = (\phi_1, \phi_2)^T$ satisfying

$$\Phi(x) = \sqrt{2} \left(\begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{3}{4\sqrt{2}} \\ -\frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} \end{bmatrix} \Phi(2x+1) + \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2\sqrt{2}} \end{bmatrix} \Phi(2x) + \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{3}{4\sqrt{2}} \\ \frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} \end{bmatrix} \Phi(2x-1) \right),$$

where $S(\Phi) = \mathcal{S}_3^1(\mathbf{Z})$. Both functions are illustrated in Figure 4.

2.2 An Orthogonal, Refinable Extension

We may extend Φ to a refinable vector of length 3 by adding a single m_1 or by adding l_3 and r_3 functions to the macroelement Λ . However, as we show in the following theorem, neither method will produce an orthogonal extension. Finding a length-4 orthogonal extension of Φ is nontrivial. The proof of the following theorem gives a construction of one such scaling vector using the fractal function idea on the C^1 macroelement Λ defined above.

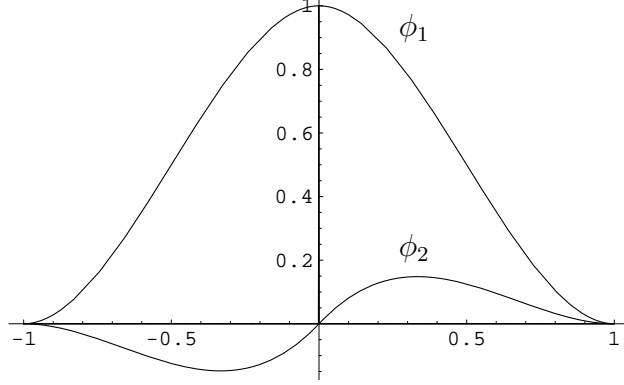


Figure 4: The scaling vector Φ generating $\mathcal{S}_3^1(\mathbf{Z})$.

Theorem 1. *The orthogonal scaling vector of least length that extends the length-2 scaling vector which generates $\mathcal{S}_3^1(\mathbf{Z})$ has length 4.*

Proof. There are two parts to the proof: we first show that a single m_1 or l_3 - r_3 pair added to the macroelement, thereby adding one function to the scaling vector, can not be found such that all of the necessary orthogonality conditions are satisfied, and then we show that a 2-vector of fractal functions (with symmetry properties, no less) can be found so that the necessary conditions are satisfied.

Suppose that the single normalized function m_1 added to the macroelement Λ gives a C^1 extension of the macroelement. Since we are assuming no symmetry properties for m_1 , we have four orthogonality conditions from (6), which reduce to the the four equations

$$\langle l_i, r_j \rangle = \langle l_i, m_1 \rangle \langle r_j, m_1 \rangle, \quad i, j \in \{1, 2\}.$$

The system of four equations with the unknowns $\langle l_1, m_1 \rangle$, $\langle l_2, m_1 \rangle$, $\langle r_1, m_1 \rangle$, and $\langle r_2, m_1 \rangle$ is inconsistent, so no single function m_1 can satisfy all of the necessary conditions.

Now suppose that two orthonormal functions l_3 and r_3 added to the macroelement Λ give a C^1 extension of the macroelement. To be an orthogonal extension, l_3 and r_3 will need to satisfy the four orthogonality conditions

$$\langle l_i - \langle l_i, l_3 \rangle l_3, r_j - \langle r_j, r_3 \rangle r_3 \rangle = 0 \quad i, j \in \{1, 2\},$$

while satisfying the endpoint conditions

$$l_i(1) - \langle l_i, l_3 \rangle l_3(1) = r_i(0) - \langle r_i, r_3 \rangle r_3(0) \quad i = 1, 2.$$

The endpoint conditions force $\langle l_1, l_3 \rangle = \langle r_1, r_3 \rangle$ and $\langle l_2, l_3 \rangle = \langle r_2, r_3 \rangle$, the values of which we will denote α and β , respectively. Then the four orthogonality conditions become

$$\begin{cases} \alpha(\langle l_1, r_3 \rangle + \langle r_1, l_3 \rangle) &= \frac{9}{70} \\ \alpha \langle r_2, l_3 \rangle + \beta \langle l_1, r_3 \rangle &= \frac{13}{420} \\ \alpha \langle l_2, r_3 \rangle + \beta \langle r_1, l_3 \rangle &= -\frac{13}{420} \\ \beta(\langle l_2, r_3 \rangle + \langle r_2, l_3 \rangle) &= -\frac{1}{140} \end{cases}$$

with unknowns $\langle l_1, r_3 \rangle$, $\langle l_2, r_3 \rangle$, $\langle r_1, l_3 \rangle$, and $\langle r_2, l_3 \rangle$. Again, the system is inconsistent, so no pair of functions l_3 and r_3 can satisfy all of the necessary conditions.

To show that a length-4 orthogonal scaling vector is possible, we will actually construct one by adding two functions m_1 and m_2 to the macroelement Λ . Consider two FIF m_1 and m_2 and function vector $\Gamma = (m_1, m_2)^T$ satisfying

$$\Gamma(x) = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \Phi(2x-1) + \begin{bmatrix} 0 & q \\ s & t \end{bmatrix} \Gamma(2x) + \begin{bmatrix} 0 & -q \\ -s & t \end{bmatrix} \Gamma(2x-1)$$

where the maximum ∞ -norm of the matrix coefficients of $\Gamma(2x)$ and $\Gamma(2x-1)$ is 1 and $\alpha, \beta \neq 0$ are chosen so that $\|m_1\| = \|m_2\| = 1$. The extended macroelement $\hat{\Lambda} = (l_1, l_2, r_1, r_2, m_1, m_2)^T$ is refinable, with

$$\hat{\Lambda}(x) = \sqrt{2} \begin{cases} \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{3}{4\sqrt{2}} & 0 & 0 & 0 & 0 \\ -\frac{1}{8\sqrt{2}} & -\frac{3}{8\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{2\sqrt{2}} & \frac{3}{4\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{8\sqrt{2}} & -\frac{3}{8\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & 0 & 0 \\ \frac{\alpha}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{q}{\sqrt{2}} \\ 0 & \frac{\beta}{\sqrt{2}} & 0 & 0 & \frac{s}{\sqrt{2}} & \frac{t}{\sqrt{2}} \end{bmatrix} \hat{\Lambda}(2x) \text{ if } x \in [0, \frac{1}{2}], \\ \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & \frac{3}{4\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{2}} & -\frac{3}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{2}} & \frac{3}{4\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{8\sqrt{2}} & -\frac{3}{8\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{\alpha}{\sqrt{2}} & 0 & 0 & -\frac{q}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{\beta}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & \frac{t}{\sqrt{2}} \end{bmatrix} \hat{\Lambda}(2x-1) \text{ if } x \in [\frac{1}{2}, 1]. \end{cases} \quad (8)$$

Let $M = \{m_1, m_2\}$. In order to construct an orthonormal macroelement equivalent to $\hat{\Lambda}$, the elements of M must satisfy the following conditions:

$$\langle (I - P_M)l_i, (I - P_M)r_j \rangle = 0, \quad i, j \in \{1, 2\}.$$

One can verify that m_1 and m_2 are symmetric and antisymmetric, respectively, about $x = \frac{1}{2}$, so that $\langle m_1, m_2 \rangle = 0$. Also, due to symmetry,

$$\langle r_1, m_1 \rangle = \langle l_1, m_1 \rangle, \quad \langle r_1, m_2 \rangle = -\langle l_1, m_2 \rangle, \quad \langle r_2, m_1 \rangle = -\langle l_2, m_1 \rangle, \quad \text{and} \quad \langle r_2, m_2 \rangle = \langle l_2, m_2 \rangle.$$

Thus, the four orthogonality conditions reduce to the three equations

$$\begin{cases} \frac{9}{70} - \langle r_1, m_1 \rangle^2 + \langle r_1, m_2 \rangle^2 = 0 \\ \frac{1}{140} - \langle r_2, m_1 \rangle^2 + \langle r_2, m_2 \rangle^2 = 0 \\ \frac{13}{420} - \langle r_1, m_1 \rangle \langle r_2, m_1 \rangle + \langle r_1, m_2 \rangle \langle r_2, m_2 \rangle = 0. \end{cases} \quad (9)$$

Using direct computation, we know that

$$\langle r_1, r_1 \rangle = \langle l_1, l_1 \rangle = \frac{13}{35}, \quad \langle r_2, r_2 \rangle = \langle l_2, l_2 \rangle = \frac{1}{105}, \quad \text{and} \quad \langle r_1, r_2 \rangle = -\langle l_1, l_2 \rangle = \frac{11}{210}.$$

Using the coefficients in (8), we may expand and solve for $\langle r_1, m_1 \rangle$, $\langle r_1, m_2 \rangle$, $\langle r_2, m_1 \rangle$, and $\langle r_2, m_2 \rangle$ in the following system:

$$\begin{cases} 2\langle r_1, m_1 \rangle = \frac{\alpha}{2} \\ 2\langle r_2, m_1 \rangle = -\frac{1}{4}q(\langle r_1, m_2 \rangle - 2\langle r_2, m_2 \rangle) + \frac{13\alpha}{120} \\ 2\langle r_1, m_2 \rangle = s\langle r_1, m_1 \rangle + \frac{3s}{2}\langle r_2, m_1 \rangle + t\langle r_1, m_2 \rangle - \frac{3t}{2}\langle r_2, m_2 \rangle - \frac{19\beta}{420} \\ 2\langle r_2, m_2 \rangle = \frac{3s}{4}\langle r_2, m_1 \rangle + \frac{t}{4}\langle r_2, m_2 \rangle - \frac{\beta}{168}. \end{cases} \quad (10)$$

The systems (9) and (10) have the solutions

$$\begin{aligned} \langle r_1, m_1 \rangle &= \frac{\alpha}{4}, \quad \langle r_1, m_2 \rangle = -\frac{1}{4}\sqrt{\alpha^2 - \frac{72}{35}}, \quad \langle r_2, m_1 \rangle = \frac{1}{216} \left(13\alpha - \sqrt{7\alpha^2 - \frac{72}{5}} \right), \\ \langle r_2, m_2 \rangle &= \frac{1}{216} \left(\sqrt{7}\alpha - 13\sqrt{\alpha^2 - \frac{72}{35}} \right), \quad q = \frac{26\alpha - 20\sqrt{7\alpha^2 - \frac{72}{5}}}{5\sqrt{7}\alpha + 70\sqrt{\alpha^2 - \frac{72}{35}}}, \\ s &= \frac{13104 - 6125\alpha^2 + 35\alpha\sqrt{7\alpha^2 - \frac{72}{5}} + 51\sqrt{7}\alpha\beta + 147\beta\sqrt{\alpha^2 - \frac{72}{35}}}{\sqrt{7}(504 + 140\alpha^2 + 29\sqrt{5}\alpha\sqrt{35\alpha^2 - 72})}, \text{ and} \\ t &= \frac{35280 - 7\sqrt{5}\sqrt{35\alpha^2 - 72}(1465\alpha - 9\sqrt{7}\beta) + 5\alpha(8575\alpha - 9\sqrt{7}\beta)}{35(504 + 140\alpha^2 + 29\sqrt{5}\alpha\sqrt{35\alpha^2 - 72})}. \end{aligned}$$

Again using the coefficients in (8), we may expand $\langle m_1, m_1 \rangle$ and $\langle m_2, m_2 \rangle$ and numerically solve for α and β . After substituting the above results into the initial expansions

$$q^2 - 2q\alpha\langle r_1, m_2 \rangle + \frac{13}{35}\alpha^2 = 1 \text{ and } s^2 + t^2 - 2s\beta\langle r_2, m_1 \rangle + 2t\beta\langle r_2, m_2 \rangle + \frac{\beta^2}{105} = 1,$$

we find the approximate solutions $\alpha \approx 1.63240645$ and $\beta \approx 14.19575451$, so that

$$\langle r_1, m_1 \rangle \approx 0.40810161, \quad \langle r_1, m_2 \rangle \approx -0.19487303, \quad \langle r_2, m_1 \rangle \approx 0.08869880,$$

$$\langle r_2, m_2 \rangle \approx -0.02691878, \quad q \approx 0.01570030, \quad s \approx 0.45783086, \quad \text{and } t \approx -0.03034240.$$

Since

$$\left\| \begin{bmatrix} 0 & q \\ s & t \end{bmatrix} \right\|_{\infty} = \left\| \begin{bmatrix} 0 & -q \\ -s & t \end{bmatrix} \right\|_{\infty} \approx 0.48817326 < \frac{1}{2},$$

then by Lemma 3, $m_1, m_2 \in C^1([0, 1])$. By setting

$$\tilde{l}_1 = \frac{(I - P_M)l_1}{\|(I - P_M)l_1\|}, \quad \tilde{l}_2 = \frac{(I - P_M)l_2}{\|(I - P_M)l_2\|}, \quad \tilde{r}_1 = \frac{(I - P_M)r_1}{\|(I - P_M)r_1\|}, \quad \text{and } \tilde{r}_2 = \frac{(I - P_M)r_2}{\|(I - P_M)r_2\|},$$

we have the orthonormal, refinable C^1 macroelement $\tilde{\Lambda} = (\tilde{l}_1, \tilde{l}_2, \tilde{r}_1, \tilde{r}_2, m_1, m_2)^T$ that is equivalent to $\hat{\Lambda}$ and an extension of Λ . By Lemma 2, we have the scaling vector $\tilde{\Phi}$ of length 4 as defined in (3) and (4) that is an extension of Φ . Since $\langle \tilde{\phi}_1, \tilde{\phi}_2 \rangle = 0$ due to their symmetry-antisymmetry about $x = 0$, $\tilde{\Phi}$ is an orthogonal scaling vector. \square

The elements of the orthogonal scaling vector constructed in the above proof are illustrated in Figure 5.

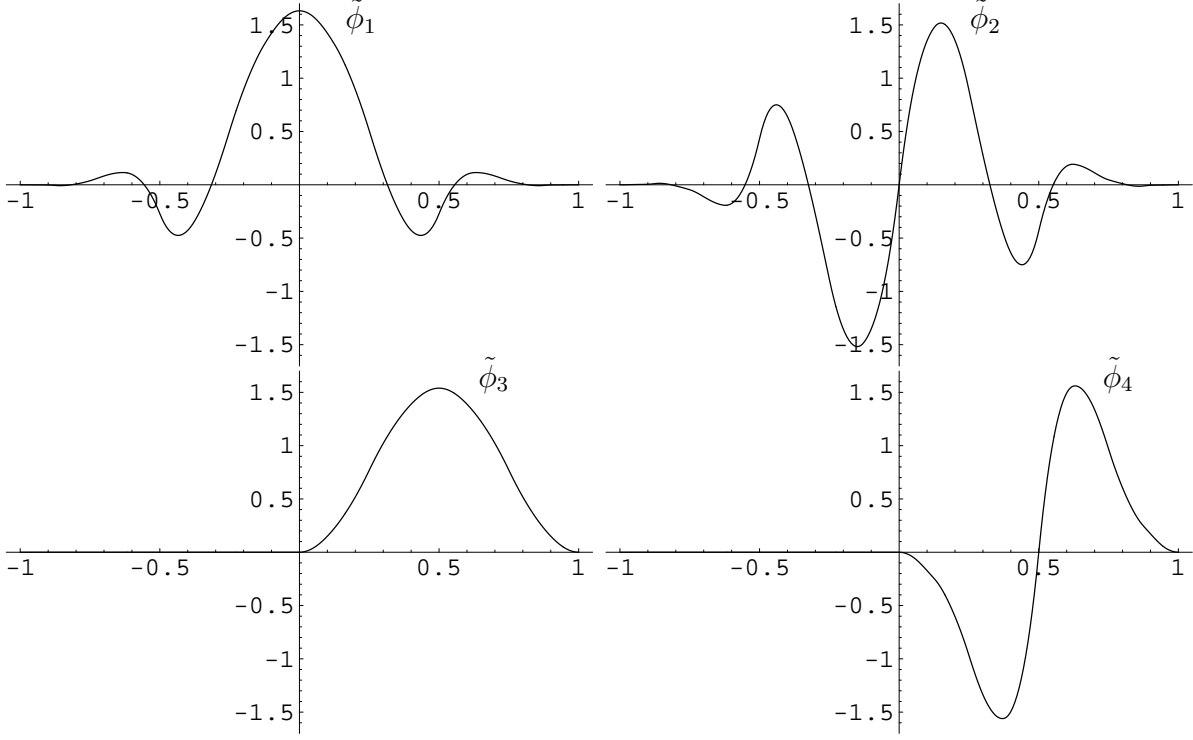


Figure 5: The orthogonal scaling vector $\tilde{\Phi} = (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\phi}_4)^T$, where $S(\tilde{\Phi}) \supset \mathcal{S}_3^1(\mathbf{Z})$.

2.3 The Associated Multiwavelets

A general construction for a multiwavelet associated with a scaling vector defined on \mathbf{R} was found in [16] (see also [10]), and that technique could be used here. However, I will use an equivalent approach that will produce wavelets with symmetry properties. From Jia in [14], we know that, since $\tilde{\Phi}$ was of length 4, then $\tilde{\Psi}$ will also have length $4(2-1) = 4$.

Define the 6-dimensional space $V = \{f : f \in V_{-1}, \text{supp } f \subseteq [0, 1]\}$ and let P_V denote the orthogonal projection onto V . Note that, in this construction, $V \cap V_0^\perp = \emptyset$, since there are 6 independent orthogonality conditions affecting elements of V . Recall the original macroelement $\tilde{\Lambda} = (\tilde{l}_1, \tilde{l}_2, \tilde{r}_1, \tilde{r}_2, m_1, m_2)^T$. Let

$$\bar{l}_i(x) = P_V \tilde{l}_i(x) + a_i \tilde{\phi}_1(Nx - N) + b_i \tilde{\phi}_2(Nx - N) \text{ for } i = 1, 2,$$

where a_i and b_i are chosen so that $\langle \bar{l}_i(x), \phi_1(x-1) \rangle = 0$ and $\langle \bar{l}_i(x), \phi_2(x-1) \rangle = 0$. Likewise, let

$$\bar{r}_i(x) = P_V \tilde{r}_i(x) + c_i \tilde{\phi}_1(Nx) + d_i \tilde{\phi}_2(Nx) \text{ for } i = 1, 2,$$

where c_i and d_i are chosen so that $\langle \bar{r}_i(x), \phi_j(x) \rangle = 0$. Note that the properties $\langle \bar{l}_i, \bar{r}_j \rangle = 0$, $\langle \bar{l}_i, m_j \rangle = 0$, and $\langle \bar{r}_j, m_j \rangle = 0$ for $i, j \in \{1, 2\}$ are maintained from the original macroelement, and that $a_1 = c_1$, $b_1 = -d_1$, $a_2 = -c_2$, and $b_2 = d_2$ due to symmetry properties.

We may construct functions

$$f_1(x) = \bar{l}_1(x+1) + \bar{r}_1(x) \quad \text{and} \quad g_1(x) = -\bar{l}_2(x+1) + \bar{r}_2(x)$$

that are symmetric with respect to $x = 0$, and

$$f_3(x) = -\bar{l}_1(x+1) + \bar{r}_1(x) \quad \text{and} \quad g_3(x) = \bar{l}_2(x+1) + \bar{r}_2(x)$$

that are antisymmetric with respect to $x = 0$, such that all are orthogonal to V_0 . Set

$$f_2 = g_1 - \frac{\langle g_1, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 \quad \text{and} \quad f_4 = g_3 - \frac{\langle g_3, f_3 \rangle}{\langle f_3, f_3 \rangle} f_3$$

to handle the last remaining orthogonalities, and set $\tilde{\psi}_i = \frac{f_i}{\|f_i\|}$ for $i = 1, \dots, 4$. Then $\tilde{\Psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4)^T$ is a multiwavelet that generates W_0 , and is illustrated in Figure 6.

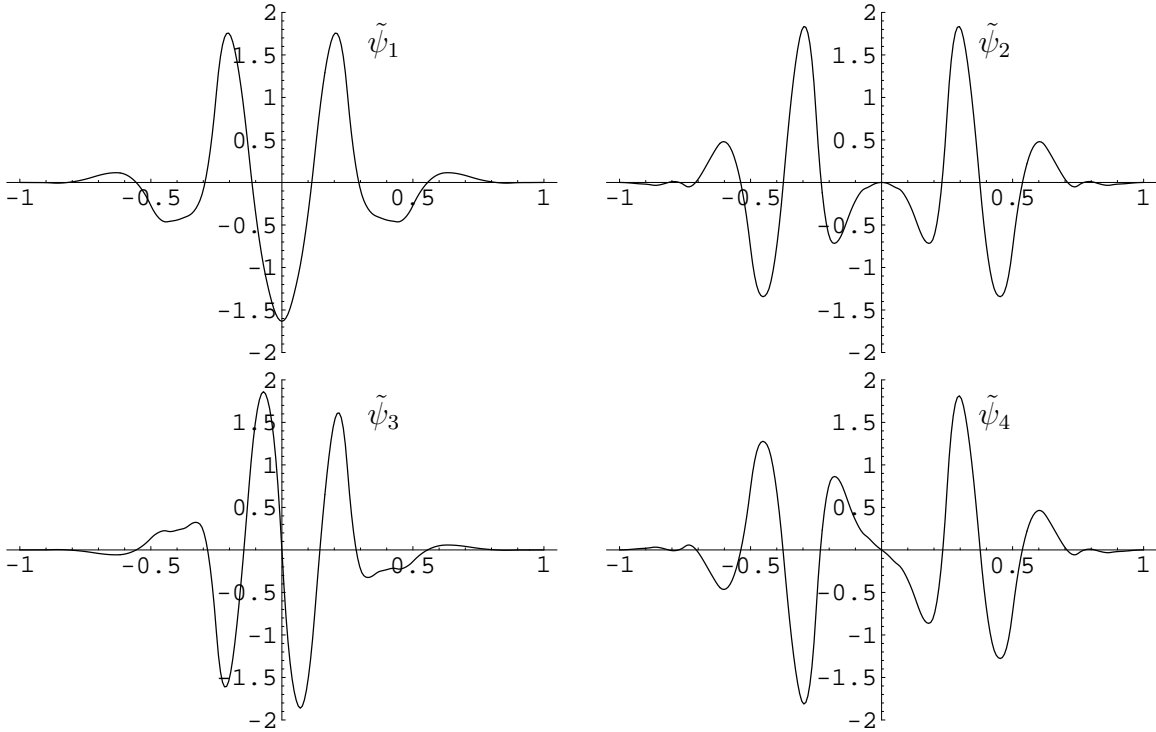


Figure 6: The multiwavelet $\tilde{\Psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4)^T$, where $S(\tilde{\Psi}) = W_0$.

3 An Application of the Bases

One possible use for this smoother scaling vector is in image compression applications. The original version of JPEG used the discrete cosine transform, a non-wavelet technique, on disjoint 8×8 blocks of pixels to decompose the image data. This caused noticeable distortions, or “artifacts,” to appear in the reconstructed image at higher compression ratios. The discontinuous Haar wavelet basis can also be used to decompose the image data, but with similar results at higher compression ratios. Even continuous bases like Daubechies’s D4 scaling function or the GHM scaling vector constructed in Example 2, which are not differentiable everywhere, leave sharp distortions in the image at higher compression ratios. We would expect that by using a smoother, differentiable basis like we have constructed in this paper, we should be able to produce a smoother compressed image with no sharp artifacts.

Let c_i denote the sequence of scaling function coefficients in V_i , and let d_i denote the sequence of wavelet coefficients in W_i . The wavelet approach to producing a compressed

image is to take a signal (or function) in V_0 and find its best approximation in the smoother, nested function space V_1 , keeping the error in the wavelet space W_1 . If the function is close to being in V_1 , then the wavelet coefficients will be very small (close to zero). We may then repeat this process with the function in V_1 , and then V_2 , etc. The process is illustrated in Figure 7 for a decomposition to the fourth level. The image may be reconstructed exactly from c_n , where n is the last level of decomposition, and from the wavelet coefficients d_1, \dots, d_n . However, wavelet schemes typically trade accuracy of the image for more efficient storage

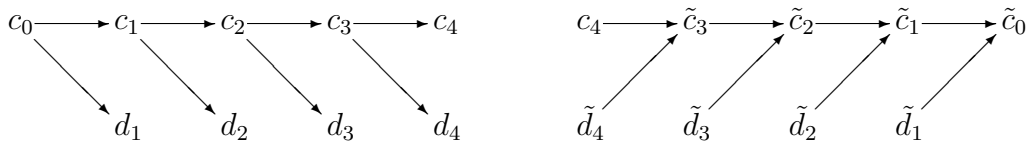


Figure 7: The decomposition of the signal c_0 and the reconstruction of \tilde{c}_0 .

and/or transmission of the image. In order to achieve compression of the signal, we *quantize* the wavelet coefficients, replacing coefficients with values in a certain range with a central value. This creates a signal with lower *entropy*, a measure of the smallest average bits per character needed to store a signal without losing information. This measure is quantifiable, using the formula

$$E = - \sum_{i=1}^N p(i) \log_2 p(i),$$

where N is the number of distinct characters in the signal, and $p(i)$ is the relative frequency of the i^{th} character. The altered signal is then stored near this bit-rate using an entropy encoder. This is called *lossless* compression. A less accurate version of the original image can then be reconstructed from the quantized wavelet coefficients, as illustrated in Figure 7. This is called *lossy* compression.

3.1 Prefiltering

The process of turning discrete data into a function in V_0 is called *prefiltering*. Ideally, a prefilter (usually a set of matrices) should be orthogonal (norm-preserving) and send data sampled from a polynomial of degree n to a multiple of the same polynomial in V_0 , up to the approximation order of V_0 , using as few matrices in the prefilter as possible. Prefiltering is not an issue when using a single scaling function, as using the raw data as the basis coefficients (the identity prefilter) accomplishes each of these goals. However, prefiltering becomes vitally important when using multiple scaling functions: the best basis will perform poorly in applications if the data is not prefiltered efficiently. See [12] for a more thorough discussion on prefiltering.

Multiple-matrix prefilters that accomplish all of the above goals can be difficult to find. We may always find an orthogonal single-matrix prefilter that preserves constant data by using the following method. Consider a scaling vector Φ of length r that generates the function space V_0 of approximation order $n \leq r$. Let a_k be the r -vector of function values sampled uniformly over $[0, 1]$ from $f_k(x) = x^k$, $k = 0, \dots, n - 1$, and let a_k fill out the space of r -vectors for $k = n, \dots, r - 1$. Likewise, let α_k be the r -vector of basis coefficients of $\Phi(x)$

needed to construct $f_k(x)$, $k = 0, \dots, n - 1$, and let α_k fill out the space of r -vectors for $k = n, \dots, r - 1$. Then use the Gram-Schmidt process on $\{a_0, \dots, a_{r-1}\}$ and $\{\alpha_0, \dots, \alpha_{r-1}\}$ in increasing order of degrees, and normalize to get $\{\tilde{a}_0, \dots, \tilde{a}_{r-1}\}$ and $\{\tilde{\alpha}_0, \dots, \tilde{\alpha}_{r-1}\}$. We may then construct the orthogonal, single-matrix prefilter

$$Q = \begin{bmatrix} \tilde{\alpha}_0 & \cdots & \tilde{\alpha}_{r-1} \end{bmatrix} \begin{bmatrix} \tilde{a}_0^T \\ \vdots \\ \tilde{a}_{r-1}^T \end{bmatrix}$$

which maps \tilde{a}_k to $\tilde{\alpha}_k$. While the prefilter Q does not preserve polynomial order above constants, it gets fairly close, and has the advantages of having very short support and being orthogonal. We have observed that this type of prefilter works well in applications, and so, we will use this type of prefilter for the GHM scaling vector and the new scaling vector in the following example. We concede that better prefilters may be found for both scaling vectors. (In fact, see [12] for more elaborate prefilters for the GHM scaling vector.)

Postfiltering, turning the basis coefficients back into discrete data, is usually just an inversion of the prefiltering process. For the prefilter Q described above, we use the postfilter Q^T , since $Q^T Q = I$.

3.2 Comparison of Reconstructed Images

In this section, we consider a digital 512×512 grayscale image called ‘‘Zelda,’’ basically a rectangular data set ranging in value from 0 to 255, requiring 256 Kb of memory (1 byte per pixel) for storage as raw data. The original image is shown in Figure 8. We decompose the image using three different bases: the single D4 scaling function, the GHM scaling vector constructed in Example 2 from a C^0 macroelement, and the scaling vector from Theorem 1 constructed from a C^1 macroelement. We quantize the wavelet coefficients uniformly, with 0 at the center of the zero-bin, to achieve a moderate 25:1 compressed image (file size 10.24 Kb) and an extreme 50:1 compressed image (file size 5.12 Kb), based on the entropy of the quantized signal. Particularly in the 50:1 compressed images, the reader will detect distortions specific to the basis being used. The error in the images will be measured visually by the reader and with the *root mean square error (RMSE)*, defined by

$$RMSE = \sqrt{\frac{\sum_{i,j} (\text{original}_{i,j} - \text{new}_{i,j})^2}{\text{rows} \times \text{columns}}}.$$

Reconstructions from the quantized data are shown in Figure 9 for the D4 basis, in Figure 10 for the GHM scaling vector, and in Figure 11 for the new scaling vector constructed in this paper. The reconstructions for the smooth scaling vector constructed in this paper have the lowest *RMSE*’s of the three methods. Hopefully, the reader also finds the artifacts are less noticeable in the images where the new basis is used.



Figure 8: The original image 512×512 grayscale image Zelda.



$RMSE \approx 4.88688$

$RMSE \approx 6.88849$

Figure 9: A 25:1 and 50:1 compression using the D4 scaling function.

4 Appendix

4.1 Scaling Vector Coefficients

The matrix coefficients of the scaling vector constructed in Section 2.2 satisfying

$$\tilde{\Phi}(x) = \sqrt{2} \sum_{i=-2}^1 g_i \tilde{\Phi}(2x - i)$$



$RMSE \approx 4.37044$

$RMSE \approx 6.23848$

Figure 10: A 25:1 and 50:1 compression using the GHM scaling vector.



$RMSE \approx 4.13475$

$RMSE \approx 5.84525$

Figure 11: A 25:1 and 50:1 compression using the new scaling vector.

are given below.

$$g_{-2} = \begin{bmatrix} 0 & 0 & 0.02669163201881291 & 0.026188780570349884 \\ 0 & 0 & -0.03940748766209954 & -0.040769035522339014 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
g_{-1} &= \begin{bmatrix} -0.1175124743509559 & -0.10652668959666897 & 0.30676124077120465 & 0.35964167838854366 \\ 0.18731879383595215 & 0.18181954032277048 & -0.4224381824999862 & -0.433225475337907 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
g_0 &= \begin{bmatrix} 0.7071067811865841 & 0 & 0.3067612407712861 & -0.3596416783882024 \\ 0 & 0.35355339059336993 & 0.42243818250022613 & -0.43322547533688205 \\ 0 & 0 & 0.47106586494422936 & 0.23604093395898632 \\ 0 & 0 & -0.5666158791523569 & -0.2916640296146426 \end{bmatrix} \\
g_1 &= \begin{bmatrix} -0.11751247435904282 & 0.10652668958949803 & 0.026691633234707347 & -0.026188779541901065 \\ -0.1873187938600181 & 0.18181954030143205 & 0.03940749128037211 & -0.040769032461869345 \\ 0.6669057455800359 & 0 & 0.47106586494422936 & -0.23604093395898632 \\ 0 & 0.4333094490577162 & 0.5666158791523569 & -0.2916640296146426 \end{bmatrix}
\end{aligned}$$

4.2 Multiwavelet Coefficients

The matrix coefficients of the multiwavelet constructed in Section 2.3 satisfying

$$\tilde{\Psi}(x) = \sqrt{2} \sum_{i=-2}^1 h_i \tilde{\Phi}(2x - i)$$

are given below.

$$\begin{aligned}
h_{-2} &= \begin{bmatrix} 0 & 0 & 0.0266916313500821 & 0.02618877991421749 \\ 0 & 0 & 0.04374365300600389 & 0.06435982959198171 \\ 0 & 0 & -0.013410540810749855 & -0.013157895717088848 \\ 0 & 0 & -0.04006995259666773 & -0.06065185761803572 \end{bmatrix} \\
h_{-1} &= \begin{bmatrix} -0.11751247140680435 & -0.10652668692775463 & 0.3067612330856239 & 0.3596416693780991 \\ -0.3334538093583893 & -0.42469284946317537 & 0.19229779050087667 & -0.40670678109361624 \\ 0.05904119433180234 & 0.053521662417019424 & -0.15412448873935367 & -0.18069293784197155 \\ 0.3166000901995356 & 0.40882389594960616 & -0.15157506646288127 & 0.4513983241700974 \end{bmatrix} \\
h_0 &= \begin{bmatrix} -0.7071067870918081 & 0 & 0.30676123308570535 & -0.3596416693777578 \\ 0 & 0 & 0.19229779050222978 & 0.40670678109948577 \\ 0 & -0.9347644627208598 & 0.15412448873939458 & -0.18069293784180007 \\ 0 & -0.034862829700229456 & 0.15157506646421717 & 0.45139832417589415 \end{bmatrix} \\
h_1 &= \begin{bmatrix} -0.11751247141489118 & 0.10652668692058369 & 0.02669163256597651 & -0.026188778885768697 \\ -0.3334538094952175 & 0.424692849341863 & 0.043743673577174125 & -0.06435981219212034 \\ -0.05904119433586537 & 0.05352166241341656 & 0.013410541421645506 & -0.013157895200370542 \\ -0.31660009033465436 & 0.40882389582980966 & 0.04006997291081938 & -0.060651840435570266 \end{bmatrix}
\end{aligned}$$

4.3 Prefilter Matrices

The prefilter used in the GHM scaling vector image compression examples in Section 3.2 is given below. This prefilter originally appeared in [12].

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}} \end{bmatrix}$$

The prefilter used in the new scaling vector image compression examples in Section 3.2 is given below.

$$Q = \begin{bmatrix} 0.843806675746766 & 0.48588947587427395 & -0.22260947332787062 & 0.04844309635543219 \\ -0.5359722073168798 & 0.7840831502767046 & -0.30731469556949864 & 0.05920375260967379 \\ 0.015288019449165685 & 0.26864647184429297 & 0.7701708393722542 & 0.5783011566714802 \\ -0.022144150703698316 & -0.27740602706575324 & -0.5126788258279672 & 0.8122290035974187 \end{bmatrix}$$

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