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# Bounds on $k$ -Regular Ramanujan Graphs and Separator Theorems

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BOUNDS ON  $k$ -REGULAR RAMANUJAN GRAPHS AND SEPARATOR  
THEOREMS

A Thesis  
Presented to  
The Faculty of the Department of Mathematics  
Western Kentucky University  
Bowling Green, Kentucky

In Partial Fulfillment  
Of the Requirements for the Degree  
Master of Science

By  
James Skees  
May, 2007

BOUNDS ON  $k$ -REGULAR RAMANUJAN GRAPHS AND SEPARATOR  
THEOREMS

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# BOUNDS ON $k$ -REGULAR RAMANUJAN GRAPHS AND SEPARATOR THEOREMS

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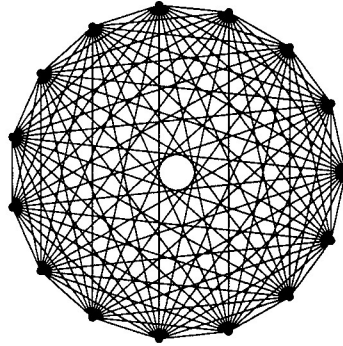
Western Kentucky University

## Abstract

Expander graphs are a family of graphs that are highly connected. Finding explicit examples of expander graphs which are also sparse is a difficult problem. The best type of expander graph in a certain sense is a Ramanujan graph. Families of graphs that have separator theorems fail to be Ramanujan if the vertex set gets sufficiently large. Using separator theorems to get an estimate on the expanding constant of graphs, we get bounds on the number of vertices for such  $k$ -regular graphs in order for them to be Ramanujan.

## CHAPTER 1

### Introduction



Expander graphs are one of the deepest tools of theoretical computer science and one of the main objects of study in discrete mathematics, showing up in many contexts since their introduction in the 1970s by Bassalygo and Pinsker, see [2]. Expander graphs are families of graphs that are highly connected. In computer science, expanders are useful when dealing with communications networks. Finding explicit examples of expander graphs which are also sparse is a difficult problem. Ramanujan graphs are the best expanders, for reasons that will be discussed later. One can determine if an individual graph is Ramanujan.

An interesting property of some graph families are separator theorems. This is the ability to separate graphs into parts making them easily disconnectable. Graphs with separator theorems are generally not expanders. We use the notion of separator theorems to get an explicit bound on the number of vertices of Ramanujan graphs.

In doing this we help narrow the search for finding explicit examples of expander graphs.

### 1.1. Introduction To Graphs

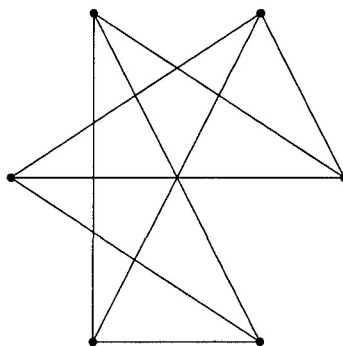


FIGURE 1. 3-regular graph on 6 vertices

A **graph** consists of a set of points, called **vertices**, and line segments, called **edges**, connecting some of the vertices. Let  $V(G)$  denote the set of vertices and  $E(G)$  the set of edges. So a graph  $G$  is characterized by the pair  $(V(G), E(G))$ .

A **path** in  $G$  is a sequence  $v_1, v_2, \dots, v_k$  of vertices where  $v_i$  is adjacent to  $v_{i+1}$ . A graph  $G$  is **connected** if every two vertices can be joined by a path. The vertices at ends of an edge are said to be **incident** with the edge. Two vertices which are incident with a common edge are adjacent, as are two edges which are incident with a common vertex. An edge incident with a single vertex is called a **loop** and an edge with distinct ends is a **link**. If a graph has no loops and no two of its links join the same pair of vertices then we call the graph **simple**.

A graph is **finite** if both its vertex set and edge set are finite. The **degree** of a vertex  $v$  is the number of edges incident with  $v$ .  $G$  is  **$k$ -regular** if every vertex is exactly of degree  $k$ .

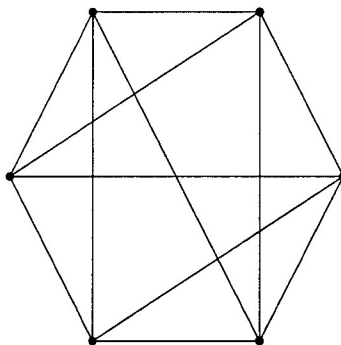


FIGURE 2. 4-regular graph on 6 vertices

A **bipartite** graph is one whose vertex set can be partitioned into two subsets  $V(G) = A \cup B$ , so that each edge has one end in  $A$  and one end in  $B$ . A graph  $H$  is a **subgraph** of  $G$  (written  $H \subset G$ ) if  $V(H) \subset V(G)$  and  $E(H) \subset E(G)$ . Two graphs  $G_1$  and  $G_2$  are **isomorphic** if there is a one-to-one correspondence between the vertices of  $G_1$  and those of  $G_2$  with the property that the number of edges joining any two vertices of  $G_1$  is equal to the number of edges joining the corresponding vertices of  $G_2$ . A graph  $H$  is called a **minor** of the graph  $G$  if  $H$  is isomorphic to a graph that can be obtained by zero or more edge contractions on a subgraph of  $G$ . Edge contraction is the process of removing an edge and identifying its two endpoints. A **complete graph** is defined by requiring every vertex to be connected to every other vertex. A **cycle** is a simple closed path, with no repeated vertices other than the starting and ending vertices. A **cycle graph** is a graph that consists of a single cycle.

## CHAPTER 2

# Planar Graphs and Separator Theorems

### 2.1. Planar Graphs

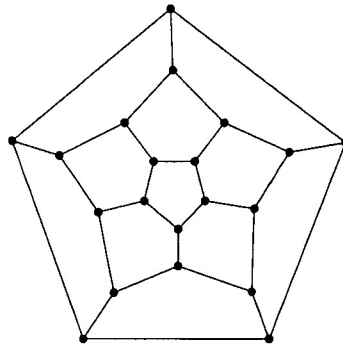


FIGURE 1. A Planar Graph

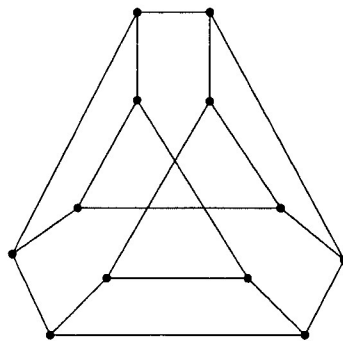


FIGURE 2. A Nonplanar Graph

A graph is said to be **planar** if it can be drawn in the plane so that its edges intersect only at their endpoints. Planar graphs have two key properties. The first is the following theorem.

**THEOREM 1.** *If  $G$  is  $k$ -regular and planar, then  $k=2,3,4$  or  $5$ .*

PROOF. Let  $G$  be a planar  $k$ -regular graph with  $n$  vertices. Embed  $G$  in the plane so that no two edges intersect and an edge intersects vertices only at endpoints. Let  $V$  be the number of vertices,  $E$  be the number of edges, and  $F$  be the number of faces (regions bounded by edges) of the graph. If all the faces were triangles, then as there are  $k$  faces per vertex and 3 vertices per face we have  $F = \frac{nk}{3}$ . As any face has at least 3 vertices per face then we have  $F \leq \frac{nk}{3}$  in general. To get the number of edges consider that every vertex is of  $k^{th}$  degree, so every vertex has  $k$  edges, so by counting every edge on every vertex we get a total of  $nk$  edges. But we end up counting every edge twice since every edge is shared by two vertices, so there are  $\frac{nk}{2}$  edges.

We have  $V = n$  and by Euler's formula  $V - E + F = 2$ .

So  $F = 2 - V + E = 2 - n + \frac{nk}{2}$ .

Thus as  $F \leq \frac{nk}{3}$  we get  $2 - n + \frac{nk}{2} \leq \frac{nk}{3}$  and

this gives  $k \leq 6(1 - \frac{2}{n})$  so  $k < 6$ . □

## 2.2. Separator Theorems

The second key property is the fact that one can partition the vertices of planar graphs into three components, two of which are relatively large and have no adjacent vertices, and we may have edges between the small and large components. Figure 3 illustrates this Planar Separator Theorem.

Originally due to Lipton and Tarjan [8], it was later improved by Alon, Seymour, and Thomas[1].

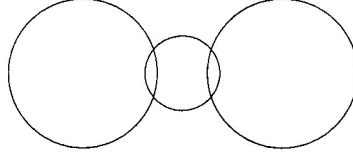


FIGURE 3.

THEOREM 2. [1] *[Planar Separator Theorem] Let  $G$  be a planar graph with  $n > 0$  vertices. Then there is a partition  $(A, B, C)$  of  $V(G)$  such that  $|A|, |B| \leq \frac{2}{3}n$ ,  $|C| \leq 2\sqrt{2n}$ , and no vertex in  $A$  is adjacent to any vertex in  $B$ .*

For our purposes, we want sets  $A$  and  $B$  to be no larger than  $\frac{n}{2}$ . This can be achieved if the value of  $|C|$  is allowed to be larger. However,  $|C|$  can still be on the order of  $\sqrt{n}$ . The following is a corollary from Lipton and Tarjan [8].

COROLLARY 1. [1] *[Planar Separator Corollary] Let  $G$  be a planar graph with  $n > 0$  vertices. Then there is a partition  $(A, B, C)$  of  $V(G)$  such that  $|A|, |B| \leq \frac{n}{2}$ ,  $|C| \leq \frac{3\sqrt{2}}{2(1-\sqrt{\frac{2}{3}})}\sqrt{n}$ , and no vertex in  $A$  is adjacent to any vertex in  $B$ .*

There are other separator theorems that apply to other classes of graphs. The following proposition is a version of the above corollary for a more general class of separator theorems.

PROPOSITION 1. [3] *Let  $0 \leq \alpha < 1$  and  $c > 0$ . Let  $\mathcal{G}_\alpha(c)$  be a family of graphs so that for  $G \in \mathcal{G}_\alpha(c)$  with  $|V(G)| = n$  there exists a vertex partition  $V(G) = A \cup B \cup C$  where  $|A|, |B| \leq \frac{2n}{3}$ ,  $|C| \leq cn^\alpha$ , and no vertex in  $A$  is adjacent to a vertex in  $B$ .*

*Suppose further that if  $G \in \mathcal{G}_\alpha(c)$  then every subgraph of  $G$  is in  $\mathcal{G}_\alpha(c)$ . Then any  $G \in \mathcal{G}_\alpha(c)$  can be partitioned  $V(G) = A' \cup B' \cup C'$  where  $|A'|, |B'| \leq \frac{n}{2}$ ,  $|C'| \leq \frac{c}{1-(\frac{2}{3})^\alpha}n^\alpha$ , and no vertex in  $A'$  is adjacent to a vertex in  $B'$ .*

The following is a separator theorem for simple graphs with no  $K_h$ -minors, where  $K_h$  is referring to the complete graph with  $h$  vertices.

COROLLARY 2. [3] *Let  $G$  be a simple graph with  $n$  vertices and no  $K_h$ -minor. Then there exists a vertex partition  $V(G) = A \cup B \cup C$  where  $|A|, |B| \leq \frac{n}{2}$ ,  $|C| \leq \frac{h^{\frac{1}{2}}}{1-\sqrt{2/3}}n^{1/2}$ , and no vertex in  $A$  is adjacent to a vertex in  $B$ .*

The **genus** of a graph is the minimum number of handles (see [7]) that must be added to the sphere to embed the graph without any crossings.

COROLLARY 3. [3] *Let  $G$  be a graph with  $n$  vertices and genus less than or equal to  $g$ . Then there exists a vertex partition  $V(G) = A \cup B \cup C$  where  $|A|, |B| \leq \frac{n}{2}$ ,  $|C| \leq \frac{6\sqrt{g}+2\sqrt{2}+1}{1-\sqrt{2/3}}n^{1/2}$ , and no vertex in  $A$  is adjacent to a vertex in  $B$ .*



## CHAPTER 3

# Expander Graphs, Estimating The Expander Constant, and Ramanujan Graphs

### 3.1. Expander Graphs

For  $X \subseteq V(G)$ , the boundary  $\partial X$  is the set of edges connecting  $X$  to  $V(G) - X$ .

DEFINITION 1. The **expanding constant** of  $G$  is

$$h(G) = \inf \left\{ \frac{|\partial X|}{\min \{|X|, |V(G)-X|\}} : X \subseteq V(G), 0 < |X| < \infty \right\}$$

Note that, if  $G$  is finite on  $n$  vertices, this can be rephrased as

$$h(G) = \min \left\{ \frac{|\partial X|}{|X|} : X \subseteq V(G), 0 < |X| \leq \frac{n}{2} \right\}$$

The expanding constant measures the connectivity of a graph. If we view  $G$  as a network transmitting information, then  $h(G)$  measures the “quality” of  $G$  as a communications network. That is, if  $h(G)$  is large then information propagates well because the graph is well connected. If  $h(G)$  is small then information does not propagate well. Consider the following (extreme) examples.

- 1) Consider  $K_m$ , the complete graph on  $m$  vertices, with  $m \geq 2$ .

PROPOSITION 2.  $h(K_m) = \lceil \frac{m+1}{2} \rceil$ , the greatest integer function of  $\frac{m+1}{2}$ .

PROOF. Let  $X \subset K_m$  with  $|X| = \ell \leq \frac{m}{2}$ .

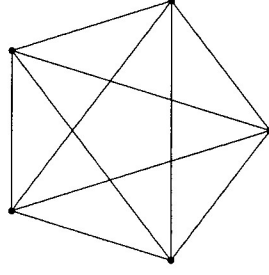


FIGURE 1. The complete graph on 5 vertices.

Note that removing the edges  $\partial X$  from  $K_m$  decomposes  $K_m$  into two smaller complete graphs,  $K_\ell$  and  $K_{m-\ell}$ .

$$\text{As } |E(K_m)| = \binom{m}{2} \text{ then } |\partial X| = \binom{m}{2} - \binom{\ell}{2} - \binom{m-\ell}{2} = m\ell - \ell^2.$$

So  $\frac{|\partial X|}{|X|} = m - \ell$ . Since  $\ell \leq \frac{m}{2}$  this is minimized

for  $\ell = \frac{m}{2}$  with  $m$  even and  $\ell = \frac{m-1}{2}$  with  $m$  odd.  $\square$

Note that  $h(K_m) \rightarrow \infty$  as  $m \rightarrow \infty$ . But also note that  $K_m$  has  $\binom{m}{2}$  edges since every pair of vertices are adjacent. Thus as a communications network,  $K_m$  would be well connected but very expensive, as there are a lot of edges in  $K_m$ .

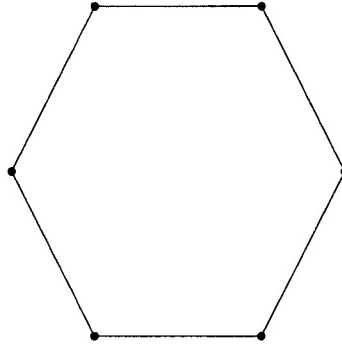


FIGURE 2. The cycle graph on 6 vertices.

2) Consider  $C_n$ , the cycle graph on  $n$  vertices. If  $X$  is half of the cycle, then  $|\partial X| = 2$ , so  $h(C_n) \leq \frac{2}{\lfloor \frac{n}{2} \rfloor} \sim \frac{4}{n}$ ; in particular  $h(C_n) \rightarrow 0$  for  $n \rightarrow \infty$ .

From these two examples, we see that the highly connected complete graph has a large expanding constant that grows proportionately with the number of vertices. On the other hand, the minimally connected cycle graph has a small expanding constant that decreases to zero as the number of vertices grows. In this sense,  $h(G)$  does indeed provide a measure of the “quality”, or connectivity, of  $G$  as a network.

Informally, an expander graph is a graph  $G$  in which every subset  $X$  of vertices *expands* quickly, in the sense that it is connected to many vertices in the set  $V - X$  of complementary vertices. When we refer to expander graphs we are actually referring to a family of graphs.

DEFINITION 2. *Let  $(G_m)_{m \geq 1}$  be a family of graphs indexed by  $m \in \mathbb{N}$ . Such a family  $(G_m)_{m \geq 1}$  of finite, connected,  $k$ -regular graphs is a family of **expanders** if  $|V(G_m)| \rightarrow \infty$  for  $m \rightarrow \infty$ , and if there exists  $\epsilon > 0$ , such that  $h(G_m) \geq \epsilon$  for every  $m \geq 1$ .*

It is an interesting and difficult problem to find explicit examples of families of expanders. The following are some examples of families of expanders.

1) This family of graphs  $G_m$  lies on a grid. Let the vertex set be  $V_m = \mathbb{Z}_m \times \mathbb{Z}_m$ . The degree is  $k = 4$  and the edges are described as follows:  
Vertex  $(x, y)$  has edges to  $(x + y, y), (x - y, y), (x, y + x)$  and  $(x, y - x)$  (where all operations are done modulo  $m$ ).

Margulis showed that this is an expander family, see [10].

Gaber and Galil showed that this is an expander family, see [5].

**Remark:** This is interesting because it is well-known that usual grid graphs are not expanders! In fact, grid graphs are planar graphs so we can apply the Planar Separator Theorem. Usual grid graphs are of degree 4 and the edges are described as follows:

Vertex  $(x, y)$  has edges to  $(x + 1, y)$ ,  $(x - 1, y)$ ,  $(x, y + 1)$ , and  $(x, y - 1)$ .

2) This family has graphs of size  $p$  (for all prime  $p$ ). Here  $V_p = \mathbb{Z}_p$  and  $k = 3$ . Each vertex  $x$  is connected to its neighbors and its inverse (i.e.  $x + 1, x - 1$  and  $x^{-1}$ ). This was shown to be an  $\epsilon$ -expander family by Lubotsky, Philips and Sarnak, see [9].

### 3.2. Estimating The Expanding Constant

Computing the expanding constant for a graph is not easy to do. Consider that for a large graph  $G$ , calculating the ratio  $\frac{|\partial F|}{|F|}$  for every subset  $F$  of vertices containing no more than half the vertices in the graph. Doing this would be a time-consuming task (an **NP**-problem), since if there are  $n$  vertices in the graph then there are exponentially many such subsets  $F$ . Finding an estimate for  $h(G)$  is much easier and often suffices for specific situations. In particular, finding an upper bound for  $h(G)$  is generally much simpler than finding a lower bound. Fortunately, there is a relationship between the expanding constant and another number related to the graph that allows one to estimate  $h(G)$  indirectly. This is the first eigenvalue of the adjacency matrix of the graph.

The **adjacency matrix**  $A(G) = (A_{xy})$  of graph  $G$  with  $n$  vertices is an  $n \times n$  matrix where the entries are indexed by pairs of vertices  $x, y$  of  $G$ . and  $A_{xy}$  is the number of edges between vertices  $x$  and  $y$ .

Consider the 4-regular graph on 6 vertices.

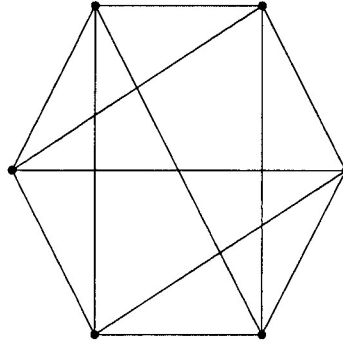


FIGURE 3

The adjacency matrix of this graph is the following,

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Using Mathematica, we get that the eigenvalues are 4, -2 with multiplicity 2, and 0 with multiplicity 3.

Notice that the matrix is symmetric because each entry is the number of edges between two vertices,  $x$  and  $y$ . So if there is an edge from  $x$  to  $y$  then there is an edge from  $y$  to  $x$ . Further the eigenvalues are real numbers, 4 is an eigenvalue, and the other eigenvalues are in  $[-4, 4]$ . These are examples of more general theorems.

**THEOREM 3.** *If  $A$  is an  $n$ -by- $n$  real symmetric matrix, then all of the eigenvalues of  $A$  are real.*

PROOF. This follows from the real spectral theorem, see [11].  $\square$

Therefore the adjacency matrix  $A(G)$  of a graph with  $n$  vertices has  $n$  real eigenvalues, repeated according to multiplicities that we list in decreasing order:

$$\mu_0 \geq \mu_1 \geq \dots \geq \mu_{n-1}.$$

The collection of eigenvalues of  $A(G)$  is called the **spectrum** of the graph  $G$ . So the spectrum of the 4-regular graph on 6 vertices, given by Figure 3, is  $\{-2, 0, 4\}$ .

The following theorem demonstrates the connection between the spectrum of a  $k$ -regular graph and its properties:

THEOREM 4. [4] *Let  $G$  be a  $k$ -regular simple graph on  $n$  vertices and  $\{\mu_0, \dots, \mu_{n-1}\}$  be the spectrum of  $G$ . Then the following hold:*

- $\mu_0 = k$
- *The graph is connected if and only if  $\mu_0 > \mu_1$*
- *The graph is bipartite if and only if  $\mu_0 = -\mu_{n-1}$*

Again from the spectrum of Figure 3 we see these facts to be true since  $\mu_0 = 4$  and we had a 4-regular graph, and  $\mu_0 = 4 > \mu_1 = 3$  and the graph was connected. Note that since -4 is not an eigenvalue, the graph is not bipartite.

We focus on the second eigenvalue because it is related to the expanding constant of the graph. This relationship is given by the following theorem.

THEOREM 5. [4] *Let  $G$  be a simple, finite, connected,  $k$ -regular graph. Then*

$$\frac{k - \mu_1}{2} \leq h(G) \leq \sqrt{2k(k - \mu_1)}.$$

The theorem actually proves that  $k - \mu_1$ , also known as the **spectral gap**, can give a good estimate on the expanding constant of a graph. Since  $h(G) \geq \frac{k - \mu_1}{2}$ , as the spectral gap becomes large so does  $h(G)$ . It follows that the graph  $G$  is well connected. Moreover, for  $k$ -regular families of graphs, a graph is an expander ( $h(G) > \epsilon$  for all  $G$  in the family) if the spectral gap is bounded ( $k - \mu_1 > \epsilon'$ ). This is because  $h(G) \geq \frac{k - \mu_1}{2} > \frac{\epsilon'}{2}$  so there exists some  $\epsilon = \frac{\epsilon'}{2}$  such that  $h(G) > \epsilon$ , for all graphs  $G$  in the family which means the graphs are expanders.

### 3.3. Ramanujan Graphs

The spectral gap has an asymptotic lower and upper bound.

**THEOREM 6.** [4] *Let  $(G_m)_{m \geq 1}$  be a family of connected,  $k$ -regular, finite graphs, with  $|V_m| \rightarrow \infty$  as  $m \rightarrow \infty$ . Then,  $\lim_{m \rightarrow \infty} \inf \mu_1(G_m) \geq 2\sqrt{k-1}$ .*

The **girth** of a connected graph  $G$  is the length of the shortest cycle in  $G$ . For a finite, connected,  $k$ -regular graph, let  $\mu(G)$  be the smallest eigenvalue of  $G$  with  $|\mu(G)| \neq k$ .

**THEOREM 7.** [4] *Let  $(G_m)_{m \geq 1}$  be a family of connected,  $k$ -regular, finite graphs, with the girth of  $G_m \rightarrow \infty$  as  $m \rightarrow \infty$ . Then,  $\lim_{m \rightarrow \infty} \sup \mu(G_m) \leq -2\sqrt{k-1}$ .*

Ramanujan graphs are a specific type of expander graphs that have essentially have the largest possible spectral gap. The definition of Ramanujan graphs is a result of the previous two theorems.

DEFINITION 3. A finite, connected,  $k$ -regular graph  $G$  is **Ramanujan** if for every eigenvalue  $\mu$  of  $A(G)$  with  $|\mu| \neq k$ , we have  $|\mu| \leq 2\sqrt{k-1}$ .

Because of the way Ramanujan graphs are defined they make the best expander graphs. Note that we have included the property of regularity in the definition of Ramanujan graphs. This is done for simplicity. There is a more general definition of Ramanujan graphs that includes non-regular graphs.



## CHAPTER 4

### Bounds On $k$ -regular Ramanujan Graphs

#### 4.1. Bounds On Planar $k$ -regular Ramanujan Graphs

For a planar finite  $k$ -regular graph  $G$ , we want to find a lower bound on the number of vertices needed to guarantee a graph not to be Ramanujan. To do this we estimate  $h(G)$  using the separator theorems, and then relate that estimate to the eigenvalues to show that  $G$  is not a Ramanujan graph.

Let  $G$  be a finite connected  $k$ -regular, planar graph with  $n > 0$  vertices.

Since  $G$  is planar with  $n > 0$  vertices, there exists a partition  $(A, B, C)$  such that  $|A|, |B| < \frac{n}{2}$ ,  $|C| \leq m\sqrt{n}$ , where  $m = \frac{3\sqrt{2}}{2(1 - \sqrt{\frac{2}{3}})}$ , and no vertex in  $A$  is adjacent to any in  $B$ . From the (Planar Separator Corollary).

We will cut the edges in  $C$  to disconnect the graph. Then we use the number of these edges and the cardinality of the smaller component to estimate  $h(G)$ .

**LEMMA 1.** *Let  $G$  be a  $k$ -regular planar graph. with  $k \geq 3$ . Then the number of edges cut is not more than  $C(k)m\sqrt{n}$  edges, where  $C(3) = 1$  and  $C(4) = C(5) = 2$ .  $C(k)$  is the most number of edges cut from a vertex of  $k^{th}$  degree.*

**PROOF.** Let  $G$  be a  $k$ -regular planar graph, with  $n$  vertices. Applying the Planar Separator Corollary we partition  $V(G) = A \cup B \cup C$  as in the Corollary. Note that any vertex in  $C$  connects to  $A$ ,  $B$ , or  $C$ . We choose to disconnect vertices in  $C$ .

This then disconnects the graph  $G$ . We do this by considering the cases  $k = 3, 4, 5$  individually.

Consider  $k = 3$ .

If  $G$  is 3-regular then for any vertex  $v$  in  $C$  we have 4 cases.

Case 1: The vertex  $v$  has no edges connecting to  $A$  and it has 3 edges going to  $B$  or  $C$ . So we remove no edges to disconnect  $A$  from  $B$ .

Case 2: The vertex  $v$  has 1 edge connecting to  $A$  and it has 2 edges going to  $B$  or  $C$ . We remove the single edge connecting to  $A$  we disconnect  $A$  from  $B$ .

Case 3: The vertex  $v$  has 2 edges connecting to  $A$  and it has 1 edge going to  $B$  or  $C$ . We remove the single edge connecting to  $B$  or  $C$  we disconnect  $A$  from  $B$ .

Case 4: The vertex  $v$  has 3 edges connecting to  $A$  and it has no edges going to  $B$  or  $C$ . We remove no edges to disconnect  $A$  from  $B$ .

So in the worst case we remove 1 edge from every vertex in  $C$  and this disconnects  $A$  from  $B$ . Since there are  $m\sqrt{n}$  vertices in  $C$ , we remove at most  $m\sqrt{n}$  edges.

If  $G$  is 4-regular then for any vertex  $v$  in  $C$  we have 5 cases.

Case 1: The vertex  $v$  has no edges connecting to  $A$  and it has 4 edges going to  $B$  or  $C$ . We remove no edges to disconnect  $A$  from  $B$ .

Case 2: The vertex  $v$  has 1 edge connecting to  $A$  and it has 3 edges connecting to  $B$  or  $C$ . We remove the single edge connecting to  $A$  we disconnect  $A$  from  $B$ .

Case 3: The vertex  $v$  has 2 edges connecting to  $A$  and it has 2 edges connecting to  $B$  or  $C$ . We remove the 2 edges connecting to  $A$  or the 2 edges connecting to  $B$  or  $C$  we disconnect  $A$  from  $B$ .

Case 4: The vertex  $v$  has 3 edges connecting to  $A$  and it has 1 edge connecting to  $B$ . We remove the single edge connecting to  $B$  or  $C$  we disconnect  $A$  from  $B$ .

Case 5: The vertex  $v$  has 4 edges connecting to  $A$  and it has no edges going to  $B$  or  $C$ . We remove no edges to disconnect  $A$  from  $B$ .

So in the worst case we remove 2 edges from every vertex in  $C$  and this disconnects  $A$  from  $B$ . Since there are  $m\sqrt{n}$  vertices in  $C$ , we remove at most  $2m\sqrt{n}$  edges.

If  $G$  is 5-regular then for any vertex  $v$  in  $C$  we have 6 cases.

Case 1: The vertex  $v$  has no edges connecting to  $A$  and it has 5 edges going to  $B$  or  $C$ . We remove no edges to disconnect  $A$  from  $B$ .

Case 2: The vertex  $v$  has 1 edge connecting to  $A$  and it has 4 edges connecting to  $B$  or  $C$ . We remove the single edge connecting to  $A$  we disconnect  $A$  from  $B$ .

Case 3: The vertex  $v$  has 2 edges connecting to  $A$  and it has 3 edges connecting to  $B$  or  $C$ . We remove the 2 edges connecting to  $A$  we disconnect  $A$  from  $B$  and  $C$ .

Case 4: The vertex  $v$  has 3 edges connecting to  $A$  and it has 2 edges connecting to  $B$  or  $C$ . We remove the 2 edges connecting to  $B$  or  $C$  we disconnect  $A$  from  $B$ .

Case 5: The vertex  $v$  has 4 edges connecting to  $A$  and it has 1 edge connecting to  $B$  or  $C$ . We remove the single edge connecting to  $B$  or  $C$  we disconnect  $A$  from  $B$ .

Case 6: The vertex  $v$  has 5 edges connecting to  $A$  and it has no edges going to  $B$  or  $C$ . We remove no edges to disconnect  $A$  from  $B$ .

So in the worst case we remove 2 edges from every vertex in  $C$  and this disconnects  $A$  from  $B$ . Since there are  $m\sqrt{n}$  vertices in  $C$ , we remove at most  $2m\sqrt{n}$  edges.  $\square$

To find the lower bound we need the following lemma.

LEMMA 2. *Let  $G$  be a  $k$ -regular graph. If  $h(G) < \frac{k}{2} - \sqrt{k-1}$  then  $G$  is not Ramanujan.*

PROOF. Let  $G$  be a finite, connected  $k$ -regular graph. Then

$$\frac{k - \mu_1}{2} \leq h(G) \leq \sqrt{2k(k - \mu_1)}.$$

If  $h(G) < \frac{k}{2} - \sqrt{k-1}$  then this gives us  $\frac{k - \mu_1}{2} < \frac{k}{2} - \sqrt{k-1}$ . Therefore we compute that

$$k - \mu_1 < k - 2\sqrt{k-1}$$

$$-\mu_1 < -2\sqrt{k-1}$$

$$\mu_1 > 2\sqrt{k-1}$$

Therefore  $G$  is not Ramanujan.  $\square$

The expanding constant will be the ratio of the number of edges cut in  $C$  and the number of vertices in the smaller of the two pieces.

LEMMA 3. *Let  $G$  be a  $k$ -regular planar graph, with  $n$  vertices. Applying the Planar Separator Corollary we get a vertex partition  $V(G) = A \cup C \cup B$  where  $|A|, |B| \leq \frac{n}{2}$  and  $|C| \leq m\sqrt{n}$ . Then  $|A|, |B| \geq \frac{n}{2} - m\sqrt{n}$ .*

PROOF. Let  $G$  be  $k$ -regular planar graph, with  $n$  vertices. Applying the Planar Separator Corollary we get a vertex partition  $V(G) = A \cup C \cup B$ , where  $|A|, |B| \leq \frac{n}{2}$  and  $|C| \leq m\sqrt{n}$ .

Assume that  $|A| < \frac{n}{2} - m\sqrt{n}$ .

Then  $|A| + |B| + |C| < \frac{n}{2} - m\sqrt{n} + \frac{n}{2} + m\sqrt{n} = n$

But  $n = |G| = |A \cup B \cup C| \leq |A| + |B| + |C| < n$ . This gives a contradiction!

So  $|A| \geq \frac{n}{2} - m\sqrt{n}$ .

Similarly, it can be shown that  $|B| \geq \frac{n}{2} - m\sqrt{n}$ . □

From Lemma 1 and Lemma 3 we can estimate the expander constant to be:

$$h(G) \leq \frac{C(k)m\sqrt{n}}{\frac{n}{2} - m\sqrt{n}} = \frac{2C(k)m\sqrt{n}}{n - 2m\sqrt{n}} = \frac{2C(k)m}{\sqrt{n} - 2m}$$

So from the Lemma 2, if  $\frac{2C(k)m}{\sqrt{n} - 2m} < \frac{k}{2} - \sqrt{k-1}$  then  $G$  is not Ramanujan.

From  $\frac{2C(k)m}{\sqrt{n} - 2m} < \frac{k}{2} - \sqrt{k-1}$  we get

$$\frac{2C(k)m}{\frac{k}{2} - \sqrt{k-1}} < \sqrt{n} - 2m, \text{ Note that } \sqrt{n} - 2m > 0 \text{ as is } \frac{k}{2} - \sqrt{k-1}.$$

$$\frac{2C(k)m}{\frac{k}{2} - \sqrt{k-1}} + 2m < \sqrt{n}$$

$$\sqrt{n} > \frac{4C(k)m}{k - 2\sqrt{k-1}} + 2m$$

$$n > \left( \frac{4C(k)m}{k - 2\sqrt{k-1}} + 2m \right)^2$$

$$n > 4m^2 \left( \frac{2C(k)}{k - 2\sqrt{k-1}} + 1 \right)^2$$

So if  $n > 4m^2 \left( \frac{2C(k)}{k - 2\sqrt{k-1}} + 1 \right)^2$  then  $G$  is not Ramanujan.

Applying this to  $k = 3, 4, 5$ -regular graphs we use Lemma 1 to get the following explicit estimates:

$$k = 3 \text{ then } n > 4m^2 \left( \frac{2(1)}{3 - 2\sqrt{3-1}} + 1 \right)^2 \approx 85631.93$$

$$k = 4 \text{ then } n > 4m^2 \left( \frac{2(2)}{4 - 2\sqrt{4-1}} + 1 \right)^2 \approx 38295.34$$

$$k = 5 \text{ then } n > 4m^2 \left( \frac{2(2)}{5 - 2\sqrt{5-1}} + 1 \right)^2 \approx 13363.62$$

Which gives us the following theorem.

**THEOREM 8.** *Let  $G$  be a finite connected  $k$ -regular, planar graph with  $n > 0$  vertices. Then  $G$  is not Ramanujan for*

$$k = 3 \text{ if } n \geq 85632$$

$$k = 4 \text{ if } n \geq 38296$$

$$k = 5 \text{ if } n \geq 13364$$

## 4.2. Bounds On Other Classes Of Graphs

The following is a more general result.

THEOREM 9. Let  $\mathcal{G}_{\frac{1}{2}}$  be a family of graphs satisfying the assumptions of Proposition 1. with  $\alpha = \frac{1}{2}$ . If  $G \in \mathcal{G}_{\frac{1}{2}}(c)$  is  $k$ -regular and  $n > \frac{4c^2}{(1 - \sqrt{\frac{2}{3}})^2} (\frac{2C(k)}{k - 2\sqrt{k-1}} + 1)^2$  then  $G$  is not Ramanujan.

PROOF. Let  $G \in \mathcal{G}$  with  $|V(G)| = n$  and for any subgraph  $H \subset G$ ,  $H \in \mathcal{G}$ . Let  $G$  be a  $k$ -regular graph. Let there exist a vertex partition  $V(G) = A \cup B \cup C$  where  $|A|, |B| \leq \frac{n}{2}$ ,  $|C| \leq c\sqrt{n}$ .

From Proposition 1 then there exists a vertex partition  $V(G) = A \cup B \cup C$  where  $|A|, |B| \leq \frac{n}{2}$ ,  $|C| \leq \frac{c}{1 - \sqrt{\frac{2}{3}}} \sqrt{n}$ , and no vertex in  $A$  is adjacent to a vertex in  $B$ .

We will cut the edges in  $C$  to disconnect the graph.

The expansion constant will be estimated by the ratio of the number of edges cut in  $C$  and the number of vertices in the smaller of the two pieces.

LEMMA 4. Let  $G \in \mathcal{G}_{\frac{1}{2}}(c)$  be a  $k$ -regular graph, where  $\mathcal{G}_{\frac{1}{2}}(c)$  satisfies the assumptions of Proposition 1 with  $n$  vertices. Applying Proposition 1 we get a vertex partition  $V(G) = A \cup B \cup C$ . With  $|A|, |B| \leq \frac{n}{2}$  and  $|C| \leq \frac{c}{1 - \sqrt{\frac{2}{3}}} \sqrt{n}$ . Then  $|A|, |B| \geq \frac{n}{2} - \frac{c}{1 - \sqrt{\frac{2}{3}}} \sqrt{n}$ .

PROOF. Let  $G \in \mathcal{G}_{\frac{1}{2}}(c)$  be  $k$ -regular graph, where  $\mathcal{G}_{\frac{1}{2}}(c)$  satisfies the assumptions of Proposition 1 with  $n$  vertices. Applying Proposition 1 we get a vertex partition  $V(G) = A \cup C \cup B$ , where  $|A|, |B| \leq \frac{n}{2}$  and  $|C| \leq \frac{c}{1 - \sqrt{\frac{2}{3}}} \sqrt{n}$ .

Assume that  $|A| < \frac{n}{2} - \frac{c}{1 - \sqrt{\frac{2}{3}}} \sqrt{n}$ .

Then  $|A| + |B| + |C| < \frac{n}{2} - \frac{c}{1 - \sqrt{\frac{2}{3}}} \sqrt{n} + \frac{n}{2} + \frac{c}{1 - \sqrt{\frac{2}{3}}} \sqrt{n} = n$

But  $n = |G| = |A \cup B \cup C| \leq |A| + |B| + |C| < n$ . This gives a contradiction!

$$\text{So } |A| \geq \frac{n}{2} - \frac{c}{1 - \sqrt{\frac{2}{3}}} \sqrt{n}.$$

$$\text{Similarly, it can be shown that } |B| \geq \frac{n}{2} - \frac{c}{1 - \sqrt{\frac{2}{3}}} \sqrt{n}. \quad \square$$

We get the estimate of the expander constant to be:

$$h(G) \leq \frac{C(k) \frac{c}{1 - \sqrt{\frac{2}{3}}} \sqrt{n}}{\frac{n}{2} - \frac{c}{1 - \sqrt{\frac{2}{3}}} \sqrt{n}}, \text{ where } C(k) \text{ is the number of edges cut.}$$

$$h(G) \leq \frac{C(k) \frac{c}{1 - \sqrt{\frac{2}{3}}}}{\frac{\sqrt{n}}{2} - \frac{c}{1 - \sqrt{\frac{2}{3}}}}$$

So from the Lemma 2 if  $\frac{C(k) \frac{c}{1 - \sqrt{\frac{2}{3}}}}{\frac{\sqrt{n}}{2} - \frac{c}{1 - \sqrt{\frac{2}{3}}}} < \frac{k}{2} - \sqrt{k-1}$  then  $G$  is not Ramanujan.

From  $\frac{C(k) \frac{c}{1 - \sqrt{\frac{2}{3}}}}{\frac{\sqrt{n}}{2} - \frac{c}{1 - \sqrt{\frac{2}{3}}}} < \frac{k}{2} - \sqrt{k-1}$  we get

$$\frac{C(k) \frac{c}{1 - \sqrt{\frac{2}{3}}}}{\frac{k}{2} - \sqrt{k-1}} < \frac{\sqrt{n}}{2} - \frac{c}{1 - \sqrt{\frac{2}{3}}}, \text{ Note that } \frac{\sqrt{n}}{2} - \frac{c}{1 - \sqrt{\frac{2}{3}}} > 0 \text{ as is } \frac{k}{2} - \sqrt{k-1}.$$

$$\frac{2C(k) \frac{c}{1 - \sqrt{\frac{2}{3}}}}{k - 2\sqrt{k-1}} + \frac{c}{1 - \sqrt{\frac{2}{3}}} < \frac{\sqrt{n}}{2}$$

$$\sqrt{n} > \frac{2C(k) \frac{2c}{1 - \sqrt{\frac{2}{3}}}}{k - 2\sqrt{k-1}} + \frac{2c}{1 - \sqrt{\frac{2}{3}}}$$

$$n > \left( \frac{2C(k) \frac{2c}{1 - \sqrt{\frac{2}{3}}}}{k - 2\sqrt{k-1}} + \frac{2c}{1 - \sqrt{\frac{2}{3}}} \right)^2$$

$$n > \frac{4c^2}{(1 - \sqrt{\frac{2}{3}})^2} \left( \frac{2C(k)}{k - 2\sqrt{k-1}} + 1 \right)^2$$

So if  $n > \frac{4c^2}{(1 - \sqrt{\frac{2}{3}})^2} \left( \frac{2C(k)}{k - 2\sqrt{k-1}} + 1 \right)^2$  then  $G$  is not Ramanujan.  $\square$



An application of Theorem 9 and Corollary 2 gives us the following.

COROLLARY 4. *Let  $G$  be a simple graph with  $n$  vertices and no  $K_h$ -minor. If  $G$  is  $k$ -regular and  $n > \frac{4h^3}{(1 - \sqrt{\frac{2}{3}})^2} (\frac{2C(k)}{k - 2\sqrt{k-1}} + 1)^2$  then  $G$  is not Ramanujan.*

An application of Theorem 9 and Corollary 3 gives us the following.

COROLLARY 5. *Let  $G$  be a graph with  $n$  vertices and genus less than or equal to  $g$ . If  $G$  is  $k$ -regular and  $n > \frac{4(6\sqrt{g} + 2\sqrt{2} + 1)^2}{(1 - \sqrt{\frac{2}{3}})^2} (\frac{2C(k)}{k - 2\sqrt{k-1}} + 1)^2$  then  $G$  is not Ramanujan.*

## CHAPTER 5

### Applications of Expander Graphs

Here are some applications for expanders graphs.

- *Constructing good error-correcting codes* [6]: An error correcting code is a set of words in  $\{0, 1\}^n$ . Its distance is the minimal Hamming distance between two codewords. Therefore, if we transmit a codeword through a noisy channel that flips some of the bits, then we can correct errors, as long as the number of bits flipped is bounded by half the distance. There is a trade-off between the size of the code and the number of errors it can correct. Expanders can be used to build error correcting codes that have large size and distance. These codes are also efficiently decodable.

- *Amplifying the success probability of random algorithms* [6]: Let  $L$  be some language in  $RP$ , randomized polynomial time, and assume that  $A$  is a randomized algorithm that decides whether  $x \in L$  with a one sided error. Assume that  $A$  tosses  $m$  coins and has an error probability of  $\beta$ . Build an expander graph such that  $V = \{0, 1\}^m$ ; i.e. the vertex set of the graph is the probability domain of  $A$ 's coin tosses. Fix some input  $x$  and let  $B$  be all the coin tosses for which  $A(x)$  is wrong. Now let  $A'$  be the following algorithm:

1. pick a vertex  $v_0 \in V$  uniformly and at random.
2. perform a random walk of length  $t$  resulting with the set of vertices  $(v_0, v_1, \dots, v_t)$ .

3. return  $\bigcup_{i=0}^t A(x, v_i)$

The error probability is reduced exponentially while the number of random bits is only  $m + t \log d = m + O(t)$ . The same trick can be used to amplify the success of the probability of a two-sided error algorithm.

- *Designing computer networks:* An ideal computer network is both well connected and inexpensive. For a network to be well connected you want enough connections to ensure a large proportion of systems on the network can communicate if some systems on the network fail. For a network to be inexpensive you want as few direct connections as possible. Consider a network in a metropolitan area. If every computer had a direct line to every other computer in that area it would be extremely expensive. Finding a good balance between connectivity and expense can be very hard to do but with the introduction of expander graphs it has become easier because expander graphs are well connected and sparse. A network designed by using an expander graph is well connected and inexpensive.

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