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Random Walks with Elastic and Reflective Lower Boundaries

Lucas Clay Devore
Western Kentucky University, lucas.devore639@wku.edu

RANDOM WALKS WITH ELASTIC AND REFLECTIVE LOWER BOUNDARIES

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David K. Neal

Director of Thesis

Jonathan Quinton

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Dean, Graduate Studies and Research   Date

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RANDOM WALKS WITH ELASTIC AND REFLECTIVE LOWER BOUNDARIES

Lucas Clay Devore December 2009 80 Pages

Directed by: David K. Neal, Jonathan Quiton, and Jemal Gishe

Department of Mathematics Western Kentucky University

This thesis provides a study of one-dimensional random walks which move in one-step intervals and are bounded below by an elastic or reflective lower boundary. The random walks start at a given starting height j on the y-axis. They then move one unit to the right and upward with probability p, downward with probability q, or stay at the same height with probability r. The walk ceases when a given upper boundary $n > j$ is reached. If an elastic lower boundary $m < j$ is reached, the walk automatically has one unit of height restored without an extra step taken. If a reflective lower boundary $m < j$ is reached, the next step taken is automatically upward.

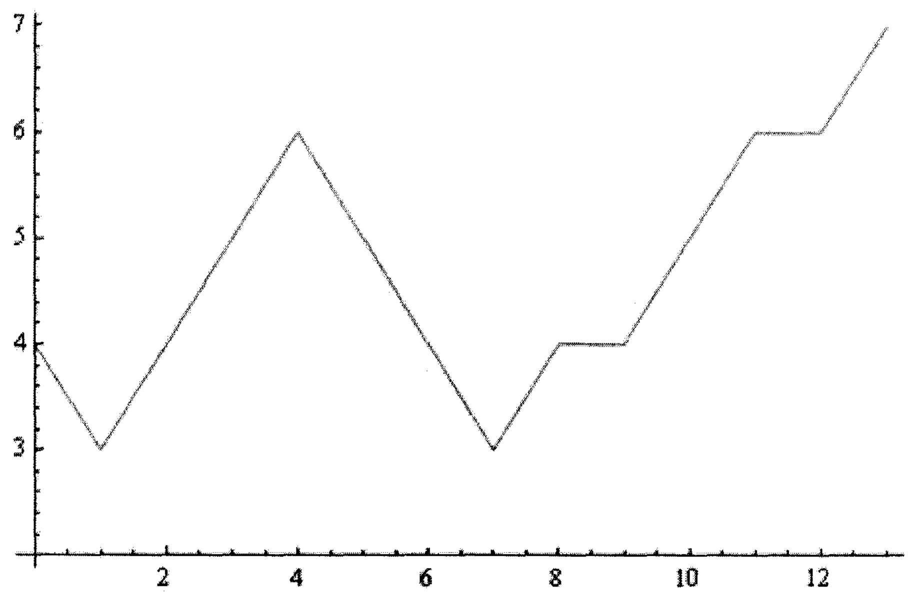
We first show that, given $p > 0$, a random walk containing an elastic or reflective lower boundary will reach any upper boundary n with probability 1. We then derive formulas for the average number of steps needed to reach n for both cases. Next, we use Markov Chain methods and Systems of Equations to analyze scenarios where our walk moves upward a units on an upward step for some integer $a > 1$. For our conclusion, we use the results obtained to analyze a scenario in which a gambler wagers at a House that gives the bettor one chip when he or she goes broke. Throughout, computer simulations are used to verify many of the results obtained in the thesis.

Chapter 1: Introduction

In this thesis, we will study two special cases of a lower boundary in a random walk: an elastic lower boundary and a reflective lower boundary. Both of these special lower boundaries automatically restore one unit of height whenever the walk reaches them. For the case of the elastic lower boundary, we restore one unit of height without an extra step taken. The reflective lower boundary, however, restores one unit of height on the following step after it is reached.

We first provide background information on what a random walk is. The type of random walk we will consider throughout the course of this thesis may be described as a process which measures the height of a particle that randomly moves upward, moves downward, or stays at the same height with each unit step. We often present a random walk as a two-dimensional graph where the units on the x-axis represent a unit step, and units on the y-axis represent the height of the particle after each step. Some often consider cases where the random walk ends after a given number of steps. For our example, however, we will demonstrate a random walk which only ceases when a given upper or lower boundary is reached.

Before we demonstrate such a walk, we first provide some notations which we will use throughout this thesis. We let j denote the initial starting height of the walk on the y-axis. We let a represent the size of upward steps taken, and b represent the size of downward steps taken. We let p be the probability of taking an upward step, q be the probability of taking a downward step, and $r = 1 - p - q$ be the probability of staying at the same height. Finally, our walk is bounded above by upper boundary n, and below by lower boundary m. We now provide the following example of a random walk path:



A random walk path with $j = 4$, $p = 0.5$, $q = 0.4$, $r = 0.1$, $n = 7$, $m = 0$, and $a = b = 1$.

We note that this walk began with initial height $j = 4$ and proceeded to take a downward step with probability $q = 0.4$. The second, third and fourth steps were each upward with probability $p = 0.5$, and the walk reached a height of 6. The walk then proceeded to take three downward steps and one upward step until step 8, when it then took a step and remained at the same height with probability $r = 0.1$. We see that the walk stops upon reaching the upper boundary of $n = 7$ after thirteen steps.

Two important questions one might ask regarding random walks of this form are as follows: Given j, p, q, n and m, what is the probability of reaching an upper or lower boundary? Also, given that an upper or lower boundary is reached, what is the average number of steps needed to reach one of the boundaries? The answer to these questions was first given by William Feller in 1968. In his book **An Introduction to Probability Theory and Its Applications** [1], he shows that for the classic boundary problem, where $a = b = 1$, any random walk in which $r \neq 1$ will reach an upper or lower boundary with probability 1. He also proves that the average number of steps needed for the classic boundary problem is given by the following formula:

$$E\left[{}_j S_m^n\right] = \begin{cases} \dfrac{(n-j)(j-m)}{2p} & \text{if } p = q \\ \dfrac{n-m}{p-q} \times \left(\dfrac{1-(q/p)^{j-m}}{1-(q/p)^{n-m}} \right) - \dfrac{j-m}{p-q} & \text{if } p \neq q. \end{cases}$$

Henceforth we will refer to the case where $a = b = 1$ as the $1:1$ case.

In Section 1 of Chapter 2 of this thesis, we adjust Feller's argument for the classic boundary problem to prove that, when $p > 0$, random walks with an elastic lower boundary will reach any given upper boundary n almost surely. This argument applies to the $1:1$ case. Next, we proceed to adjust Fellers argument for determining $E[{}_j S_m^n]$ to obtain a formula for the average number of steps needed to reach n for the elastic lower boundary case when the step sizes are $1:1$. We denote this average by $E[{}_j S_{(m)}^n]$.

In Section 3, we provide two specific examples of random walks with elastic lower boundaries for the $1:1$ case. Numerous computer simulations are performed for each example using *Mathematica*, and the results are used to verify the conclusions of Section 2 for a case where $p = q$ and a case where $p \neq q$.

In Section 4 of Chapter 2, we begin by discussing cases where our walk may move up a units of height on an upward step where a is an integer greater than 1. We will refer to this as the $a:1$ case. First we demonstrate that obtaining a closed-form solution for the average number of steps using a similar argument as with the $1:1$ case is impossible. We then give the results of numerous computer simulations on two different $a:1$ scenarios in order to obtain an approximation for the average number of steps needed to reach n for both.

In Section 5 we describe how to adjust the Markov Chain method described by David Neal in his paper *Generalized Boundary Problem for One-Dimensional Random Walks* [2] to provide an approximation for the average number of steps needed to reach n for the $a:1$ case. We also use the method to analyze probability states for our walk after a given number of steps. Finally, we use a Markov Chain on the same examples used in Section 4 to see how the results compare to the computer simulations performed in that section.

Section 6 describes how to adjust the Systems of Equations method, also given by Neal in [2], to obtain exact solutions for the average number of steps needed to reach n in an $a:1$ case. To conclude the chapter, we apply the Systems of Equations solution to the two scenarios used in Sections 2.4 and 2.5 and compare the results to those obtained in those sections.

In Chapter 3 we perform the same steps as in Chapter 2 except we now analyze the reflective lower boundary. Each section in Chapter 3 mirrors its counterpart in Chapter 2; however the results obtained are significantly different due to the extra step taken in the reflective lower boundary case when our random walk reaches m. This includes determining the average number of steps needed to reach the upper boundary for the $1:1$ case which we denote by $E[{}_jR^n_{(m)}]$.

Random walks are naturally conducive to modeling gambling scenarios; thus in our conclusion, Chapter 4, we apply the results obtained in Chapters 2 and 3 to a gambling scenario involving a gambling House that spots the bettor one chip whenever he or she goes broke. However, the bettor must repay the House any chips he or she was spotted at the end of the gambling session. We first demonstrate how to calculate the average number of chips the bettor is spotted and must pay back. Determining this value allows us to find the average net gain of the bettor at the end of the gambling session, which we denote $E[{}_jG^n_{(0)}]$. Next, we prove that a gambler at such a House will break even on average if $p=q=0.5$ and the payouts are $1:1$. In Section 4.2, we proceed to show by induction that if $p<q$ and the payouts are $1:1$, the bettor is guaranteed to lose money on average. We then compute $E[{}_jG^n_{(0)}]$ for a specific case. Finally, in Section 4.3, we describe how we may use the Markov Chain method and Systems of Equations to determine $E[{}_jG^n_{(0)}]$ for a gambling scenario with an $a:1$ payout.

Chapter 2: The Elastic Lower Boundary Scenario

For this random-walk scenario, we assume that whenever our walk drops to height 0, we automatically have one unit of height restored with no extra step taken. We will call this an "elastic lower boundary." A good application of this problem would be a gambling scenario where we have a generous house that always spots the bettor one chip when he or she goes broke. For this scenario, we will do the following:

1) Prove that, for $p > 0$, the particle will eventually reach any height $n \geq j$ with probability 1.

2) Derive the average number of steps needed to reach height n given $a = b = 1$ (the $1:1$ case).

3) Provide numerical examples and computer simulations of the $1:1$ case.

4) Discuss the $a:1$ case ($a > b = 1$), and use simulations to approximate the average number of steps needed to reach height n.

5) Use a Markov Chain Method to derive the probability states after k steps for the $a:1$ case, as well as to approximate the average number of steps needed to reach height n.

6) Use a System of Equations Method to obtain numerical solutions for the average number of steps needed to reach height n in the $a:1$ case.

2.1 The Probability of Reaching Height *n*

Using induction, we will prove that a bettor at the generous casino will reach any given maximum almost surely. Assume for now that we have an elastic lower boundary of $m = 0$. Let e_j represent the probability of reaching height n from height j, and let $p > 0$. Clearly $e_n = 1$, and for $1 < j < n$, we have

$$e_j = p \times e_{j+1} + q \times e_{j-1} + r \times e_j.$$

For $j = 1$, however, going down is the same as staying even due to the elastic boundary. So for $j = 1$, we have

$$e_1 = p \times e_2 + q \times e_1 + r \times e_1. \tag{1}$$

We now wish to solve for e_1 in terms of e_2. Because $p + q + r = 1$, we may rewrite Equation (1) as

$$(p + q + r)e_1 = p \times e_2 + q \times e_1 + r \times e_1.$$

Then we have

$$p \times e_1 + q \times e_1 + r \times e_1 = p \times e_2 + q \times e_1 + r \times e_1,$$

which gives $p \times e_1 = p \times e_2$. Thus, $e_1 = e_2$, because $p \neq 0$. Thus, the probability of reaching height n from height 1 is the same as reaching height n from height 2. That is, $e_1 = e_2$.

Now assume that $e_j = e_{j+1}$ for some $j \geq 1$. Then

$$e_{j+1} = p \times e_{j+2} + q \times e_j + r \times e_{j+1}.$$

However, $e_j = e_{j+1}$, so e_{j+1} may be rewritten as

$$e_{j+1} = p \times e_{j+2} + q \times e_{j+1} + r \times e_{j+1}$$

Thus,

$$(p+q+r)e_{j+1} = p \times e_{j+2} + q \times e_{j+1} + r \times e_{j+1},$$

which gives $p \times e_{j+1} = p \times e_{j+2}$, and then $e_{j+1} = e_{j+2}$.

By induction, we may conclude that

$$e_j = e_{j+1} \text{ for } 1 \leq j \leq n-1.$$

Because $e_n = 1$, we may conclude that $e_j = 1$ for *all* j from 1 to n.

The previous result was obtained given a lower bound of $m = 0$. However, we may generalize this result for any lower bound m, given $p > 0$ and starting position j, where $m+1 \leq j \leq n-1$. Note that a random walk beginning at height j with an elastic lower boundary m and upper boundary n is equivalent to a random walk beginning at height $j-m$ with upper boundary $n-m$ and elastic lower boundary 0. Therefore, we may adjust our result to accommodate any lower bound $m < j$ by replacing n with $n-m$ and j with $j-m$. This allows us to state:

Theorem 2.1. *Given $p > 0$, any simple random walk beginning at height j with an elastic lower boundary of $m < j$ will reach any upper boundary $n > j$ with probability 1.*

Therefore, with an elastic lower boundary of m, as long as there is *some* chance of moving upward, then we will reach any upper boundary n with probability 1 from any starting height j between m and n.

2.2 Average Number of Steps to Reach Height *n*

We wish to determine the average number of steps needed to reach height n when starting from height j, where $0 < j < n$, and 0 is an elastic lower boundary. Let e_j now represent the average number of steps needed to reach height n when starting at height j. Starting at height j, we take a step and go up 1 with probability p, down 1 with probability q, or stay at the same height with probability r. Thus, we may express e_j as follows:

$$e_j = 1 + p \times e_{j+1} + q \times e_{j-1} + r \times e_j.$$

Note that our walk stops upon reaching height n; therefore, $e_n = 0$.

We will now set up a difference equations argument to find a solution for e_j in terms of p, q, n, and j. To do this, we will first establish a recursive relationship for e_j in terms of e_{j+1}. For $j = 1$, we have

$$e_1 = 1 + p \times e_2 + q \times e_1 + r \times e_1$$

because going down from height 1 is the same as staying even. We now rewrite e_1 as before:

$$p \times e_1 + q \times e_1 + r \times e_1 = 1 + p \times e_2 + q \times e_1 + r \times e_1.$$

Then we have $p \times e_1 = 1 + p \times e_2$, or

$$e_1 = \frac{1}{p} + e_2. \qquad (2)$$

Next, we have

$$e_2 = 1 + p \times e_3 + q \times e_1 + r \times e_2,$$

then

$$p \times e_2 + q \times e_2 + r \times e_2 = 1 + p \times e_3 + q \times e_1 + r \times e_2, \text{ and}$$

$$p \times e_2 + q \times e_2 = 1 + p \times e_3 + q \times e_1.$$

We now substitute for e_1 using Equation (2) to obtain the following equations:

$$\begin{aligned} p \times e_2 + q \times e_2 &= 1 + p \times e_3 + q \times \left(\frac{1}{p} + e_2 \right) \\ p \times e_2 + q \times e_2 &= 1 + p \times e_3 + \frac{q}{p} + q \times e_2 \\ p \times e_2 &= 1 + p \times e_3 + \frac{q}{p} \\ e_2 &= e_3 + \frac{1}{p} + \frac{q}{p^2} \end{aligned}$$

Continuing in this fashion, we see that, in general,

$$e_j = \sum_{i=1}^{j} \frac{q^{i-1}}{p^i} + e_{j+1}.$$

We now define the *forward difference*, f_j, as $e_j - e_{j+1}$. Thus

$$f_j = e_j - e_{j+1} = \sum_{i=1}^{j} \frac{q^{i-1}}{p^i} = \frac{1}{p} \sum_{i=1}^{j} \frac{q^{i-1}}{p^{i-1}} = \frac{1}{p} \sum_{i=0}^{j-1} \frac{q^i}{p^i} = \frac{1}{p} \sum_{i=0}^{j-1} \left(\frac{q}{p} \right)^i.$$

If $p = q$, we have

$$f_j = \frac{1}{p}\sum_{i=0}^{j-1}\left(\frac{q}{p}\right)^i = \frac{1}{p}\sum_{i=0}^{j-1}(1)^i = \frac{1}{p}\sum_{i=0}^{j-1}1 = \frac{1}{p}\times j = \frac{j}{p}.$$

If $p \neq q$, then

$$f_j = \frac{1}{p}\sum_{i=0}^{j-1}\frac{q^i}{p^i} = \frac{1}{p}\sum_{i=0}^{j-1}\left(\frac{q}{p}\right)^i = \frac{1}{p}\left(\frac{1-\left(\frac{q}{p}\right)^j}{1-\frac{q}{p}}\right) = \left(\frac{1-\left(\frac{q}{p}\right)^j}{p-q}\right) = \frac{\left(\frac{q}{p}\right)^j - 1}{q-p}.$$

We will now use the forward differences to obtain a closed–form solution for e_j. First, because $e_n = 0$, we have

$$\sum_{i=1}^{n-1} f_i = (e_1 - e_2) + (e_2 - e_3) + ... + (e_{n-1} - e_n) = e_1.$$

Also,

$$\sum_{i=1}^{j-1} f_i = (e_1 - e_2) + (e_2 - e_3) + ... + (e_{j-1} - e_j) = e_1 - e_j.$$

Therefore,

$$e_j = e_1 - \sum_{i=1}^{j-1} f_i = \sum_{i=1}^{n-1} f_i - \sum_{i=1}^{j-1} f_i = \sum_{i=j}^{n-1} f_i.$$

If $p = q$, we then obtain

$$e_j = \sum_{i=j}^{n-1} f_i = \sum_{i=j}^{n-1} \frac{i}{p} = \frac{1}{p}\sum_{i=j}^{n-1} i = \frac{1}{p}\left(\sum_{i=1}^{n-1} i - \sum_{i=1}^{j-1} i\right) = \frac{1}{p}\left(\frac{(n-1)n}{2} - \frac{(j-1)j}{2}\right) = \frac{(n-j)(n+j-1)}{2p}.$$

If $p \neq q$, we have

$$e_j = \sum_{i=j}^{n-1} f_i = \sum_{i=j}^{n-1} \frac{\left(\frac{q}{p}\right)^i - 1}{q-p} = \frac{1}{q-p}\sum_{i=j}^{n-1}\left[\left(\frac{q}{p}\right)^i - 1\right] = \frac{1}{q-p}\left(\sum_{i=j}^{n-1}\left(\frac{q}{p}\right)^i - \sum_{i=j}^{n-1} 1\right)$$

$$= \frac{1}{q-p}\left(\frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^j}{\frac{q}{p} - 1} - (n-j)\right) = \frac{1}{q-p}\left(\frac{p\left(\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^j\right)}{q-p} - (n-j)\right)$$

$$= \left(\frac{p\left(\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^j\right)}{(q-p)^2} - \frac{(n-j)}{q-p}\right) = \frac{n-j}{p-q} + \frac{p\left(\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^j\right)}{(p-q)^2}.$$

Our results so far have been based on having a lower bound of 0. However, we may generalize these results for any lower bound $m < j$ by replacing n with $n-m$ and j with $j-m$ as we did at the end of Section 2.1. Making these substitutions, we can state:

Theorem 2.2. *Let $p > 0$. For a simple random walk beginning at height j with an elastic lower boundary of $m < j$, the average number of steps needed to reach the boundary height of $n \geq j$ is given by*

$$E\left[{}_{j}S^{n}_{(m)}\right] = \begin{cases} \dfrac{(n-j)(n+j-2m-1)}{2p} & \text{if } p = q \\[2ex] \dfrac{n-j}{p-q} + \dfrac{p\left(\left(\dfrac{q}{p}\right)^{n-m} - \left(\dfrac{q}{p}\right)^{j-m}\right)}{(p-q)^2} & \text{if } p \neq q. \end{cases}$$

2.3 Numerical Examples and Computer Simulations for the 1:1 Case

We will now examine two specific examples for the 1:1 case, and then use simulations done with *Mathematica* to verify the accuracy of our results in Theorem 2.2.

Scenario 1: p = q

Suppose we have a random walk with the following properties:

Upper Boundary: $n = 8$

Initial Height: $j = 2$

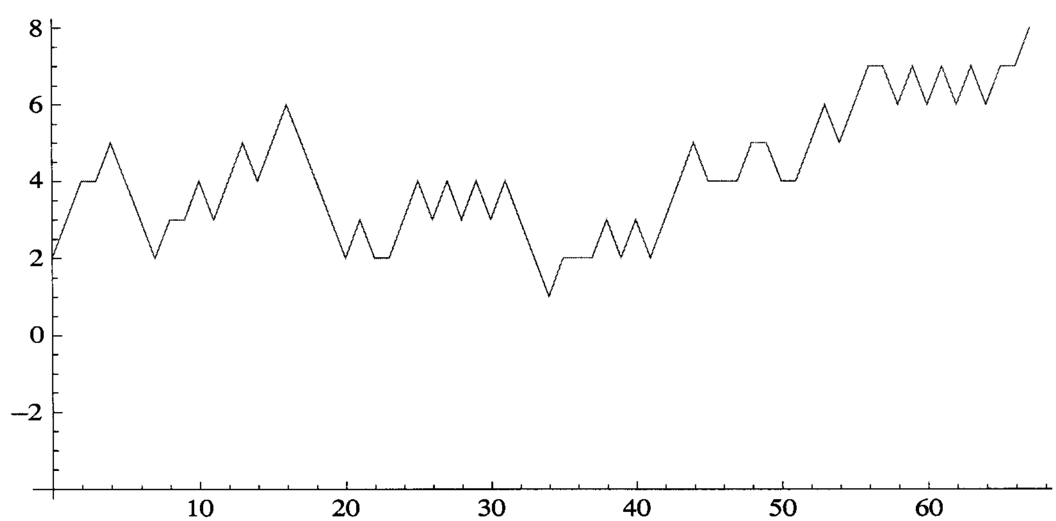
Elastic Lower Boundary: $m = -4$

Probability of Going Up: $p = 0.4$

Probability of Going Down: $q = 0.4$

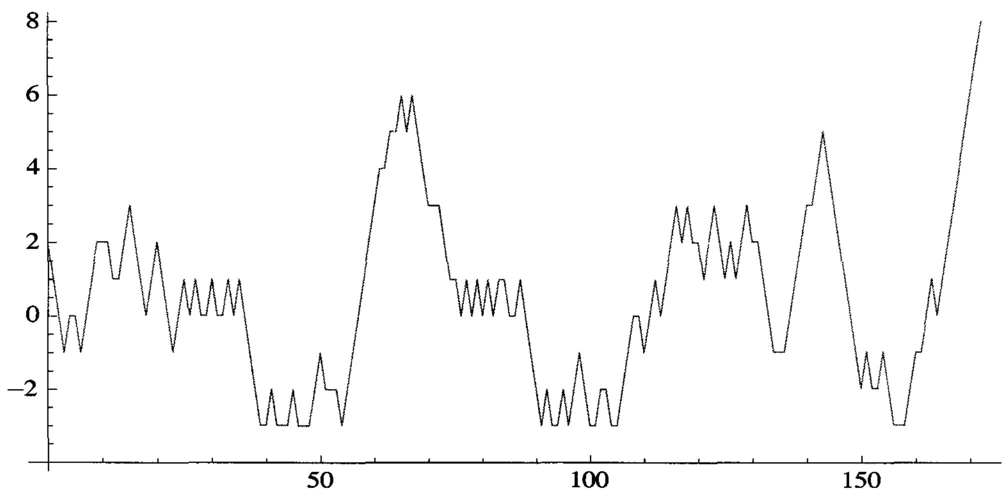
According to Theorem 2.2, our walk should reach height n in an average of $\frac{(n-j)(n+j-2m-1)}{2p} = \frac{(8-2)(8+2-2(-4)-1)}{2(.4)} = 127.5$ steps. We will now take a look at some simulations done with *Mathematica* to see how our results hold up. To view the code used to generate these results, refer to the Appendix.

Simulation 1:



This simulation took 69 steps to complete, well under the expected average of 127.5. Because $p = q$ and we started at $j = 2$, which is located directly between the upper and lower bounds, we had an equal chance of hitting the lower bound or the upper bound first. However, because this particular walk never reached the lower bound, it is not surprising that the number of steps taken was below average.

Simulation 2:



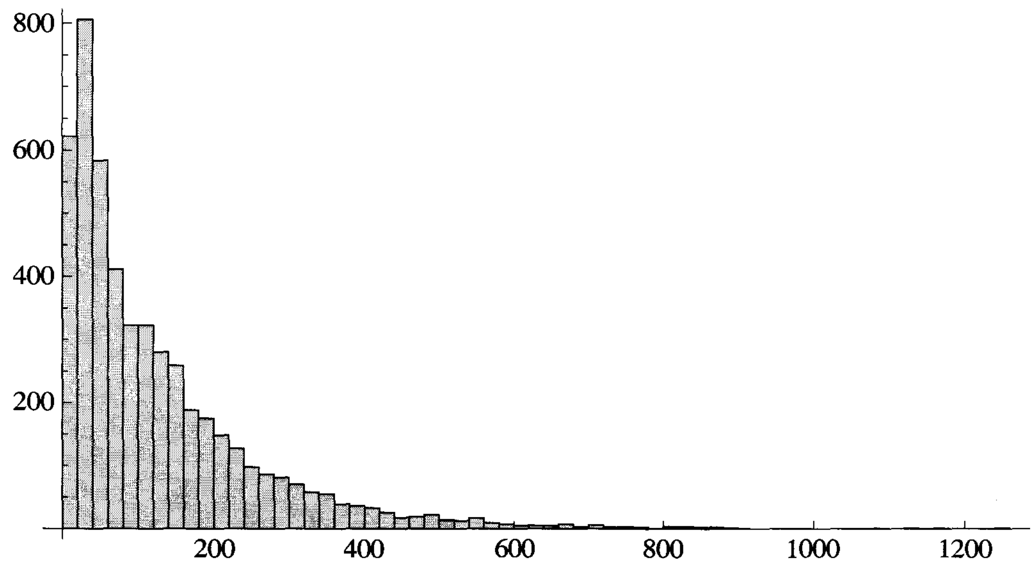
Simulation 2 took 172 steps to reach n, which is above the expected average. Note that this walk hit the lower bound at most 10 times.

Clearly we cannot ascertain the validity of our results based on a small number of simulations. However, we may run a few thousand simulations and see if our sample average for the walk matches our result from Theorem 2.2. Using code included in Appendix A, Section 2.3, we obtain the following result after 5000 simulations:

SampleMean	TrueAvgSteps
127.249	127.5

Hence our resultant average of 127.249 steps is quite close to our predicted average of 127.5 steps, which supports our result for the $p = q$ case of Theorem 2.2.

The following histogram displays the distribution of the number of steps taken to complete the simulated walks:



Scenario 2: $p \neq q$

Suppose we have a random walk with the following properties:

Upper Boundary: $n = 8$

Probability of Going Up: $p = 0.4$

Initial Height: $j = 2$

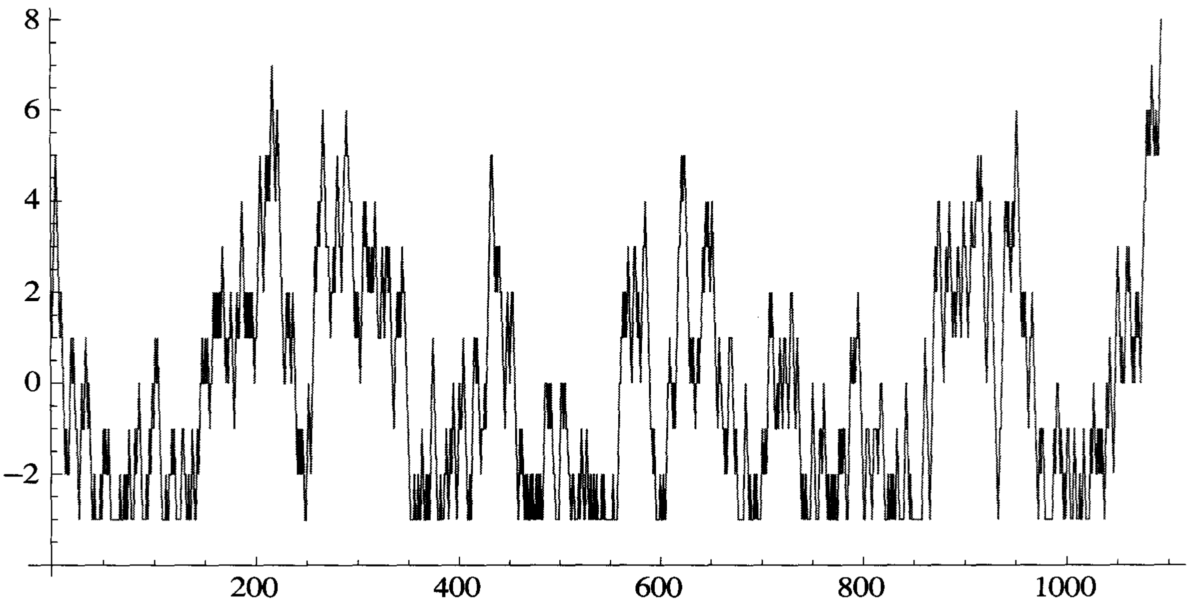
Probability of Going Down: $q = 0.5$

Elastic Lower Boundary: $m = -4$

Note that the only change in these values from Scenario 1 is that q is raised from 0.4 to 0.5.

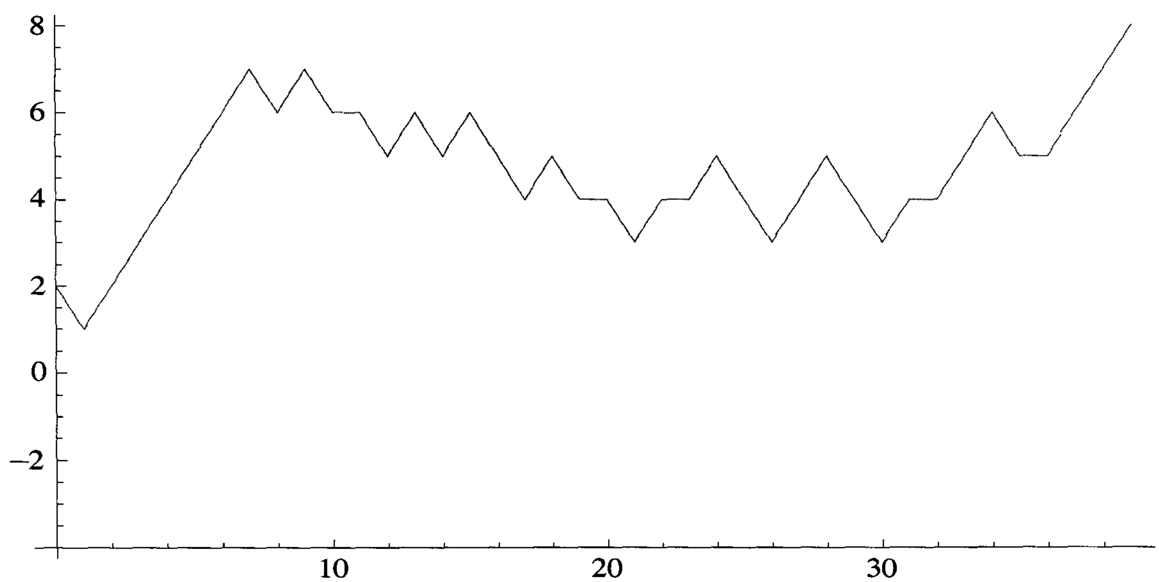
According to Theorem 2.2, random walks should reach height n in an average of $\frac{8-2}{.4-.5} + \frac{.4\left((.5/.4)^{8-(-4)} - (.5/.4)^{2-(-4)}\right)}{(.4-.5)^2} \approx 369.489$ steps. Once again we will run simulations with *Mathematica* to verify the validity of our result. The code for this scenario is the same as for Scenario 1 except the value of p should not equal the value of q.

Simulation 1:



This particular simulation took over 1000 steps to reach the upper boundary, well over our predicted average of 369.489.

Simulation 2:

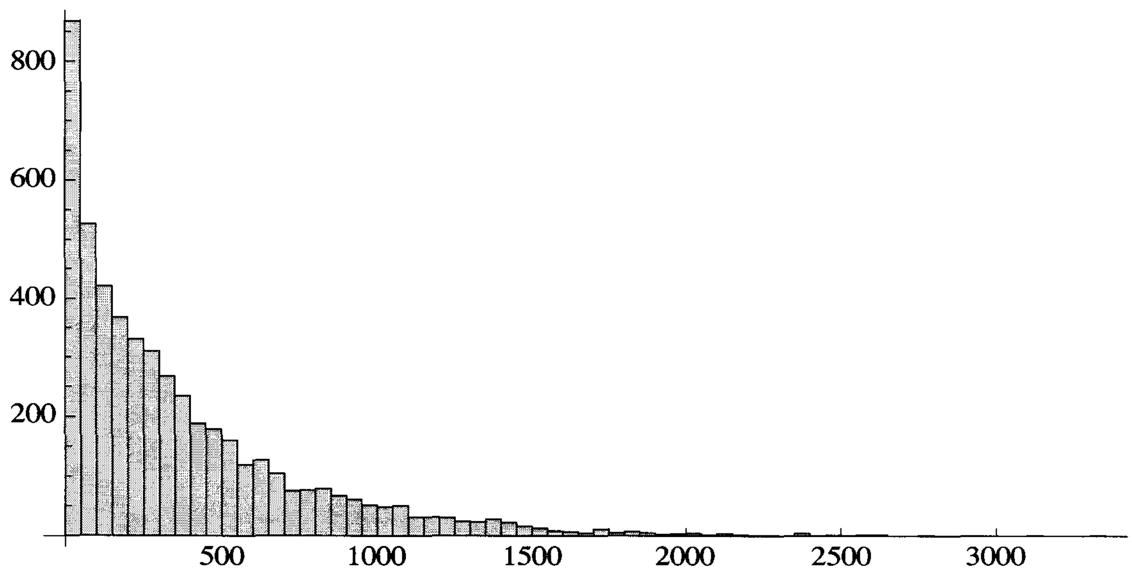


This time, our walk took only 40 steps to reach the upper boundary.

Upon simulating Scenario 2 5000 times, *Mathematica* provides us with the following result for the average number of steps:

SampleMean	TrueAvgSteps
370.928	369.489

Our average of 370.928 steps for the 5000 simulations agrees quite nicely with our expected average of 369.489. Thus our result for the $p \neq q$ case of Theorem 2.2 is supported by our results. The following histogram shows the distribution of the number of steps taken for the simulated random walks:



2.4 The a : 1 Case (a > 1)

Our work thus far with the elastic lower boundary has involved taking a step and going up 1 unit with probability $p > 0$, down 1 unit with probability q, or remaining at the same height with probability $r = 1 - p - q$ (the 1:1 case). We now consider the $a:1$ case of the elastic lower boundary where we can move upward a units at a time where $a > 1$. Any such random walk will still reach any upper boundary $n > j$ with probability 1; clearly if we are guaranteed to hit the boundary when going up 1 step with probability p, then we will hit it if we go up a steps with probability p.

We will not be able to use difference-equations to establish a closed-form solution for the average number of steps needed to reach the upper-boundary for the a: 1 case. The reason difference-equations worked in the 1:1 case was that we could establish a recursive relationship between e_j and e_{j+1} in terms of q, p, and j. However, with the a: 1 case we cannot solve for e_j solely in terms of e_{j+1} and the aforementioned variables, making a forward-difference argument impossible to implement. To illustrate this, take for example a 2 : 1 case, and let e_j once again represent the average number of steps needed to reach an upper boundary n from height j with elastic lower boundary 0. In general, we have

$$e_j = 1 + p \times e_{j+2} + q \times e_{j-1} + r \times e_j .$$

For $j = 1$, due to the elastic lower boundary, we have

$$\begin{aligned} e_1 &= 1 + p \times e_3 + q \times e_1 + r \times e_1 \\ p \times e_1 + q \times e_1 + r \times e_1 &= 1 + p \times e_3 + q \times e_1 + r \times e_1 \\ p \times e_1 &= 1 + p \times e_3 \\ e_1 &= \frac{1}{p} + e_3 \end{aligned} \tag{3}$$

Note that our result with the 1 : 1 case was $e_1 = \frac{1}{p} + e_2$. Continuing, we have

$$
\begin{aligned}
e_2 &= 1 + p \times e_4 + q \times e_1 + r \times e_2 \\
p \times e_2 + q \times e_2 + r \times e_2 &= 1 + p \times e_4 + q \times e_1 + r \times e_2 \\
p \times e_2 + q \times e_2 &= 1 + p \times e_4 + q \times e_1 \\
(p+q) \times e_2 &= 1 + p \times e_4 + q \times e_1 \\
e_2 &= \frac{1 + p \times e_4 + q \times e_1}{p+q}
\end{aligned}
$$

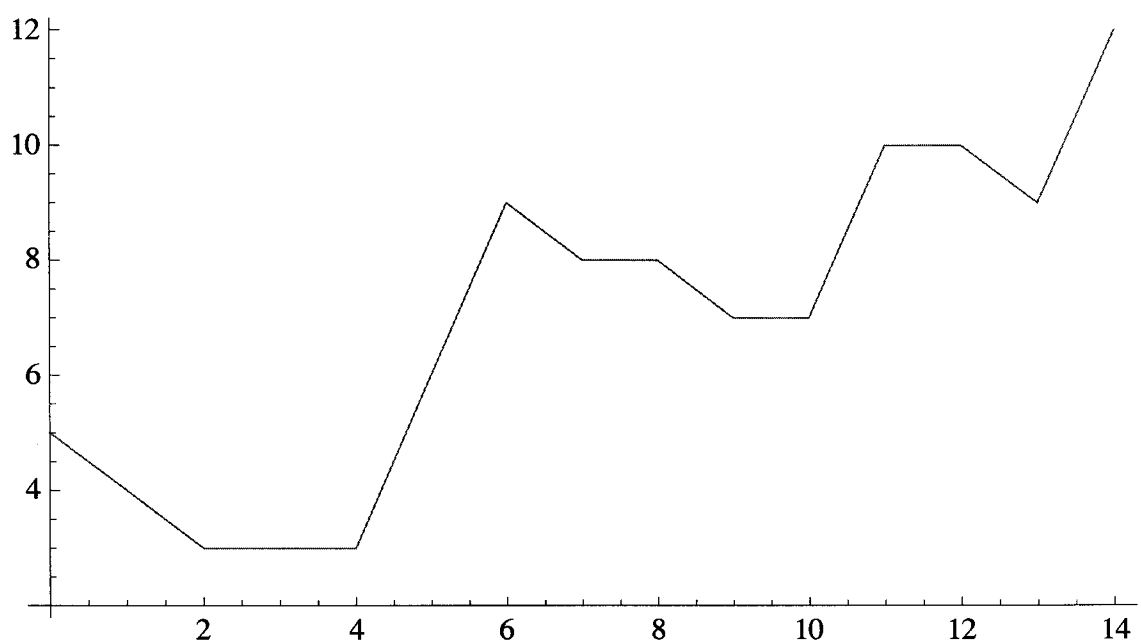
Clearly from this point we will not be able to solve for e_2 solely in terms of e_4 or e_3, even by substituting using Equation (3).

While we will not be able to provide a closed-form solution for the average number of steps needed to reach the upper boundary for the $a:1$ case (as we did in the $1:1$ case), later in this chapter we will discuss how to use a Markov Chain Method to give probability states after k steps. Also, we will describe how to use systems of equations to determine exact numerical solutions for the average number of steps needed to reach the upper boundary. First, however, we will run some simulations with *Mathematica* to obtain approximations for the average number of steps needed to reach height n for some different examples of the $a:1$ case. Note that because our upward jumps are greater than 1, there is now the possibility of *exceeding* the upper boundary on the last step. At this point the walk will end as if it had hit the upper boundary.

Scenario 1

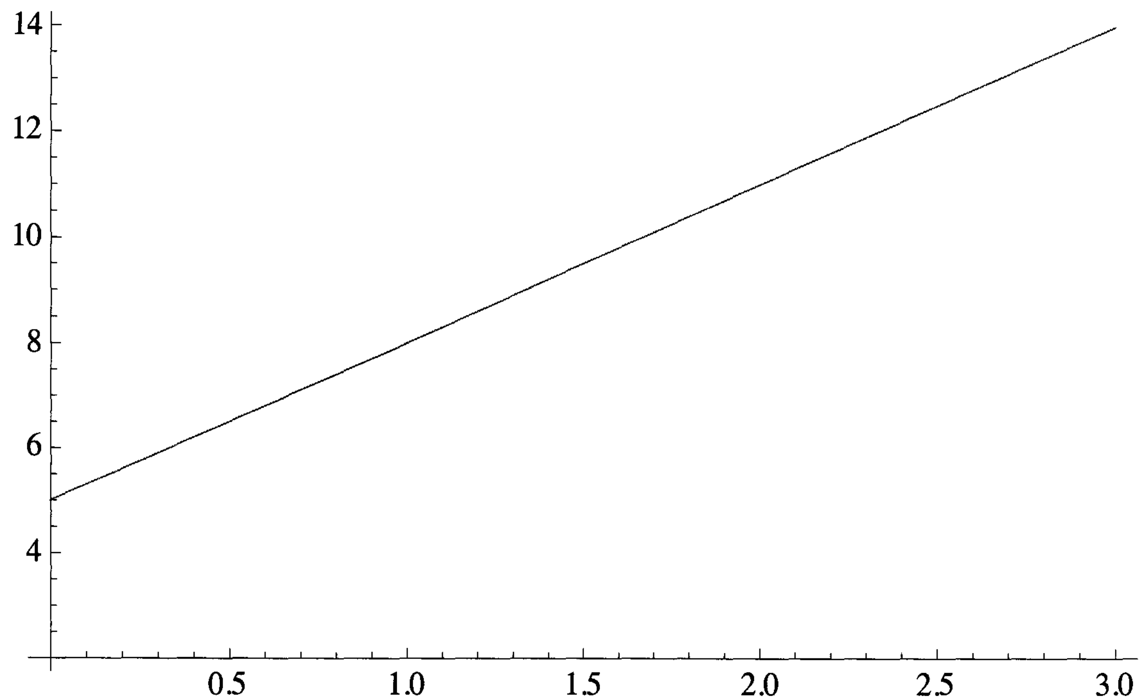
Upper Boundary:	$n = 12$	Probability of Going Up:	$p = 0.3$
Initial Height:	$j = 5$	Probability of Going Down:	$q = 0.5$
Elastic Lower Boundary:	$m = 2$	Size of Upward Jumps:	$a = 3$

Simulation 1:



This simulation took 14 steps to complete. We see that after 13 steps, the walk was at height 9. On the 14^{th} step, the walk took an upward jump and reached the upper boundary of 12, thus ending the walk.

Simulation 2:



This time our walk took 3 upward jumps in a row and actually surpassed the upper boundary by going from 5 to 8 to 11 to 14. The walk then halted because we stop whenever the upper boundary is met *or* exceeded.

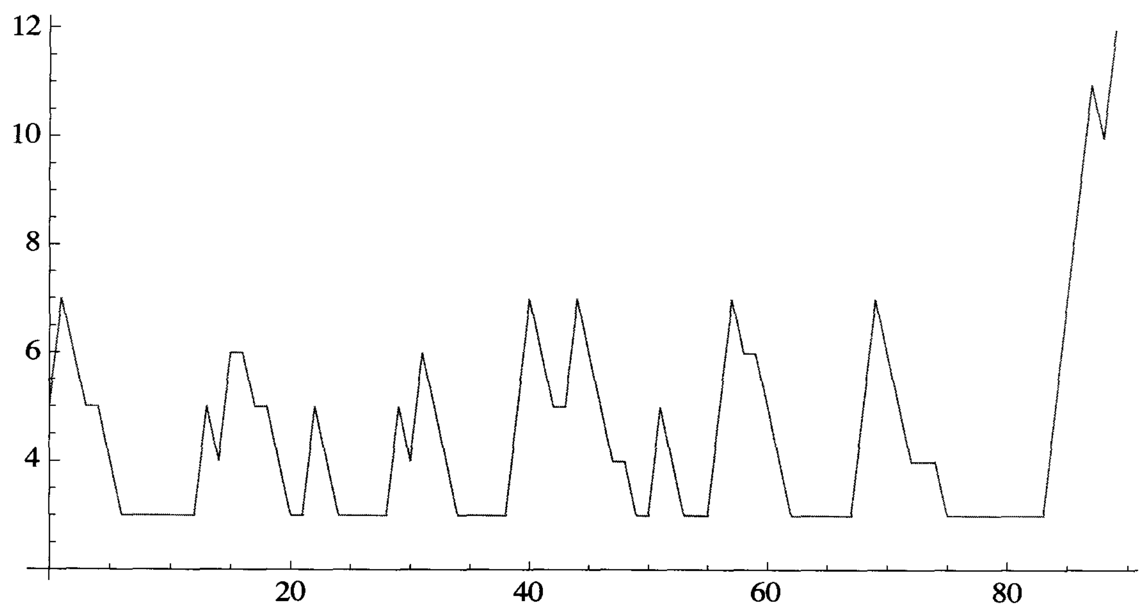
After running 500000 simulations with the above scenario, our average number of steps to reach the upper boundary was 16.0463.

Scenario 2

Upper Boundary:	$n = 12$	Probability of Going Up:	$p = 0.3$
Initial Height:	$j = 5$	Probability of Going Down:	$q = 0.5$
Elastic Lower Boundary:	$m = 2$	Size of Upward Jumps:	$a = 2$

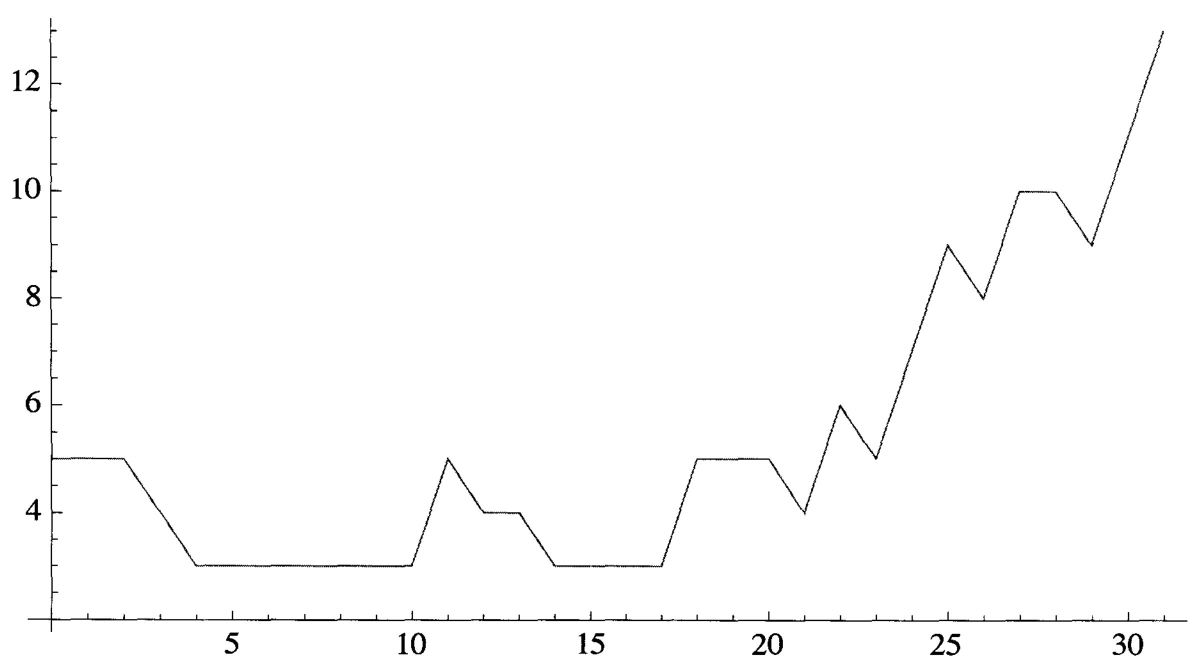
For the second scenario, all variables are kept the same is in the first scenario except the size of upward jumps is lowered from 3 to 2.

Simulation 1:



This simulation took 89 steps to complete.

Simulation 2:



This walk took 31 steps to complete. Our average number of steps appears to be increasing from Scenario 1, which we would expect by lowering the size of upward jumps. After 250000 simulations, the average number of steps needed to reach the upper boundary was 37.975.

2.5 Markov Chain Method for the a : 1 Case

We will now discuss how to use Markov Chains to analyze the probability states after a given number of steps. Let k be a positive integer. Using this method, we shall do the following:

a) Give the probability states after k steps;
b) Determine the probability of reaching the upper boundary within k steps;
c) Determine the probability of reaching the upper boundary in exactly k steps;
d) Approximate the average number of steps needed to reach the upper boundary.

To implement our Markov Chain method, we will need to construct two matrices, the first of which is the *initial state vector (I).* This initial state matrix designates the starting position of our walk (j) among all possible ending heights that may be achieved. Due to the elastic lower boundary, the lowest ending height we may achieve is $m+1$. Because we now go up a steps at a time, we could reach height $n-1$ and go up a steps for a maximum ending height of $n+a-1$. Thus our walk will always have $(n-1+a)-m$ possible ending heights. However, we will never start our walk on a boundary point, so j will always correspond to a height between $m+1$ and $n-1$. Because our walk will begin at j with probability 1, our initial state matrix will have a 1 in the column corresponding to j. All other columns will have a value of 0. The initial state matrix I will look as follows. A matrix displaying all possible heights is shown below I for comparison:

$$I = [0 \quad \dots \quad \dots \quad 1 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad 0]$$
$$[m+1 \quad \dots \quad j \quad \dots \quad n \quad \dots \quad n+a-1]$$

We also need a *transition matrix* $A = (a_{ih})$, for $m+1 \le i,h \le n+a-1$. This matrix gives the probabilities of going to height h from any height i after one step. For example, assume we are at height i, with $i \ge m+2$. Then we take a step and reach height $i-1$

with probability q, height $i+a$ with probability p, or remain at i with probability r. The probability of reaching any other height from i is 0.

There are two exceptions to the above. One is when we *are* at height $m+1$. In this case, due to the elastic lower boundary, we take a step and stay at height $m+1$ with probability $q+r$ or move up a steps with probability p. The other exception occurs when we meet or surpass the upper boundary n. When this occurs, we remain at that height with probability 1.

For our example, assume the following values:

Upper Boundary: $n=7$ Size of Upward Jumps: $a=3$

Elastic Lower Boundary: $m=2$

Our transition matrix A will look as follows:

$$A=\left[\begin{array}{c|ccccccc} & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 3 & q+r & 0 & 0 & p & 0 & 0 & 0 \\ 4 & q & r & 0 & 0 & p & 0 & 0 \\ 5 & 0 & q & r & 0 & 0 & p & 0 \\ 6 & 0 & 0 & q & r & 0 & 0 & p \\ 7 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 9 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Note that the values in the leftmost column are placeholders and represent the ending height of the previous state; thus this column is not a part of the actual matrix. Likewise, the top row represents the possible heights after a step is taken and is also not a part of the actual matrix. Matrix A always has dimensions $(n-1+a-m)\times(n-1+a-m)$.

a) Determining probability states after k steps:

To determine the probability of being at each possible ending height after k steps, we multiply $I \times A^k$. The result will be a $1 \times (n-1+a-m)$ matrix where the first entry represents the probability of ending at height $m+1$ after k steps, the second entry represents the probability of ending at height $m+2$, and so on until the last entry gives the probability of ending at height $n+a-1$ after k steps.

b) Determining the probability F(k) of reaching the upper boundary within k steps:

To determine the probability of reaching the upper boundary within k steps, we simply multiply $I \times A^k$ as above and sum the probabilities of each entry of the resulting matrix that correspond to part of the upper boundary. In other words, we multiply $I \times A^k$ and sum the n^{th} through $(n+a-1)^{\text{th}}$ entries in the resulting matrix. Henceforth we will denote this result, known as the *cumulative distribution function (cdf)*, by $F(k)$.

c) Determining the probability f(k) of reaching the upper boundary in exactly k steps:

To determine the probability of reaching the upper boundary in *exactly* k steps, we take the probability of reaching the upper boundary *within* k steps and subtract from it the probability of reaching the upper boundary within $k-1$ steps. Thus the *probability distribution function (pdf)*, denoted $f(k)$, is given by

$$f(k) = F(k) - F(k-1) \text{ for } k \geq 1.$$

Note that $F(k) = 0$ when $k < 1$; thus $f(k) = F(k)$ when $k = 1$.

d) Approximating the average number of steps needed to reach the upper boundary:

To determine the average number of steps needed to reach the upper boundary, we take each possible number of steps *k*, beginning with $k=1$, and multiply k by the probability *f(k)* of reaching the upper boundary in exactly k steps. We then take the sum of each product $k \times f(k)$ from $k=1$ to ∞. However, because it is impossible to compute infinitely many definite sums, the best we can do is obtain an approximation of the average number of steps needed by choosing a large integer *N* and summing the product $k \times f(k)$ from $k=1$ to *N*. Thus the approximate average number of steps needed to reach or surpass the upper boundary of n is given by

$$E\left[{}_jS_{(m)}^{n}\right] \approx \sum_{k=1}^{N} k \times f(k) \text{ for } N \text{ large.}$$

We will now simulate parts *a)* through *d)* above using *Mathematica*. We will use the same values we used in Section 2.4, Scenario 1 so that we may compare our result in *d)* with the result we obtained previously for the average number of steps needed to reach the upper boundary.

Scenario 1

Upper Boundary:	$n=12$	Probability of Going Up:	$p=0.3$
Initial Height:	$j=5$	Probability of Going Down:	$q=0.5$
Elastic Lower Boundary:	$m=2$	Size of Upward Jumps:	$a=3$

Based on these values, our initial height matrix *I* and our transition matrix *A* will look as follows:

$$I = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} .7 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .5 & .2 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .5 & .2 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .5 & .2 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & .2 & 0 & 0 & .3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & .2 & 0 & 0 & .3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5 & .2 & 0 & 0 & .3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .5 & .2 & 0 & 0 & .3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .5 & .2 & 0 & 0 & .3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

a) First we will examine the probability states after 1 step is taken. To do this, we multiply $I \times A^1$ and obtain:

$$\begin{bmatrix} 0 & 0.5 & 0.2 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that this matrix corresponds to the 3rd row of our transition matrix. We obtained this result because we started at the 3rd lowest possible height. Now we will examine the probability states after 5 steps, given by $I \times A^5$:

$$\begin{bmatrix} 0.164 & 0.060 & 0.132 & 0.146 & 0.024 & 0.104 & 0.146 & 0.054 & 0.007 & 0.040 & 0.073 & 0.050 \end{bmatrix}$$

The probability states after 6 steps, given by $I \times A^6$, are as follows:

$$\begin{bmatrix} 0.145 & 0.078 & 0.099 & 0.090 & 0.075 & 0.133 & 0.100 & 0.022 & 0.033 & 0.084 & 0.089 & 0.052 \end{bmatrix}$$

Finally, to approximate the final ending probabilities of our walk, we find $I \times A^k$ for a very large value of k. Here we compute $I \times A^{1000}$:

$$\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.450 & 0.336 & 0.214\end{bmatrix}$$

Hence after 1000 steps our walk will almost assuredly have reached the upper boundary. According to our result, we will eventually reach height 12 with probability 0.450, height 13 with probability 0.336, or height 14 with probability 0.214.

b) We will now examine the probability *F(k)* of reaching the top within *k* steps. We will use the values for *k* we used in part *a)*. First we consider $k = 1$. Because we start at height 5, clearly there is no chance of reaching the upper boundary of 12 in 1 step. Thus we would expect the sum of the probability states that correspond to the upper boundary to be 0, which is the case. Hence *F(0)* = 0.

For the k =5 case, we see that there is a 0.040 probability of reaching height 12, a 0.073 probability of reaching height 13, and a 0.05 probability of reaching height 14. Thus *F(5)* is approximately $0.040 + 0.073 + 0.050 = 0.163$.

Similarly, the cumulative probability of reaching the upper boundary within 6 steps is given by

$$F(6) \approx 0.084 + 0.089 + 0.052 = 0.225.$$

For the k =1000 case, we see that our probability of reaching the top within 1000 steps is

$$F(1000) \approx 0.450 + 0.336 + 0.214 = 1.$$

c) Now we will examine the probability of reaching the upper boundary in *exactly k* steps. For example, we know that *F(5)* is approximately 0.163, and *F(6)* is approximately 0.225. Therefore we can conclude that the probability of reaching the upper boundary in *exactly* 6 steps, given by *F(6) – F(5)*, is

$$f(6) = F(6) - F(5) \approx 0.225 - 0.163 = 0.062$$

Next we will use *Mathematica* to provide approximations of *f(k)* for $k = 1$ to $k = 20$.

k	f(k)	k	f(k)
1	0.000	11	0.045
2	0.000	12	0.040
3	0.027	13	0.037
4	0.057	14	0.036
5	0.079	15	0.034
6	0.062	16	0.031
7	0.046	17	0.028
8	0.044	18	0.026
9	0.050	19	0.024
10	0.051	20	0.022

Using the method in part *b)*, we may determine that there is a cumulative probability of approximately 0.739 that we will hit the upper boundary within 20 steps. Summing the above figures may give a slightly different result due to rounding.

d) Using *Mathematica* to compute the sum $\sum_{k=1}^{15000} k \times f(k)$, we obtain a value 16.0498 for the average number of steps needed to reach the top. This result is close to our sample average of 16.020 that we obtained in Section 2.4, Scenario 1.

2.6 System of Equations Solution for the a : 1 Case

In this section, we will use a System of Equations method to determine exact numerical solutions for the average number of steps needed to reach the upper boundary for the $a:1$ case. Let x_j represent the average number of steps needed to reach or surpass the upper boundary n from starting position j, where $m+1 \le j \le n+a-1$. If $n \le j \le n+a-1$, then clearly $x_j = 0$ because our walk has reached the upper boundary.

Now suppose $m+2 \le j \le n-1$. Then one step is taken and we go up a units with probability p, down 1 unit with probability q, or stay at the same height with probability r. Thus we may express x_j in terms of x_{j-1} and x_{j+a} as follows:

$$x_j = 1 + q \times x_{j-1} + r \times x_j + p \times x_{j+a}.$$

To establish our system of equations, we will need to organize the equations so that the variables appear on one side of the equation and the constant on the other. Thus we take the preceding equation and subtract x_j and 1 from both sides to obtain:

$$q \times x_{j-1} + (r-1) \times x_j + p \times x_{j+a} = -1.$$

For the case when $j = m+1$, we must take the elastic lower boundary into consideration. In this case taking a downward step is equivalent to remaining at the same height; thus, for $j = m+1$, we have:

$$x_{m+1} = 1 + (q+r) \times x_{m+1} + p \times x_{m+1+a}$$

which we rewrite as:

$$(q + r - 1) \times x_{m+1} + p \times x_{m+1+a} = -1.$$

We now have the following system of equations:

$$
\begin{aligned}
(q+r-1)\times x_{m+1} + p\times x_{m+1+a} &= -1 \\
q\times x_{m+1} + (r-1)\times x_{m+2} + p\times x_{m+2+a} &= -1 \\
&\vdots \\
q\times x_{n-2} + (r-1)\times x_{n-1} + p\times x_{n+a-1} &= -1 \\
x_n &= 0 \\
&\vdots \\
x_{n+a-1} &= 0
\end{aligned}
$$

To solve the system of equations, we first create the *matrix of coefficients T*, a $(n-1+a-m)\times(n-1+a-m)$ matrix that is almost identical to the transition matrix *A* used in the previous section. The only difference is that where *r* is located in matrix *A*, we now have $r-1$ in matrix *T*. Next, let *F* be the $1\times(n-1+a-m)$ column matrix that contains the constants that appear on the right hand side of each equation. Finally, let *X* be the $1\times(n-1+a-m)$ matrix that contains the variables $x_{m+1}, x_{m+2}, \ldots, x_{n+a-1}$. Our goal is to solve for each x_j by solving the system $TX = F$.

For our example, we will use the following values:

Upper Boundary: $n = 7$

Elastic Lower Boundary: $m = 2$

Size of Upward Jumps: $a = 3$

For a given p, q, and r, the system $TX = F$ will look as follows:

$$\begin{bmatrix} q+r-1 & 0 & 0 & p & 0 & 0 & 0 \\ q & r-1 & 0 & 0 & p & 0 & 0 \\ 0 & q & r-1 & 0 & 0 & p & 0 \\ 0 & 0 & q & r-1 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly, we can use row operations to eliminate q in each of the second, third, and fourth rows. Thus, T can be converted into a non - singular, upper - triangular matrix, which shows that the determinant of T is not zero and T is invertible. Because T will always be invertible, the solution for X is given by:

$$X = T^{-1}F .$$

We will now use the system of equations method to determine the average steps needed to hit the upper boundary for the same walk we used in Sections 2.4 and 2.5. We will then compare our results to the results obtained in those sections.

Scenario 1

Upper Boundary: $n = 12$

Probability of Going Up: $p = 0.3$

Elastic Lower Boundary: $m = 2$

Probability of Going Down: $q = 0.5$

Size of Upward Jumps: $a = 3$

Given this scenario, the system $TX = F$ will be as follows:

$$\begin{bmatrix} -.3 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .5 & -.8 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .5 & -.8 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .5 & -.8 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & -.8 & 0 & 0 & .3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & -.8 & 0 & 0 & .3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5 & -.8 & 0 & 0 & .3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .5 & -.8 & 0 & 0 & .3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .5 & -.8 & 0 & 0 & .3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we calculate the solutions for $x_3, x_4, \ldots, x_{14}$, given by $X = T^{-1}F$:

$$\begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix} = X = T^{-1}F \approx \begin{bmatrix} 17.6818 \\ 17.0274 \\ 16.0498 \\ 14.3484 \\ 12.6036 \\ 11.0871 \\ 8.1795 \\ 6.3622 \\ 5.2264 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We may conclude that the *exact* numerical solution for x_3 to four decimal places is 17.6818 steps, the exact solution for x_4 is 17.0274 steps, and so on. Note that the solution for x_5, the average number of steps to reach the upper boundary from starting position 5, is 16.0498. This is precisely the same solution we obtained with the Markov Chain method from Section 2.5, and very close to the value of 16.020 we obtained from running numerous simulations in Section 2.4.

Chapter 3: The Reflective Lower Boundary Scenario

For this random walk scenario, we will assume that whenever our walk drops to height 0, the next step will be automatically upward. We will call this a "reflective lower boundary." For this scenario, we will do the following:

1) Prove that, for $p > 0$, the particle will eventually reach any height $n \geq j$ with probability 1.

2) Derive the average number of steps needed to reach height n, given $a = b = 1$ (the $1:1$ case).

3) Provide numerical examples and computer simulations of the $1:1$ case.

4) Discuss the $a:1$ case ($a > b = 1$), and use simulations to approximate the average number of steps needed to reach height n.

5) Use a Markov Chain Method to derive the probability states after k steps for the $a:1$ case, and to approximate the average number of steps needed to reach height n.

6) Use a System of Equations Method to obtain numerical solutions for the average number of steps to reach height n in the $a:1$ case.

3.1 The Probability of Reaching Height *n*

We will show that a simple random walk with a reflective lower bound will almost surely reach any upper bound n. Let e_j represent the probability of reaching height n from height j. As with our scenario with the elastic lower bound, we have $e_1 = e_0$, because when our walk reaches the lower boundary the next step is automatically upward. By the same argument used in the proof of Theorem 2.1, we then have

$$e_j = e_{j+1} \text{ for } 1 \le j \le n-1.$$

Because $e_n = 1$, we must have $e_j = 1$ for $1 \le j \le n-1$. We may also generalize this result for any lower bound $m < j$ by our vertical translation argument we used in Section 2.1. This allows us to state:

Theorem 3.1. *Given $p > 0$, any simple random walk beginning at height j with a reflective lower boundary of $m < j$ will reach any upper boundary $n > j$ with probability 1.*

3.2 Average Number of Steps to Reach Height *n*

Let e_j represent the average number of steps needed to reach height n from beginning height j, where $0 < j < n$. Starting at height j, we take a step and go up 1 with probability p, down 1 with probability q, or stay at the same height with probability r. Thus, we may express e_j as follows:

$$e_j = 1 + p \times e_{j+1} + q \times e_{j-1} + r \times e_j.$$

We note that our walk stops upon reaching height n; therefore, $e_n = 0$.

We will now set up a difference equations argument to find a solution for e_j in terms of p, q, n, and j. First we note that

$$e_0 = 1 + e_1$$

because from $j = 0$ the next step is automatically upward. We now write e_1 as follows:

$$e_1 = 1 + p \times e_2 + q \times e_0 + r \times e_1.$$

Substituting $e_1 = (p+q+r)e_1$ and $e_0 = 1 + e_1$, we obtain the following equations:

$$\begin{aligned} p \times e_1 + q \times e_1 + r \times e_1 &= 1 + p \times e_2 + q \times (1 + e_1) + r \times e_1 \\ p \times e_1 + q \times e_1 &= 1 + p \times e_2 + q \times (1 + e_1) \\ p \times e_1 + q \times e_1 &= 1 + p \times e_2 + q + q \times e_1 \\ p \times e_1 &= 1 + q + p \times e_2 \\ e_1 &= \frac{1+q}{p} + e_2 \qquad (2) \end{aligned}$$

Now consider the following equation for e_2:

$$p \times e_2 + q \times e_2 + r \times e_2 = 1 + p \times e_3 + q \times e_1 + r \times e_2$$

This expression may be simplified to

$$p \times e_2 + q \times e_2 = 1 + p \times e_3 + q \times e_1.$$

We now substitute for e_1 using Equation (2) to obtain

$$p \times e_2 + q \times e_2 = 1 + p \times e_3 + q \times \left(\frac{1+q}{p} + e_2 \right)$$

Then we have

$$\begin{aligned} p \times e_2 + q \times e_2 &= 1 + p \times e_3 + q \times \left(\frac{1+q}{p} \right) + q \times e_2 \\ p \times e_2 &= 1 + p \times e_3 + q \times \left(\frac{1+q}{p} \right) \\ e_2 &= \frac{1}{p} + \frac{q(1+q)}{p^2} + e_3 \\ e_2 &= \frac{1}{p} + \frac{q}{p^2} + \frac{q^2}{p^2} + e_3 \end{aligned}$$

Continuing in this fashion, we see that, in general,

$$e_j = e_{j+1} + \frac{q^j}{p^j} + \frac{1}{q} \sum_{i=1}^{j} \left(\frac{q}{p} \right)^i.$$

Thus we have $f_j = e_j - e_{j+1} = \dfrac{q^j}{p^j} + \dfrac{1}{q}\sum_{i=1}^{j}\left(\dfrac{q}{p}\right)^i$.

If $p = q$, then

$$f_j = \frac{q^j}{p^j} + \frac{1}{q}\sum_{i=1}^{j}\left(\frac{q}{p}\right)^i = \frac{p^j}{p^j} + \frac{1}{p}\sum_{i=1}^{j}\left(\frac{p}{p}\right)^i = 1 + \frac{1}{p}\sum_{i=1}^{j}1 = 1 + \frac{j}{p}.$$

Otherwise, if $p \neq q$, we have

$$f_j = \frac{q^j}{p^j} + \frac{1}{q}\sum_{i=1}^{j}\left(\frac{q}{p}\right)^i = \left(\frac{q}{p}\right)^j + \frac{1}{q}\left(\frac{\left(\frac{q}{p}\right)^{j+1} - \left(\frac{q}{p}\right)}{\left(\frac{q}{p}\right) - 1}\right)$$

$$= \left(\frac{q}{p}\right)^j + \left(\frac{\left(\frac{q}{p}\right)^{j+1} - \left(\frac{q}{p}\right)}{\left(\frac{q^2}{p}\right) - q}\right) = \left(\frac{q}{p}\right)^j + \left(\frac{\left(\frac{q}{p}\right)^{j} - 1}{q - p}\right)$$

$$= \left(\frac{q}{p}\right)^j + \left(\frac{1 - \left(\frac{q}{p}\right)^{j}}{p - q}\right).$$

Hence

$$f_j = e_j - e_{j+1} = \frac{q^j}{p^j} + \frac{1}{q}\sum_{i=1}^{j}\left(\frac{q}{p}\right)^i = \begin{cases} 1 + \dfrac{j}{p} & \text{if } p = q \\ \left(\dfrac{q}{p}\right)^j + \dfrac{1 - \left(\frac{q}{p}\right)^j}{p - q} & \text{if } p \neq q \end{cases}$$

To obtain a closed-form solution for e_j, we will use the forward differences formula we obtained in Chapter 2. So once again we have

$$e_j = \sum_{i=j}^{n-1} f_i.$$

We evaluate the summand for the $p = q$ case:

$$e_j = \sum_{i=j}^{n-1}\left[1+\frac{i}{p}\right] = (n-j)+\frac{1}{p}\sum_{i=j}^{n-1} i = (n-j)+\frac{(n-j)(n+j-1)}{2p} = \frac{(n-j)(n+j+2p-1)}{2p}$$

If $p \neq q$, we obtain

$$e_j = \sum_{i=j}^{n-1}\left[\left(\frac{q}{p}\right)^i + \frac{1-\left(\frac{q}{p}\right)^i}{p-q}\right]$$

$$= \sum_{i=j}^{n-1}\left(\frac{q}{p}\right)^i + \frac{1}{p-q}\sum_{i=j}^{n-1}\left[1-\left(\frac{q}{p}\right)^i\right]$$

$$= \sum_{i=j}^{n-1}\left(\frac{q}{p}\right)^i + \frac{1}{p-q}\sum_{i=j}^{n-1}1 - \frac{1}{p-q}\sum_{i=j}^{n-1}\left(\frac{q}{p}\right)^i$$

$$= \frac{p\left(\left(\frac{q}{p}\right)^j - \left(\frac{q}{p}\right)^n\right)}{p-q} + \frac{n-j}{p-q} - \frac{p\left(\left(\frac{q}{p}\right)^j - \left(\frac{q}{p}\right)^n\right)}{(p-q)^2}$$

$$= \frac{p(p-q-1)\left(\left(\frac{q}{p}\right)^j - \left(\frac{q}{p}\right)^n\right) + (p-q)(n-j)}{(p-q)^2}.$$

Summarizing, we have

$$e_j = \sum_{i=j}^{n-1} f_i = \begin{cases} \dfrac{(n-j)(n+j+2p-1)}{2\text{p}} & \text{if } p = q \\ \dfrac{p(p-q-1)\left(\left(\dfrac{q}{p}\right)^j - \left(\dfrac{q}{p}\right)^n\right) + (p-q)(n-j)}{(p-q)^2} & \text{if } p \neq q \end{cases}$$

We may once again generalize these results for *any* lower bound m as we did in Section 2.2 by replacing j with $j-m$ and n with $n-m$. Making these substitutions into our result for e_j, we state:

Theorem 3.2. *Let $p > 0$. For a simple random walk beginning at height j with a reflective lower boundary of $m < j$, the average number of steps needed to reach the boundary height of $n \geq j$ is given by*

$$E\left[{}_j R_{(m)}^n\right] = \begin{cases} \dfrac{(n-j)(n+j-2m+2p-1)}{2p} & \text{if } p = q \\ \dfrac{p(p-q-1)\left(\left(\dfrac{q}{p}\right)^{j-m} - \left(\dfrac{q}{p}\right)^{n-m}\right) + (p-q)(n-j)}{(p-q)^2} & \text{if } p \neq q. \end{cases}$$

3.3 Numerical Examples and Computer Simulations for the 1:1 Case

Similar to Section 2.3, we will now test the result of Theorem 3.2 by running simulations for both the $p = q$ and $p \neq q$ cases and seeing if the averages match our expected result.

Scenario 1: p = q

Suppose we have a random walk with the following properties:

Upper Boundary: $n = 8$

Initial Height: $j = 5$

Reflective Lower Boundary: $m = 2$

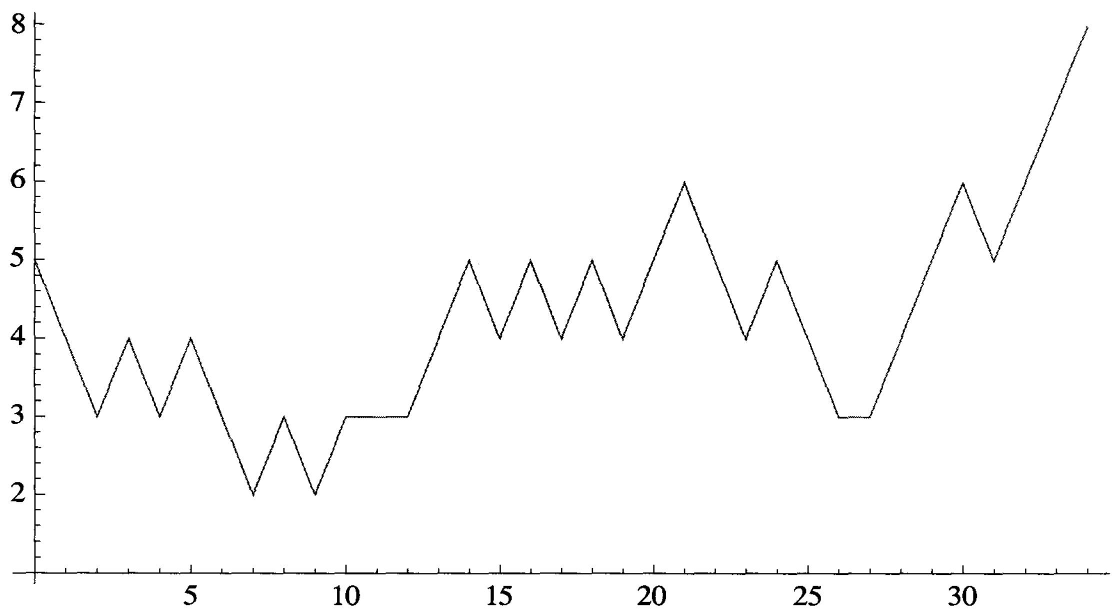
Probability of Going Up: $p = 0.4$

Probability of Going Down: $q = 0.4$

According to Theorem 3.2, our walk should reach n in an average of

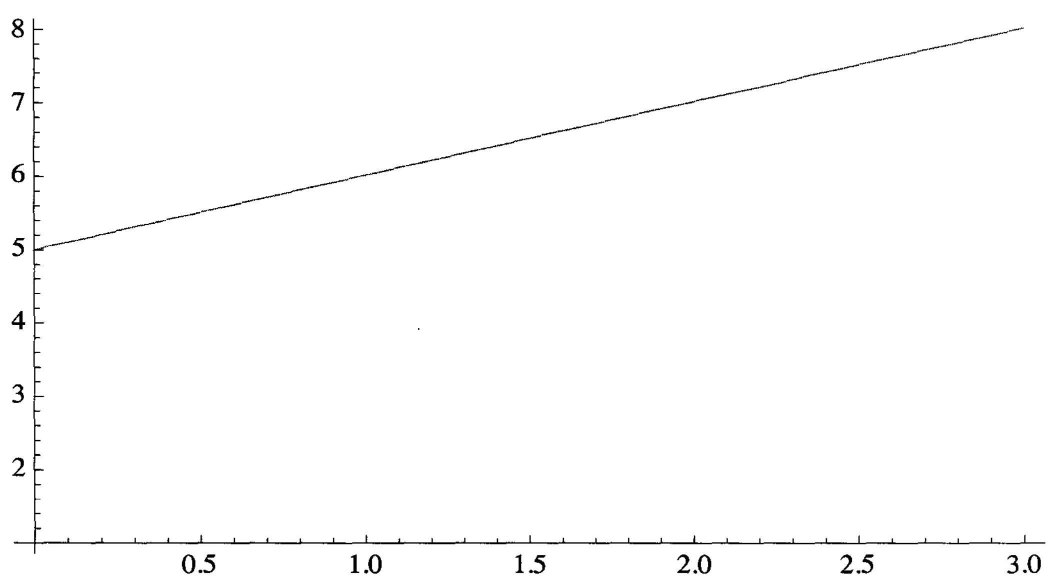
$\frac{(n-j)(n+j-2m+2p-1)}{2p} = \frac{(8-5)(8+5-2(2)+2(.4)-1)}{2(.4)} = 33$ steps. Once again, we will take a look at some simulations done with *Mathematica* to see how our results hold up. To view the code used to generate these results, refer to the Appendix.

Simulation 1:



Simulation 1 took 34 steps to complete, very close to our expected average of 33 steps.

Simulation 2:



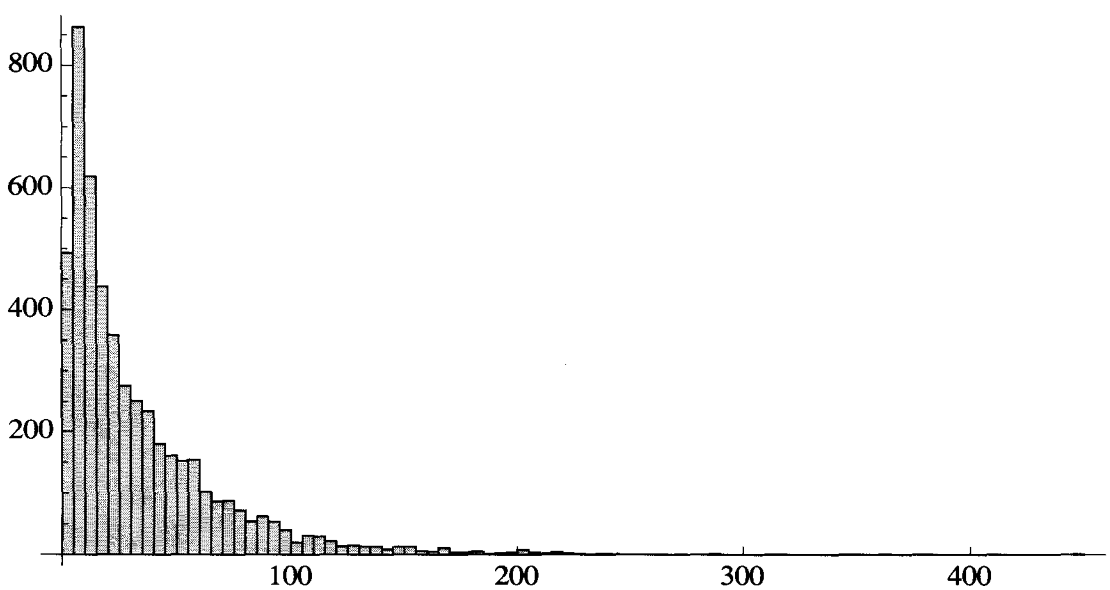
Simulation 2 reaches the upper boundary in 3 steps, which is the minimum for this scenario. This is not entirely unusual as the probability of going up the first three steps is $(.4)^3 = 0.064$ or 6.4%.

After 200000 simulations, *Mathematica* provides us with the following result for the average number of steps in this scenario:

SampleMean	TrueAvgSteps
33.042	33

The sample average of 33.042 steps is quite close to our expected average of 33 steps. Thus our result for the $p = q$ case of Theorem 3.2 is supported by the data.

The following histogram shows the distribution of the number of steps in the simulated walks:



Scenario 2: p ≠ q

Suppose we have a random walk with the following properties:

Upper Boundary: $n = 8$

Initial Height: $j = 5$

Reflective Lower Boundary: $m = 2$

Probability of Going Up: $p = 0.4$

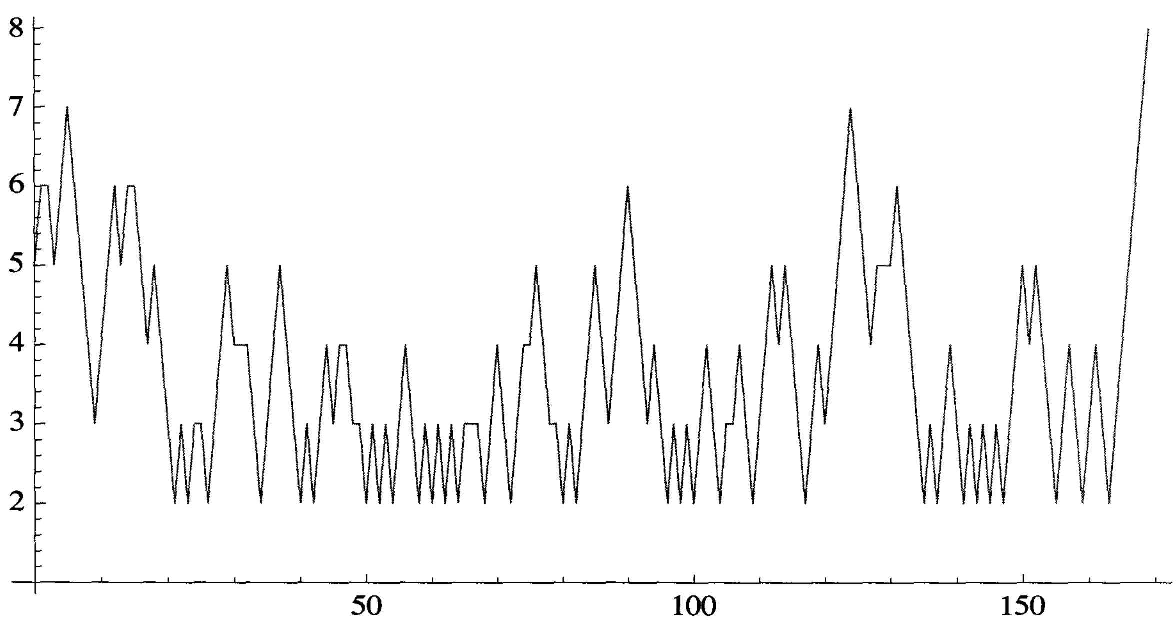
Probability of Going Down: $q = 0.5$

According to Theorem 3.2, we would expect this walk to be completed in an average of

$$\frac{.4(.4-.5-1)\left(\left(\frac{.5}{.4}\right)^{5-2}-\left(\frac{.5}{.4}\right)^{8-2}\right)+(.4-.5)(8-5)}{(.4-.5)^2} \approx 51.91$$

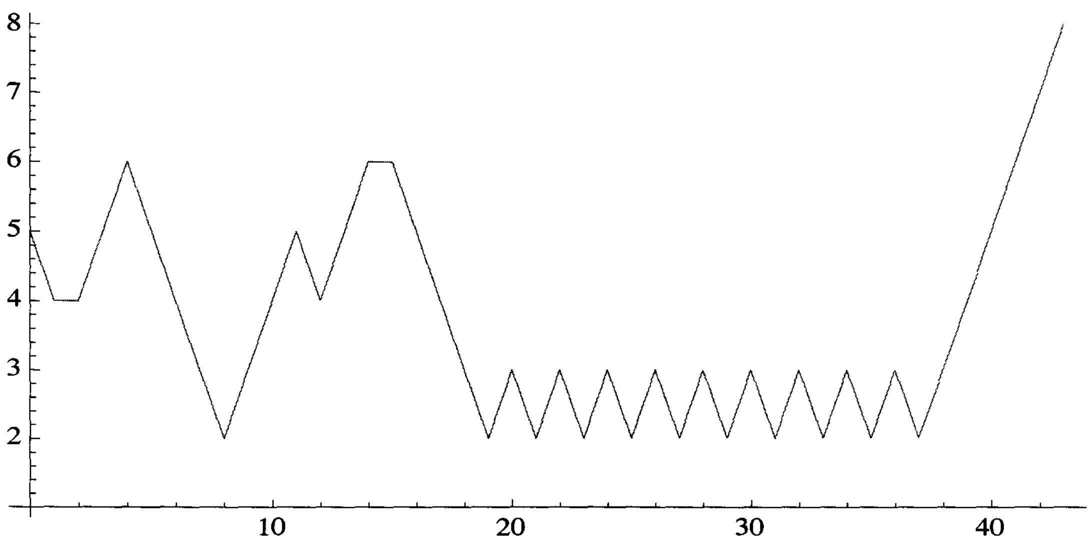
steps. Two simulations of Scenario 2 follow:

Simulation 1:



This simulation took nearly 170 steps to reach the upper bound, well over our expected average of 51.91 steps.

Simulation 2:



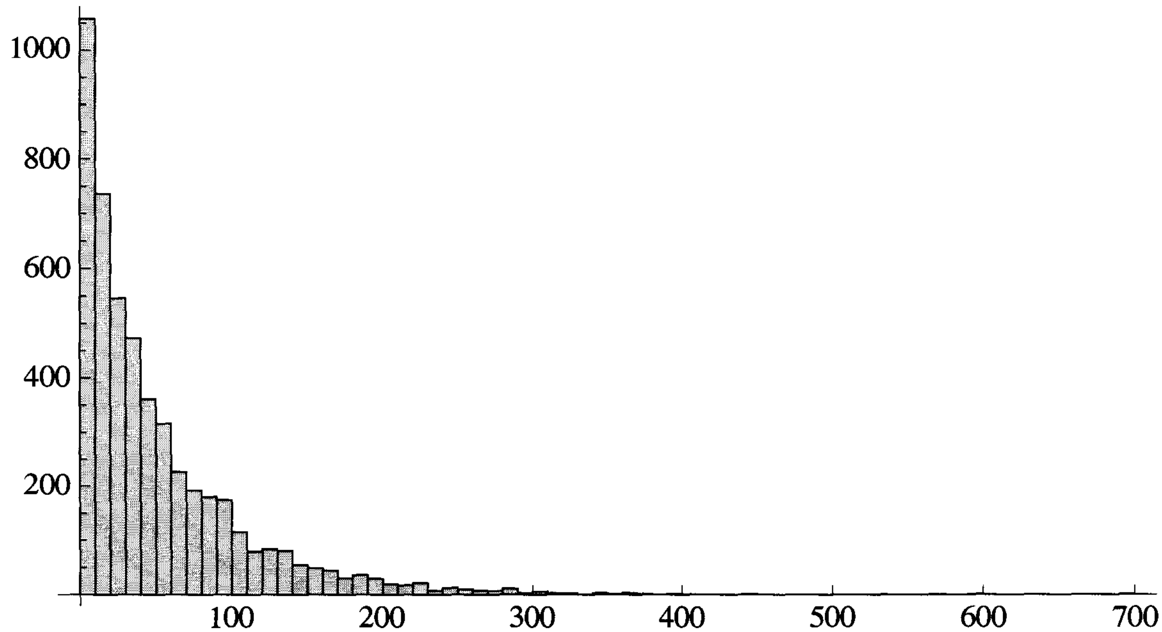
This time our walk took only 43 steps to complete.

After performing 5000 simulations of the scenario with *Mathematica*, we obtain the following data:

SampleMean	and	TrueAvgSteps
52.00		51.91

Our result agrees with our expected average.

The histogram displaying the distribution of the total number of steps follows:


1000
800
600
400
200
100
200
300
400
500
600
700

3.4 The a : 1 Case (a > 1)

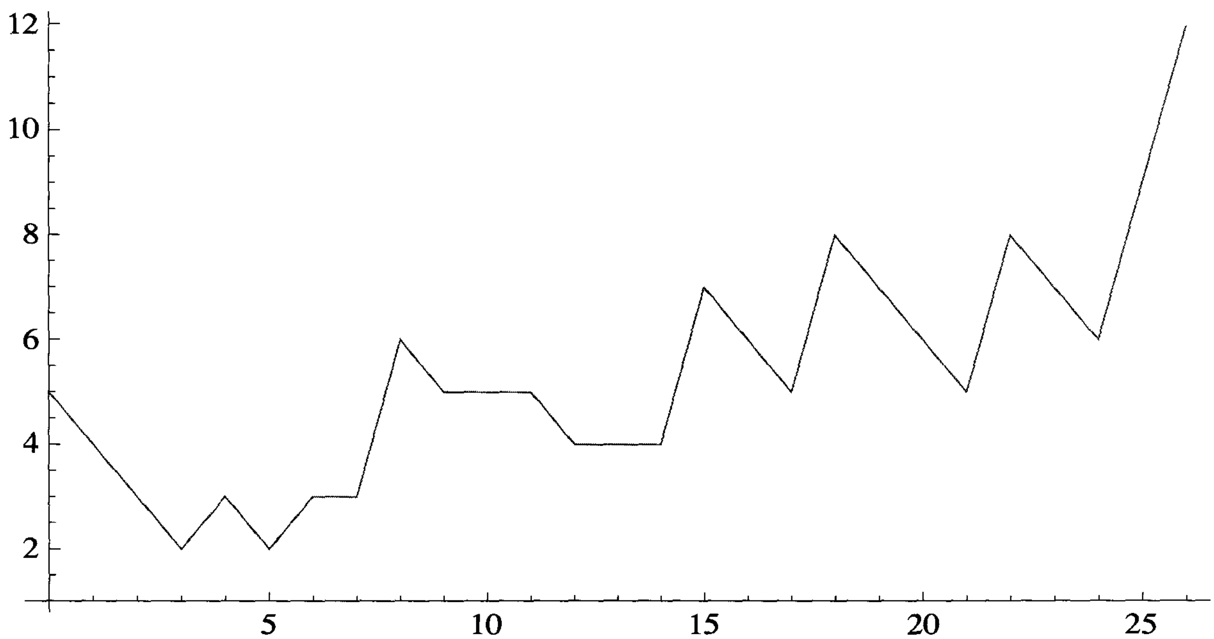
As we did with the elastic lower boundary, we will now analyze a scenario for the reflective lower boundary where we take a step and go up a units with probability $p > 0$ (where a is an integer such that $a > 1$), down 1 unit with probability q, or remain at the same height with probability $r = 1 - p - q$. Once again, we are guaranteed to hit any upper boundary *eventually*. We have proven already that with $p > 0$ we will hit any upper boundary with probability 1 in the 1 : 1 case, and clearly going up a steps instead of 1 will not hurt our chances of reaching the top.

The difference-equations argument we used to obtain a closed-form solution for the average number of steps needed to reach the top will not work with the a : 1 case. The reasoning is very similar to that used for the elastic lower boundary; the only difference is that with the reflective boundary we have $e_0 = 1 + e_1$ instead of having $e_0 = e_1$. So once again we will use Markov Chains to determine probability states after k steps and systems of equations to determine exact solutions for the average number of steps needed to reach the upper boundary n. First, however, we will run a few simulations with *Mathematica* to obtain some estimates for the average number of steps needed, then compare our results with the solutions we obtain using systems of equations. Note that once again our walks will come to a halt upon meeting *or exceeding* the upper boundary.

Scenario 1

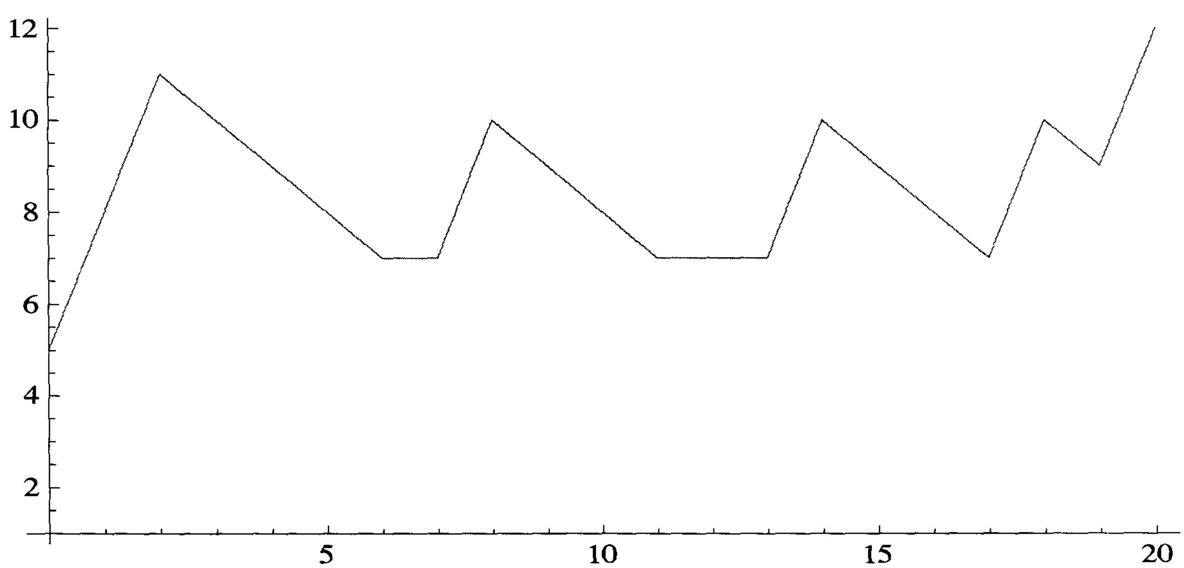
Upper Boundary:	$n = 12$	Probability of Going Up:	$p = 0.3$
Initial Height:	$j = 5$	Probability of Going Down:	$q = 0.5$
Reflective Lower Boundary:	$m = 2$	Size of Upward Jumps:	$a = 3$

Simulation 1:



This simulation took 26 steps to complete, coming to a stop upon reaching height 12.

Simulation 2:



This time our simulation took 20 steps before stopping upon reaching height 12.

After running 500000 simulations with the above scenario, our sample average number of steps to reach the upper boundary is 17.3776. Note that this is an increase over the simulated average time of 16.0463. This difference is due to the fact that with the reflective lower boundary, an extra step is taken each time the lower boundary is reached.

Scenario 2

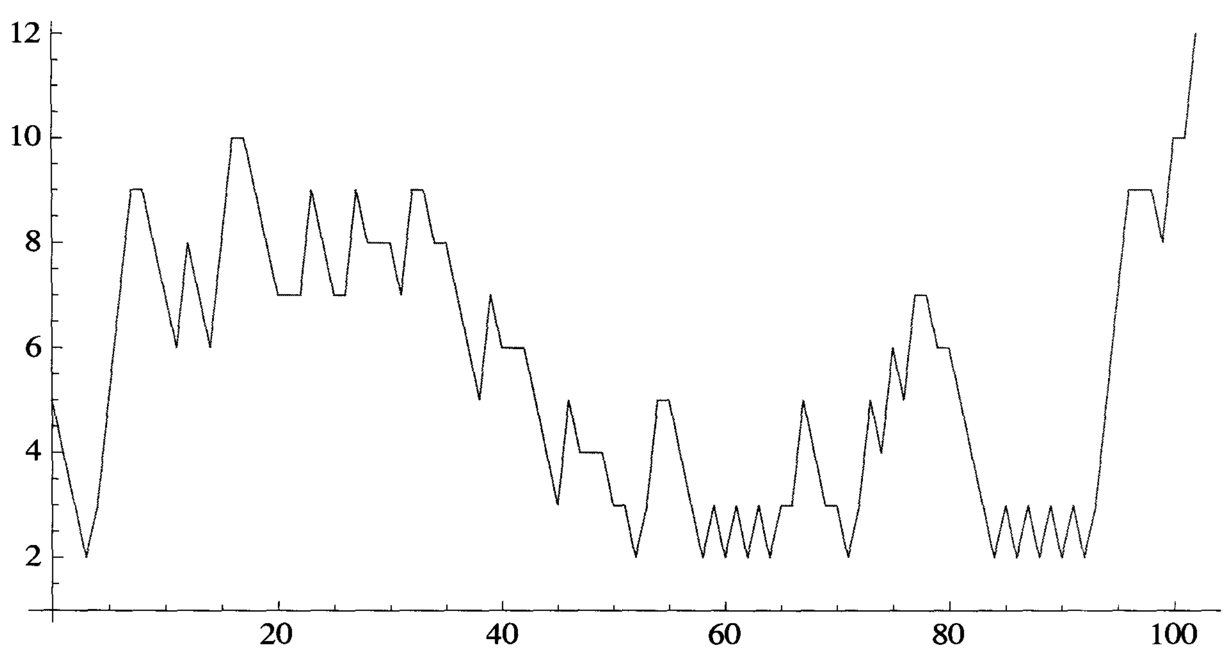
Upper Boundary: $n = 12$ Probability of Going Up: $p = 0.3$

Initial Height: $j = 5$ Probability of Going Down: $q = 0.5$

Reflective Lower Boundary: $m = 2$ Size of Upward Jumps: $a = 2$

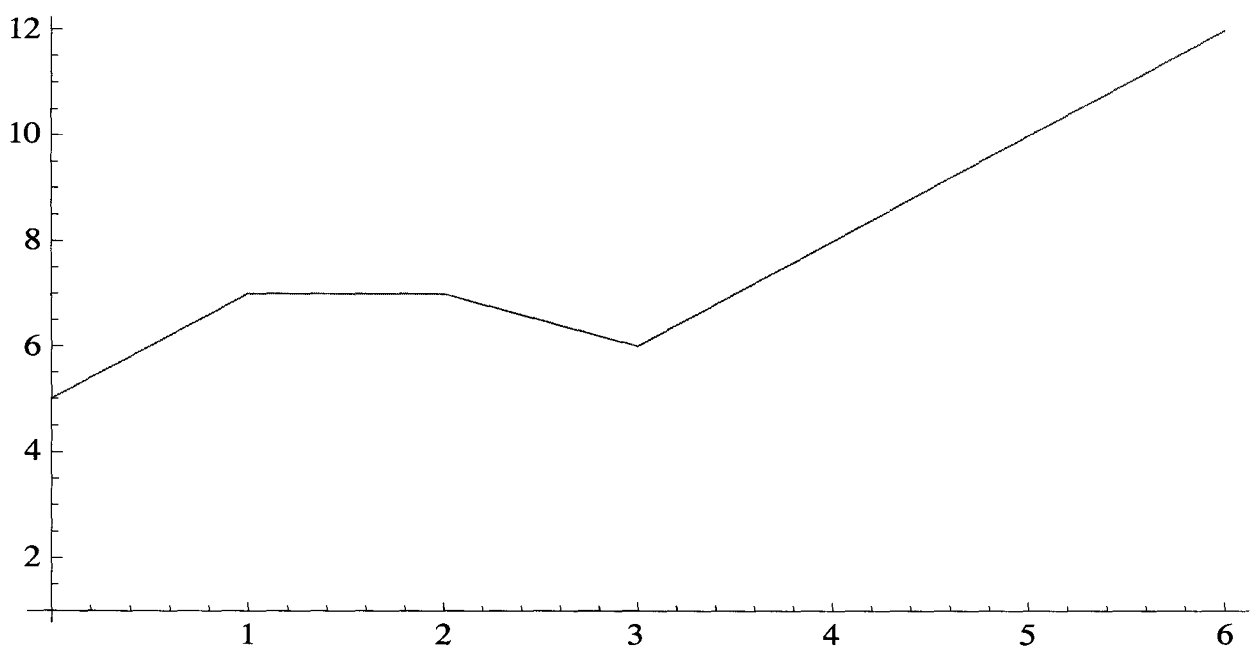
Once again, we change the size of upward jumps from 3 to 2, keeping all other variables the same.

Simulation 1:



The previous simulation took 102 steps before reaching the upper boundary.

Simulation 2:



This time our simulation took only 6 steps to reach the upper boundary.

After running 150000 simulations with the above scenario, our average number of steps to reach the upper boundary is 41.481, slightly higher than the average of 37.975 we obtained for the elastic lower boundary.

3.5 Markov Chain Method for the a : 1 Case

In this section, we will use Markov Chains to analyze the probability states after a given number of steps. Given a positive integer k, we will use this method to do the following:

a) Give the probability states after k steps;
b) Determine the probability of reaching the upper boundary within k steps;
c) Determine the probability of reaching the upper boundary in exactly k steps;
d) Approximate the average number of steps needed to reach the upper boundary.

As we did for the elastic lower boundary case, we will first need to construct an *initial state matrix* (I) that designates the starting position (j) of our walk among all possible ending heights. Unlike with the elastic lower boundary, however, our walk may attain the lower boundary m, so I must include an entry corresponding to m. Thus m is the lowest possible ending height that may be achieved, while the greatest possible ending height remains $n+a-1$. We will assume once again that our walk will not start on a boundary point, so j will correspond to a height between $m+1$ and $n-1$. I will again have a 1 in the column corresponding to j, while all other entries in the matrix will be 0. The matrix I will look as follows, with a matrix displaying the possible heights shown below for comparison:

$$I = \begin{bmatrix} 0 & \dots & 1 & \dots & \dots & \dots & 0 \end{bmatrix}$$
$$\begin{bmatrix} m & \dots & j & \dots & n & \dots & n+a-1 \end{bmatrix}$$

Next we need to construct the transition matrix A. Because m is a possible height, the transition matrix is given by $A = (a_{ih})$, for $m \le i, h \le n+a-1$. In other words, we will need to include a row and column for the lower boundary m where as with the elastic lower boundary none was needed. Due to the reflective lower boundary, whenever height m is reached the next step is automatically upward 1 unit. Otherwise, our walk takes a step and moves up a units with probability p, down 1 unit with probability q, or stays at the same height with probability $r = 1 - p - q$. For our example, we will use the same values as in Section 2.5 so we may compare the similarities and differences in the transition matrix for the two cases.

Given the following values:

Upper Boundary: $n = 7$ Size of Upward Jumps: $a = 3$

Reflective Lower Boundary: $m = 2$

The transition matrix A will be as follows:

$$A = \left[\begin{array}{c|cccccccc} & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & q & r & 0 & 0 & p & 0 & 0 & 0 \\ 4 & 0 & q & r & 0 & 0 & p & 0 & 0 \\ 5 & 0 & 0 & q & r & 0 & 0 & p & 0 \\ 6 & 0 & 0 & 0 & q & r & 0 & 0 & p \\ 7 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Note that the top row and left column serve as placeholders and are not a part of the actual matrix. Matrix A has dimensions $(n+a-m)\times(n+a-m)$.

a) Determining probability states after k steps:

To determine the probability states after k steps, we multiply $I \times A^k$ as described in Section 2.5. The only difference is that the resulting matrix will have dimensions

$1 \times (n+a-m)$, and the leftmost entry will now represent the probability of ending at height m instead of height $m+1$ as was the case in the elastic lower boundary scenario. The determination of parts *b)*, *c)*, and *d)* for the cdf *F(k)*, the pdf *f(k)*, and the approximate average number of steps needed to reach n are performed precisely as in Section 2.5.

Again we will simulate parts *a)* through *d)* using *Mathematica*. The values used in Section 3.4 will be used again so that we may compare our result for part *d)* with the result we obtained previously. Also, we will compare this result with the results obtained in the elastic lower boundary case (Sections 2.4 and 2.5) for the approximate number of steps needed to reach the upper boundary.

Scenario 1

Upper Boundary: $n = 12$ Probability of Going Up: $p = 0.3$

Initial Height: $j = 5$ Probability of Going Down: $q = 0.5$

Reflective Lower Boundary: $m = 2$ Size of Upward Jumps: $a = 3$

Based on these values, our initial height matrix *I* and our transition matrix *A* will look as follows:

$$I = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .5 & .2 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .5 & .2 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .5 & .2 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & .2 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & .2 & 0 & 0 & .3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5 & .2 & 0 & 0 & .3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .5 & .2 & 0 & 0 & .3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .5 & .2 & 0 & 0 & .3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .5 & .2 & 0 & 0 & .3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

a) First we will examine the probability states after 5 steps are taken. To do this, we multiply $I \times A^5$ and obtain:

$$[0.093 \quad 0.120 \quad 0.060 \quad 0.113 \quad 0.128 \quad 0.024 \quad 0.104 \quad 0.135 \quad 0.054 \quad 0.007 \quad 0.040 \quad 0.073 \quad 0.050].$$

Next we determine the probability states after 6 steps:

$$[0.060 \quad 0.147 \quad 0.068 \quad 0.086 \quad 0.074 \quad 0.075 \quad 0.122 \quad 0.092 \quad 0.022 \quad 0.033 \quad 0.081 \quad 0.089 \quad 0.052].$$

To approximate the final ending probabilities for the walk, we compute $I \times A^{1000}$:

$$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.450 \quad 0.336 \quad 0.214]$$

Thus after 1000 steps our walk will almost surely reach the upper boundary, with a 45.0% chance of having stopped at height 12, a 33.6% chance of having stopped at height 13, and a 21.4% chance of having stopped at height 14. These probabilities are precisely the same as with the elastic lower boundary case.

b) We will now examine the probability $F(k)$ of reaching the top within k steps. We will use the values for k we used in part *a)*. For the $k = 5$ case, we see that there is a 0.040 probability of reaching height 12, a 0.073 probability of reaching height 13, and a 0.05 probability of reaching height 14. Thus $F(5) \approx 0.040 + 0.073 + 0.050 = 0.163$

Similarly, the cumulative probability of reaching the upper boundary within 6 steps is given by

$$F(6) \approx 0.081 + 0.089 + 0.052 = 0.222\,.$$

For the $k = 1000$ case, we see that our probability of reaching the top is

$$F(1000) \approx 0.450 + 0.336 + 0.214 = 1.$$

c) Now we will examine the probability of reaching the upper boundary in *exactly* k steps. For example, we know that $F(5)$ is approximately 0.163, and $F(6)$ is approximately 0.222. Therefore we can conclude that the probability of reaching the upper boundary in *exactly* 6 steps, given by $F(6) - F(5)$, is

$$f(6) = F(6) - F(5) \approx 0.222 - 0.163 = 0.059$$

Next we will use *Mathematica* to provide approximations for $f(k)$ for $1 \le k \le 20$.

k	$f(k)$	k	$f(k)$
1	0.000	11	0.041
2	0.000	12	0.036
3	0.027	13	0.034
4	0.057	14	0.033
5	0.079	15	0.031
6	0.059	16	0.029
7	0.044	17	0.027
8	0.041	18	0.025
9	0.047	19	0.023
10	0.046	20	0.022

Using the method in part *b)*, we may determine that there is a cumulative probability of approximately 0.703 that we will hit the upper boundary within 20 steps. Summing the above figures may produce a slightly different result due to rounding.

d) Using *Mathematica* to compute the sum $\sum_{k=1}^{15000} k \times f(k)$, we obtain a value 17.3945 for the average number of steps needed to reach the top. This result is close to our sample average of 17.3776 that we obtained in Section 3.4, Scenario 1.

3.6 System of Equations Solutions for the a : 1 Case

In this section, we will use the systems of equations method described in Section 2.5 to determine exact numerical solutions for the average number of steps needed to reach the upper boundary for the $a:1$ case of the reflective lower boundary. Let x_j represent the average number of steps needed to reach or surpass the upper boundary n from starting position j, where $m \le j \le n+a-1$. If $n \le j \le n+a-1$, then clearly $x_j = 0$ because our walk has reached the upper boundary.

Now suppose $m+1 \le j \le n-1$. Then one step is taken and we go up a units with probability p, down 1 unit with probability q, or stay at the same height with probability r. Thus we may express x_j in terms of x_{j-1} and x_{j+a} as follows:

$$x_j = 1 + q \times x_{j-1} + r \times x_j + p \times x_{j+a}.$$

To establish our system of equations, we once again organize the equations so that the variables appear on one side of the equation and the constant on the other. Thus we take the preceding equation and subtract x_j and 1 from both sides to obtain:

$$q \times x_{j-1} + (r-1) \times x_j + p \times x_{j+a} = -1.$$

For the case when $j = m$, we must take the reflective lower boundary into consideration. When our walk reaches height m, the next step is automatically upward, so we have:

$$x_m = 1 + x_{m+1}$$

which we rewrite as:

$$-x_m + x_{m+1} = -1$$

We now have the following system of equations:

$$
\begin{aligned}
-x_m + x_{m+1} &= -1 \\
q \times x_m + (r-1) \times x_{m+1} + p \times x_{m+1+a} &= -1 \\
q \times x_{m+1} + (r-1) \times x_{m+2} + p \times x_{m+2+a} &= -1 \\
&\vdots \\
q \times x_{n-2} + (r-1) \times x_{n-1} + p \times x_{n+a-1} &= -1 \\
x_n &= 0 \\
&\vdots \\
x_{n+a-1} &= 0
\end{aligned}
$$

To solve the system of equations, we again create T, the $(n+a-m)\times(n+a-m)$ matrix of coefficients, F, the $1\times(n+a-m)$ matrix that contains the constants that appear on the right hand side of each equation, and X, the $1\times(n+a-m)$ matrix that contains the variables $x_m, x_{m+1}, \ldots, x_{n+a-1}$. Our goal is to solve for each x_j by solving the system $TX = F$.

For our example, we will use the following values:

Upper Boundary: $n = 7$

Size of Upward Jumps: $a = 3$

Reflective Lower Boundary: $m = 2$

For a given p, q, and r, the system $TX = F$ will look as follows:

$$
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
q & r-1 & 0 & 0 & p & 0 & 0 & 0 \\
0 & q & r-1 & 0 & 0 & p & 0 & 0 \\
0 & 0 & q & r-1 & 0 & 0 & p & 0 \\
0 & 0 & 0 & q & r-1 & 0 & 0 & p \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix}
=
\begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

By the same reasoning used in Section 2.6, matrix T will be invertible, and the solution for X is given by:

$$X = T^{-1}F \, .$$

We will now use the system of equations method to determine the average steps needed to hit the upper boundary for the same walk we used in Sections 3.4 and 3.5. We will then compare our results to the results obtained in those Sections.

Scenario 1

Upper Boundary: $n = 12$ Probability of Going Up: $p = 0.3$

Reflective Lower Boundary: $m = 2$ Probability of Going Down: $q = 0.5$

Size of Upward Jumps: $a = 3$

Given this scenario, the system $TX = F$ will be as follows:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .5 & -.8 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .5 & -.8 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .5 & -.8 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & -.8 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & -.8 & 0 & 0 & .3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5 & -.8 & 0 & 0 & .3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .5 & -.8 & 0 & 0 & .3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .5 & -.8 & 0 & 0 & .3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .5 & -.8 & 0 & 0 & .3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

Now we calculate the solutions for $x_3, x_4, \ldots, x_{14}$, given by $X = T^{-1}F$:

$$\begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix} = X = T^{-1}F \approx \begin{bmatrix} 21.2949 \\ 20.2949 \\ 18.9074 \\ 17.3945 \\ 15.2949 \\ 13.2614 \\ 11.5397 \\ 8.4623 \\ 6.5390 \\ 5.3369 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We may conclude that the *exact* numerical solution for x_2 to four decimal places is 21.2949 steps, the exact solution for x_3 is 20.2949 steps, and so on. Note that the solution for x_5, the average number of steps to reach the upper boundary from starting position 5, is 17.3945. This is precisely the same solution we obtained with the Markov Chain method from Section 3.5, and very close to the value of 17.3776 we obtained from running numerous simulations in Section 3.4.

Chapter 4: Conclusions and Results

For our conclusion, we will now apply the results we obtained in Chapters 2 and 3 to a gambling scenario with a generous House. Throughout, let $r = 0$ and let $m = 0$. Let $p > 0$ be the probability of winning a bet and let $q = 1 - p$. Assume that $p \leq 0.50$. We start with j chips and will keep betting one chip at a time (of the same value) until we reach n chips, where $n > j$. Whenever we go broke, the House spots us a chip so that we can keep betting. When we reach n chips, we quit, but we have to pay back the House all the chips that were given to us. Let ${}_jG^n_{(0)}$ be the final amount of chips we have gained at the conclusion after paying back the House. Here, ${}_jS^n_{(0)}$ and ${}_jR^n_{(0)}$ are respectively the number of bets needed to reach n chips with elastic and reflective lower boundaries of $m = 0$.

Because ${}_jR^n_{(0)} - {}_jS^n_{(0)}$ is the number of times we go broke before attaining n chips, it is also the number of chips given to us that we must pay back. In the $1:1$ case, we end with n chips, so our net gain is $(n-j) - ({}_jR^n_{(0)} - {}_jS^n_{(0)})$, which may be negative. We may now conclude that the average net gain is given by

$$E[{}_jG^n_{(0)}] = E[(n-j) - ({}_jR^n_{(0)} - {}_jS^n_{(0)})] = n - j + E[{}_jS^n_{(0)}] - E[{}_jR^n_{(0)}].$$

In this chapter, we will use this result as well as results obtained in previous chapters to do the following:

1) Show that $E[{}_jG^n_{(0)}] = 0$ for the $1:1$ case when $p = q = 0.5$.

2) Prove by induction that when $p < q$, $E[{}_jG^n_{(0)}] < 0$ for all $n \geq j+1$ for the $1:1$ case.

3) Compute $E[{}_jG^n_{(0)}]$ for a specific example for the $1:1$ case.

4) Analyze a specific example for the $a:1$ case, and compare the result for $E[{}_jG^n_{(0)}]$ with the result obtained in Part 3.

4.1 Average Net Gain for the 1 : 1 Case when $p = q = 0.5$

Using our result for $E[{}_jG^n_{(0)}]$ and our formulas for $E[{}_jR^n_{(m)}]$ and $E[{}_jS^n_{(m)}]$ with $m = 0$ and $p = q = 0.5$, our average net gain in the 1:1 case is given by

$$\begin{aligned}
E[{}_jG^n_{(0)}] &= E[(n-j)-({}_jR^n_{(0)}-{}_jS^n_{(0)})] \\
&= n-j+E[{}_jS^n_{(0)}]-E[{}_jR^n_{(0)}] \\
&= n-j+\frac{(n-j)(n+j-2m-1)}{2p}-\frac{(n-j)(n+j+2p-1)}{2p} \\
&= n-j+\frac{(n-j)(n+j-1)}{2(.5)}-\frac{(n-j)(n+j+2(.5)-1)}{2(.5)} \\
&= n-j+(n-j)(n+j-1)-(n-j)(n+j) \\
&= (n-j)(1+n+j-1-n-j) \\
&= (n-j)(0) \\
&= 0.
\end{aligned}$$

Hence when $p = q = 0.5$ with 1:1 payoffs, we would expect to break even on average for any gambling scenario when the House spots us an additional chip whenever we go broke.

4.2 Average Net Gain for the 1 : 1 Case when $p < q$

Previously we showed that when $p = q$, we would have an average net gain of 0 for a given gambling scenario where we are given one chip whenever we go broke. However, any gambling house will want $p < q$ for any wager for which there is a 1:1 payout. We will now prove by induction that, when $p < q$ and the payout is 1:1, $E[{}_jG_{(0)}^n] < 0$ for all $n \geq j+1 > 1$.

Step 1: Suppose $n = j+1$. Then $E[{}_jG_{(0)}^{j+1}] =$

$$(j+1)-j+\frac{(j+1)-j}{p-q}+\frac{p\left(\left(\frac{q}{p}\right)^{j+1}-\left(\frac{q}{p}\right)^{j}\right)}{(p-q)^2}-\frac{p(p-q-1)\left(\left(\frac{q}{p}\right)^{j}-\left(\frac{q}{p}\right)^{j+1}\right)+(p-q)((j+1)-j)}{(p-q)^2}$$

$$=1+\frac{1}{p-q}+\frac{p\left(\left(\frac{q}{p}\right)^{j}\left(\frac{q}{p}-1\right)\right)}{(p-q)^2}-\frac{p(p-q-1)\left(\left(\frac{q}{p}\right)^{j}\left(1-\frac{q}{p}\right)\right)+(p-q)}{(p-q)^2}$$

$$=1+\frac{p(q-p)\left(\frac{q}{p}\right)^{j}\left(1-\frac{q}{p}\right)}{(p-q)^2}$$

$$=1-\left(\frac{q}{p}\right)^{j}.$$

Because $q > p$, this result is clearly negative for any $j > 0$. If $j = 0$ and $n = 1$, then we enter the casino with no chips, the casino then spots us a chip so we then reach our goal of gaining one chip, and then we pay back the one chip that the casino gave us. Thus we would break even, so $E[{}_0G_{(0)}^1] = 0$. For any other case when $n = j+1$, clearly $E[{}_jG_{(0)}^{j+1}] < 0$.

Step 2: Assume that $E[{}_jG_{(0)}^{n+1}] < 0$ for some $n \geq j+1 > 1$.

Step 3: We claim that $E[{}_jG_{(0)}^{n+1}] = E[{}_jG_{(0)}^n] + E[{}_nG_{(0)}^{n+1}]$. First,

$$E[{}_jG_{(0)}^{n+1}] = n+1-j+E[{}_jS_{(0)}^{n+1}] - E[{}_jR_{(0)}^{n+1}]$$

$$= n+1-j+\frac{n+1-j}{p-q}+\frac{p\left(\left(\frac{q}{p}\right)^{n+1}-\left(\frac{q}{p}\right)^{j}\right)}{(p-q)^2}-\frac{p(p-q-1)\left(\left(\frac{q}{p}\right)^{j}-\left(\frac{q}{p}\right)^{n+1}\right)+(p-q)(n+1-j)}{(p-q)^2}.$$

Also, we have

$$E[{}_jG_{(0)}^{n}] + E[{}_nG_{(0)}^{n+1}] =$$

$$n-j+\frac{n-j}{p-q}+\frac{p\left(\left(\frac{q}{p}\right)^{n}-\left(\frac{q}{p}\right)^{j}\right)}{(p-q)^2}-\frac{(p-q-1)p\left(\left(\frac{q}{p}\right)^{j}-\left(\frac{q}{p}\right)^{n}\right)+(p-q)(n-j)}{(p-q)^2}+$$

$$n+1-n+\frac{n+1-n}{p-q}+\frac{p\left(\left(\frac{q}{p}\right)^{n+1}-\left(\frac{q}{p}\right)^{n}\right)}{(p-q)^2}-\frac{(p-q-1)p\left(\left(\frac{q}{p}\right)^{n}-\left(\frac{q}{p}\right)^{n+1}\right)+(p-q)}{(p-q)^2}$$

$$= n+1-j+\frac{n+1-j}{p-q}+\frac{p\left(\frac{q}{p}\right)^{n}-p\left(\frac{q}{p}\right)^{j}}{(p-q)^2}-\frac{(p-q-1)p\left(\frac{q}{p}\right)^{j}-(p-q-1)p\left(\frac{q}{p}\right)^{n}}{(p-q)^2}+$$

$$\frac{p\left(\frac{q}{p}\right)^{n+1}-p\left(\frac{q}{p}\right)^{n}}{(p-q)^2}-\frac{(p-q-1)p\left(\frac{q}{p}\right)^{n}-(p-q-1)p\left(\frac{q}{p}\right)^{n+1}+(p-q)(n-j+1)}{(p-q)^2}$$

$$= n+1-j+\frac{n+1-j}{p-q}+\frac{p\left(\left(\frac{q}{p}\right)^{n+1}-\left(\frac{q}{p}\right)^{j}\right)}{(p-q)^2}-\frac{p(p-q-1)\left(\left(\frac{q}{p}\right)^{j}-\left(\frac{q}{p}\right)^{n+1}\right)+(p-q)(n+1-j)}{(p-q)^2}$$

$$= E[{}_jG_{(0)}^{n+1}].$$

Therefore, $E[{}_jG_{(0)}^{n+1}] = E[{}_jG_{(0)}^{n}] + E[{}_nG_{(0)}^{n+1}]$. By Steps 2 and 1, we know that $E[{}_jG_{(0)}^{n}]$ and $E[{}_nG_{(0)}^{n+1}]$ are both negative; thus $E[{}_jG_{(0)}^{n+1}] < 0$. We may now conclude by induction that $E[{}_jG_{(0)}^{n}] < 0$ for all $n \geq j+1 > 1$.

Scenario 1: Average Net Gain for 1 : 1 Case

We will now compute a numerical example for the average net gain for a 1:1 case. Suppose we are playing roulette at a generous House and betting on red. We start with 10 chips and stop playing when we reach 20 chips. Our values for p , q , j , n , and m are then as follows:

$$p = \frac{18}{38} \qquad q = \frac{20}{38} \qquad j = 10 \qquad n = 20 \qquad m = 0 .$$

The average net gain in this scenario, given by $E[{}_jG^n_{(0)}] = n - j + E[{}_jS^n_{(0)}] - E[{}_jR^n_{(0)}]$,

is then $20 - 10 + E[{}_{10}S^{20}_{(0)}] - E[{}_{10}R^{20}_{(0)}] \approx 10 + 726.097 - 774.312 \approx -38.215$.

Therefore, after reaching our goal of 20 chips, on average we would be down about 38 chips after paying the House back all the chips it spotted us.

4.3 Average Net Gain for the a : 1 Case

We will now determine the average net gain in an $a:1$ scenario. Suppose that we play roulette starting with 10 chips and stop when we reach 20 chips as we did in Scenario 1. This time, however, we will make column bets instead of betting on red. The amount we may win now increases from 1 to 2 chips, but our odds of winning decrease from 18/38 to12/38. Our values for for p, q, j, n, m, and a are now

$$p=\frac{12}{38} \qquad q=\frac{26}{38} \qquad j=10 \qquad n=20 \qquad m=0 \qquad a=2.$$

Because we now gain 2 chips when we win, we must take into consideration the fact that we may reach 20 or 21 chips when our gambling session ends. To account for this, we will apply the Markov Chain method from Section 2.5 to compute the end probability states of being at 20 or 21 chips. We then obtain $p_{20} \approx 0.673$ and $p_{21} \approx 0.327$. Thus our final average chip count before paying back the House, given by $20 \times p_{20} + 21 \times p_{21}$, is approximately 20.327. Because we start the session with 10 chips, our average gain before paying back the house will be approximately $20.327 - 10 = 10.327$.

Next we determine the average amount of chips we must give back to the House for our scenario. Using the system of equations solutions from Sections 2.6 and 3.6, we obtain

$$E[{}_jS^n_{(0)}] \approx 237.606 \text{ and } E[{}_jR^n_{(0)}] \approx 259.864.$$

The average number of times we hit the lower boundary, and therefore the average number of chips we must pay back, is then $E[{}_jR^n_{(0)}] - E[{}_jS^n_{(0)}] \approx 22.258$. Because our average gain is about 10.327 chips and the average we must pay back is around 22.258, we may conclude that our average net gain after paying back the House is approximately $10.327 - 22.258 = -11.931$ chips. Comparing our result to the result from Scenario 1, we may conclude that making column bets is more advantageous than betting red. Not only is our average net gain of -11.931 in the column betting scenario greater than the -38.215 average net in Scenario 1, but the average number of steps needed in the column betting scenario is much less than the average number of steps in Scenario 1. So not only do we lose less on average making the column bets, we also reach our goal more quickly.

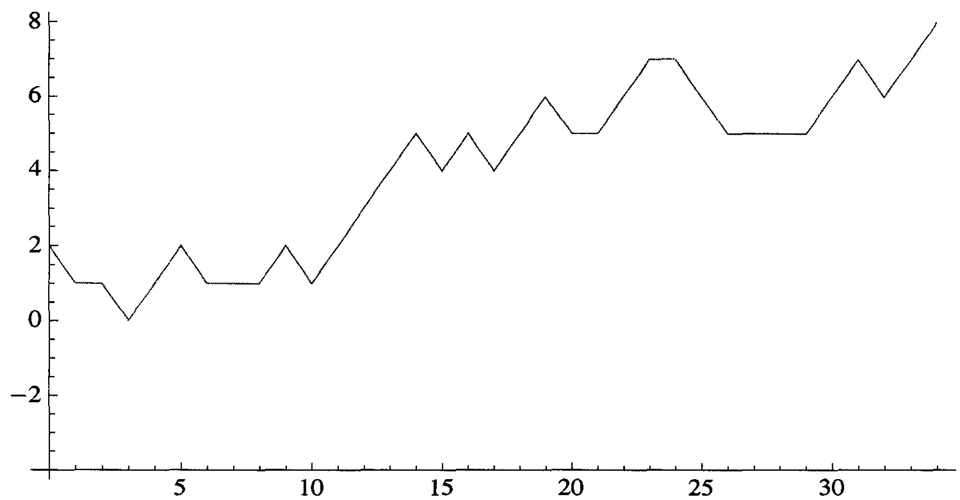
Appendix

Below we present the *Mathematica* code used to generate the random walk simulations used throughout the course of the thesis.

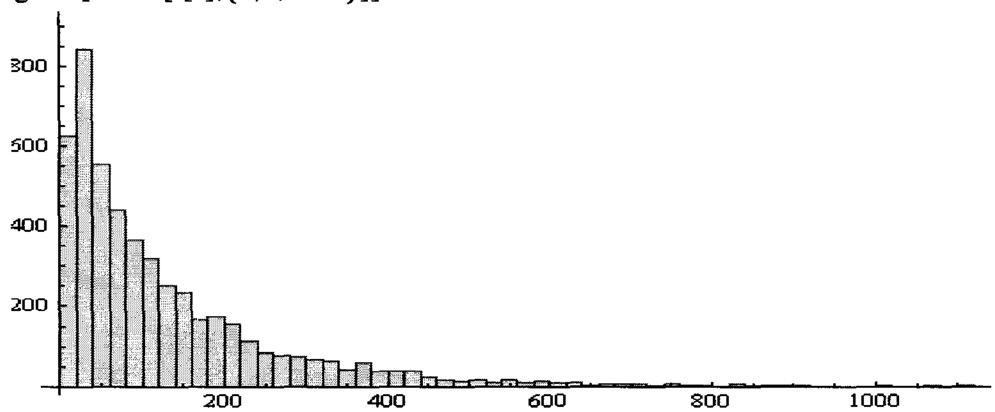
Section 2.3

Elastic Lower Boundary – 1:1 steps

```
ProbUp = p = 0.4;
ProbDown = q = 0.5;
r = 1 – q – q;
SizeOfUpwardJumps = a = 1;
SizeofDownwardJumps = b = 1;
StartingHeight = j = 2;
UpperBound = n = 8;
LowerBound = m = -4;
NumberOfIterations = num = 5000;
Do[x[s,0] = j,{s,1,num}]
 Do[i = 0; While[x[s,i] < n,
y[s,i + 1] = If[x[s,i] > m + 1,1,0];
z[i + 1] = Random[ ];
x[s,i + 1] = If[y[s,i + 1] == 1,If[z[i + 1] < p,x[s,i] + a,If[z[i + 1] >= 1 - q,x[s,i] - b,x[s,i]]],
If[z[i + 1] < p,x[s,i] + a,x[s,i]]];i + +];t[s] = i,{s,1,num}]
 For[s = 1,s ≤ 10,s + +,Print[ListPlot[Table[{k,x[s,k]},{k,0,t[s]}],
PlotJoined → True,AxesOrigin → {0,m}]]]
```



Histogram[Table[t[s],{s,1,num}]]



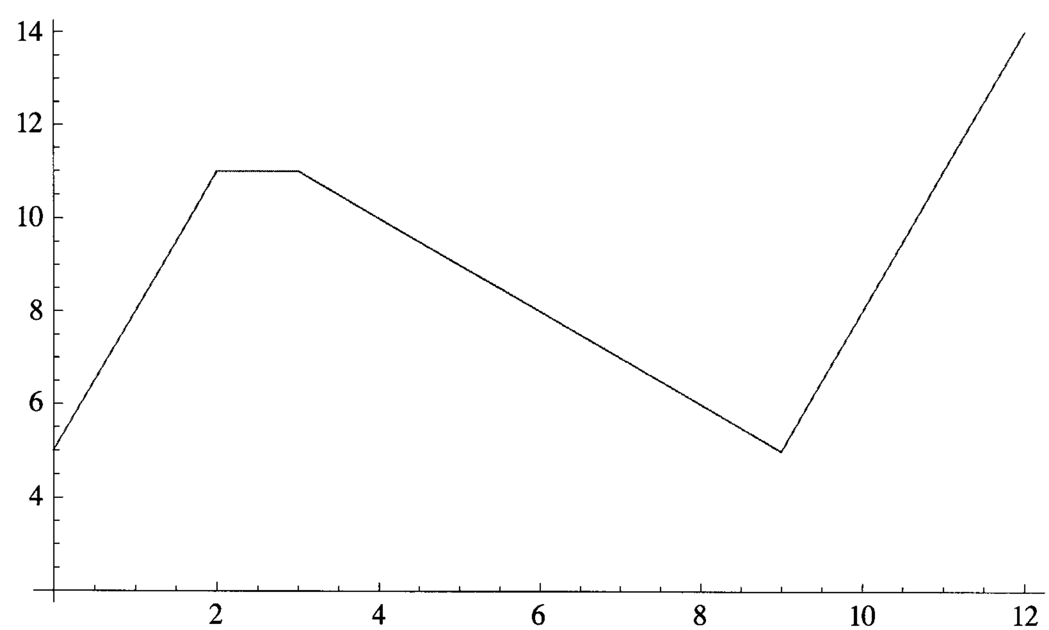

```
MeanSteps = N[Mean[Table[t[s],{s,1,num}]]];
MatrixForm[{{"SampleMean","TrueAvgSteps"},
{MeanSteps,If[p==q,(n - j)(n+ j - 2m - 1)/(2p),
(n - j)/(p - q) + p ((q/p)^(n - m) - (q/p)^(j - m))/(p - q)^2]}}]
```

Sample Mean	True Average Steps
127.249	127.5

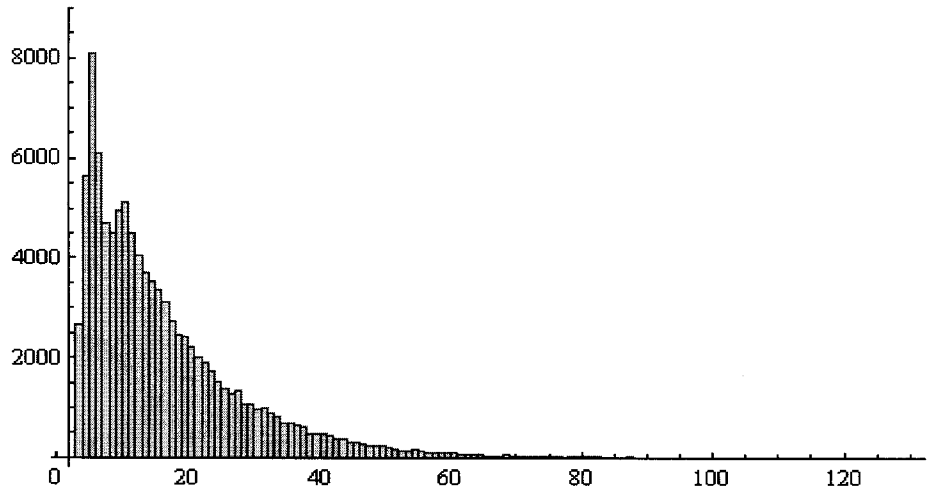
Section 2.4

Elastic Lower Boundary – *a*:1 steps

```
ProbUp = p = 0.3;
ProbDown = q = 0.5;
r = 1 - q - p;
SizeOfUpwardJumps = a = 3;
SizeOfDownwardJumps = b = 1;
StartingHeight = j = 5;
UpperBound = n = 12;
LowerBound = m = 2;
NumberOfIterations = num = 500000;
Do[x[s,0] = j,{s,1,num}]
Do[i = 0; While[x[s,i] < n,
y[s,i+1] = If[x[s,i] > m+1,1,0];
z[i+1] = Random[ ];
x[s,i+1] = If[y[s,i+1] == 1,If[z[i+1] < p,x[s,i]+a,If[z[i+1] >= 1-q,x[s,i]-b,x[s,i]]],
If[z[i+1] < p,x[s,i]+a,x[s,i]]];i++];t[s] = i,{s,1,num}]
For[s = 1,s ≤ 10,s++,Print[ListPlot[Table[{k,x[s,k]},{k,0,t[s]}],
PlotJoined → True,AxesOrigin → {0,m}]]]
```



Histogram[Table[t[s],{s,1,num}]]



MeanSteps = N[Mean[Table[t[s],{s,1,num}]]]

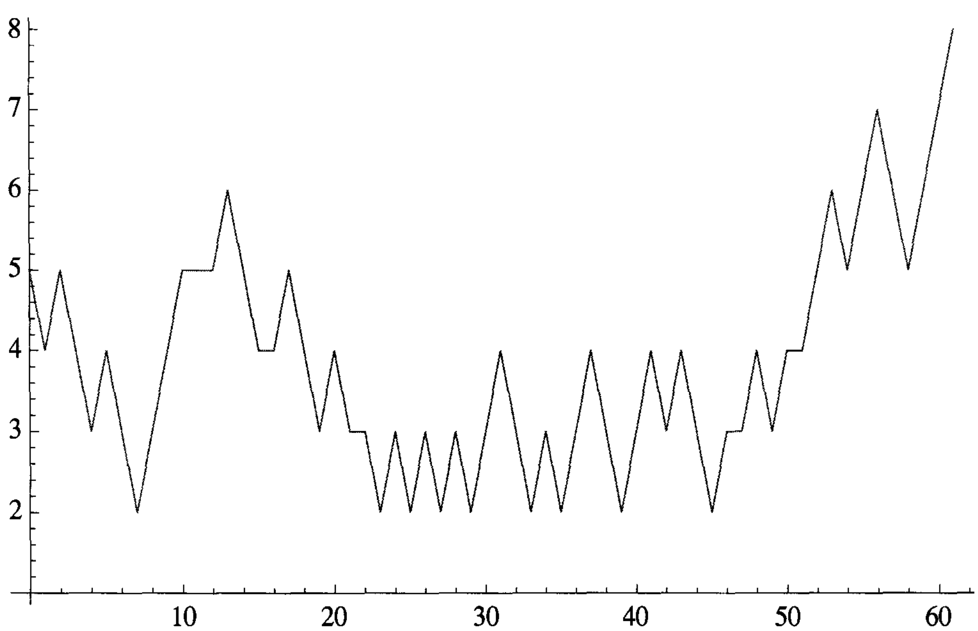
16.0463

Section 3.3

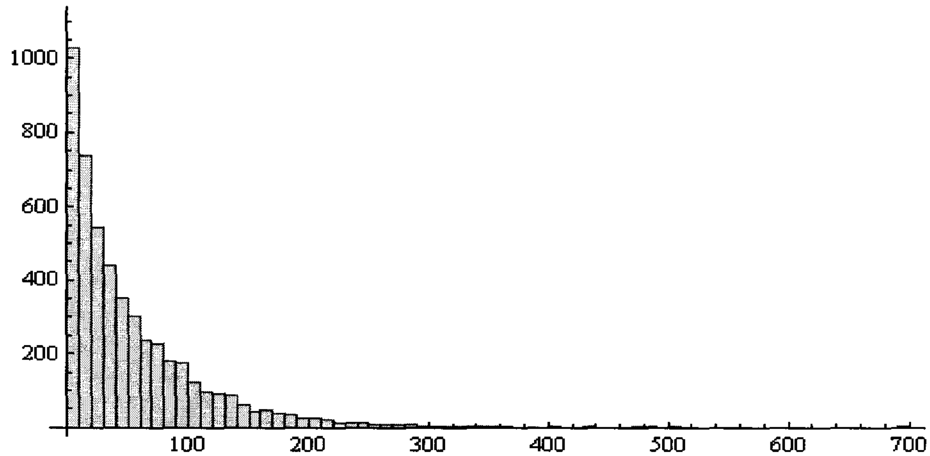
Reflective Lower Boundary - 1:1 steps

```
ProbUp = p = 0.40;
ProbDown = q = 0.50;
r = 1 - q - p;
SizeOfUpwardJumps = a = 1;
SizeOfDownwardJumps = b = 1;
StartingHeight = j = 5;
UpperBound = n = 8;
LowerBound = m = 2;
NumberOfIterations = num = 5000;
Do[x[s,0] = j,{s,1,num}]
Do[i = 0;While[x[s,i] < n,
y[s,i + 1] = If[x[s,i] > m,1,0];
z[i + 1] = Random[];
x[s,i + 1] = If[y[s,i + 1] == 1,If[z[i + 1] < p,x[s,i] + a,If[z[i + 1] >= 1 - q,x[s,i] - b,x[s,i]]],x[s,i] + a];
i + +];t[s] = i,{s,1,num}]

For[s = 1,s ≤ 10,s + +,Print[ListPlot[Table[{k,x[s,k]},{k,0,t[s]}],
PlotJoined → True,AxesOrigin → {0,m - 1}]]]
```



```
Histogram[Table[t[s],{s,1,num}]]
```




```
MeanTime = N[Mean[Table[t[s],{s,1,num}]]];
MatrixForm[{{"SampleMean","TrueAvgSteps"},
{MeanTime,If[p == q,n - j + (n - j)(n + j - 2m - 1)/(2p),
(n - j)/(p - q) + (2*p*q + p*r)*((q/p)^(n - m) - (q/p)^(j - m))/(p - q)^2]}}]
```

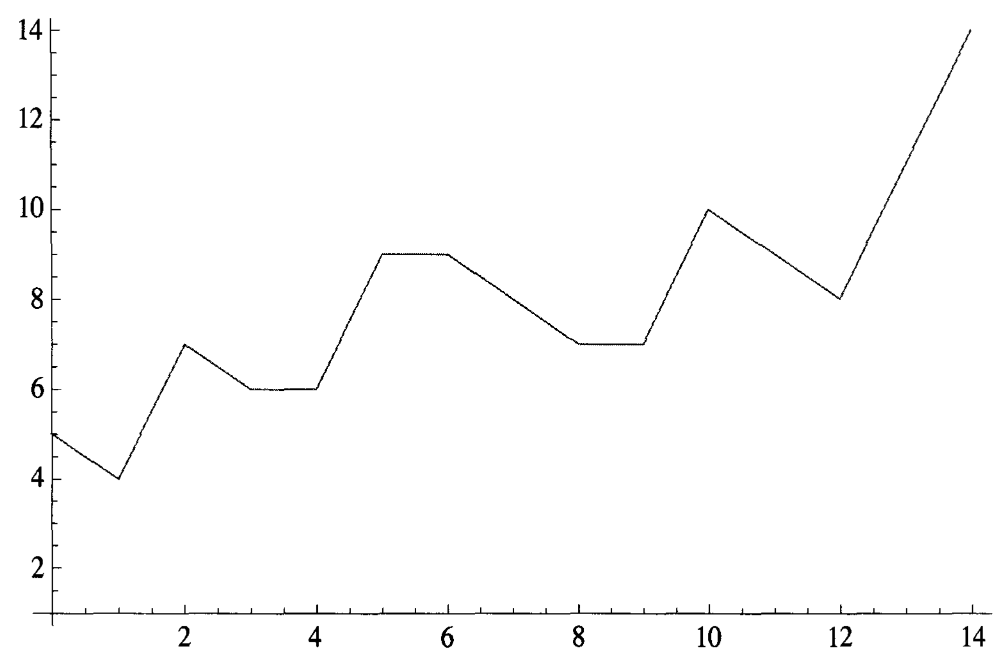
Sample Mean	True Average Steps
52.00	51.91

Section 3.4

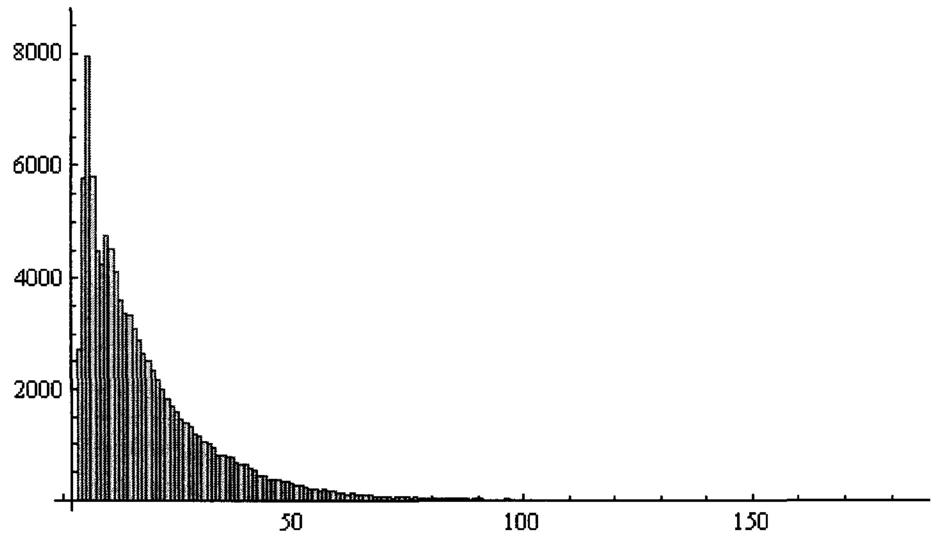
Reflective Lower Boundary - *a*:1 steps

```
ProbUp = p = 0.3;
ProbDown = q = 0.5;
r = 1 - q - p;
SizeOfUpwardJumps = a = 3;
SizeOfDownwardJumps = b = 1;
StartingHeight = j = 5;
UpperBound = n = 12;
LowerBound = m = 2;
NumberOfIterations = num = 500000;
Do[x[s,0] = j,{s,1,num}]
 Do[i = 0;While[x[s,i] < n,
y[s,i+1] = If[x[s,i] > m,1,0];
z[i+1] = Random[];
x[s,i+1] = If[y[s,i+1] == 1,If[z[i+1] < p,x[s,i] + a,If[z[i+1] >= 1 - q,x[s,i] - b,x[s,i]]],x[s,i] + 1];
i++];t[s] = i,{s,1,num}]

For[s = 1,s ≤ 10,s++,Print[ListPlot[Table[{k,x[s,k]},{k,0,t[s]}],
PlotJoined → True,AxesOrigin → {0,m - 1}]]]
```



Histogram[Table[t[s],{s,1,num}]]



MeanTime = N[Mean[Table[t[s],{s,1,num}]]]

17.3776

Bibliography

1. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. I, 3rd Edition, John Wiley and Sons, 1968.

2. D. Neal, Generalized Boundary Problem For One-Dimensional Random Walks. *Mathematica in Education*, Vol. 3, No. 4, Fall 1994, p. 11 – 16.