COPYRIGHT AND CITATION CONSIDERATIONS FOR THIS THESIS/ DISSERTATION

- Attribution — You must give appropriate credit, provide a link to the license, and indicate if changes were made. You may do so in any reasonable manner, but not in any way that suggests the licensor endorses you or your use.

- NonCommercial — You may not use the material for commercial purposes.

- ShareAlike — If you remix, transform, or build upon the material, you must distribute your contributions under the same license as the original.

How to cite this thesis

Misconceptions and Resulting Errors Displayed by Grade 11 Learners in the Learning of Geometry

By

YEYISANI EVANS MAHUBELE

A dissertation submitted in fulfilment of the requirements for

THE MASTERS DEGREE IN MATHEMATICS EDUCATION

in the

Faculty of Education

Department of Mathematics, Science, Technology and Computer Education

at the

University of Johannesburg

Supervisor: Dr. Kakoma Luneta

2014
DECLARATION

I declare that the dissertation hereby handed in for the qualification Masters’ Degree in Mathematics Education at the University of Johannesburg is my own independent work and that I have not previously submitted the same work for a qualification at another university/faculty.

20 MAY 2015

MAKHUBELE Y.E

DATE

STUDENT NO: 909500179
ACKNOWLEDGEMENTS

My sincere thanks to all involved in the completion of this research report and in particular the following:

- My supervisor and mentor, Dr Kakoma Luneta for his motivation, guidance, constant support, professional advice and support throughout the study. Dr Luneta is thanked for the assistance and support that he gave me during the research process and the writing of this dissertation. I thank him especially for his expertise that he generously shared with me and ensuring that I complete my dissertation.

- Dr J.P. Makonye, my B.ED Honours degree supervisor at Wits University who laid the foundation by teaching me the methods and techniques of conducting a research.

- My special gratitude goes to my wife, Irene, for her continued support and encouragement when I was disillusioned and wanted to quit my M. ED. research degree. She is the brains behind the design of all the tables, figures and graphs used in this research.

- My daughter, Lusanda, is thanked for her understanding when me and her mother had to leave her behind for days to attend courses and conferences.

- This study would not be possible if it were not my mother, Thembi Ana Mathebula for the sincere care and love from infancy.

- The principal, educators and learners of the school where this study was undertaken.

- My fellow masters students through their constructive criticisms during our project presentations.

- My late father Naison Gavaza Makhubele and spiritual pastor Mavovo Senias Lebese. They passed away while I was still busy with my study.

- Finally, I want to thank the Almighty God for giving me the strength, wisdom and courage to persevere in the completion of this study. May the glory and honour be unto the Lord.
ABSTRACT

This research report explored the misconceptions and resulting errors displayed by grade 11 learners while learning geometry. The aim of this study was to establish the kind of misconceptions and the resulting errors learners display when learning Euclidean Geometry. This research was also aimed at identifying the causes of these misconceptions and errors and how to handle them.

The main question which this research sought to answer was: What types of misconceptions and the resulting errors do learners display in Euclidean Geometry? It also sought to answer this sub-question: What are the causes of these misconceptions and errors?

It was significant to conduct this research because misconceptions and errors are a big impediment to meaningful learning. They create great difficulties for learners.

The phenomenological research design was employed in this research. Phenomenological methods are particularly effective at bringing to the fore the experiences and perceptions of individuals from their own perspectives. In this study, 30 learners were sampled purposefully (10 high achievers, 10 average achievers, 10 low achievers). Data collection instruments used for this study were learners’ class workbooks, a test and interviews.

This research found a lot of misapplication of concepts. This was a result of learning concepts without understanding. This research also found that learners have problems in understanding the features and properties of shapes. They also experienced challenges with proof questions. Most learners were found to be operating at levels 1 and 2 of van Hiele.

This study recommends that misconceptions and errors should not be viewed from negative perspective, but should be treated as catalysts for the learning. They should be viewed as natural stage of conceptual development.
**TABLE OF CONTENTS**

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Declaration</td>
<td>ii</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>iii</td>
</tr>
<tr>
<td>Abstract</td>
<td>iv</td>
</tr>
<tr>
<td>List of tables</td>
<td>viii</td>
</tr>
<tr>
<td>List of figures</td>
<td>ix</td>
</tr>
<tr>
<td><strong>CHAPTER 1: INTRODUCTION</strong></td>
<td>1</td>
</tr>
<tr>
<td>1.1 Background</td>
<td>1</td>
</tr>
<tr>
<td>1.2 The research problem</td>
<td>5</td>
</tr>
<tr>
<td>1.3 Aims and objectives of this study</td>
<td>13</td>
</tr>
<tr>
<td>1.4 Significance of the study</td>
<td>13</td>
</tr>
<tr>
<td>1.5 Research questions</td>
<td>17</td>
</tr>
<tr>
<td>1.6 Definitions of key geometric concepts</td>
<td>17</td>
</tr>
<tr>
<td>1.7 Limitations of the study</td>
<td>19</td>
</tr>
<tr>
<td><strong>CHAPTER 2: THEORETICAL FRAMEWORK AND LITERATURE REVIEW</strong></td>
<td>20</td>
</tr>
<tr>
<td>2.1 THEORETICAL FRAMEWORK</td>
<td>20</td>
</tr>
<tr>
<td>2.1.1 Background of the South African curriculum</td>
<td>20</td>
</tr>
<tr>
<td>2.1.2 The theory that is promoted by the South African curriculum</td>
<td>20</td>
</tr>
<tr>
<td>2.1.3 The theory that underpins this research</td>
<td>22</td>
</tr>
<tr>
<td>2.1.4 What is constructivism</td>
<td>22</td>
</tr>
<tr>
<td>2.2 Literature review</td>
<td>27</td>
</tr>
<tr>
<td>2.2.1 The history and foundation of geometry</td>
<td>27</td>
</tr>
<tr>
<td>2.2.2 The importance of geometry</td>
<td>28</td>
</tr>
<tr>
<td>2.2.3 The teaching and learning situation (the didactic situation) of geometry</td>
<td>30</td>
</tr>
<tr>
<td>2.2.3.1 The constructivist geometry classroom</td>
<td>31</td>
</tr>
<tr>
<td>2.2.3.2 Constructivist geometry educators and their roles</td>
<td>32</td>
</tr>
<tr>
<td>2.2.3.3 A constructivist learner</td>
<td>33</td>
</tr>
<tr>
<td>2.2.4 Methods and theories for teaching geometry</td>
<td>34</td>
</tr>
<tr>
<td>2.2.4.1 The Harkness discussion method</td>
<td>34</td>
</tr>
<tr>
<td>2.2.4.2 The inquiry centred instruction</td>
<td>36</td>
</tr>
<tr>
<td>2.2.4.3 Collaboration among learners</td>
<td>36</td>
</tr>
<tr>
<td>2.2.4.4 The van Hiele model</td>
<td>37</td>
</tr>
<tr>
<td>2.2.5 Misconceptions and errors</td>
<td>42</td>
</tr>
</tbody>
</table>
2.2.5.1 Misconceptions 42
2.2.5.2 Errors 43
2.2.5.3 Link between misconceptions and errors 49
2.2.6 Identified misconceptions and errors in geometry by various researchers 50
2.2.6.1 Identification/Classification of basic shapes 50
2.2.6.2 Concepts and incorrect terminology 56
2.2.6.3 Markings on the diagram 58
2.2.7 Sources of misconceptions and errors 58
2.2.7.1 Curricular factor 59
2.2.7.2 Faulty reasoning 59
2.2.7.3 Prior knowledge 60
2.2.7.4 Knowledge acquisition 61
2.2.7.5 Procedural and conceptual knowledge 62
2.2.7.6 Faulty schema 65
2.2.7.7 Mathematical tasks 68
2.2.7.8 Negative attitude 69
2.2.7.9 Mathematical anxiety 70
2.2.7.10 Language 71
2.2.7.11 Educators 73
2.2.7.12 Learning concepts 76
2.2.8 Analysis of errors and misconceptions 77
2.2.9 Handling of errors and misconceptions 81
2.2.9.1 General handling 81
2.2.9.2 Handling learners’ errors and misconceptions in terms of a theory 83
2.2.10 Recognising mistakes: the better option 87
2.2.11 The confrontation of misconceptions and errors 91
2.2.12 Error feedback 92
2.2.13 Replacement of misconceptions and errors 93
2.2.14 How to get past errors, from an emotional perspective 96
2.2.15 The role played by errors 97
2.2.16 Learning from erroneous examples 102
2.2.17 Primary school geometry as a base for minimizing misconceptions and errors 103
CHAPTER 3: RESEARCH METHODOLOGY

3.1 Research paradigm
3.2 Research design
3.3 Population and sampling
3.4 Data collection instruments and procedures
3.4.1 Learners’ written work
3.4.2 Interviews
3.5 Data analysis procedures
3.6 Ethical considerations
3.7 Quality assurance strategies

CHAPTER 4: DATA ANALYSIS

4.1 Categorization of misconceptions and errors
4.2 Classwork and test discussions
4.2.1 Classwork
4.2.1.1 Question 1
4.2.1.2 Question 2
4.2.1.3 Question 3
4.2.1.4 Question 4
4.2.1.5 Total marks of learners’ performance in classwork and test
4.2.1.6 Analysis of learners’ classwork responses
4.2.2 Test
4.2.2.1 Question 1
4.2.2.2 Question 2
4.2.2.3 Question 3
4.2.2.4 Question 4
4.2.2.5 Question 5
4.2.2.6 Analysis of learners’ test responses

CHAPTER 5: DISCUSSION, RECOMMENDATIONS AND CONCLUSION

5.1 Discussion
5.1.1 Introduction
5.1.2 Research findings
5.2 Recommendations
5.3 Conclusion
REFERENCES 217
APPENDIXES 239
LIST OF TABLES
Table 1.1 Provincial comparison of percentage of grade 12 learners that obtained admission to further studies, 2011 to 2011 1
Table 1.2 Learners’ performance in mathematics by province and level of achievement from 2009 to 2011 2
Table 1.3 2012 Mpumalanga education district comparison of the pass rate in 11 most popular grade 12 subjects. 3
Table 1.4 Selected schools’ 2012 grade 12 Mathematics’ results per province 3
Table 1.5 Achievement in grade 9 Mathematics by province in 2012 9
Table 1.6 Seven key levels of achievement 9
Table 1.7 Percentage of grade 9 learners in achievement levels in Mathematics by province in 2012 10
Table 2.1 Radatz’s classification of errors 46
Table 2.2 Watson’s classification of errors 46
Table 2.3 Movshovitz-Hadar, Zaslavsky, and Inbar’s classification of errors 47
Table 2.4 Elbrink’s classification of errors 48
Table 2.5 Luneta and Makonye’s coding and categorizing of errors 48
Table 2.6 NS and SAS learners’ performance on the naming of shapes and stating of reasons 51
Table 2.7 Mean scores of Nigerian schools and South African schools on the TPGT per school for terminology associated with a circle 56
Table 2.8 Some of the learners’ incorrect spellings 57
Table 2.9 The first level’s schematic categories of errors 67
Table 2.10 Total number and percentage of different types of errors the Chinese and the Singapore students made in solving 11 problems 68
Table 2.11 Steps that describe the process for completing error pattern analysis 81
Table 4.1 Categorizing learners’ errors from C1 to C5 133
Table 4.2 Categorizing learners’ errors in terms of slips, knowledge based errors, misapplication errors and order of operations errors 133
Table 4.3 Categorizing learners’ responses in terms of Van Hiele Levels 134
Table 4.4 Cognitive levels 135
Table 4.5 Summary of learners’ marks per question 152
Table 4.6 Analysis of learners’ classwork responses per question in terms of the outlined codes and categories 153
Table 4.7 Analysis of learners’ test responses per question in terms of the outlined codes and categories 167
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Comparing South Africa &amp; Mpumalanga achievement in Mathematics, 2012. Gr. 1, 2, 3, 4, 5, 6 &amp; 9</td>
<td>8</td>
</tr>
<tr>
<td>1.2</td>
<td>Comparing Mpumalanga’s districts’ achievement in Mathematics, 2012</td>
<td>10</td>
</tr>
<tr>
<td>1.3</td>
<td>A diagrammatic representation of the key concepts of circle geometry</td>
<td>17</td>
</tr>
<tr>
<td>1.4</td>
<td>A diagrammatical illustration of the concept subtend</td>
<td>18</td>
</tr>
<tr>
<td>2.1</td>
<td>An example of learners sitting around an oval table in a typical Harkness discussion model</td>
<td>34</td>
</tr>
<tr>
<td>2.2</td>
<td>The van Hiele theory of geometric thought</td>
<td>37</td>
</tr>
<tr>
<td>2.3</td>
<td>Illustrating learners’ difficulty with identifying and naming shapes</td>
<td>52</td>
</tr>
<tr>
<td>2.4</td>
<td>A diagrammatic illustration of the properties of the various quadrilaterals</td>
<td>52</td>
</tr>
<tr>
<td>2.5</td>
<td>A diagrammatical illustration of the relationships of the various quadrilaterals</td>
<td>53</td>
</tr>
<tr>
<td>2.6</td>
<td>Quadrilaterals possibly not recognized when reasoning with a concept image and not a concept definitions</td>
<td>53</td>
</tr>
<tr>
<td>2.7</td>
<td>A square which may be recognized as a diamond or a kite</td>
<td>54</td>
</tr>
<tr>
<td>2.8</td>
<td>Figures which may not be recognized as right angles</td>
<td>54</td>
</tr>
<tr>
<td>2.9</td>
<td>Misconceptions concerning markings on the diagram</td>
<td>58</td>
</tr>
<tr>
<td>2.10</td>
<td>A typical learner response on the identification of 2-D shapes and 3-D objects</td>
<td>104</td>
</tr>
<tr>
<td>3.1</td>
<td>Components of data analysis: Flow model</td>
<td>123</td>
</tr>
<tr>
<td>3.2</td>
<td>Data analysis process as proposed by Ragpot</td>
<td>124</td>
</tr>
<tr>
<td>4.1</td>
<td>Coding process according to Creswell</td>
<td>132</td>
</tr>
<tr>
<td>4.2</td>
<td>Question 1.1 (classwork)</td>
<td>135</td>
</tr>
<tr>
<td>4.3</td>
<td>L1 response to 1.1 (classwork)</td>
<td>136</td>
</tr>
<tr>
<td>4.4</td>
<td>L9 response to 1.1 (classwork)</td>
<td>136</td>
</tr>
<tr>
<td>4.5</td>
<td>L7 response to 1.1 (classwork)</td>
<td>137</td>
</tr>
<tr>
<td>4.6</td>
<td>L16 response to 1.1 (classwork)</td>
<td>137</td>
</tr>
<tr>
<td>4.7</td>
<td>Question 1.2 (classwork)</td>
<td>137</td>
</tr>
<tr>
<td>4.8</td>
<td>L21 response to 1.2 (classwork)</td>
<td>138</td>
</tr>
<tr>
<td>4.9</td>
<td>L8 response to 1.2 (classwork)</td>
<td>138</td>
</tr>
<tr>
<td>4.10</td>
<td>L16 response to 1.2 (classwork)</td>
<td>138</td>
</tr>
<tr>
<td>4.11</td>
<td>Question 2 (classwork)</td>
<td>139</td>
</tr>
<tr>
<td>4.12</td>
<td>L5 response to 2.1 (classwork)</td>
<td>140</td>
</tr>
<tr>
<td>4.13</td>
<td>L14 response to 2.2 (classwork)</td>
<td>140</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>4.14</td>
<td>L1 response to 2.2 (classwork)</td>
<td>140</td>
</tr>
<tr>
<td>4.15</td>
<td>L15 response to 2.2 (classwork)</td>
<td>141</td>
</tr>
<tr>
<td>4.16</td>
<td>L12 response to 2.2 (classwork)</td>
<td>141</td>
</tr>
<tr>
<td>4.17</td>
<td>L6 response to 2.2 (classwork)</td>
<td>141</td>
</tr>
<tr>
<td>4.18</td>
<td>L27 response to 2.3 (classwork)</td>
<td>141</td>
</tr>
<tr>
<td>4.19</td>
<td>L19 response to 2.3 (classwork)</td>
<td>142</td>
</tr>
<tr>
<td>4.20</td>
<td>L16 response to 2.3 (classwork)</td>
<td>142</td>
</tr>
<tr>
<td>4.21</td>
<td>L13 response to 2.3 (classwork)</td>
<td>142</td>
</tr>
<tr>
<td>4.22</td>
<td>L1 response to 2.4 (classwork)</td>
<td>143</td>
</tr>
<tr>
<td>4.23</td>
<td>Question 3 (classwork)</td>
<td>143</td>
</tr>
<tr>
<td>4.24</td>
<td>L11 response to 3a (classwork)</td>
<td>144</td>
</tr>
<tr>
<td>4.25</td>
<td>L1 response to 3a (classwork)</td>
<td>144</td>
</tr>
<tr>
<td>4.26</td>
<td>L4 response to 3a (classwork)</td>
<td>144</td>
</tr>
<tr>
<td>4.27</td>
<td>L6 response to 3a (classwork)</td>
<td>144</td>
</tr>
<tr>
<td>4.28</td>
<td>L7 response to 3a (classwork)</td>
<td>145</td>
</tr>
<tr>
<td>4.29</td>
<td>L8 response to 3b (classwork)</td>
<td>145</td>
</tr>
<tr>
<td>4.30</td>
<td>L7 response to 3b (classwork)</td>
<td>145</td>
</tr>
<tr>
<td>4.31</td>
<td>L3 response to 3c (classwork)</td>
<td>146</td>
</tr>
<tr>
<td>4.32</td>
<td>L10 response to 3c (classwork)</td>
<td>146</td>
</tr>
<tr>
<td>4.33</td>
<td>L6 response to 3c (classwork)</td>
<td>146</td>
</tr>
<tr>
<td>4.34</td>
<td>Question 4 (classwork)</td>
<td>146</td>
</tr>
<tr>
<td>4.35</td>
<td>L15 response to 4a (classwork)</td>
<td>147</td>
</tr>
<tr>
<td>4.36</td>
<td>L6 response to 4a (classwork)</td>
<td>148</td>
</tr>
<tr>
<td>4.37</td>
<td>L27 response to 4b (classwork)</td>
<td>148</td>
</tr>
<tr>
<td>4.38</td>
<td>L7 response to 4b (classwork)</td>
<td>148</td>
</tr>
<tr>
<td>4.39</td>
<td>L4 response to 4b (classwork)</td>
<td>149</td>
</tr>
<tr>
<td>4.40</td>
<td>L6 response to 4b (classwork)</td>
<td>149</td>
</tr>
<tr>
<td>4.41</td>
<td>L2 response to 4b (classwork)</td>
<td>149</td>
</tr>
<tr>
<td>4.42</td>
<td>L26 response to 4c (classwork)</td>
<td>149</td>
</tr>
<tr>
<td>4.43</td>
<td>L12 response to 4c (classwork)</td>
<td>150</td>
</tr>
<tr>
<td>4.44</td>
<td>L9 response to 4c (classwork)</td>
<td>150</td>
</tr>
<tr>
<td>4.45</td>
<td>L15 response to 4a (classwork)</td>
<td>150</td>
</tr>
<tr>
<td>4.46</td>
<td>L23 response to 4b (classwork)</td>
<td>151</td>
</tr>
</tbody>
</table>
Figure 4.47  Question 1 (test)  154
Figure 4.48  L18 response to question 1 (test)  155
Figure 4.49  L4 response to question 1 (test)  155
Figure 4.50  L5 response to question 1 (test)  155
Figure 4.51  L8 response to question 1 (test)  156
Figure 4.52  Question 2 (test)  156
Figure 4.53  L17 response to question 2.1 (test)  157
Figure 4.54  L7 response to question 2.1 (test)  157
Figure 4.55  L16 response to question 2.2 (test)  158
Figure 4.56  L6 response to question 2.2 (test)  158
Figure 4.57  L16 response to question 2.3 (test)  158
Figure 4.58  L8 response to question 2.3 (test)  159
Figure 4.59  L7 response to question 2.3 (test)  159
Figure 4.60  Question 3 (test)  159
Figure 4.61  L9 response to question 3.1 (test)  160
Figure 4.62  L20 response to question 3.1 (test)  160
Figure 4.63  L18 response to question 3.2 (test)  161
Figure 4.64  L9 response to question 3.2 (test)  161
Figure 4.65  L9 response to question 3.3 (test)  161
Figure 4.66  L29 response to question 3.4 (test)  162
Figure 4.67  L11 response to question 3.4 (test)  162
Figure 4.68  L15 response to question 3.4 (test)  163
Figure 4.69  L4 response to question 3.4 (test)  163
Figure 4.70  Question 4 (test)  163
Figure 4.71  L4 response to question 4 (test)  164
Figure 4.72  L30 response to question 4 (test)  164
Figure 4.73  Question 1 (test)  165
Figure 4.74  L6 response to question 5.1 (test)  165
Figure 4.75  L30 response to question 5.1 (test)  166
Figure 4.76  L24 incorrect response to question 5.2 (test)  166
Figure 4.77  L30 response to question 5.2 (test)  166
Figure 4.78  L4 response to question 5.2 (test)  166
Figure 5.1  An illustration of how to prove that a line is a tangent using the tan-chord theorem  173
| Figure 5.2 | An illustration of how to prove that a quadrilateral is cyclic | 175 |
| Figure 5.3 | Possible factors involved in understanding a geometrical fact | 180 |
| Figure 5.4 | L1 response to 1.2 (classwork) | 181 |
| Figure 5.5 | Typical geometry problem involving tangent-chord theorem and a learner’s response to such a problem | 182 |
| Figure 5.6 | A typical proof which is presented to Grade 11 learners as a finished product | 184 |
| Figure 5.7 | L3 response to question 2 (test) | 186 |
| Figure 5.8 | L15 (incorrect awarding of marks) | 190 |
| Figure 5.9 | L19 (incorrect awarding of marks) | 190 |
| Figure 5.10 | L23 (incorrectly marked) | 191 |
| Figure 5.11 | L23 (wrong marking and incorrect awarding of marks) | 191 |
| Figure 5.12 | Number of errors made by learners in terms of slips, knowledge-based, conceptual, procedural and order of operations | 194 |
| Figure 5.13 | L1 response to question 1.1 (classwork) | 195 |
| Figure 5.14 | L8 response to question 2 (test) | 195 |
| Figure 5.15 | Number of errors made by learners in terms of statements and reasons | 196 |
| Figure 5.16 | Learners’ van Hiele’s levels | 198 |
| Figure 5.17 | An example of descriptive (a posterior) defining of a concept | 203 |
| Figure 5.18 | An example of constructive (a priori) defining of a concept | 203 |
| Figure 5.19 | Examples of disorientated figures | 205 |
| Figure 5.20 | A guided problem solving teaching model | 206 |
CHAPTER 1: INTRODUCTION

1.1 BACKGROUND

This study investigates the misconceptions and the errors associated with the learning of geometry. This research was conducted in grade 11 at a high school in Bushbuckridge (Bohlabela district, Manyeleti Circuit). Bohlabela is one of the four districts of Mpumalanga Province in South Africa, the other three being Ehlanzeni, Nkangala and Gert Sibande. Bohlabela is a district which is situated on the north-eastern side of Mpumalanga Province. It is mostly rural. The village where this school is built is therefore rural and underdeveloped. About 150 learners are doing mathematics and the rest mathematical literacy. Learners at this school are not doing well in mathematics. Generally Mpumalanga province has not been doing well in terms of grade 12 results. The analysis of the 2011 and 2012 grade 12 results in terms of certificates, diplomas and bachelors clearly shows that South Africa and of course Mpumalanga, where this research was conducted, is not producing enough learners who qualify to enroll for university courses. In 2012, only 19.8% of Mpumalanga learners achieved bachelors (see table 1.1)

Table 1.1: Provincial comparison of percentage of grade 12 learners that obtained admission to further studies, 2011 to 2012, (DBE, 2012f).

<table>
<thead>
<tr>
<th>Province</th>
<th>Higher Certificates</th>
<th>Diploma Studies</th>
<th>Bachelor Studies</th>
</tr>
</thead>
<tbody>
<tr>
<td>EC</td>
<td>18.5</td>
<td>18.8</td>
<td>23.8</td>
</tr>
<tr>
<td>FS</td>
<td>17.0</td>
<td>17.2</td>
<td>32.3</td>
</tr>
<tr>
<td>GP</td>
<td>13.3</td>
<td>13.8</td>
<td>32.5</td>
</tr>
<tr>
<td>KZN</td>
<td>17.5</td>
<td>16.7</td>
<td>28.0</td>
</tr>
<tr>
<td>LP</td>
<td>20.7</td>
<td>21.1</td>
<td>25.6</td>
</tr>
<tr>
<td>MP</td>
<td>18.8</td>
<td>20.1</td>
<td>27.4</td>
</tr>
<tr>
<td>NW</td>
<td>16.5</td>
<td>18.4</td>
<td>33.0</td>
</tr>
<tr>
<td>NC</td>
<td>20.5</td>
<td>20.4</td>
<td>28.4</td>
</tr>
<tr>
<td>WC</td>
<td>13.7</td>
<td>13.6</td>
<td>31.1</td>
</tr>
<tr>
<td>National</td>
<td>17.2</td>
<td>17.3</td>
<td>28.5</td>
</tr>
</tbody>
</table>

The general poor performance of learners is also displayed in Mathematics. Mpumalanga as a province has also not been doing well in mathematics. Statistics from 2009 to 2011 reflect poor performance in mathematics. This province is not producing quality mathematics results
needed for mathematical related careers. From 2009 to 2011 the average performance of learners in mathematics is below 50% (refer to table 1.2).

Table 1.2: Learners’ performance in mathematics by province and level of achievement from 2009 to 2011, (DBE, 2011b).

<table>
<thead>
<tr>
<th>Province</th>
<th>% achieved at 30% and above</th>
<th>% achieved at 40% and above</th>
</tr>
</thead>
<tbody>
<tr>
<td>EC</td>
<td>37.5</td>
<td>37.3</td>
</tr>
<tr>
<td>FS</td>
<td>53.3</td>
<td>48.4</td>
</tr>
<tr>
<td>GP</td>
<td>55.7</td>
<td>59.6</td>
</tr>
<tr>
<td>KZN</td>
<td>44.6</td>
<td>47.6</td>
</tr>
<tr>
<td>LP</td>
<td>39.2</td>
<td>39.6</td>
</tr>
<tr>
<td>MP</td>
<td>38.6</td>
<td>41.4</td>
</tr>
<tr>
<td>NW</td>
<td>52.0</td>
<td>53.4</td>
</tr>
<tr>
<td>NC</td>
<td>43.6</td>
<td>52.3</td>
</tr>
<tr>
<td>WC</td>
<td>64.9</td>
<td>66.0</td>
</tr>
<tr>
<td>NATIONAL</td>
<td>46.0</td>
<td>47.4</td>
</tr>
</tbody>
</table>

In 2012, Mathematics had the lowest average percentage pass when compared to the other subjects. The average percentage in mathematics nationally was 54%, and in Mpumalanga it was 53.1% which is just below the national average (DBE, 2012e).

Mpumalanga’s districts’ results of the 11 most popular Grade 12 subjects in 2012 show that Bohlabela district, where this research was conducted, has been the worst performing district in mathematics (see table 1.3). In 2012, Grade 12 learners in Bohlabela registered the lowest pass rate in 10 of the 11 most popular subjects. Less than 50 per cent of Grade 12 learners in Bohlabela achieved the pass rate of 30 per cent in Mathematics. This shows that half of Bohlabela learners got levels 1 and 2 which are not enough to be admitted for mathematically related courses like engineering and medicine.
Table 1.3: 2012 Mpumalanga education district comparison of the pass rate in 11 most popular Grade 12 subjects, (Mpumalanga Department of Basic Education, 2013).

<table>
<thead>
<tr>
<th>Subject</th>
<th>Bohlabela</th>
<th>Ehlanzeni</th>
<th>Gert Sibande</th>
<th>Nkangala</th>
<th>Mpumalanga</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accounting</td>
<td>43.4%</td>
<td>63.2%</td>
<td>52.9%</td>
<td>57.1%</td>
<td>54.9%</td>
</tr>
<tr>
<td>Agricultural Sciences</td>
<td>55.6%</td>
<td>69.2%</td>
<td>73.1%</td>
<td>80.4%</td>
<td>65.9%</td>
</tr>
<tr>
<td>Business Studies</td>
<td>63.8%</td>
<td>72.5%</td>
<td>62.7%</td>
<td>72.2%</td>
<td>68.1%</td>
</tr>
<tr>
<td>Economics</td>
<td>59.7%</td>
<td>64.3%</td>
<td>58.6%</td>
<td>56.0%</td>
<td>59.8%</td>
</tr>
<tr>
<td>English/First Additional</td>
<td>91.8%</td>
<td>97.2%</td>
<td>96.7%</td>
<td>99.5%</td>
<td>96.3%</td>
</tr>
<tr>
<td>History</td>
<td>63.4%</td>
<td>77.0%</td>
<td>81.4%</td>
<td>69.0%</td>
<td>71.3%</td>
</tr>
<tr>
<td>Life Sciences</td>
<td>57.4%</td>
<td>66.2%</td>
<td>67.0%</td>
<td>69.1%</td>
<td>65.3%</td>
</tr>
<tr>
<td>Mathematics Literacy</td>
<td>69.8%</td>
<td>86.8%</td>
<td>88.1%</td>
<td>94.6%</td>
<td>84.9%</td>
</tr>
<tr>
<td>Mathematics</td>
<td>36.9%</td>
<td>56.6%</td>
<td>59.2%</td>
<td>58.8%</td>
<td>53.1%</td>
</tr>
<tr>
<td>Physical Sciences</td>
<td>52.2%</td>
<td>67.0%</td>
<td>62.7%</td>
<td>69.0%</td>
<td>63.2%</td>
</tr>
</tbody>
</table>

This lower trend of Bohlabela’s learners’ performance in mathematics is the reflection of South African results as a whole where learners are not performing well in Mathematics, most especially in Euclidean geometry (Maree, Aldous, Hattingh, Swanepoel, & van der Linde, 2006; DBE, 2010; DBE, 2011b; DBE, 2012e; DBE, 2012f).

A closer look at individual schools reveals poor performance in mathematics. The Department of Basic Education 2012 National Senior Certificate Examination school subject report provides an analysis of results per school in the whole country. Results revealed many schools which achieved an average of less than 20% with some achieving as low as 1.6% in mathematics

Table 1.4: Selected schools’ 2012 grade 12 Mathematics’ results per province, (DBE, 2012e).

<table>
<thead>
<tr>
<th>SCHOOL</th>
<th>PROVINCE</th>
<th>SUBJECT</th>
<th>% achieved 30% and above</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elukhanyisweni College</td>
<td>Eastern Cape</td>
<td>1.6</td>
<td></td>
</tr>
<tr>
<td>Phehellang Senior Secondary</td>
<td>Free state</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Illinge Secondary</td>
<td>Gauteng</td>
<td>15.9</td>
<td></td>
</tr>
<tr>
<td>Siphuthando Public Combined</td>
<td>Kwazulu-Natal</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Botsikana Secondary</td>
<td>Limpopo</td>
<td>10.7</td>
<td></td>
</tr>
<tr>
<td>German Chiloane High</td>
<td>Mpumalanga</td>
<td>10.7</td>
<td></td>
</tr>
<tr>
<td>Lobang High</td>
<td>North-West</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Hoërskool Boesmanland</td>
<td>Northern Cape</td>
<td>7.1</td>
<td></td>
</tr>
<tr>
<td>Desmond Mpilo Tutu Secondary</td>
<td>Western Cape</td>
<td>10.3</td>
<td></td>
</tr>
</tbody>
</table>
Most schools in Bohlabela had not been teaching geometry. Now that geometry is compulsory, learners seem to have many misconceptions on geometrical concepts resulting in a variety of errors. Most educators who are teaching mathematics are not qualified to teach this subject. To try to address the situation, educators have been outsourced from neighbouring African countries as well as India.

The C2005 and the RNCS mathematics syllabus which was used in grades 10 –12, did little to advance the improvement teaching of Geometry. Learning Outcome number 3, Specific Outcome number 2 for grades 10, 11 and 12 were considered “optional mathematics assessment standards for examination” (DOE, 2003). This is a geometry section. By making geometry optional, the department stroke at the heart of mathematical teaching and the future careers which needs geometry. Consequently learners were given a choice of writing a geometry question paper, which was paper 3. According to Bowie (2009), because geometry was difficult for learners and educators, and that the grade 12 mathematics pass rate for most schools was below average, most schools chose not to write this section to boost their results. This negatively affected careers like engineering which requires the background knowledge of geometry.

Geometry is now a compulsory examinable section of mathematics in the FET Band (DBE, 2011a). According to the DBE (2011a, p. 12), geometry is to be written as a section of paper 2, with 20 marks out of 100 (20%) in grade 10, 40 out of 150 (26.7%) in both grades 11 and 12. This will indeed provide a big challenge for both educators and learners. Most educators had not been teaching this section since C2005 was introduced and learners in grade 12, and therefore have no firm background knowledge of this section. InMpumalanga province, workshops for educators that were and are still being conducted with regard to the newly launched Curriculum and Assessment Policy Statement (CAPS) do not focus on the content which will be a challenge for educators.

Various researchers and examination reports reveal that the failure rate in mathematics at schools in South Africa remains unacceptably high (Maree, Pretorius & Eiselen, 2003; Steyn & Maree, 2003; Maree, Aldous, Hattingh, Swanepoel, & van der Linde, 2006; DBE, 2010; DBE, 2011b; DBE, 2012e; DBE, 2012f). This is a clear indication that learners harbour misconceptions which results in errors, which in turn results in learners not performing well.
A research therefore needs to be done to contribute to the pool of knowledge on these misconceptions and errors as well as their causes and strategies to handle them. Our learners’ poor performance in mathematics is an indication that there are serious misconceptions and errors that need to be identified and exposed. This research was therefore conducted against the background of learners’ poor performance in mathematics, with a special focus on misconceptions and errors in geometry.

1.2 THE RESEARCH PROBLEM

Geometry has always been a difficult section for learners in mathematics. Cassim (2006), for example, commented on learners’ poor performance in Euclidean Geometry, indicating that learners’ performance in school geometry, especially in grade 12 has been inadequate. This is alluded to by the 2006 TIMSS report on learners’ poor performance in geometry which reported that South African learners performed worst in Mathematics out of 50 participating countries and that the weakest area of performance was geometry (Reddy 2005). This suggests that geometry education requires urgent attention and hence this study is timely.

Learners in the school where I teach are displaying misconceptions resulting in a variety of errors in geometry. This seems to be the general trend in South Africa. This is confirmed by various researchers who also did researches on geometry. De Villiers (1997) alluded to this fact: “South African learners are displaying misconceptions resulting in variety of errors in Geometry”. According to de Villiers (1997, p.42), “in South Africa, learners’ performance in Grade 12 geometry is far worse than in algebra”. In 2001, the study by Howie (2001, p.11) found that South African learners had “difficulty dealing with geometry questions, and in some cases were successfully distracted by questions testing misconceptions” in geometry. The study by Roux (2003, p.362) also confirms that high school learners’ mathematical performance in South Africa appears to be unimpressive, more especially in geometry. Roux (2003) found that learners’ performance is poorer when it comes to items involving understanding of features and properties of shapes. Mathematics educators have therefore expressed concern over learners’ dislike for geometry and their inability to comprehend the deductive logical system of the subject (de Villiers, 1997).

In South Africa, Geometry has been an optional section of mathematics in the FET Band. Making geometry an optional subject was one of the worst educational blunders made by the
education authorities. This is because geometry, according to Blumenfed (2002), is the most useful of all the sciences. Blumenfed argues that geometry is an important section of mathematics that makes one’s eye quicker in seeing things, and one’s head steadier in doing things. A person can draw better, write better, cut out clothes, make boots and shoes, work at any mechanical trade, or earn any art. By making geometry an optional subject, the department is opening a gap for geometry not to be done and thereby depriving learners of these essential skills. The Department of Education had been suggesting to universities that, especially for engineering, they should make paper 3 a part of their entrance criteria (Kearsley, in ASSAF, 2009). The problem with this is that the majority of schools did not teach the work examined in paper 3 at all (Roux, 2003). This is because paper 3 was optional. This was a challenge for universities because this meant that students who were pursuing careers that required a background knowledge of geometry, like engineering courses, had no geometry knowledge at all. For example, Prof. Kearsley (in ASSAF, 2009) indicated that of the 42 060 candidates that obtained 65% or more for mathematics in 2008, 64% did not write paper 3. She expressed the concern that schools that in the past achieved 100% pass rates had chosen not to offer tuition for mathematics paper 3. Prof. Kearsley (in ASSAF, 2009) revealed that a survey which was done among the approximately 1 700 first-year engineering students at the University of Pretoria to establish how many of them had written mathematics paper 3, found that only 18% had done so. This simply means that if the university insisted that learners only be admitted to engineering if they had written mathematics paper 3, they would have admitted only 18% of our current first year class. This would indeed be doing South Africa a disservice (Bowie, 2009). Most universities are therefore forced to make the intervention of presenting a six week course in geometry for first-year students.

Fransman (2009), the then chairman of the Committee on Higher Education and Training in Parliament, indicated that we need an education and skills revolution in South Africa. The Labour Force Survey (Statistics South Africa, 2006) found that almost 50% of people in the age group 15 to 24 had no jobs and were not currently in the education system. Statistics South Africa stated that for the second quarter of 2009, 71.4% of 15- to 24- year-olds were economically inactive (Statistics South Africa, 2009). The Minister of Higher Education and Training, Dr. Blade Nzimande, indicated that about 2.8 million 18 to 24 year old South Africans were neither employed nor in education or training (Serrao, 2009). The aim of OBE was to provide the average school-leaver with the skills required to earn a living. Geometry is therefore important in this respect because knowledge of geometry is an essential life skill
and should be linked to practical applications to ensure that learners and educators appreciate why and where theorems can be used.

Making geometry a non-compulsory section of the mathematics curriculum was totally not a good idea. Learners are denied the problem-solving thinking process that is developed through solving geometry problems that are essential in developing innovative engineers and scientists. Professor Kearsley compares the making of geometry optional to taking an important life skill from learners (Kearsley, in ASSAF, 2009). In the discussion during the proceedings of an Academy of Science of South Africa Form (ASSAF), Addler (in ASSAF, 2009) indicated his concern was whether South Africa clearly understood what was lost by making geometry an optional section of mathematics. He argued that in the past, high school mathematics learners got most of their problem-solving experience in geometry.

According to Addler (in ASSAF, 2009), anything in a curriculum that is optional causes confusion among educators and schools, who consider how they can best make use of the optional component. Those that wish to maintain high marks simply cut geometry out in their syllabus to achieve high pass rate in mathematics. It is therefore stimulating that South Africa has realized the need for correcting its past mistakes. Geometry is back. The decision to make geometry an optional section has been reversed. From 2012, this section is compulsory in grade 10, then in grade 11 in 2013 and finally in grade 12 in 2014. In 2014 it will therefore be compulsory for all schools in South Africa to offer Geometry and write this section as part of paper 2 in Mathematics. Most educators have not been teaching this section for quite some time since the time it was made optional. This section will therefore pose serious challenges for both educators and learners which will result in misconceptions leading to errors. In the circuit where I am currently teaching, most schools were not teaching this section. Educators view geometry as difficult and will consequently lower their mathematics grade 12 results. A closer look at the schools which teach geometry reveals that learners are displaying misconceptions and errors. This is a general trend in the whole country where learners’ performance in geometry is generally poor (Maree, Aldous, Hattingh, Swanepoel, & van der Linde, (2006); DBE, 2010; DBE, 2011b; DBE, 2011d; DBE, 2012e; DBE, 2012f). Learners seem to have many misconceptions on geometrical concepts resulting in a variety of errors. The complex and abstract nature of geometry makes the learning of this section of mathematics difficult for learners. This will result in misconceptions and errors in geometry.
The 1981 de Lange report Howie (2003), revealed that many of the common errors made in answering mathematics can be traced back to misconceptions regarding work studied in the GET phase or earlier. The 2012 ANA results confirm the results of this report (see figure 1.1). These results show a worrying decreasing trend from grade 1 to grade 9. The results show that grade 9, which is the last grade of the GET Band, and of course the transit and feeder grade for FET Band achieved a mere 13% average in the 2012 ANA results (figure 1.1). This could be the reason why learners are displaying misconceptions and errors in grade 11 geometry.

![Figure 1.1: Comparing South Africa & Mpumalanga achievement in Mathematics, 2012. Gr. 1, 2, 3, 4, 5, 6 & 9, (DBE, 2012c).](image)

Mpumalanga province, where this research was conducted, achieved lower average percentage marks in mathematics across all grades when compared to South Africa (figure 1.1). The average percentage mark dropped progressively up to Grade 5 after which an increase, both nationally and provincially, is noticeable. When compared with other provinces, Mpumalanga ranked in 7th position for all grades with the exception of 5th position for Grade 2 and 6th position for Grade 9.

In terms of quality, the ANA results are not good. The national South Africa average mark for mathematics is 13%. 2.3% of the learners achieved an average mark of 50% or more (refer to table 1.5 below). The average mark for Mpumalanga is 11.9%. 1% achieved 50% or
more. This clearly indicates why learners are having misconceptions in mathematics in senior classes.

Table 1.5: Achievement in grade 9 Mathematics by province in 2012, (DBE, 2012d).

<table>
<thead>
<tr>
<th>Province</th>
<th>Average% mark</th>
<th>% Learners achieving 50% or more</th>
</tr>
</thead>
<tbody>
<tr>
<td>EC</td>
<td>14.6</td>
<td>2.6</td>
</tr>
<tr>
<td>FS</td>
<td>14.0</td>
<td>3.1</td>
</tr>
<tr>
<td>GP</td>
<td>14.7</td>
<td>3.7</td>
</tr>
<tr>
<td>KZN</td>
<td>12.0</td>
<td>1.9</td>
</tr>
<tr>
<td>LP</td>
<td>8.5</td>
<td>0.5</td>
</tr>
<tr>
<td>MP</td>
<td>11.9</td>
<td>1.0</td>
</tr>
<tr>
<td>NC</td>
<td>13.2</td>
<td>2.0</td>
</tr>
<tr>
<td>NW</td>
<td>11.2</td>
<td>1.4</td>
</tr>
<tr>
<td>WC</td>
<td>16.7</td>
<td>5.0</td>
</tr>
<tr>
<td>National</td>
<td>12.7</td>
<td>2.3</td>
</tr>
</tbody>
</table>

Learner achievement in South Africa is expressed in terms of the seven levels of achievement as shown below:

Table 1.6: Seven key levels of achievement, (DBE, 2012d).

<table>
<thead>
<tr>
<th>Rating code</th>
<th>Percentage</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1</td>
<td>0–29</td>
<td>Not achieved</td>
</tr>
<tr>
<td>Level 2</td>
<td>30–39</td>
<td>Elementary achievement</td>
</tr>
<tr>
<td>Level 3</td>
<td>40–49</td>
<td>Moderate achievement</td>
</tr>
<tr>
<td>Level 4</td>
<td>50–59</td>
<td>Adequate achievement</td>
</tr>
<tr>
<td>Level 5</td>
<td>60–69</td>
<td>Substantial achievement</td>
</tr>
<tr>
<td>Level 6</td>
<td>70–79</td>
<td>Meritorious achievement</td>
</tr>
<tr>
<td>Level 7</td>
<td>80–100</td>
<td>Outstanding achievement</td>
</tr>
</tbody>
</table>

The table below shows the percentage of grade 9 learners in achievement levels in Mathematics by province in 2012. It clearly shows that learners are getting marks below 29%, which is Level 1. About three percent of learners are in the highest category of achievement and 43 percent in the lowest. Less than one percent of Grade 9 learners are in the highest category of achievement and 92 percent in the lowest. Less than one percent of Grade 9 learners are in the highest category of achievement and 92 percent in the lowest. The achievement in levels 1 and 2 is a clear indication that learners are having challenges in mathematics, which are likely to cause misconceptions and errors.
Table 1.7: Percentage of grade 9 learners in achievement levels in Mathematics by province in 2012, (DBE, 2012d).

<table>
<thead>
<tr>
<th>Province</th>
<th>L1</th>
<th>L2</th>
<th>L3</th>
<th>L4</th>
<th>L5</th>
<th>L6</th>
<th>L7</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>97.9</td>
<td>1.2</td>
<td>0.5</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
<td>0.0</td>
</tr>
<tr>
<td>MP</td>
<td>95.0</td>
<td>2.7</td>
<td>1.3</td>
<td>0.6</td>
<td>0.3</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>NW</td>
<td>94.7</td>
<td>2.7</td>
<td>1.2</td>
<td>0.7</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>NC</td>
<td>93.3</td>
<td>3.2</td>
<td>1.5</td>
<td>1.0</td>
<td>0.5</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>KZN</td>
<td>92.4</td>
<td>3.6</td>
<td>2.0</td>
<td>1.0</td>
<td>0.5</td>
<td>0.3</td>
<td>0.1</td>
</tr>
<tr>
<td>FS</td>
<td>90.6</td>
<td>4.1</td>
<td>2.3</td>
<td>1.5</td>
<td>0.9</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>EC</td>
<td>88.3</td>
<td>5.8</td>
<td>3.4</td>
<td>1.5</td>
<td>0.7</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>GP</td>
<td>88.1</td>
<td>5.3</td>
<td>3.0</td>
<td>1.7</td>
<td>1.1</td>
<td>0.6</td>
<td>0.3</td>
</tr>
<tr>
<td>WC</td>
<td>86.1</td>
<td>5.7</td>
<td>3.3</td>
<td>2.2</td>
<td>1.4</td>
<td>0.9</td>
<td>0.4</td>
</tr>
<tr>
<td>National</td>
<td>91.9</td>
<td>3.8</td>
<td>2.1</td>
<td>1.1</td>
<td>0.6</td>
<td>0.3</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Unfortunately this poor performance is reflected in the lower grades as shown by figure 1.2. The 2012 ANA Mathematics results for Mpumalanga’s four education districts are presented in Figure 1.5 below. Ehlanzeni attained the highest average percentage mark for Grade 3 and Grade 6, whereas Nkangala achieved the highest mark for Grade 9. Bohlabela, where this research was conducted, registered lower marks than Mpumalanga’s average marks in all three Grades. The most worrying aspect is the grade 9 results because these are the learners who will be doing mathematics in grade 10 in 2013.

Figure 1.2: Comparing Mpumalanga’s districts’ achievement in Mathematics, 2012, (DBE, 2012c).
The poor performance by learners is not only in mathematics, but also in the home language and the first additional language (DBE, 2012d). This is very much worrying because learners need a good command and understanding of language to perform in mathematics. Thus many of the errors made by learners in mathematics have their origins in a poor understanding of the basics and foundational competencies taught in the earlier grades. Many learners struggle with concepts in the curriculum that requires deeper conceptual understanding, which should have been mastered in the earlier grades.

Learners are therefore making lots of errors in mathematics as a result of misconceptions that they hold. Mathematical errors are a world-wide phenomenon. Learners of any age, any country, any era, irrespective of their performance in mathematics, have experienced getting mathematics wrong. This has inspired researchers to do research on misconceptions and errors. This interest resulted in the formation of many theories about the nature of mathematical errors, their interpretation and the ways of overcoming them. In one psychological approach the errors were initially conceptualized negatively: an error was considered a digression, the result of confusion and it should be avoided (Rouche, 1988). A reversal of the traditional view on errors is found in the work of Piaget and in the Geneva School (van de Walle, 2007). For the first time, errors were viewed positively since they allow the tracing of a reasoning mechanism adopted by the learner.

The reasons for the poor performance of learners in mathematics were explored by various researchers like Stacey & Macgregor, (2000) and Olivier (1989). In terms of South Africa, Howie (2002) indicates that international benchmark studies, such as the Third International Maths and Science Study (TIMSS) clearly indicated a systemic problem linked to the consistent weak performance of South African participants which causes mathematical errors. The reasons are broadly grouped into issues linked to language of tuition, learner specific socio-economic situations and linked to educators’ attitude, level of qualification and competence (Howie, 2002).

Research indicates that learners who spoke either English or Afrikaans at home achieved higher scores than those who did not (Ndlovu, 1989). Most South African learners are second language speakers of English which is used as a medium of instruction. The main focus of this research is not a language issue, but language may be a contributing factor to learners’ misconceptions. The National 2011 and 2012 Home Language, First Additional
Language results for grades 4, 5, 6 and 9 are not good. The average percentage for all the grades is below 44% (DBE, 2012d). This does not augur well for mathematics in the FET Band since mathematics depends on language for meaningful teaching and learning.

The recurring poor performance of our learners in mathematics over the past years is a clear indication that learners are having problems, and that there are serious problems and gaps in our education system that may result in learners’ misconception which might lead to errors. This is the backbone of this research. The research problem that is driving my research is the grade 11 learners who often display misconceptions when engaging with geometry tasks. According to Fricke (2008, p. 65) South Africa has a shortage of learners matriculating with mathematics and science marks that qualify them for further study in Science, Technology, Engineering and Maths (STEM). According to him, international benchmark studies confirm that school mathematics and science in South Africa are weak and suffer from systemic problems. Howie (2003, p. 1) gives a similar report when he writes; “International measures indicate that South African learners are performing poorly in mathematics. For example, of the 38 and 50 countries that participated in the Trends in Mathematics and Science Study (TIMSS) in 2001 and 2003, respectively, some of which are developing countries, South African learners came last in Mathematics and Science”. According to Reddy (2005, p. 125) this is huge South African problem because competency in these gateway subjects at a school level, opens up opportunities for empowerment through an understanding of common technologies, and provides better access to tertiary education and higher skilled jobs and livelihoods. The National Council of Teachers of Mathematics (NCTM, 2000), international benchmark studies confirm that school education of mathematics in South Africa is weak and suffers from systemic problems. The number of students in civil engineering has declined steadily in South Africa for the past decade. All possible help to promote mathematics is needed to attract potential learners to engineering and other studies to address the shortage of these mathematically related careers. This makes the study of misconceptions and errors in geometry a significant step towards addressing this problem. This is what made it necessary for this research to be done.
1.3 AIMS AND OBJECTIVES OF THIS STUDY

This study aims to:

- establish the kind of misconceptions and errors learners display when learning Euclidean Geometry.
- identify the causes of these misconceptions and errors.

This study is done because learners are performing badly in Geometry as a result of these errors and misconceptions. Learners find it difficult to apply the knowledge of angles and theorems in solving riders. Proofs of problem in Geometry have to be written following the logical deductive system. This enables a learner to write a well structured and coherent proof. This research is therefore done specifically to help students develop logical deductive thinking skills which will enable them to improve their performance in geometry.

1.4 SIGNIFICANCE OF THE STUDY

“if students’ flawed mathematical reasoning and thinking are not deconstructed, unpacked and unveiled to both teachers and students for reconsideration and reflection, then little progress in learning will result, because teaching will continue to be misdirected away from the most important challenges facing students” (Makonye, 2011, p. 18).

This quotation summarizes the significance of this research. It is important for mathematicians to study and understand learners’ misconceptions and errors as this has a potential of improving learners’ performance in mathematics. Teaching becomes irrelevant if it does not correct learners’ misconceptions and errors. According to Nesher (1987), a single misconception produces a cluster of errors since mathematical errors ideas are hierarchical and closely connected. Misconceptions are therefore like bacteria which multiply rapidly and can be fatal if not treated early. This makes the study of misconceptions and errors extremely important in mathematics, more especially in geometry, considered to be one of the most difficult sections of mathematics. According to Makonye (2011), analysis of errors provides teachers with insight regarding the learners’ procedural and conceptual misunderstanding. These errors can sometimes be more informative to teachers as errors often provide insight into learners’ misunderstandings about a particular mathematics concept or skill.
Many educators in schools have tried different teaching methods and programs to make learners understand geometry, sometimes with success and sometimes not. It is thus significant to explore the reasons why learners are having serious misconceptions which result in errors and what could educators possibly do to help the situation. Geometry is an important section of mathematics. According to NCTM (2000), Geometry helps us represent and describe in an orderly manner the world in which we live. Educators could give their learners an edge in the competitive global economy by strengthening their geometry knowledge. It is therefore significant for this research to be done as it will improve the performance of our learners in geometry. It would then be easier for them to pursue careers in engineering and other important fields which require mathematics, a subject in which most of our learners are not performing well.

The problem that we are experiencing in South Africa is that learners are not performing well in geometry (Atebe & Schäfer, 2011). This has forced the government to make geometry optional in the FET Band, in order to boost grade 12 mathematical results. Atebe & Schäfer (2011) report that as learners progresses through the school system, it seems that geometry becomes a problematic section of mathematics. The effects of inadequate geometry teaching had been documented in the TIMSS 1999 report (TIMMS, 1999). This report revealed that South African learners fared poorly in mathematics. High school learners struggled to understand geometry concepts. Many learners are therefore having geometry misconceptions resulting in errors.

Shannon (2002) argues that misconceptions are a big impediment to meaningful learning. Setati (2002) is of the same opinion by arguing that mistakes create great difficulties for learners. According to Confrey (1987), misconceptions results in mathematics being viewed as difficult and dull. This generates in learners a negative attitude towards mathematics. This is the reason why Shaughnessy & Burger (1985) associated mistakes with negative feelings. The study by Mestre (1989) found that learners are emotionally and intellectually attached to their misconceptions, because they have actively constructed them. Misconceptions and errors results in the emotional disposition of a set of emotions like fear, anxiety, frustration, rage. This threatens both performance and participation in mathematics and this makes studies in misconceptions and errors of extreme importance. It is for this reason that misconceptions and errors should be studied because they develop in learners, a negative attitude towards mathematics.
It is important for this study to be conducted because Geometry is now compulsory in grade 10 to 12. For the first time it will be written by all learners as a section of paper 2 with approximately 40 marks. This will be a challenge for mathematics educators who had not been teaching this section of mathematics. The identification of misconceptions and errors, as well as their causes will go down a long way in closing the gap on how to best to use these misconception and errors to help learners achieve good results in mathematics. The outcome of this research could benefit the socio-economic dispensation of the country by contributing to an increase in the number of learners entering into science and mathematics based courses at university level and subsequently an increase in the number of graduates in mathematics and science fields, as well as an increase in the number and quality of mathematics educators. The outcome of this research has the possibility of contributing to the upliftment of the overall standard of mathematics in this country. Knowledge gained from this study may contribute to alleviating the mathematics crisis experienced in the country.

Lastly the report presented by Professor Chisholm to the minister of basic education identifies poor educational outcomes as the second biggest challenge for South African development, following unemployment (DBE, 2012b). The department is well aware of the challenges in education: international and regional studies, as well as its own Annual National Assessments (ANA) all confirm that South African learners are performing well below par, and well below their own talents and abilities (DBE, 2012b, and refer to tables 1.2, 1.3, 1.4, 1.5, 1.7, and figures 1.1, 1.2, ). The department is also well aware that potentials of learners are not being realized, talents are not being developed and abilities are not being stretched. The department reveals that the reasons and causes are constantly researched and probed and the results inform specific strategies developed. This research is therefore worth carrying out as it will assist the department in addressing these problems and will hopefully; help to turn this situation around.

According to NCTM (2000, p. 50), those who understand and can do mathematics, in this changing world, will have significantly enhanced opportunities and options for shaping their futures. Mathematical competence opens doors to productive futures. A lack of mathematical competence keeps those doors closed. All learners should have the opportunity and the support necessary to learn significant mathematics withy depth and understanding. The study of misconceptions and errors will go a long way in addressing this issue.
Major employers in the engineering, construction, pharmaceutical, financial and retail sectors all need people with appropriate geometry skills. The bottom line is that South Africa is not producing enough grade 12 learners with Maths and Science to meet the needs of our growing economy or producing enough skilled professionals, with technical skills founded in Maths and Science, to meet the staffing requirements of industry, commerce, health and education. The role of the South African government is to create million jobs and service delivery of its people. However, without educated and skilled people to fill these positions, this is not a formula that will drive prosperity for our country. South Africa’s future success depends on greatly improved education for all South Africa’s future success depends on greatly improved mathematical education for all our young people.

Mathematics and science are key areas of knowledge and competence for the development of an individual, and the social and economic development of South Africa in a globalizing world (Reddy, 2005, p. 125). Since 1994, the new democratic government in South Africa has emphasized the centrality of mathematics and science as part of the human development strategy for South Africa. Performance in this area is one of the indicators of the health of the South African educational system. It makes an important contribution to the economy and has been a contributor to inequalities of access and income (Reddy, 2005, p. 125). A greater number of mathematics graduates results in a more skilled, and therefore a more productive workforce, which in turn contributes to an internationally more competitive nation and redressing the balance of trade problems. Mathematics is considered to be among the requirements for creating wealth and improving the quality of life. If mathematics is considered to be among the requirements for creating wealth and improving the quality of life, the importance of quality mathematics education is obvious. The low success rate of grade 12 learners in mathematics has a negative influence on the future prospects of learners who cannot successfully pursue careers of their choice. The poor grades achieved in this subject discourage learners from pursuing key careers that are critical in contributing to the well-being of the country. The lack of fundamental professionals such as teachers, doctors, scientists, and other scientifically oriented professionals is also impeding the growth of the economy.

Learners who set their hopes on entering fields of study that require mathematics are being discouraged because they are failing mathematics at grade 12 level thus disallowing them to pursue careers of their choice. This indeed makes this study an important endeavour to take.
1.5 RESEARCH QUESTIONS

This research seeks to answer the following main question:

❖ What types of misconceptions and errors do learners have on Euclidean geometry?

It also seeks to answer the following sub-question:

❖ What are the causes of these misconceptions and errors?

1.6 DEFINITIONS OF KEY GEOMETRIC CONCEPTS

This research focuses on circle geometry as done in grade 11 in terms of the Curriculum and Assessment Policy (CAPS). Below is a diagrammatic representation of the key concepts which learners are expected to know.

![Diagram of circle geometry concepts](attachment:image.png)

**FIGURE 1.3:** A diagrammatic representation of the key concepts of circle geometry, (Aird, & Van Duyn, 2012).

**Chord:** a chord is a straight line through the centre of the circle that joins two points on the circumference of a circle (see figure 1.3 above)

**Diameter:** a diameter is a chord that passes through the centre of a circle (see figure 1.3 above).

**Radius:** a radius is a line from the centre of the circle to a point on the circumference. It is half the length of the diameter. Two or more radius are called radii (see figure 1.10 above).

**Tangent:** a tangent is a straight line that touches an outside part of a circle at one point only (See figure 1.3 above).
**Arc:** An arc is a portion on the circumference of the circle. It is a set of two points that lie in the on the circumference of the circle (see figure 1.3 above).

**Centre:** in circle geometry, centre means the middle point of a circle (see figure 1.6 above).

**Segment:** a segment is the region bounded by a chord and the arc subtended by the chord (see figure 1.3 above).

**A circle:** a circle is a plane figure whose boundary points are equidistant from a fixed point. It is simple closed curve which divides the plane into two regions, an interior and an exterior. Half of a circle is called a semi-circle (see figure 1.3 above)

**Quadrilateral:** a quadrilateral is a polygon with four sides and four vertices or corners. It can also be defined as a four-sided polygon with four angles (Soni, & Cronje, 2011)

**Cyclic quadrilateral: A quadrilateral** is said to be cyclic if all four of its vertices lie on the circumference of a circle (Aird, & Van Duyn, 2012).

**Perpendicular Lines:** In geometry, two lines are considered perpendicular to each other if they are at right angles to each other.

**Parallel lines:** In geometry lines are parallel if they are always the same distance apart and will never meet, and they point in the same direction. In geometric figures, where there are parallel lines, there are always corresponding angles, alternate angles and co-interior angles.

**Subtend:** Subtend is closely related to the words hanging and support. For example in Figure 1.4, the minor arc AB (or chord AB) subtends $\angle O_1$ at the centre O and $\angle C$ at point C on the circumference. It is like AB is supporting/hanging $\angle O_1$ at the centre or and $\angle C$ at the circumference of the circle. In other words $\angle O_1$ and $\angle C$ stand on chord AB

![Diagram of subtend](image)

**Figure 1.4: A diagrammatical illustration of the concept subtend (Aird, & Van Duyn, 2012).**

**Theorem:** the word theorem is derived from the Greek word “theoria”, which means “vision” (Mariott, Bartolini Bussi, Boero, Ferri & Garuti, 1997). A mathematical theorem consists of a system of statement, proof and theory.
1.7 LIMITATIONS OF THE STUDY

The population and sample were selected from only one high school in Manyeleti circuit in Bushbuckridge. The problem under investigation focuses more on learners and not educators. Only grade 11 learners were selected. This is a low performing school and the results and findings will therefore not be generalizable to high performing schools, but can only be attributed to low performing schools. Also the external validity of the findings could be questioned because of a small sample from only one school.

The fact that data was collected using learners’ written work without going into the classroom was a limiting factor. A more balanced technique would have been to observe teachers teaching to have a clearer insight into teachers’ classroom practices, their preparations, teaching strategies and the way they handle learners’ misconceptions and errors and how feedback is given to learners.
CHAPTER 2: THEORETICAL FRAMEWORK AND LITERATURE REVIEW

The chapter begins the description of the theoretical framework that guided this research. This is followed by a literature review, which hinges mainly on misconceptions and errors in Euclidean geometry.

2.1 THEORETICAL FRAMEWORK

2.1.1 Background of the South African curriculum

The National Curriculum Statement for Grades R - 12 aims to produce learners that are able “to identify and solve problems and make decisions using critical and creative thinking; work effectively as individuals and with others as members of a team; organise and manage themselves and their activities responsibly and effectively; collect, analyse, organise and critically evaluate information; communicate effectively using visual, symbolic and/or language skills in various modes” (DBE, 2011a, p. 3). It is for this reason that this curriculum statement articulates that teaching should not only be limited to the how of questions, but should also feature the when and why of problem types. Problem solving and cognitive development are therefore central to all mathematics teaching. Collaboration and working together are the recommended ways of learning. Learners need to spend time with each other where they are allowed to talk with one another, either in a group of two to five learners.

2.1.2 The theory that is promoted by the South African curriculum

The principles and aims of the South African curriculum stated in 2.1.1 above support and concur with the philosophy of constructivism. The learning content emphasizes the importance of using a range of teaching and learning strategies in a variety of contexts, which is also promoted by constructivism. These characteristics of education, as interpreted in the South African context, are underpinned by the principles of constructivism.
The philosophy underlying this curriculum is therefore based on a socio-constructivist theory. The learner is not viewed as a passive receiver of knowledge, an empty vessel into which the facilitator must pour knowledge. Rather, the learner is viewed as an active participant who constructs his/her own knowledge. The learner comes to the learning situation with his own existing knowledge; new ideas are understood and interpreted in the light of the learner's existing knowledge, built up out of his previous experience. In terms of our curriculum, learners have work effectively as individuals and with others as members of a team. This is a social process learning that is promoted by constructivism. Learners learn from each other (and the facilitator) through discussion, communication and sharing of ideas, by actively comparing different ideas, reflecting on their own thinking and trying to understand other people's thinking by negotiating a shared meaning.

According to Sapiere and Mays (2008), in South Africa, the constructivist theory of mathematics learning has been strongly supported by researchers, by educators and by the education department. The problem-centred approach of Curriculum 2005, NCS and the newly revised curriculum, CAPS, which is dubbed Action Plan 2014, are all based on constructivist principles. The new CAPS therefore promote a constructivist approach to teaching and learning mathematics.

Current mathematics education reforms supporting a constructivist perspective suggest that the automation of skills and passive intellectual involvement should be replaced by active learning processes (Hiebert, 1992). According to Hiebert (1992), active learning denotes learning activities in which learners are given considerable autonomy and control of the direction of the learning activities. Learning activities commonly identified in this manner include investigational work, problem solving, small group work, collaborative learning and experiential learning. CAPS promote this type of learning. According to DBE (2011a), mathematics is most effectively learned through learners’ active participation in mathematical situations, rather than through passive acceptance and repetition of knowledge. Outcomes Based Education (OBE) formed the foundation of the new curriculum in South Africa. The outcomes encouraged a learner-centred and activity based approach to education. Such form of learning is also encouraged by the revised Curriculum and Assessment Policy Statement. Inquiry based teaching and learning which is underpinned by constructivism is therefore greatly promoted by the South African National Curriculum.
2.1.3 The theory that underpins this research

This study is informed by the theory of constructivism. This is because the national curriculum is based on the theory of constructivism. Constructivism derives from the cognitive school of psychology and the theories of Piaget and Vygotsky. According to Ernest (2007), constructivism “is a theory of knowledge with roots in philosophy, psychology and cybernetics”. Cybernetics simply denotes it is a science of communication. Ernest calls this theory a new philosophy of Mathematics (Ernest, 2007). According to van de Walle (2007, p. 22) the basic tenet of constructivism is simply that “children construct their own knowledge”. They construct or give meaning to things they perceive or think. The verb construct implies that the mental structures (schemas) the child ultimately possesses are built up gradually from separate components in a manner initially different from that of an adult (Njisane, 1992).

According to Dikgomo (1994, p. 18), a number of researchers “have appealed to the social domain of constructivism to be considered indispensable to the learning of Mathematics”. Njisane (1992) highlighted the social construction of meaning as a significant development of mathematics education. This has a bearing in understanding misconception and errors of learners in geometry. Learners are social beings and their misconceptions and errors can be traced from their influence and interactions with their educators and adults. Learners naturally tend to consult each other and work together. They prefer to ask their peers rather than their educators. It is for this reason that this research is guided by the theory of constructivism.

2.1.4 What is constructivism

According to Watson (2001) constructivism is a theory of knowledge that argues that humans generate knowledge and meaning from an interaction between their experiences and their idea. Llewellyn (2005) describes constructivism as a philosophy about how an individual learns, one in which the learner is embedded in active engagement and is constantly constructing and reconstructing knowledge through environmental interactions. Constructivism is underpinned by the belief that the best way learners gain knowledge is by inventing it and that they construct knowledge for themselves. Constructivists therefore hold the view that "learners construct their own reality or at least interpret it based upon their
perceptions of experiences, so an individual’s knowledge is a function of prior experiences, mental structures, and beliefs used to interpret objects and events” (van de Walle, 2007). According to Watson (2001), what someone knows is grounded in a perception of the physical and social experiences which are comprehended by the mind. This view is in contrast with the view that sees learning as the passive transmission of information from one individual to another. This is supported by von Glasersfeld (1995, p. 120) who argues that constructivism embraces the basic principle that “learning is not a passive receiving of ready-made knowledge but a process of construction in which the students themselves have to be the primary actors”. The assumptions of constructivism according to Piaget (1970) are that knowledge is constructed from experience, that learning is a personal interpretation of the world and an active ongoing process, in which meaning is developed on the basis of experience. Conceptual growth comes from the negotiation of meaning, the sharing of multiple perspectives and the changing of our internal representations through collaborative learning. Another assumption is that learning should be situated in realistic settings where testing should be integrated with the task and should not be undertaken as a separate activity. Constructivism acknowledges that each learner has a unique background and needs, and as such is a unique individual. In constructivism, the responsibility for learning therefore lies with the learner (Watson, 2001). Emphasis then is placed on the learner being actively involved in the learning process.

Constructivism guides educators to interact with learners through questioning and discussion, skillfully responding to the learners’ ideas, and allowing children to discover relationships and predict future events (Piaget, 1970). A constructivist framework challenges educators “to create environments in which they and their students are encouraged to think and explore” (Watson, 2001).

Vygotsky (1979) emphasized the role of culture and of social interaction in education and advocated teaching that encourages and guides learners but never forces or dictates. He believed that learners learn through active participation in a collaborative effort with educators, parents or older learners. Vygotsky focused on social interaction as a key component in the development of knowledge. He believed that mental processes exist between people in social settings, and that from these social settings the learner moves ideas into his or her own psychological realm. This transfer of ideas from those that are external to the individual-ideas exchanged in the social setting-to those that are internal, personal constructs. He calls this process internalization. According to him, this process of
internalization occurs within each learner’s Zone of Proximal Development (ZPD). The ZPD is not a physical space, but a symbolic space created through the interaction of learners with more knowledgeable adults and peers and their culture. Vygotsky (1979, p. 16), describe the ZPD as “the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers”. Lately van de Walle (2007) defined the Zone of Proximal Development as “the distance between the child’s actual development, determined with the help of independently solved tasks, and the level of potential development of the child, determined with the help of tasks solved by the child under the guidance of adults and in cooperation with his more intelligent partners”. Vygotsky’s theory proposes that all learning takes place through mediation by the teacher in the ZPD. The ZPD controls how a child learns. The ZPD is therefore a zone where learning occurs when a child is helped in learning a concept in the classroom (Vygotsky, 1979). By assisting children in learning, many theorists and educators have proven that Vygotsky’s theory works. Often children will learn easiest within this zone when others are involved. An example would be an activity where a learner works on the assignment with aid from the educator. Once learners achieve the goal of the initial activity, their zone grows and the students can do more. This involves the social constructivist method where learners act first on what they can do on their own and then with assistance from the educator, they learn the new concept based on what they were doing individually. Nickolai Veraksa (in Dolya & Palmer, 2004), one Vygotsky’s contemporary scholars, therefore refers to the ZPD as “the place where the child and adult meet.” It is for this reason that intelligence is measured not by what a child already knows but by what he or she can learn under adult or peer guidance through collaboration and imitation.

From the constructivist perspective, learners must therefore construct their own knowledge, irrespective of how they are taught. Even in the case of direct teaching (telling), learners cannot absorb an idea exactly as it is taught, but must interpret it and give meaning to what the educator says in terms of their existing knowledge. In this way they would be constructing their knowledge (Watson, 2001). The theory of constructivism, suggests that learners must be active participants in the development of their own understanding. Each learner, it is believed, constructs his/her own meaning in his/her own special way. This happens as learners interact with their environment, as they process different experiences and as they build on the knowledge or schema which they already have (Sapire & Mays, 2008).
According to Piaget (1970), when a person interacts with an experience/situation/idea, one of two things happens. Either the new experience is integrated into his existing schema (a process called assimilation) or the existing schema has to be adapted to accommodate the new idea/experience (a process called accommodation) (Watson, 2001). He suggested that through processes of accommodation and assimilation, individuals construct new knowledge from their experiences. Assimilation refers to the use of an existing schema to give meaning to new experiences (Llewellyn, 2005). Assimilation is based on the learner’s ability to notice similarities among objects and match the new ideas to those he/she already possesses. When individuals assimilate, they incorporate the new experience into an already existing framework without changing that framework. This may occur when individuals' experiences are aligned with their internal representations of the world, but may also occur as a failure to change a faulty understanding; for example, they may not notice events, may misunderstand input from others, or may decide that an event is a fluke and is therefore unimportant as information about the world. In contrast, when individuals' experiences contradict their internal representations, they may change their perceptions of the experiences to fit their internal representations (Watson, 2001). This is called accommodation. Accommodation is therefore the process of altering existing ways of seeing things or ideas that do not fit into existing schemata. It is the process of reframing one's mental representation of the external world to fit new experiences Accommodation is facilitated by reflective thought and results in the changing or modification of existing schemata. Accommodation can be understood as the mechanism by which failure leads to learning: when we act on the expectation that the world operates in one way and it violates our expectations, we often fail, but by accommodating this new experience and reframing our model of the way the world works, we learn from the experience of failure, or others' failures.

In terms of Mathematics teaching, the constructivist view therefore requires a shift from the traditional approach of direct teaching to facilitation of learning by the educator. Teaching by negotiation has to replace teaching by imposition and learners have to be actively involved in doing mathematics. This doing may involve peer discussion and group work. Learners have to engage in constructive learning on their own, working quietly through set tasks, allowing their minds to sift through the materials they are working with, and consolidate new ideas together with existing ideas.
The theory of constructivism promotes the construction of mathematical understanding. Skemp’s theory on the psychology of learning mathematics identifies the importance of learning mathematics with understanding, which he describes as reflective intelligence (Skemp, 1979). He points out that an individual schema is very important in the formation of conceptual structures. If the early schemas were inappropriately made the learner may later have difficulties with assimilation of more complex ideas. In his argument Skemp points out that if schemas are well formed, they are likely to last longer. These schemas take into account the bigger picture, not just the immediate task. So if learners have well formed schemas they are able to combine relevant mathematical ideas into logical reasoning, creating a network of ideas that could be called upon and used appropriately. Kim (2001) suggests that when learning with understanding occurs, a mathematical idea, procedure, or fact that is understood thoroughly is linked to existing networks with stronger or more numerous connections where mental representations are enriched by being connected with a network of ideas.

The growth of mathematical understandings involves learners in the construction and/or assimilation of new schemas into existing schemas in order to create a network of reference points, which they in turn use to connect and give meaning to new ideas (Williams, 2011). In order for learners to construct, link and assimilate their schemas they must be located in social settings where they can construct useful and powerful connections.

The constructivist view requires a shift from the traditional approach of direct teaching to facilitation of learning by the educator. Teaching by negotiation has to replace teaching by imposition; learners have to be actively involved in doing mathematics (Williams, 2011). This doing need not always be active and involve peer discussion, though it often does. Learners will also engage in constructive learning on their own, working quietly through set tasks, allowing their minds to sift through the materials they are working with, and consolidate new ideas together with existing ideas.
2.2 LITERATURE REVIEW

A literature review was used in this study as one of the data gathering techniques. Mouton (2001, p. 86) states that a literature review represents the first phase of the empirical study and entails reviewing other authors’ work in a specific field of study.

This section provides an account of the literature reviewed on misconceptions and errors made by grade 11 learners in geometry. The purpose of the literature review was to outline the factors that contribute towards misconceptions and errors as researched internationally and within South Africa. In the literature study, I therefore looked into the brief history and importance of geometry as well as the sources of misconceptions and errors in geometry as identified by various researchers, both locally and internationally.

According to Azalia College (2007, p. 120), literature study is carried out to accomplish some of the following purposes: to become familiar with the background and history of the problem, to identify possible ways to study the problem, to assess the strength and weaknesses of the previous study, and to develop a conceptual framework and rationale for the present study. The researcher therefore used literature study to understand the background and the nature of geometry misconceptions and errors in South Africa. The researcher was able to understand how deep this problem is and its consequences. Literature study was therefore used as a base of this research.

2.2.1 The history and foundation of geometry

Geometry is the branch of mathematics concerned with the shape of individual objects, spatial relationships among various objects, and the properties of surrounding space. It is a branch of mathematics concerned with the properties of space and shape. Geometry is therefore basically the study of shapes.

Geometry is one of the most ancient forms of mathematics. Geometry is associated with the ancient Greeks. According to Elias Loomis’ (in Blumenfeld, 2002), geometry is derived from a Greek word, meaning the “science of land – measuring”. The Greek word from which the name is derived therefore means earth measurement (Stahl, 1993).
Geometry is a very old subject that grew out of practical need. Geometry was first discovered among the Egyptians and originated in the remeasuring of their lands (Jones, 2000). This was necessary for them because the Nile overflows and obliterates the boundaries between their properties.

### 2.2.2 The importance of geometry

Standard 3 of the NCTM Draft “Standards 2000” document suggests that mathematics instruction programmes should pay attention to geometry and spatial sense so that all students, among other things, “use visualization and spatial reasoning to solve problems both within and outside of mathematics” (NCTM, 1998). This is one of the most important reasons why geometry is included in the school curriculum. The study of geometry contributes to helping learners develop the skills of visualization, critical thinking, intuition, perspective, problem-solving, conjecturing, deductive reasoning, logical argument and proof.

There are many demands and problems of the 21st century which can be solved geometrically. In terms of food production, Ivester (2008) argues that geometry technological advancement has led to dramatic improvement in the production of food and the processing of food.

In terms of housing, geometry plays an important role when it comes to plans and building quality houses. The possibility of a sudden rise in water levels at coastal areas as well as unstable surfaces because of flooding and earthquakes has heightened the necessity of dealing with housing geometrically.

There is a huge demand for clothes because of the growing population. Very few of us living in this century, still clothes ourselves with garments that are made entirely by hand. Most 21st century citizens are dependent on factory-made garments. Most modern garments are designed through the use of advanced technologies (also called CAD) that employ basic geometric concepts.

Health in the 21st century also needs geometry designs. Dentist, plastic surgeries and many operations are performed by employing geometry strategies.
Geometry also plays an important role in terms of transport. All car designs and other modes of transport are geometrically designed.

Geometry involves the study of shapes. Shapes play an important role in our every-day life. From home improvements to architecture we see and use geometric shapes in a variety of ways. According to Srinivasan (2005), this makes geometry to be an important strand of mathematics because it is an essential part of our lives.

The above explained roles of geometry in our society justify the inclusion of geometry in the school curriculum. Geometry has an important place in school Mathematics curricula. It develops students’ spatial ability, logical reasoning skills (French, 2004), and the ability to solve real-world problems in which geometrical terminology and properties occur (Jones & Mooney, 2003; Presmeg, 2006).

According the National Council of Teachers Mathematics geometry is an important school subject because it provides perspectives for developing learners’ deductive reasoning abilities and the acquisition of spatial awareness (NCTM, 1989). This is alluded to by Clements & Battista (1992, p. 420) who argue that understanding geometry is an important mathematical skill since the world in which we live is "inherently geometric". Improving learners’ geometric thinking levels is one of the major aims of mathematics education since geometric thinking is very important in many specific, technical, and occupational areas (Olkun, Sinoplu & Deryakulu, 2005).

Generally geometry is used in many professions involving physics, engineering, architecture, space exploration, and anything that requires measurement. According to Leopold (2003), geometry is the most useful of all the sciences. The understanding of geometry helps in the learning all other sciences; and no other science can be learned unless you know something of geometry (Blumenfed, 2002). Studying geometry will make people’s eyes quicker in seeing things, and tour hands steadier in doing things. People can draw better, write better; cut clothes, make boots and shoes, work at any mechanical trade, or learn any art. Geometry provides learners with the means to analyse, understand and describe their world and to deepen their understanding while adding to the ability to solve real-world problems (Jones, 2002). As such geometry is important for the economy and nation building.
According to Leopold (2005), the rationale for including formal geometry in the school curriculum is twofold: it is seen as a vehicle for teaching and learning deductive thinking (proof), and also as a first encounter with a formal axiomatic system. According to Brown, Jones & Taylor (2003), the important objectives in teaching mathematics at the secondary school level include developing knowledge and understanding of, and the ability to use, geometrical properties and theorems and encouraging the development and use of conjecture, deductive reasoning and proof. One of the rationales of mathematics teaching as suggested by the Southampton/Hampshire Group of mathematicians and mathematics educators is for designing and developing suitable teaching materials that support the teaching and learning of geometrical reasoning (Brown, Jones & Taylor, 2003). Geometry is therefore a natural setting for developing students’ reasoning and justification skills.

The above explained importance of geometry has made the South African government to include geometry in the CAPS, and most importantly, make it to be compulsory in grades 10 to 12.

2.2.3 The teaching and learning situation (the didactic situation) of geometry

The teaching and learning of geometry should be learner centred. The learners’ needs and interests are the central focus of learner centred learning. This change from teacher centredness to learner centredness may be attributed to the introduction of technologies which encourage a constructivist approach to learning (Van Niekerk, 2010). These authors consider the dimensions of learner centred instruction as being engagement, effectiveness and viability. Engagement refers to the capacity to provide prompt and convincing interaction and feedback to learners and is that which motivates learners. Effectiveness requires one to question how much learners actually learn and do tests necessarily indicate the depth of their understanding or the skills they have acquired. Viability refers to the feasibility of using tools, software improved distribution and the possibility of integration into classroom activities. All these aspects of the learning process are critical.
2.2.3.1 The constructivist geometry classroom

Geometry classrooms should be constructivist classrooms where all learners are involved “in sharing and socially interacting (cooperative learning), inventing and investigating new ideas, challenging, negotiating, solving problems, conjecturing, generalizing and testing” (Dolya & Palmer, 2004).

Constructivists are of the belief that learners need to be challenged with tasks that lie just beyond their level of mastery. This increases motivation and builds on their previous successes (Brownstein, 2001). Geometry classrooms should therefore be inquiry-based. Inquiry-based classrooms are described as learner friendly, where learners feel that their educator and peers value their ideas, thoughts and opinions (Lichakane, 2005). The classroom provides opportunities for active involvement in the learning process. Educators in geometry class need to develop psychological or strategic tools to create a constructivist classroom environment for all students. According to (Bigge & Shermis, 2004), examples of these tools are conversation, discussion and inquiry. The more prepared and comfortable educators are in using these effective tools, the more they will use them and learners can become adept at thinking and communicating. Learning in a constructivist class occurs when learners are challenged, open, and comfortable, while giving their full attention (Jones, Mooney & Harries, 2002). It is important that educators and learners develop trust and openness in the classroom for all learners to become engaged and attentive. When learners are not engaged, an ineffective classroom can be subject to disruptive by learners, and learning will not occur. Effective teaching methods that can be used in constructivist classroom include creating an environment where learners feel free to create unique concepts and structures to place in their memory for further retrieval (Lichakane, 2005). The components of a constructivist environment include providing means for learners to experience real world or meaningful practices. Learners learn through examples that they can relate to on an emotional, or on a cognitive basis (Van Niekerk, 2010). Learners can experience their world using meaningful practices that connect emotional or affective, as well as thinking or cognitive parts of self. Effective educators beget effective learners. A common ground must be secured in the classroom where an educator and learner discussions are free and where the earners feel comfortable to discuss their ideas or concepts without inhibitions or fear. This common ground, where the dynamic flows, allows learning or individual construction to become liberated and this contagious atmosphere can fill the classroom (Bigge & Shermis, 2004).
constructivist classroom allows effective learning where light can be shed so that imagination, knowledge and inspiration can glow within each individual learner.

The theory of constructivism suggest that teaching is not a matter of transferring information to learners and that learning is not a matter of passively absorbing information from a book or educator. Effective educators must help learners construct their own ideas using ideas they already have. The manner in which a class is conducted, the social climate that is established within the classroom and the materials available for learners to work with all have an enormous impact on what is learned and how well it is understood. A worthwhile goal is to transform a classroom into a mathematical community of learners or an environment in which learners interact with each other and with the educator (Lichakane, 2005). In such environment learners share ideas and results, compare and evaluate strategies, challenge results, determine the validity of answers, and negotiate ideas on which all can agree. The rich interaction in such a classroom significantly raises the chances that productive reflective thinking about relevant mathematical ideas will happen. This will help eliminate misconceptions and errors.

2.2.3.2 Constructivist geometry educators and their roles.

Within the social constructivist geometry classroom, the educator’s role is that of a facilitator and mediator. The educator is the mediator asking questions, requesting, paraphrases ideas, managing and focusing the discussion as needed, but avoiding comment on the correctness or the value of particular ideas (Lichakane, 2005). According to constructivists a facilitator is the person who provides an environment conducive for the learner to learn on his/her own (Kukla, 2000). This supports the notion that minds work by themselves. The mediator according to the constructivist is the person who guides the learner to achieve a certain prescribed goal. In terms of the new education policy of CAPS, facilitator refers to mediator in that sense of the word. The term facilitator as is used in our education system supports Vygotsky's notion of placing great emphasis on social and linguistic influences on learning and the role the teacher plays.

The role of the facilitator, according to constructivists, is to initiate constructive activity (Holt & Willard-Holt, 2000). An educator as a mediator should be able to establish a sound emotional environment conducive to interaction between mediator and the learners. The
mediator's duty is to develop a great deal of trust and encourage confidence and openness. This can be achieved through respect on the part of the mediator and by showing empathy with his/her learners as far as the learning of geometry is concerned.

According to the social constructivist approach, educators have to adapt to the role of facilitators and not teachers (Hiebert, Morris, & Glass, 2003). Whereas a teacher gives a didactic lecture that covers the subject matter, a facilitator helps the learner to get to his or her own understanding of the content (Kim, 2001). In the former scenario the learner plays a passive role and in the latter scenario the learner plays an active role in the learning process. The emphasis thus turns away from the instructor and the content, and towards the learner (Woolfolk, 2004). This change of role implies that an educator needs to display a totally different set of skills than a teacher (Hiebert, Morris & Glass, 2003). The educator needs to act as a facilitator. A teacher tells, a facilitator asks; a teacher lectures from the front, a facilitator supports from the back; a teacher gives answers according to a set curriculum, a facilitator provides guidelines and creates the environment for the learner to arrive at his or her own conclusions; a teacher mostly gives a monologue, a facilitator is in continuous dialogue with the learners. A facilitator should also be able to adapt the learning experience in mid-air by taking the initiative to steer the learning experience to where the learners want to create value (Lichakane, 2005).

Educators should make it clear that each learner must take responsibility for his or her work. Educators should also create opportunities for learners’ own learning, individually and as members of groups. Educators should do so by supporting learners’ ideas and questions and by encouraging them to pursue their dreams. They should give individual learners active roles in the design and implementation of investigations, in the work with their peers, and in learners’ assessment of their own work.

2.2.3.3 A constructivist learner

Constructivism encourages, acknowledges and accepts the fact that learners are independent (Kim, 2001). Social constructivism views each learner as a unique individual with unique needs and backgrounds. Teaching and learning should therefore be learner-centred. The learners’ needs and interests are the central focus of learner-centred learning. The learner is also seen as complex and multidimensional. Social constructivism not only acknowledges the
uniqueness and complexity of the learner, but actually encourages, utilizes and rewards it as an integral part of the learning process (Lichakane, 2005). Emphasis then is placed on the learner being actively involved in the learning process.

### 2.2.4 Methods and theories for teaching geometry

According to Woolfolk (2004) the constructivist does not prescribe explicit instructional strategies, however the sense of learning and understanding does imply a new set of goals for the classroom. Teaching geometry is to be understood as providing learners with the opportunity and stimulation to construct powerful geometrical ideas for themselves and to come to know their own power as mathematics thinkers and learners.

Various approaches in pedagogy derive from constructivist theory. They usually suggest that learning is accomplished best using a hands-on approach. Learners learn by experimentation, and not by being told what will happen, and are left to make their own inferences, discoveries and conclusions. Below are few examples of teaching method based on constructivism that can be used in the teaching of geometry.

#### 2.2.4.1 The Harkness discussion method

![Figure 2.1: An example of learners sitting around an oval table in a typical Harkness discussion model.](image)
According to Tingley (2002) this method of teaching is called the "Harkness" discussion method because it was developed at the Phillips Exeter Academy with funds donated in the 1930s by Edward Harkness. It is a teaching method whereby 12 learners and their educator sit in a circle around a Harkness table which is oval in shape. The classes are called “Harkness” classes and the teachers “Harkness” teachers. The aim is to make class more “real” and therefore more involving. Learners learn by discussing their thoughts and ideas rather than just by taking notes. Learners act as a team, cooperatively, to make these discussions work. They all participate, but not in a competitive way. Rather, they all share in the responsibility and the goals, much as any members share in any team sport. Discussion skills are important. Everyone must be aware of how to get this discussion rolling and keep it rolling and interesting.

The Harkness discussion model teaches learners to collaborate rather than compete with each other. Learning is a cooperative enterprise in which the learners and educator work together as partners. The Harkness teaching method fosters a sense of collaboration and encouragement that continues even when class is over (Tingley, 2002). Learners learn just as much from each other after class as they do in class, whether they’re the one giving the help or getting it. This model makes learners respectful of one another’s ideas. There is a notion of democracy that is characterized by the quality of thoughts, efforts and enthusiasm. According to Tingley (2002) the respect learners feel for one another grows out of being together at the Harkness Table and extends to every aspect of their lives. No one is allowed to hide during lessons. Everyone speaks his or her mind. Everyone is expected to contribute this discussion by doing any of the following: organizing, leading, summarizing, restating or clarifying the text, citing specific quotations, passages or pages from the text, asking a question about the text, commenting on the text, giving an opinion or reaction, making a suggestion about text or discussion, summarizing discussion up to that point, analyzing text or comment or whole discussion, reacting to comments, answering comments, restarting discussions, filling in a gap, arguing a point, asking for new information, and asking for other comments or reactions. According to Makonye (2011), sometimes it may be an appropriate teaching strategy for learners to be directed to reflect on their own errors and misconceptions or that of their peers. This approach helps if learners are exposed to wrongly-worked mathematical examples and they correct them themselves. The Harkness discussion method provides learners with such opportunities. This will benefit learners because according to Makonye (2011), when learners produce errors in the process of engaging with mathematics, it can be a moment of
exploration and learning if the error sets up an occasion of serious dialogue, exchange and consideration of the ideas involved in making the mistake.

Educators are participants in a Harkness discussion. The educator is a facilitator rather than a leader, and the learning style encourages everyone in a class to think independently, to express oneself articulately, and to understand the beliefs and viewpoints of others. The educator demonstrates to learners how to learn rather than just what to learn. The educator acts as little as possible. Educators guide learners in significant ways without dominating the lesson.

2.2.4.2 The inquiry centred instruction

Layman (1996) proposes an inquiry-centred instruction as a learning theory and as an approach to teaching and learning in mathematics. This instruction is characterized by educators who allow learner responses to drive lessons, shift instructional strategies and alter content, Engage learners in experiences that pose contradictions to their initial hypotheses and then encourage discussion, Encourage learners to engage in dialogue, both with the teacher and with one another, Encourage learner inquiry by posing thoughtful, open-ended questions and asking learners to question each other, encourage and accept student autonomy and initiative, use raw data and primary sources, along with manipulative, interactive, and physical materials, allow time after posing questions and nurture learners’ natural curiosity

2.2.4.3 Collaboration among learners

This is a form of pedagogy that requires learners with different skills and backgrounds to collaborate in tasks and discussions in order to arrive at a shared understanding of the truth in a specific field. Most social constructivist models, such as that proposed by Lichakane (2005), also stress the need for collaboration among learners, in direct contradiction to traditional competitive approaches.

Current mathematics education reforms supporting a constructivist perspective suggest that the automation of skills and passive intellectual involvement should be replaced by active learning processes (Jones, 2002). According to Even & Tirosh (2008) active learning denotes learning activities in which learners work together, and are given considerable autonomy and control of the direction of the learning activities, such as investigational work, problem
solving, small group work, collaborative learning and experiential learning. In contrast, passive learning activities denotes learning activities in which learners work alone and are passive receivers of information, include listening to the educator’s exposition, being asked a series of closed questions, and practice and application of information already presented.

The report presented by the Southampton/Hampshire Group of mathematicians and mathematics educators sponsored by the Qualifications and Curriculum Authority (QCA) to develop and trial some teaching/learning materials for use in schools that focus on the development of geometrical reasoning at the secondary school level found that learners benefit from working collaboratively in groups with the kind of discussion and argumentation that has to be used to articulate their geometrical reasoning (Brown, Jones & Taylor, 2003).

2.2.4.4 The van Hiele model

![The van Hiele Theory of Geometric Thought](image)

**Figure 2.2: The van Hiele theory of geometric thought, (Van De Walle, 2001).**

According to Senk (1989), the Van Hiele Theory of Levels of Thought in geometry is the most famous and prominent model used in the teaching of geometry. It is arguably the best-known framework presently available for studying teaching and learning processes in Geometry (de Villiers, 1996). It has made a significant difference to the world in terms of geometry education. The theory was developed in the 1950s when two Dutch mathematics educators, Pierre van Hiele and his wife Dina van Hiele-Geldof, did research on thought and concept development in geometry. They also investigated the role of instruction in assisting learners to acquire geometric knowledge and raise their thought levels (van Hiele, 1986). Their work evolved out of their experiences as educators where they found, by their own
admission, secondary school geometry difficult to teach (van Hiele, 1999). Resulting from their research, the van Hieles developed the notion of developmental levels of thinking in geometry. Yee (2006) explains that according to van Hiele there are 5 levels in children’s geometric understanding.

Each of these five levels describes the thinking processes used in geometric contexts (van Hiele, 1986). The levels describe how we think and what types of geometric ideas we think about, rather how much knowledge we have (van Hiele, 1986). A significant difference from one level to the next is the object of thought – what we are able to think about geometrically. Fuys, Geddes & Tischler (1988) characterized the levels as follows:

**Level 0: (recognition/visualization).** Learners identify, name, compare and operate on geometric figures on the basis of their appearance in a holistic manner at this level they are able to recognize and name figures based on the characteristics of the figure. The emphasis at this level is on the shapes that students can observe, feel, build, take part, or work with in some manner (Senk, 1989).

**Level 1: (analysis).** Learners analyze figures in terms of their components and discover the relationships among those components as well as derive the properties/rules of a class of shapes empirically. They are able to consider all shapes within a class rather than a single shape. Learners begin to appreciate that a collection of shapes goes together because of properties (van de Walle, 2007, p. 410).

**Level 2: (informal deduction).** Learners are able to interrelate logically previously discovered properties/rules by giving or following informal arguments. They begin to be able to think about properties of geometric objects. According to van de Walle (2007), students have now developed an understanding of various properties of shapes.

**Level 3: (deduction).** Learners are able to prove theorems deductively and establish interrelationships among networks of theorems. Learners begin to appreciate the need for a system of logic. They are able to work with abstract statements about geometric properties and make conclusions based on logic than intuition. They also prove theorems using clearly articulated logical reasoning.
**Level 4: (rigor).** Learners establish theorems in different postulation systems and analyze/compare these systems. At this level, there is an appreciation of the distinctions and relationships between different axiomatic systems. Learners can understand the necessity of precision and the interrelationships between mathematical systems or structures. This means learners can reason abstractly without any reference to a concrete model (van de Walle, 2007, p. 409).

According to Usiskin (1982) the theory of Van Hiele asserts that geometry learning takes place in a hierarchical nature. This simply means that a learner cannot be expected to master the higher levels of deduction and rigour before mastering the lower levels of recognition, analysis and abstraction. This implies that by mastering geometry in terms of these levels, the learner will avoid making errors and will be free of misconceptions (Senk, 1989). The levels are therefore sequential. To arrive at any level above the level 0, learners must move through all prior levels (van Hiele, 1999). To move through a level means that one has experienced geometric thinking appropriate for that level and has created in one’s mind the types of objects or relationships that are the focus of thought at the next level. According to van de Walle (2007, p. 413), if learners are to be adequately prepared for the deductive geometry curriculum of high school, then it is important for their thinking to have grown to level 2 by the end of the eighth grade. All educators should be aware that the experiences they provide are the single most important factor in moving children up this developmental ladder. This theory highlights the necessity of teaching at the learner’s level of thought (van Hiele, 1999). At level 0, instructional activities should involve lots of sorting and classifying of shapes.

These levels show the teaching style of moving from the unknown to the known Usiskin (1982). The learners’ knowledge and ability to work geometry problems can be improved if educators can understand this theory. This will assist in handling misconceptions and errors in geometry. According to Mogari (2002), the van Hiele theory provides guidelines for a pedagogy which will allow learners to enrich their knowledge of geometry and develop their skills in constructing proofs of theorems and riders in geometry. It deals with cognitive development in the learning of geometry. It also indicates how learners could meaningfully learn geometry or how they could effectively be taught geometry.

This model links learners’ mathematical competencies to their levels of geometric understanding (van Hiele, 1999). According to this, the learners, assisted by instructional
experience, passes through these levels in a hierarchical order, beginning with the recognition of shapes as a whole (level 1), progressing to discovery of properties of shapes and informal reasoning about these shapes and their properties (level 2 & 3), and culminating in a formal deductive and rigorous study of axiomatic geometry. In her dissertation Mayberry (1981) argues that a learner would fail not only to answer correctly but also fail to understand the intent of a question which required thought about his attained level if he has not mastered the previous level. According to her a learner who has understood concepts of a particular level, would be able to answer question on and below that level, but would fail to answer questions based on the next level. A learner cannot therefore function adequately on a given level unless she has passed through and learned to think intuitively on each of the preceding levels.

This model emphasizes advancement in learning. The learner’s progression from one level to the next is more dependent on the instruction received than the biological maturity of the learner. This simply means that, for example, for a learner cannot move from level 0 to level 2, without first experiencing level 1 (Crowley, 1987, p. 4).

This model is characterized by learning phases that help educators to plan their lessons. Each of the five phases of learning is explained below:

**Phase 1: Inquiry / information:**
At this primary stage learners and educators are engaged in conversation with each other about the topic at hand. Learners make observations related to the task, ask clarity seeking questions and the educator should introduce vocabulary pertinent to the specific level at which the task is dealt with. This phase serves as an introduction to the topic to determine learners’ prior knowledge.

**Phase 2: Directed orientation**
At this phase, the learners’ begin to explore a topic using material that has been carefully sequenced by the educator. The activities should reveal to the learners the features peculiar to that particular level. Activities at this phase should be designed to elicit specific responses (Crowley, 1987, p. 5).
Phase 3: Explication

Building on their previous experiences learners begin to express and share their views about the observations made regarding a concept. During this stage the educator plays a minimal role. The educator’s role is restricted to assisting learners acquiring and using “accurate and appropriate language” (Crowley, 1987, p. 5). It is during this phase that particular levels of systemic relations begin to become apparent.

Phase 4: Free orientation

During this period of learning, learners are exposed to and engaged in open–ended tasks that can be completed in a variety of ways. The tasks are non-routine, multisteped, complex tasks.

Phase 5: Integration

At this stage of learning, learners need to bring together (synthesize), what they have learnt, with the aim of forming an overview of the new relationships of objects and relations. By the end of phase five (integration), learners have attained a new level of thinking. This new level of thinking replaces the previous level of thinking and learners are once again ready to repeat the five phases of learning at the next level of the van Hiele model of thinking.

Since geometry teaching and learning is a problem in South Africa, the question is why educators are not trying this model in our own context. Research indicates that only few studies have utilized this model in an African context (de Villiers, 1997). It is therefore a challenge for researchers to make further studies on how best we can apply and implement this model in terms or our South African context. According to de Villiers (1997, p. 43), “unless we South Africans can embark on a major revision of the primary school geometry curriculum along van Hiele lines, it seems clear that no amount of effort at the secondary level will be useful. The most challenging thing to do is therefore to be able to determine the van Hiele geometric levels of learners and design appropriate assessments in line with learning outcomes and assessment standards. According to van de Walle (2007, p. 413), if learners are to be adequately prepared for the deductive geometry curriculum of high school, then it is important for their thinking to have grown to level 2 by the end of the eighth grade. Indications are that learners arrive in grade 8 without having reached this level. The situation is even worse in grade 11 and 12. The challenge is for educators to understand these levels and the type of activities to be given to learners.
2.2.5 Misconceptions and errors

2.2.5.1 Misconceptions

Leinhardt, Zaslavsky & Stein (1990) define misconception as incorrect features of learners’ knowledge that are repeatable and explicit. The term misconception has many synonyms. Various researchers summarized synonyms existing in the literature for this term. Champagne, Gunstone & Klopfer (1985) call them naive conceptions nonscientific beliefs; Chinn & Brewer (1993) call them pre-instructional beliefs; Vosniadou, Ioannides, Dimitrakopoulou & Papademetriou (2001) call them intuitive knowledge and phenomenological primitives; Tomita (2008) calls them facets; and Carey, Evans, Honda, Jay & Unger (1989) call them alternative frameworks.

Dikgomo (1994, p. 25) calls misconceptions certain interfering elements. Dikgomo (1994) thus view misconceptions as conceptual difficulties that learners experience which may hinder the understanding of mathematical concepts. Misconceptions are therefore impediments in meaningful learning, more especially in the learning of Mathematics. Thabane (1994, p. 6) is of the same view when he argues that “misconceptions can be a stumbling block to understanding Mathematics concepts”. Nesher (1987, p.35) on the other hand defined misconception as “a line of thinking that causes a series of errors all resulting from an incorrect underlying premise rather than sporadic unconnected and non-systematic errors”. This is clearly an indication that misconceptions are systematic errors that reappear whenever the same type of problem is presented. Michael (2001) expresses the same sentiments by indicating that misconceptions are “conceptual or reasoning difficulties that hinder learners’ mastery of any discipline”. According to the constructivist position, a misconception are “identified when a relatively stable and functional set of beliefs held by an individual comes into conflict with an alternative position held by the community of scholars, experts or teachers as a whole” (Confrey, 1987, p. 96).

Swan (2001) views misconceptions as “natural stage of conceptual development”. According to him, frequently, a misconception is not wrong thinking but is a concept in embryo or a local generalization that the learner has made. It may in fact be a natural stage of development (Swan 2001, p. 154). According to van der Sandt & Nieuwoudt (2003), for the learners to be able to confront underlying conceptual difficulties, overcoming these misconceptions is required. Van Lehn (1982) is of the view that a misconception could be a
result of a “misapplication of a rule, an over- or under-generalization, or an alternative conception of the situation”. A misconception therefore is a wrong understanding of how something works. According to Confrey (1987, p. 96), constructivists identified a misconception “when a relatively stable and functional set of beliefs held by an individual comes into conflict with an alternative position held by the community of scholars, experts and teachers as a whole”.

Van Lehn (1983) explained misconceptions in terms of a repair theory. He explained that when solvers encounter a new problem, they try to apply their preexisting knowledge to the new situation. If they fail to solve the problem, they may introduce a repair in the procedure. When the changed procedure is correct, a creative solution is obtained. However, when the changed procedure is incorrect – a misconception is manifest. Thus, according to the repair theory, a misconception is not a simple overgeneralization that occurs whenever the solver applies a concept as is to other domains in which it is incorrect; but a misconception results from a slight repair to a given procedure. It is on this ground that Resnick, Nesher, Leonard, Magone, Omanson, & Peled, (1989, p. 26) view misconceptions as “intelligent constructions based on what is more often incomplete than incorrect knowledge”. Misconceptions are a big impediment in meaningful learning. Studies have showed that it is hard to relieve misconceptions (Confrey, 1987, Van Lehn, 1983). It is therefore important for educators to know how to handle misconceptions if they want to be successful in their lessons.

Misconceptions have generally been seen as mistakes that impede learning, a view that is difficult to square with the premise that students construct their mathematical and scientific knowledge. According to Smith, DiSessa, & Roschelle (1993), a misconception is a learner’s conception that produces a systematic pattern of errors. A misconception is a conceptual structure, constructed by the learner, which makes sense in relation to her/his current knowledge, but which is not aligned with conventional mathematical knowledge (Smith, DiSessa, & Roschelle, 1993). This is why misconceptions had been, are and will always be part and parcel of the learning process.

2.2.5.2 Errors

According to Harper (2010) words for error originally meant “wander or go astray”. From a dictionary perspective, the word error is from Latin word “errare” which means to err
(Harper, 2010). Harper (2010) defines an error as a “deviation from accuracy or correctness; a mistake, as in action or speech belief in something untrue; the holding of mistaken opinions”. It is a condition of believing what is not true. Synonyms of this word are blunder, slip, oversight, mistake, fault, transgression, trespass, and misdeed. An error is a mistake or inaccuracy. It is an incorrect belief or wrong judgement. It is a condition of deviating from accuracy or correctness.

The Merriam-Webster's Medical Dictionary (2007) defines an error as a “deficiency or imperfection in structure or function”. The American Heritage Stedman's Medical Dictionary defines an error as an “act, an assertion, or a decision, especially one made in testing a hypothesis, that unintentionally deviates from what is correct, right, or true” (Houghton Mifflin Company, 2002). An error is therefore an act that through ignorance or deficiency, the learner fails to achieve what should be done.

Generally, an error means a simple lapse of care or concentration which almost everyone makes at least occasionally. In mathematics, an error means the deviation from a correct solution of a problem. An error is regarded as a mistake in the process of solving a mathematical problem algorithmically, procedurally or by any other method. Errors could be found in wrongly answered problems which have flaws in the process that generated the answers (Young & O’Shea, 1981). According to Harper (2010), errors are witnesses of a learners’ misunderstanding or habits.

Luneta (2008, p. 386) defines errors as “simple symptoms of the difficulties a student is encountering during a learning experience”. According to Swan (2005), an error could be the result of “carelessness or misinterpretation of symbols or text”. Merenluoto (2004) expresses the same sentiments as Swan when he explained that an error could be a mistake caused by a fault: the fault being misjudgement, carelessness, or forgetfulness. According to Micawber (2005) an error etymologically implies deviation; it suggests culpability but not necessarily carelessness or intention. An error is therefore a deviation from accuracy or correctness. Olivier (1989) defines errors as “wrong answers due to planning”. It is a blunder or inaccuracy and a deviation from accuracy. Errors are systematic, persistent and pervasive mistakes performed by learners across a range of contexts (Nesher, 1987). Errors are therefore mistakes made by learners. A mistake is a result of the lack of concentration control or weak memory. An error therefore reveals inadequacy of knowledge and is closely
connected with imagination and creativity in a new situation, and is caused by an insufficient mastery of basic facts, concepts and skills.

Riccomini (2005) differentiate between systematic and unsystematic errors. According to Riccomini (2005), non-systematic errors are unintended, non-recurring wrong answers which learners can readily correct by themselves without outside help. They are also called slips, lapses or unintended mistakes. They are inconsistent and are less predictable (van Lehn, 1982). They usually emanate from fatigue, carelessness, and other sources of inconsistency. They can therefore be called careless errors. Olivier (1989) views non-systematic errors as mistakes that are easily corrected when pointed out. Many slips are caused by difficulties experienced within working memory. Learners do not intend to make those errors. They are sporadic, and are carelessly made by both experts and novices. They are easily detected and are spontaneously corrected. We say a learner is careless when he/she makes a mistake performing a task that he/she already knows. A learner who is engaged in a task may be overconfident, impulsive or hurried, leading to more careless errors. Within learning, some learners become careless, working unconscientiously and making unintended errors. This can happen when an individual is overconfident in carrying out a task, or carries out a task in an impulsive or in a hurried manner. Carelessness is a common behaviour among learners, even among high-performing students.

Systematic errors on the other hand are symptomatic of a faulty line of thinking causing them to be referred to as misconceptions (Green, Piel & Flowers, 2008; Riccomini, 2005). They are systematic in that they are applied regularly in the same circumstances. They are symptoms of an underlying or incorrect hypothesis. These types of errors are procedural and often feature an incorrect routine in an otherwise correct method. They are systematic and persistent, and are not necessarily responsive to easy correction or re-explanation of concepts. Brown & Burton (1978) classified them as bugs of a procedural nature involving an incorrect routine. Blando, Kelly, Schneider & Sleeman (1989) also used the term bugs to describe these errors. These types of errors occur when a learner is faced with a difficult of unfamiliar feature of a task that leads the student to an impasse (Riccomini, 2005). This impasse is resolved by modifying a known procedure and incorrectly applying it to the task. Studies have shown that many learners make systematic errors when learning geometry.
When reviewing the errors, it is crucial that the learner understand the nature of the error and why he/she made it (them). Various researchers studied errors made by learners in mathematics. These errors were classified and coded to understand their nature and causes. Errors as part of misconception have been classified into five main categories by Radatz (1979):

**Table 2.1: Radatz’s classification of errors (Radatz, 1979).**

<table>
<thead>
<tr>
<th>Error Number</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>language difficulties</td>
</tr>
<tr>
<td>2</td>
<td>difficulties in obtaining spatial configuration</td>
</tr>
<tr>
<td>3</td>
<td>deficient mastery of prerequisite facts and concepts</td>
</tr>
<tr>
<td>4</td>
<td>incorrect associations or rigidity of thinking</td>
</tr>
<tr>
<td>5</td>
<td>application of irrelevant rules or strategies.</td>
</tr>
</tbody>
</table>

Watson (1980) conducted research using the Newman Model, which stated that all errors could be placed in one of the following categories:

**Table 2.2: Watson’s classification of errors (Watson, 1980).**

<table>
<thead>
<tr>
<th>Category</th>
<th>Type of errors</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Reading Ability</td>
<td>Can the pupil read the question?</td>
</tr>
<tr>
<td>2</td>
<td>Comprehension</td>
<td>Can the pupil understand the question?</td>
</tr>
<tr>
<td>3</td>
<td>Transformation</td>
<td>Can the pupil select the mathematical process, which is required to obtain the solution?</td>
</tr>
<tr>
<td>4</td>
<td>Process Skills</td>
<td>Can the pupil perform the mathematical operations necessary for the task?</td>
</tr>
<tr>
<td>5</td>
<td>Encoding</td>
<td>Can the pupil write the answer in an acceptable form?</td>
</tr>
<tr>
<td>6</td>
<td>Motivation</td>
<td>The pupil could have correctly solved the problem had he or she tried.</td>
</tr>
<tr>
<td>7</td>
<td>Carelessness</td>
<td>The pupil could do all the steps but made a careless error, which is unlikely to be repeated.</td>
</tr>
<tr>
<td>8</td>
<td>Question Form</td>
<td>The pupil makes an error because of the way the problem has been presented.</td>
</tr>
</tbody>
</table>
Movshovitz-Hadar, Zaslavsky, and Inbar (1987) used a similar approach to categorizing learners’ mathematical errors. Their classification included six categories as displayed in table 2.3:

**Table 2.3: Movshovitz-Hadar, Zaslavsky, and Inbar’s classification of errors (Movshovitz-Hadar, Zaslavsky, and Inbar, 1987).**

<table>
<thead>
<tr>
<th>Category</th>
<th>Type</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Misused Data</td>
<td>The examinee did not use the information in the question correctly.</td>
</tr>
<tr>
<td>2</td>
<td>Misinterpreted Language</td>
<td>The examinee incorrectly translated the mathematical facts from the written problem into symbols.</td>
</tr>
<tr>
<td>3</td>
<td>Logically Invalid Inference</td>
<td>The examinee invalidly draws new information from a piece of given information.</td>
</tr>
<tr>
<td>4</td>
<td>Distorted Theorem or Definition</td>
<td>The examinee had an incorrect perception about the definition of a principle, rule, theorem, or definition.</td>
</tr>
<tr>
<td>5</td>
<td>Unverified Solution</td>
<td>Each step taken by the examinee was correct in itself, but the final presentation of the result was not a</td>
</tr>
<tr>
<td>6</td>
<td>Technical Errors</td>
<td>The examinee made a careless error, such as incorrect computation or incorrect data extraction.</td>
</tr>
</tbody>
</table>

According to Booker (1988), errors can also be classified as mathematical errors and didactic errors. A mathematical error is made by a person (learner, educator) who in a given moment considers as true an untrue mathematical sentence or considers an untrue sentence as mathematically true. Examples of mathematical errors include the following: omission of essential characteristics in a given class of objects is made, or inclusion of inessential characteristics into the definition when defining mathematical concepts and application of definitions, using the hypothesis without testing the proposition in theorem understanding and application, too quick, unjustified generalizations made on the basis of observing a few particular cases (Booker, 1988).

Didactic errors refer to a situation when educators’ behaviours are contradictory to the didactic, methodological and common sense guidelines. Examples of the educator’s didactic errors associated with teaching a class include the following: incoherent structure of teaching content, unsuitable selection of examples used in forming a concept, unsuitable selection of problems used for aim realization, underestimating the necessity to master the basic skills by students, such as correct calculations, representation and comprehension of the data on graphic representation of geometric figures, inappropriate educators’ response to learners’
error, (e.g. irritation due to numerous recurrences of errors that have been clarified in class and inaccurate selection of methods for subject realization (Booker, 1988).

Elbrink (2008) classify errors in terms of mechanical errors, application errors, careless errors and order of operations errors (see table 2.4).

**Table 2.4: Elbrink’s classification of errors (Elbrink (2008)).**

<table>
<thead>
<tr>
<th>Type</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mechanical errors</td>
<td>Errors committed because of hurried approach, forgotten step, lack of review</td>
</tr>
<tr>
<td>Application error</td>
<td>Caused by misunderstanding of one or more of the required step(s) concept errors. Are also known as knowledge based errors - (lack of knowledge of the concept unfamiliar with the terminology, usually caused by when a learner does not understand the properties or principles covered in the textbook</td>
</tr>
<tr>
<td>Careless errors</td>
<td>Mistakes made which can be caught automatically upon reviewing one’s work</td>
</tr>
<tr>
<td>Order of Operations errors</td>
<td>Often stems from rote learning as opposed to having a true understanding</td>
</tr>
</tbody>
</table>

Luneta and Makonye (2010) categorized errors into 5 codes, namely non-systematic errors, generalization or transfer errors, ignorance of rule restrictions or symbolism, incomplete application of rules and false concepts hypothesized to form new concepts (Refer to table 2.5 below).

**Table 2.5: Luneta and Makonye’s coding and categorizing of errors (Luneta & Makonye (2010)).**

<table>
<thead>
<tr>
<th>Code</th>
<th>Description with examples chosen from this study</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Non-systematic errors. These are slips, lapses or unintended mistakes.</td>
</tr>
<tr>
<td>1</td>
<td>Generalization or transfer errors. These refer to extension of previously available strategies in new situations where they do not apply.</td>
</tr>
<tr>
<td>2</td>
<td>Ignorance of rule restrictions or symbolism. Applying rules to contexts they do not apply. Failure to understand the bounds where a rule applies.</td>
</tr>
<tr>
<td>3</td>
<td>Incomplete application of rules</td>
</tr>
<tr>
<td>4</td>
<td>False concepts hypothesized to form new concepts</td>
</tr>
</tbody>
</table>
The above tables signify the importance of coding and categorizing errors. This helps educators to identify the type of errors held by their learners. The coding and categorizing of errors may help educators foresee difficulties and obstacles and use this ability in planning their teaching so as to prevent as many of them as possible. Educators may also use this strategy to identify a persistent tendency of individual learners to make a certain type of error across several mathematical topics (Movshovitz-Hadar, Zaslavsky, and Inbar, 1987).

2.2.5.3 Link between misconceptions and errors

Misconceptions differ from errors in several ways. Olivier (1989) states that errors are wrong answers due to planning and that they are systematic, and applied regularly in the same circumstance while misconceptions are symptoms of shared cognitive structures that could in turn cause errors.

Errors usually are not random but reflections of learners’ misconceptions regarding the actions. Olivier (1989) differentiates between slips, errors and misconceptions. Slips are wrong answers due to processing: they are not systematic, but are sporadically carelessly made by both experts and novices; they are easily detected and are spontaneously corrected Olivier (1989). Errors are wrong answers due to planning; they are systematic in that they are applied regularly in the same circumstances. Errors are the symptoms of the underlying conceptual structures that are the cause of errors. It is these underlying beliefs and principles in the cognitive structure that are the cause of systematic conceptual errors that I shall call misconceptions.

Despite the differences between misconceptions and errors, there is clearly a link between the two. Misconceptions lead to errors. The definitions of misconceptions by Confrey (1990) and Micawber (2005) above attest to this. According to Confrey (1990, p. 33) misconceptions are “a line of thinking that causes a series of errors all resulting from incorrect underlying premises”. Micawber (2005) argue that “misconceptions give rise to patterns of errors, stem from learners’ prior knowledge and are very resistant to change”. Errors are therefore symptoms of misconceptions that learners have. In health terminology misconceptions may be considered to be diseases whereas errors may be considered the symptoms of these diseases. Errors are therefore symptoms of misconceptions that learners have. Misconceptions are associated with mistakes. According to Micawber (2005), mistakes implies misconceptions, misunderstanding, a
wrong but not always blameworthy judgement, or inadvertence; it expresses less severe criticism than error. There is therefore a link between errors and misconceptions. Most researchers found that misconceptions lead to errors (Micawber, 2005; Confrey, 1990; Smith, DiSessa & Roschelle, 1993; Green, Piel & Flowers, 2008; Nesher, 1987; Riccomini, 2005).

According to Engelbrecht (in Mevarech, 1983, p. 425) misconceptions are errors which result “from responding without engagement in the task”. Clement (1987) defined misconceptions as “stumbling blocks, cognition process responsible for errors in problem solving in Mathematics. Misconceptions and errors result in mathematics being viewed from a negative perspective.

Luneta & Makonye (2010), who did a research on maths and science misconceptions, are of the view that although errors and misconceptions are related, they are different. They regard an error as a mistake, slip, blunder or inaccuracy and a deviation from accuracy. According to Luneta & Makonye (2010), errors are visible in learners’ artefacts such as written text or speech. However misconceptions are often hidden from the undiscerning observer. Educators need to listen carefully to determine why learners give answers they give so that they can correctly follow learners’ reasoning.

2.2.6 Identified misconceptions and errors in geometry by various researchers

2.2.6.1 Identification/Classification of basic shapes

Researches done by de Villiers (1997), Siyepu, (2005) and Roux (2003) found that learners’ performance in South African high schools is poor when it comes to items involving understanding of features and properties of shapes – the very fundamentals of geometric understanding. This was confirmed by Siyepu (2005) whose research also found that secondary learners in South Africa cannot identify and name shapes like kite, rhombus, trapezium, parallelogram and triangle. This is in consistent with the research done by Atebe (2008) who conducted a research involving South African and Nigerian learners, with special focus on shapes in geometry. The research found that both South African and Nigerian learners had problems in naming shapes with reasons as reflected in table 2.6
Table 2.6.: NS and SAS learners’ performance on the naming of shapes and stating of reasons (Atebe, 2008).

<table>
<thead>
<tr>
<th>Name of shape</th>
<th>No. correctly naming shape</th>
<th>No. stating correct reason</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nigeria (n=18)</td>
<td>S/Africa (n=18)</td>
</tr>
<tr>
<td>Rectangle</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>Square</td>
<td>16</td>
<td>18</td>
</tr>
<tr>
<td>Isosceles triangle</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>Parallelogram</td>
<td>14</td>
<td>12</td>
</tr>
<tr>
<td>Equilateral triangle</td>
<td>17</td>
<td>17</td>
</tr>
<tr>
<td>Rhombus</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>Kite</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 2.6 indicates that more learners can name shapes than state the properties of shapes. For example, although 8 learners from the Nigerian subsample were able to name the rhombus shape, none of them could state its distinguishing properties. Similarly, only 1 out of the 12 South African learners who correctly named rhombus shape was able to correctly state its properties. The use of imprecise properties for describing the shapes is common among the learners. The research done by Atebe (2008) found that the majority of the learners described the shapes entirely by the property of sides while neglecting their angle properties. Atebe (2008) found that, for example, all 8 of the Nigerian learners who correctly identified rhombus shape used only the side property to justify their naming: “It is a polygon having four equal sides”; “It has four equal sides”; “Four sides are equal” and so forth. Among the South African learners, only 1 out of the 12 that correctly identified the rhombus shape used both side and angle properties to justify his naming. This problem is caused by learners’ inadequate knowledge of geometric terminology.

Learners who failed to correctly name the shapes were generally assigned level 0. Those who correctly named the shapes but could not state the correct reason, were assigned van Hiele level 1. Learners who succeeded in naming the shapes correctly and also succeeded in stating the correct reason were assigned level 2. Learners who correctly named the shapes and stated minimal and sufficient properties as the reason were to be assigned level 3, but no learner in the sample met this condition. The research done by Brombacher (2001) and Howie (2001) also confirm the problem that learners can’t recognize basic geometrical shapes. Learners can’t define correctly basic geometrical shapes. They don’t know the correct properties of the
shapes. For example, Clements & Battista (1992) found that many high school learners reason that “a square is not a square if its base is not horizontal”. This was also found in the research done by Atebe as indicated by the response given by the learner in the below figure:

Figure 2.3.: Illustrating learners’ difficulty with identifying and naming shapes (Atebe, 2008).

It is important for learners to be able to identify and explain relationship between shapes. Knowing the association and relationship between shapes may help learners with solving riders and improve their proof skills and competency. One way of doing this is to use diagrams as reflected in figures 2.4 and 2.5.

Figure 2.4: A diagrammatical illustration of the properties of different quadrilaterals (Soni, & Cronje, 2011).

These quadrilaterals are related. Diagrammatic illustrations may help learners to become familiar with the properties of quadrilaterals. Below is a diagrammatically illustrations of the relationships of the various quadrilaterals:
Another problem is that learners in secondary education are able to recognize shapes only in some standard orientation. This was confirmed by Mayberry (1983) whose research reports that some learners in her study “had difficulty in recognizing a square with a non-standard orientation”. Marchis (2008) also found that learners recognized a square as not a rhombus, a rectangle as not parallelogram. According to Marchis, (2008), two third of learners can’t define correctly basic geometrical shapes because they can’t recognize the geometrical shape or they don’t know the correct properties of the shapes. This confirmed the findings of Clements & Battista (1992) who found that only 64% of the 17-year-olds in the U.S. knew that a rectangle is a parallelogram. The problem is that learners sometimes reason with a concept image than concept definition. Below is an example of figures that learners might not recognize when reasoning with a concept image only:

<table>
<thead>
<tr>
<th>Shape</th>
<th>Shape Prototype</th>
<th>Figures Possibly not Recognized when Reasoning with a Concept Image and not a Concept Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallelogram</td>
<td></td>
<td><img src="image" alt="Parallelogram Figures" /></td>
</tr>
<tr>
<td>Rectangle</td>
<td></td>
<td><img src="image" alt="Rectangle Figures" /></td>
</tr>
<tr>
<td>Square</td>
<td></td>
<td><img src="image" alt="Square Figures" /></td>
</tr>
<tr>
<td>Rhombus</td>
<td></td>
<td><img src="image" alt="Rhombus Figures" /></td>
</tr>
</tbody>
</table>

Figure 2.6: Quadrilaterals possibly not recognized when reasoning with a concept image and not a concept definition.
Learners also experience problems when diagrams are disorientated. An example of such misconception is diagrammatically illustrated in figure 2.7. In this diagram some learners may not recognize the second shape as the same square, but instead a diamond or a kite.

![Figure 2.7: A square which may be recognized as a diamond or kite.](image)

Disorientated right angles might also give learners problems. Given the right angled triangles below students may identify only the pink and green triangles as right angled triangles since they can clearly identify the right angles using the sides which are vertical and horizontal. However, since the yellow triangle does not have a vertical side meeting a horizontal one, students may not be able to acknowledge the existence of the right angle.

![Figure 2.8: Figures which may not be recognized as right angles.](image)

To avoid the misconception as a result of disorientated diagrams, educators should put more emphasis on lessons based on rotations. Lessons on mentally rotating diagrams need to be emphasized in order to give learners more opportunities to perceive them more clearly. Lack of knowledge in terms of class inclusions of shapes may result in misconceptions and errors. Knowledge of class inclusions of shapes is important in geometry because it enables the learners to reason about the relationships between different geometric shapes and their
properties. According to van Hiele (1999), the ability to recognize and name shapes has been recognized as important for geometric conceptualization. Most research evidence indicates that many high school learners lack the ability to correctly identify, name, and classify many simple geometric shapes (Clements & Battista, 1992; Marchis, 2008; de Villiers, 1997; Siyepu, 2005; and Roux, 2003). This was confirmed by Feza & Webb (2005) who found that many learners have difficulties to perceive class inclusions of shapes, for example, they might think, that a square is not rectangle.

In the research done by Atebe (2008), the sorting of shapes task revealed some important misconceptions about geometric concepts among the learners. 36 students were sampled to participate (18 Nigerian and 18 South Africans). There were 8 learners (2 Nigerians and 6 South Africans) who reasoned that all 4-sided shapes were called “square”. There were 4 other learners (3 Nigerians and 1 South African) who reasoned that all 4-sided shapes were called “rectangle”. There was yet another learner (a Nigerian) who thought that all 4-sided shapes are called “parallelogram”.

Rectangles, squares and rhombuses, for example, were all excluded from the class of parallelograms by all the learners who also failed to perceive squares to be rectangles (or rhombuses). Right-angled isosceles triangles were excluded either from the class of right-angled triangles. Learners were not yet able to perceive the relationships between the properties of a shape and between different shapes.

A task was also given to learners which required learners to state with justification whether a shape belonged to a class of shapes with some more general properties. Only 1 learner (a South African) perceived a square as belonging to the class of rectangles and of rhombuses (Atebe, 2008). No learner from the Nigerian subsample perceived a square as belonging to either class of shapes. Students’ denial of a shape with the more specific properties as not belonging to the class of the one with the more general properties was usually accomplished by the listing of a few properties of the special case not shared by the more inclusive shape. For example, some of the learners reasoned that a square is not a rectangle “because all the sides [of a square] are equal”, just as some others said that a rhombus is not a parallelogram because “all four sides are equal” (Atebe, 2008).
2.2.6.2 Concepts and incorrect terminology

The research done by Siyepu (2005) found that learners’ are having misconceptions about geometric concepts. This is confirmed in various researches whose findings reveal that learners have difficulties in understanding geometry concepts (Cunningham & Roberts, 2010; Kabaca, Karadag & Aktumen, 2011). Renne (2004) and Feza & Webb (2005) argue that learners lack the appropriate vocabulary to express the distinguishing properties of a figure or compare shapes in an orderly manner. This confirms the findings of Clements and Battista (1992), namely that in the United States, learners are failing to learn basic geometric concepts, are underprepared for the study of more sophisticated geometric proofs. In Nigeria, West African Examinations Council (WAEC, 2003) found that many high school learners could not comprehend “the concepts and principles of angle in the same segment as well as alternate and corresponding angles”. A research done by Atebe (2008) confirms problems associated with geometry concepts (refer to table 2.7 below).

<table>
<thead>
<tr>
<th>Component of circle</th>
<th>NS (n = 138)</th>
<th>SAS (n = 144)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean score (%)</td>
<td>Mean score (%)</td>
</tr>
<tr>
<td>Chord</td>
<td>38</td>
<td>31</td>
</tr>
<tr>
<td>Radius</td>
<td>30</td>
<td>33</td>
</tr>
<tr>
<td>Diameter</td>
<td>54</td>
<td>63</td>
</tr>
<tr>
<td>Tangent</td>
<td>31</td>
<td>38</td>
</tr>
<tr>
<td>Arc</td>
<td>33</td>
<td>31</td>
</tr>
<tr>
<td>Sector</td>
<td>56</td>
<td>41</td>
</tr>
<tr>
<td>Cyclic quadrilateral</td>
<td>28</td>
<td>49</td>
</tr>
<tr>
<td>Concentric circles</td>
<td>46</td>
<td>38</td>
</tr>
</tbody>
</table>

The above table clearly indicates that learners have problem in terms of geometry terminology. The mean score of most concepts is below 50(%). It means learners’ geometry language is poor. According to Feza & Webb (2005), geometry language incompetence impedes progress in geometric understanding. De Villiers (as cited in Feza & Webb, 2005, p.45) affirms by arguing that “success in geometry involves acquisition of the technical terminology”. Below are some of the concepts that confuse learners:
Congruent and Equal – Frequently learners interchange the words congruent and equal.

Midpoint and Bisect – Midpoint is a location, a noun, and bisect is an action, a verb.

Intersects and Bisects – Many learners replace the word intersects with bisects.

Complementary and Supplementary

Similarity and Congruency

Renne (2004) points to the lack of understanding of mathematical concepts as a major cause of misconceptions and errors in geometry. Such misunderstanding poses a grave concern as it leads to the formation of misconceptions and false generalisations, which in turn hinder the learning of mathematics. Leinhardt, Zaslavsky, & Stein (1990) define misconception as incorrect features of student knowledge that are repeatable and explicit. They attribute the misconception to previous formal learning. They also associate it with the lack of variety of instructional examples, or a translation, which may be performed inaccurately because of the confusion over symbolic notation.

According to the constructivist position, a misconception is “identified when a relatively stable and functional set of beliefs held by an individual comes into conflict with an alternative position held by the community of scholars, experts or teachers as a whole” (Confrey, 1987, p. 96). According to the constructivists, learning is the interaction between the individuals' past experience and the individual's current experience of the world around him. Olivier (1992) maintains that misconceptions play a very important role in learning and teaching, because misconceptions form part of the pupils' conceptual structure that will interact negatively with new concepts, which then lead to further misconceptions or alternative conceptions.

According to (Atebe, 2008), learners do not only understand geometry concepts, but also spell some geometry concepts incorrectly as indicated in table 2.8 below.

Table 2.8: Some of the learners’ incorrect spellings (Atebe, 2008).

<table>
<thead>
<tr>
<th>Concept</th>
<th>Wrong spellings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilateral triangle.</td>
<td>Equalateral triangle, equadilateral triangle, equadrilateral triangle,</td>
</tr>
<tr>
<td>Isosceles triangle.</td>
<td>Isosceless, Isocelene, Isoscele, Isoscelist, Isosscilice</td>
</tr>
<tr>
<td>Scalene triangle.</td>
<td>Scarlene, Scaline, Scalelan,</td>
</tr>
<tr>
<td>Rhombus</td>
<td>Robus, rombus</td>
</tr>
<tr>
<td>Parallelogram</td>
<td>Parallelogramme</td>
</tr>
<tr>
<td>Trapezium</td>
<td>Traipyzium</td>
</tr>
</tbody>
</table>
According to Bloom (1956, p.63) “the most basic type of knowledge in any particular field is its terminology”. Oberdorf & Taylor-Cox (1999, p.340) also argues that “lack of exposure to proper vocabulary” is one of the reasons for learners’ misconceptions in geometry. For a success in learning geometry the understanding of geometrical concepts is essential.

2.2.6.3 Markings on the diagram

According to Kabaca, Karadag & Aktumen (2011), sometimes learners confuse the marks for parallel to mean equal. For example in the below figure, it is given that chords BC//AD. In this case, learners might be confused that BC=AD

Figure 2.9: Misconceptions concerning markings on the diagram.

All the identified misconceptions and errors have a negative impact on solving riders. They are likely to result in learners committing lots of errors when solving geometry problems.

2.2.7 Sources of misconceptions and errors

Various factors can be attributed to misconceptions and resultant errors. For example, Learners sometimes make errors through lapses in concentration, hasty concentration, hasty reasoning (Luneta & Makonye, 2010). An error may be due to a misconception that a learner has about a topic. This may result in the misinterpretation of a mathematical idea. A learner might have not clearly understood a mathematical problem. Errors create great difficulties for the mathematics education to reach its goals if they aren’t avoided on time. The following are some of the common sources of misconceptions and errors:
2.2.7.1 Curricular factor

There is evidence that one of the primary causes of learners’ misconceptions and errors, and therefore poor performance in geometry at the secondary school level is the lack of a rich, coherent, and well-sequenced geometry curriculum at the primary school level (de Villiers; 1997; Siyepu, 2005). Learners therefore enter secondary school with little or no encounter with geometry. As a result; learners enter high school not knowing enough geometry to succeed. de Villiers (1997, p.42) asserts that in South Africa, the geometry curriculum is “still heavily loaded in the senior secondary school with formal geometry, and with relatively little content done informally in the primary school”.

2.2.7.2 Faulty reasoning

According to Brodie (2000), it is a big problem for learners to reason and communicate mathematically when given word problems and construction of proofs. Reasoning, broadly defined, is the process of coordinating evidence, beliefs, and ideas to draw conclusions about what is accurate or true (Leighton, 2003). Good reasoning ability is prerequisite to understanding. Reasoning mathematically refers to being able to formulate and represent a given mathematics problem, explain and justify the solution or argument to the mathematics problem (Kilpatrick, Swafford & Findell, 2001). Reasoning mathematically also involves finding out what is it that is true in a mathematics conjecture, constructing an argument to convince oneself that the result is true and thereafter find out why the conjecture is true (Brodie, 2000).

Siyepu (2005) considered misconceptions to arise from faulty reasoning. He believes misconception may come from confusion or lack of knowledge. This is supported by Michael (2001) who contends that misconceptions are conceptual or reasoning difficulties that hinder learners' mastery of any discipline. Sternberg (1985) also pointed out that reasoning may consist of the manipulation of mental models that correspond to internal analogues of scenes of actors, and errors made are due to that learners fail to consider all the possible models of the premises. For the learners to be able to confront underlying conceptual difficulties requires overcoming these misconceptions.

Yackel & Hanna (2003, p.227) discuss the phenomenon of the increased emphasis on incorporating reasoning at all levels of mathematics education, and explain its occurrence as a result of “a better understanding of how individuals come to know”. They explain that
Mathematics educators should emphasize the importance of encouraging learners’ explanation and justification in mathematics as a way of furthering their mathematical knowledge and understanding. This has been influenced by the use of constructivist models of learning in mathematics. This is also because of the belief that a person’s knowledge is composed of building blocks that form mathematical ideas (Davis, 1984). These building blocks originate in a person’s experiences, and the mental images derived from previous experiences can be used to build mathematical ideas (Maher, 1998). Since experience is inherently personal and unique, learners come to individual ways of knowing mathematics, and should be provided with the opportunity to come to know mathematics in their own way (Noddings, 1990). When educators invite learners to express their ideas and treat their learners’ ideas with respect, rather than use direct instruction to teach procedural mathematics, learners are provided the opportunity to construct rich and durable mathematical ideas.

Ball & Bass (2003) discuss the importance of reasoning in school mathematics. They posit that mathematical understanding is impossible without emphasizing reasoning. They explain that without reasoning, understanding mathematics would only be procedural or instrumental. Knowledge that lacks justification can easily become unreasonable. According to Ball & Bass (2003), reasoning in mathematics serves a number of important functions. Firstly, without conceptual understanding of mathematics, the knowledge is difficult to use or to be applied to new and varied situations. Additionally, when mathematics is learned as a reasonable discipline, rather than as a set of procedures, the knowledge that has been attained can easily be reconstructed even when the memory of the accompanying procedure has faded.

2.2.7.3 Prior knowledge

Learners do not enter the classroom as blank slates. From the constructivist perspective, it is noted that students draw on previous and concurrent learning from other areas to work with geometry problems. According to Osei (1998), during learning, the learner assimilates new information into existing cognitive schemata. The learning of new information is a process of subsumption by preconceptions already possessed by the learner. The learner (regardless of age) continually retrieves the earlier learnt concepts in order to internalize or interpret the new information for him/her. Osei (1998) argues that these earlier learnt concepts might be the cause of misconceptions if they were not correctly understood and mastered. It also
became evident to Resnick (1980) that the more incomplete the students' knowledge base, the greater the likelihood that the student will generate incorrect inferences, develop misconceptions and produce inaccurate problem solutions.

In mathematics classes, research shows that learners can enter the classroom holding misconceptions that have the strong potential to derail new learning (Chiu & Liu, 2004; Kendeou & van den Broek, 2005). Misconceptions therefore may originate in prior learning. Nesher (1987, p. 33) also makes this point when she says that errors “arise within conceptual frameworks and are based on previously acquired knowledge”. Prior knowledge based on misconceptions will hinder the process of new knowledge acquisition and will cause learners to make errors during engagement with algebraic activities. This view is supported by Olivier (1989) who suggested that “previous knowledge can cause errors in mathematical problems”. This strengthens the argument that prior knowledge based on misconceptions will cause learners to make errors. Smith, DiSessa & Roschelle (1993) share the same sentiments as Nesher (1987) and Olivier (1989) because they also argue that “misconceptions give rise to patterns of errors, stem from learners’ prior knowledge”. Expressing similar views is Mullis et al in Luneta (2008, p. 387) who argues that the issue of the “learner’s background and the context within which learning takes place can be sources of misconception”.

Misconceptions may therefore arise from learners’ prior learning, either in the classroom (especially for mathematics) or from their interaction with the physical and social world. Researchers have agreed that learners' misconceptions are the result of day-to-day experiences in the physical world (Clement, 1987; McCloskey, 1983; Resnick, 1982). According to Nesher, (1987) and Kilpatrick, Swafford & Findell (2001) misconceptions usually originate in prior instruction as learners incorrectly generalize prior knowledge to grapple with new tasks. Their research found that these misconceptions emanate from prior acknowledge as learners attempt to construct mathematical meanings. It seems learners’ errors are a result of naïve concept images that do not measure up to the concept definitions.

2.2.7.4 Knowledge acquisition

Hatano (1996) identifies the learners’ acquisition of knowledge as one of the causes of misconceptions and errors. He argues that misconceptions and errors are a result of the “learners’ inability to construct knowledge”. He is of the view that procedural misconceptions are produced because learners do not swallow a given rule or algorithms but
try to construct something subjectively tenable by induction. An algorithm is a step-by-step procedure for accomplishing a task, such as solving a problem. As learners construct their own knowledge, sometimes it happens that their construction result in errors and misconception. Misconceptions may therefore be invented by learners themselves, through their attempts to make sense of their limited experiences.

In acquiring knowledge, learners are always involved in solving problems. Van Lehn (1983) explained that when solvers encounter a new problem, they try to apply their preexisting knowledge to the new situation. If they fail to solve the problem, they may introduce a repair in the procedure. When the changed procedure is correct, a creative solution is obtained. However, when the changed procedure is incorrect – a misconception is manifest. Thus, according to the repair theory, misconception is not a simple overgeneralization that occurs whenever the solver applies a concept as is to other domains in which it is incorrect; a misconception results from a slight repair to a given procedure. This is the same sentiment expressed by Resnick, Nesher, Leonard, Magone, Omanson & Peled, (1989) who pointed that misconceptions are “intelligent constructions based on what is more often incomplete than incorrect knowledge”.

2.2.7.5 Procedural and conceptual knowledge

These are two forms of mathematical knowledge, namely procedural and conceptual knowledge. Eisenhart, Borko, Underhill, Brown, Jones & Agard (1993, p. 9) defines procedural knowledge as a “mastery of computational skills and knowledge procedures for identifying mathematical components, algorithms, and definitions, i.e., knowing how to identify a problem, in its broadest and most routine sense, and how to solve correctly”. According to Hiebert & Lefevre (1986) procedural knowledge is regarded as competence of carrying out a mathematical task, the know-how of mathematics and not the know-why. Procedural knowledge is usually taught through drill and practice and so can be automated to carry out specific mathematical tasks rapidly and efficiently. This has led to Hiebert and Carpenter (1992) defining procedural knowledge as a rule-oriented and instrumental. Procedural knowledge therefore deals with to know how and not to know why.

Eisenhart, Borko, Underhill, Brown, Jones & Agard (1993, p. 9) distinguish two types of procedural knowledge. The first is the Knowledge of the format and syntax of the symbol representation system and the second is the knowledge of rules and algorithms, some of
which are symbolic, that can be used to complete mathematical tasks. For example, if one is asked to solve a geometric problem using the formula, the first thing is to remember the formula, then to know what each letter in the formula stands for, do the substitution, simplify and get the answers. This type of knowledge leads to Skemp (1976) instrumental understanding.

Schneider & Stern (2010, p. 178) define conceptual knowledge as “one proving an abstract understanding of the principles and relations between pieces of knowledge in certain domains”. Hiebert and Carpenter (1992) regard conceptual knowledge as knowledge that is rich in relationships and relates to the principles that refine understanding of mathematics and also refers to the interconnections between ideas that explain and give meaning to mathematical procedures. This is alluded to by Luneta (2013) who argues that conceptual knowledge refers to the relationships and interconnections of ideas that explain and give meaning to mathematical procedures. It includes both knowing how and knowing why. This knowledge is important to teachers of mathematics because it enables them to define and explain mathematical concepts in ways that enable learners to understand and articulate mathematics in their own but correct ways (Luneta, 2013).

According to Luneta (2013), conceptual knowledge is acquired through conceptual understanding. These concepts are provided to learners in concept names and as definitions. Morris & Mather (2007, p. 253) argue that, if learners are to be successful in mathematics, they need to understand the concepts underlying basic skills, in other words, they need to gain conceptual knowledge. As problem solving is the centerpiece of mathematics instruction in most mathematics classrooms, learners must also acquire and apply strategic knowledge for solving maths problems (Morris & Mather, 2007, p. 253).

Conceptual knowledge is based on many relationships formed in the mind. It is acquired when learners are able to make the connection between incoming information and existing knowledge. According to Hiebert & Lefevre (1986) if the creation of conceptual geometric understanding is the product of constructive and interpretive activity, then it follows that no matter how carefully and patiently educators explain to their learners, they cannot understand for their students. Once one accepts that the learner must herself/himself actively explore geometric concepts in order to build the necessary structures of understanding, it follows that teaching geometry must be reconceived as the provision of meaningful problems designed to
encourage and facilitate the constructive process. The geometric classroom designed for conceptual understanding, rather than on computational drill, promotes students’ confidence in their own mathematical abilities. For example, as we move sets of objects around, put them together, we acquire implicit understanding of commutativity, associativity, and reversibility. This type of knowledge can be associated with what Skemp (1976) calls relational understanding. It is important for students to understand that executing an algorithm or getting the right answer does not imply conceptual understanding.

The importance of conceptual understanding in mathematics has been well documented, as opposed to low-level procedural knowledge (Worlley & Proctor 2005, p. 2). According to Worlley & Proctor (2005, p. 8), conceptual knowledge in mathematics involves more than just a recall of facts for common examples. According to Kilpatrick, Swafford & Findell (2001, p. 118), conceptual understanding refers to “an integrated and functional grasp of mathematical ideas”. Learners who have conceptual understanding know more than isolated facts and methods understand why a mathematical idea is important, and also the kinds of contexts in which it is useful. Kilpatrick, Swafford & Findell (2001, p.118) argue that learners with conceptual understanding organise their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know. These learners can draw on related networks and adjust and adapt procedural knowledge to help solve unfamiliar problems. However, knowledge will be fragmented and effective problem solving will be impeded without a conceptual framework.

According to Kilpatrick, Swafford & Findell (2001, p. 118), learners who learn with understanding make a connection between facts and methods, which makes it easier to remember and use. Kilpatrick, Swafford & Findell (2001, p. 118) argue that learners are unlikely to remember a method incorrectly if they understand it. These learners monitor what they remember, and also try to figure out whether it makes sense. According to Kilpatrick, Swafford & Findell (2001, p. 118), evidence of conceptual understanding among learners is often looked for in students’ ability to verbalize connections among concepts and representations. However, conceptual understanding needs not be explicit, as learners often understand before they can verbalize that understanding. I therefore regard conceptual understanding in mathematics as well as geometry, as extremely important, as it also promotes a deep approach to learning.
Luneta & Makonye (2010) argue that errors and misconceptions result if learners fail to build procedures from conceptual knowledge. Researchers argue which knowledge ought to come first during teaching and learning of mathematics. Some for example Rittle-Johnson & Siegler (1998) believe that the knowledge that learners must learn first is of no consequence. But Vinner (1990) strongly argues that the main problem is that procedural knowledge is taught at the expense of or before conceptual knowledge. Errors and misconceptions are therefore related to learners’ over-dependence on procedural knowledge which had no conceptual basis.

Many researchers agree that the rote learning of mathematics algorithms without connecting them to the underlying semantic information may be a source of misconception [Lawson & Thompson, (1988), and Marek, Cowan & Cavallo, (2011)]. Systematic errors would then occur when the algorithms are misused (Resnick, 1982). In this case traditional teaching techniques seem to be the major reason in the occurrence of mistakes as it focuses on rote learning. Consequently memorization of rules and procedures should be discouraged. To acquire conceptual and procedural knowledge in geometry requires an appropriate usage of the correct mathematics language.

### 2.2.7.6 Faulty schema

According to Olivier (1989, p. 2) learning basically involves the interaction between a child’s schemas and new ideas. He defined schemas as “knowledge interpreted, organized and structured into large units of interrelated concepts” (Olivier, 1989). Marshall (1995) defines a schema as a mechanism in human memory that allows for the storage, synthesis, generalization, and retrieval of similar experiences. A schema allows an individual to organize similar experiences in such a way that the individual can easily recognize additional similar experiences. Schemas are triggered when an individual tries to comprehend, understand, organize, or make sense of a new situation (Greeno, Collins, & Resnick, 1996). In knowledge construction, there is always a base structure from which to begin construction and this is called a structure of assimilation. The process of continual revision of structures is called accommodation (Noddings, 1990).

Constructivist theories suggest that in order for students to be successful in solving a problem, they should select and apply the correct solving schema. There are situations where
students apply incorrect schemas while having the correct ones in their heads. One possible explanation for this stems from the neural network theory of mind. It indicates that students probably had the correct methods in their long-term memory but they could not recall the information (Matlin, 2005). The theory further says that the students probably had both the correct and wrong schemas in their long-term memory but recalled the wrong information. Despite the existence of correct information, the reason for recalling wrong information was that the correct information was covered or inhibited.

Sometimes some new ideas may be so difficult to any available schema, that it is impossible to link it to any existing schema. This will make assimilation or accommodation impossible. Olivier (1989) then argued that in this case “the learner creates a new box and tries to memorize the idea”. This is rote learning and according to Olivier it is the cause of many mistakes in Mathematics as learners try to recall partially remembered and distorted rules.

Schemas are created by incorporating concepts into its structure by the processes of assimilation and accommodation (Anderson, Reder & Simon 2000; Skemp, 1979). Assimilation occurs essentially by the addition of the new concept structure. Accommodation on the other hand occurs as a result of a conflict between the existing structure and the new one. Since schemas are also stable and resistant to change, this process can be difficult, sometimes impossible, for an individual. It is not simply a case of replacing a link or re-writing a concept/schema. The existing concept/schema may well be valid in a limited context and may also be usefully linked in with other schema as well. Accommodation may be difficult, if not impossible, if the underlying schema is already faulty. If a new concept is successfully incorporated into the existing structure we speak of understanding. If not a misconception may occur. In terms of schema, Anderson, Reder & Simon (2000) found the following to be the causes of misconceptions and errors in terms of accommodation and assimilation: The learner is not cognitively developed enough to comprehend the concept, the concept is not added properly to an existing schema, the existing schema fails to accommodate to additional conceptual knowledge, the existing schema incorrectly accommodates the new information, the existing schema is faulty, the schema is used outside its valid context, problem solving mechanisms use old, faulty schema rather than the new updated ones, many possible schema are available and the wrong one(s) is (are) selected, and new concepts/schema interferes with other correct concepts/schema and cause confusion.
Fong (1995) also supports the view that errors may be a result of faulty schema. Fong defined a schema as the network of interrelationships between different sets of knowledge that constitute a concept and schemata as data structures that represent the generic concepts stored in memory. Fong classifies errors into two levels. The first level is categorized in terms of schematic approach: (E1) complete schema with errors, (E2) incomplete schema with no errors, (E3) incomplete schema with errors. The second level of errors is categorized into four categories: (a) language, including reading and comprehension, (b) operational, including encoding and transformation, (c) mathematical themes such as basic facts, algorithms, and concepts, and (d) psychological factors including motivation and carelessness. Psychological factors are always important factors that affect students’ problem-solving activities. Fong’s (1995) model emphasizes the importance of schematic knowledge to mathematical problem solving.

Table 2.9: The first level’s schematic categories of errors (Fong, 1995).

<table>
<thead>
<tr>
<th>Type of error</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complete schema with errors</td>
<td>This type of error arises when an error is made in computation or encoding of information although the problem solver is able to connect the relevant schema to the problem’s requirement</td>
</tr>
<tr>
<td>Incomplete schema with no errors</td>
<td>In this type of error, students present some, but not all, of the correct steps in the solution. No actual error is made other than incomplete retrieval of a schema leading to a solution. The problem solver has a limited or insufficient schema or is unable to connect all the relevant information that leads to the solution.</td>
</tr>
<tr>
<td>Incomplete schema with errors</td>
<td>This category of errors differs from the above categories in that the student makes errors such as computation and/or encoding errors in addition to demonstrating an incomplete schema or an inability to connect all relevant schemata.</td>
</tr>
</tbody>
</table>
In Singapore and China, Jiang (2012) also conducted a research involving errors caused by faulty schema. The results are displayed in table 2.10 below:

Table 2.10: Total number and percentage of different types of errors the Chinese and the Singapore students made in solving the 11 problems (Jiang, 2012).

<table>
<thead>
<tr>
<th>Error type</th>
<th>China Number</th>
<th>China %</th>
<th>Singapore Number</th>
<th>Singapore %</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1</td>
<td>407</td>
<td>9</td>
<td>431</td>
<td>6</td>
</tr>
<tr>
<td>E2</td>
<td>250</td>
<td>5</td>
<td>438</td>
<td>6</td>
</tr>
<tr>
<td>E3</td>
<td>1770</td>
<td>38</td>
<td>2324</td>
<td>31</td>
</tr>
<tr>
<td>E4</td>
<td>1053</td>
<td>23</td>
<td>2730</td>
<td>36</td>
</tr>
<tr>
<td>E5</td>
<td>1155</td>
<td>25</td>
<td>1609</td>
<td>21</td>
</tr>
</tbody>
</table>

Note. E1 = Complete schema with errors; E2 = Incomplete schema with no errors; E3 = Incomplete schema with errors; E4 = Using irrelevant procedures; E5 = No solution.

The table clearly shows that learners were having challenges in terms of E1, E2 and E3 which are errors involving faulty schema.

2.2.7.7 Mathematical tasks

Research indicates that the type of mathematical tasks given to learners may be a source of misconceptions and errors. Cockburn (1999) discusses the nature of the mathematical tasks selected by the teacher as having potential for children to make errors. She suggests that consideration must be given to the complexity of the task, i.e. whether the task is sufficiently challenging or not. Consideration should also be given to the way the task is presented and the ability of the learner to translate the task.

Thus, the tasks given to learners were also found at times to be a source of misconceptions and errors. The following were identified:

- **Mathematical complexity** – If a task is too difficult, errors may occur
- **Presentational complexity** – If a task is not presented in an appropriate way, a child may become confused with what is required from them.
- **Translational complexity** - This requires the child to read and interpret problems and understand what mathematics is required as well as understanding the language used. If the task is not interpreted correctly, errors can be made (Cockburn, 1999).
There was also a link between mathematical tasks and learners which had an effect on errors and misconceptions. The following were identified:

- **Experience** – Learners bring to school different experience. Mathematical errors may occur when educators make assumptions about what learners already know.

- **Expertise** – When learners are asked to complete tasks, there is a certain understanding of the basic rules of the task. Misconceptions may occur when a learner lacks ability to understand what is required from the task (Dickson, Brown & Gibson, 1984).

- **Mathematical knowledge and understanding** – When learners make errors it may be due a lack of understanding of which strategies/procedures to apply and how those strategies work (Cockburn, 1999).

- **Imagination and creativity** – Mathematical errors may occur when a learner’s imagination or creativity, when deciding upon an approach using past experience, may contribute to a mathematically incorrect answer.

- **Mood** – The mood with which a task is tackled may affect a learner’s performance. If the learner is not in the ‘right mood for working’ or rushed through work, careless errors may be made.

- **Attitude and confidence** – The learner’s self esteem and attitude towards their ability in mathematics and their educator may impact on their performance. For example, a learner may be able in mathematics but afraid of his/her educator and therefore not have the confidence to work to their full potential in that area (Cockburn, 1999).

Educators should carefully select resources to be used because incorrect use of resources in terms of tasks given to learners may lead to children making errors.

### 2.2.7.8 Negative attitude

Learners experience challenges in mathematics. They end up developing a negative attitude towards mathematics because of the way in which it is taught as a ‘dry’ discipline which is not connected to reality (Tella, 2007, p. 150). The negative attitude and lack of interest contribute to learners’ poor performance in mathematics. According to Ignacio, Nieto and Barona (2006, p. 18) attitude is related to academic self-image and motivation for achievement. Attitude towards mathematics has been found to be directly and positively related to success in learning (Rives, 1992). Negative attitude towards geometry may be a source of errors and misconception (Nkwe, 1985). This is supported by Moja (1982), who
asserted that “negative attitudes resulted when because of unavailability of teachers, some teachers who were not interested in and having no knowledge of the subject were forced to teach the subject”. According to Nkwe (1985), educators’ negative attitudes would then be transferred to learners. Learners would then not appreciate the subject. They will then view geometry as a difficult section of mathematics. This is attested by Howson (2000), who found that “Euclid-style geometry was found extremely difficult and often uninteresting by most school students”. This attitude will result in learners performing badly and develops misconceptions towards geometry. Issues of anxiety surrounding learners own learning of mathematics in general and Geometry in particular, are often cited as problematic and a source of negative emotions towards mathematics. Some learners go so far as to attribute the difficulties they experienced with the learning of Geometry in high school as the cause of their dislike for mathematics in general.

According to Visser (1986), Mathematics is often viewed as a difficult and uninteresting subject by both learners and educators. In such schools there is always a misconception that Geometry is a difficult section.

2.2.7.9 Mathematical anxiety

According to Hlalele, (2012) mathematical anxiety is one of the sources of misconceptions and errors. Anxiety is the feeling of fear which is caused by learner’s continuous failure to successfully complete mathematics tasks (Cavanach & Sparrow, 2011). This frustrates learners and inhibits them from unleashing their innate mathematical abilities. Learners, thus, end up performing poorly in mathematics. Dörfler (2007, p. 106) throws a challenge towards mathematics educators and researchers to change to the better the phenomenon of “widespread anxiety and frustration on the part of many learners of mathematics”.

Hlalele (2012) further argued that Mathematics anxiety has been found to have an adverse effect on confidence, motivation and achievement. This is alluded to by Rossnan (2006) who argues that mathematics anxiety could develop as a result of the learners’ prior negative experiences of learning mathematics in the classroom or at home. Tsanwani (2009) views mathematics anxiety as an irrational and impedimental dread of mathematics. Mathematics anxiety refers to a person’s feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of
ordinary and academic settings (Khatoon & Mahmood, 2010; Leppavirta, 2011; Newstead, 2006; Perry, 2004).

Levine (2008) found that teachers with mathematics anxiety emphasise rule-based strategies and treat mathematics as an arbitrary collection of facts, perhaps to promote an illusion of their expertise and disciplinary power to students. Furthermore, there is often limited classroom interaction, resulting in students’ questions not being asked or answered, and knowledge presented as limited and confusing. Frequently, these teaching strategies perpetrate and perpetuate mathematics anxiety in learners. Mathematics anxiety has been found to relate to mathematics performance of learners (Karimi & Venkatesan, 2009). Mathematics anxiety has been found to decrease the efficiency of an individual’s working memory because intrusive thoughts and worries take the focus away from the mathematics tasks at hand. This makes it difficult for individuals to think logically and results in increased errors and longer processing times when solving problems mentally.

According to Hembree (1990) reducing mathematics anxiety is consistent with improving mathematics achievement and reducing misconceptions and errors. Furthermore, Nashon (2006) found that learners, whose mathematics anxiety decreased, experienced an increase in their level of self-efficacy and reduction in misconceptions. Educators should therefore strive to help their learners deal with mathematical anxiety.

2.2.7.10 Language

Dikgomo (1994, p. 29) identified language as one of the main causes of misconceptions and errors. He found that language plays a role in the construction and maintenance of misconceptions and errors more especially to second language speakers. In his research, Sanders (1994, p. 920) describes misconceptions as resulting from everyday experiences and the unscientific everyday language. This is supported by Douglas, Klentschy, Worth, & Binder (2006) as well as Adler (2004) whose research found that learners especially in Africa where the language of instruction and assessment is not the learners mother tongue perform relatively low particularly in mathematics and science than learners instructed in their mother tongue. Clerk & Rutherford (2000), also argue that misconceptions exist when the structure of an individual’s model differs from that of the situation it is meant to represent. They suggest that language confusion can be sources for misconceptions. Clerk & Rutherford (2000) argue that learners are displaying errors due to incomprehension of items
and not because they are incompetent. As a result of the language barrier learners misinterpret the question because they failed to understand the concept. Douglas, Klentschy, Worth, & Binder (2006, p. 45) assert that language is a critical ingredient for developing the necessary scientific reasoning and conceptual understanding. Language can either help learners to understand a concept or hinder their understanding. In terms of geometry, many learners do not understand geometry language correctly, and as a result, their thinking and performance are badly affected (Sanders, 1994). Representation in geometry and symbols or the language of geometry, in general, are likely to be a major factor affecting misunderstanding. Inappropriate information or lack of it, by teachers is likely to be some of the major contributing factors affecting misunderstanding of geometry in schools.

The problem is that our curriculum uses English as the medium of instruction. The use of English as the medium of instruction has left many learners in our schools struggling to cope with a language that is foreign to them. In support of the above findings, Setati & Adler (2001) argue that learning and teaching mathematics in a classroom where the language of learning and teaching (LOLT) is not the learners’ main language is complicated. This is because learning mathematics has elements that are similar to learning a language, since mathematics, with its conceptual and abstract forms, has a specific register and set of discourses (Setati & Adler, 2001, p. 247). This supports the negative perception among some South African learners and educators that the use of English as a medium of instruction makes the learning of mathematics difficult. This is because most of the learners as well as educators in South African schools are not English speakers. Setati & Adler (2001) further explain that the challenge educators face is to encourage movement in their learners from the predominantly informal spoken language to formal written mathematical language, and this includes both conceptual and calculation discourses. Therefore, Fuys, Geddes, & Tischler (1988) suggest that instruction should carefully draw distinctions between common usage and mathematical usage of language. Educators should be encouraged to consider the learners’ language when developing ideas, but there is also the need for learners to be able to use correct mathematical terminology by the end of the topic (Pegg, 1995).

If there is a language problem, two people sometimes cannot understand each other or follow the thought process of the other. This situation is sufficient to explain why at times teachers fail to help learners in geometry learning. Learners and educators have their own languages, and often educators use a language of a higher level, which learners do not understand. In
terms of the van Hieles, providing learners with information which is above their actual thought level would not help the learners to move to the next higher level. On the contrary, it will take them to a lower level.

When introducing new terms in geometry, and mathematics in general, especially if learners already have knowledge of alternative meanings for them, educators can insist on learners using them verbally in their explanations of solutions to problems etc as much as possible. When learners use them in context verbally as well as in their written solutions they become familiar with the proper terminology that is used. Educators can associate new terms with upgraded diagrams/representations/symbols etc., which learners can connect easily to. In order to enhance conceptual understanding it is important for learners to communicate (articulate) their “linguistic associations for words and symbols and that they use that vocabulary” (Crowley, 1987, p. 13). As learners advance in their geometry studies at school they should be exposed to the appropriate terminology and encouraged to use it correctly.

2.2.7.11 Educators

According to Luneta (2013), the main cause of learners’ errors and misconceptions are mathematics teachers. Luneta (2013) identifies three main reasons why teachers may be sources of misconceptions, namely because of:

- Teaching habits; ineffective teaching approaches such as persistently teacher-centred approaches
- Teachers’ lack of content knowledge can also cause errors in learners’ work because the teacher lacks confidence in the subject and therefore does not explicitly explain the concepts to the satisfaction of the learners
- Wrong questioning techniques can also cause errors, such as incomplete, ambiguous, or unnecessarily difficult questions can cause learners to make mistakes. Such questions are a result of the teacher’s lack or weak content knowledge or pedagogical content knowledge.

Jones, (2000) argues that the successful teaching of geometry depends on teachers knowing a good deal of geometry and how to teach it effectively. Regarding the situation in South Africa, Fish (1996) unequivocally concedes that Euclidean geometry education is a complete disaster ostensibly because it is badly taught. Van Niekerk (2010) categorically ascribes the failure to transform geometry instruction in South African schools to the fact that “... the
majority of mathematics teachers are poorly trained”. Similarly, Fish (1996, p. 8) laments that not all teachers are sufficiently competent to teach the mathematics prescribed in the current syllabus. That includes geometry which in the past had to been removed from the mainstream mathematics curriculum and offered as an optional paper at the National Certificate level. It is obvious that teachers cannot effectively teach topics with which they themselves are uncomfortable. In a study carried out in the North-West province in South Africa, for example, van der Sandt & Nieuwoudt (2003) report that grade 7 teachers and prospective teachers lacked the geometry content knowledge requisite for them to be successful teachers. It is therefore not surprising that many research studies have shown that some educators harbour some misunderstanding, which is eventually passed on to the learners they teach (Marek, Cowan & Cavallo, 2011).

According to Mji & Makgato (2006) learners’ errors and misconceptions have been traced to educators’ explanation and misunderstanding are therefore sometimes perpetuated by Mathematics educators. An educator may teach Geometry badly because of the limited knowledge he/she has about it. According to Atebe & Schäfer (2010), many Mathematics educators are not up to par with the Mathematics syllabus. This may be a reason why our learners have misconceptions and errors.

The recent presentation by Chisholm to the minister of Basic Education highlights inadequate content and pedagogical content knowledge and poor curriculum coverage on the part of educators, as well as educator absenteeism, strike action and union meetings during school hours (DBE, 2012b). This might become breeding fields for misconceptions and errors. The same sentiments were echoed by Hershey (2004) who fingered educators as contributing immensely to the spread of misconceptions. An educator whose own understanding of mathematics is limited but is put in a situation where he or she has to deliver instruction on a subject matter whose content he or she does not fully understand is likely to spread misconceptions. Such an educator is unlikely to identify his or her students’ misconceptions and to remedy these in time.

The research done by King (2002) found that primary school educators generally tend to spend the minimum amount of instruction time on the teaching of geometry. This is supported by Cassim (2006), who in his research found that both primary and secondary educators tend to delay the teaching of geometry to as late as possible in the school year. When the subject is taught, it is usually done using the traditional transmission model. As a
result learners have problems with conceptual understanding in the higher standards or grades where deeper knowledge of geometric concepts is expected or presupposed. Teaching geometry therefore remains problematic because it requires knowledgeable and competent educators. Due to educators’ poor mathematical backgrounds, many abstract concepts and formulas are introduced without paying much attention to aspects such as logic, reasoning, and understanding. This causes many of the learners to think that geometry is very difficult to learn.

The research done by Padayachee (2010) found the high prevalence of unqualified and under-qualified educators in urban township and rural schools could well have given rise to the serious problem of misconceptions and errors about mathematics concepts and terms among the learners propagated by educators whose grasp of mathematical terms was limited. Such misconceptions and errors could result in incorrect responses being given. One of the reasons for educators to be sources of misconceptions and errors is therefore that their mathematical competency is low.

According to de Villiers (1996), numerous studies documented reveal that educators test at low levels of the van Hiele model. Using semi-structured interviews to study the geometric reasoning of 19 educators, Mayberry found that 13% of their responses were at a pre-recognition Level 0, 20% were at Level 1, 19% were at Level 2, 24% were at Level 3, and 25% were at Level 4; and there were no responses beyond Level 4. Vinner & Hershkowitz (1980) studied 5th through 8th grade learners’ and their teachers’ geometric knowledge, reporting that both groups had low level of knowledge concerning basic geometrical figures and their attributes; moreover, both groups exhibited similar patterns of misconceptions in geometry. Similarly, Fuys (1985), who interviewed students from 6th through 9th grades and elementary in-service teachers reported similar deficiencies among the two groups. Mason & Schell (1988) combined Mayberry’s interview questions and a written protocol from Usiskin (1982) as the data source for their van Hiele level analysis of 67 pre-service elementary teachers. They found that 38% of the elementary pre-service teachers in their study were functioning below Level 4; and 8% below the lowest level – Recognition. This is evidence that educators also harbour misconceptions in geometry that might be passed to their learners. Misunderstandings and misconceptions are therefore perpetuated by mathematics educators. Some educators therefore harbour some misunderstandings and misconceptions, which are eventually passed on to the learners they teach.
2.2.7.12 Learning concepts

Misconceptions and errors may be a result of learning concepts. According to Li & Li (2008) results from educational research reveals that a learner may well be paying attention carefully to what is being said, but they are construing it in ways unintended by the expert. That is, they develop a misconception, which is a result of a learner misunderstanding the basic concept of what is taking place. Learning concepts in a non-meaningful way leads to the formation and increasing of misconceptions (Marek, Cowan & Cavallo, 2011). Misconception is the perception of concepts by learners in a different way than their scientifically accepted definitions. The difficulties for learners that can occur in learning concepts can also be related to; time, memory, strategies, concentration, culture, development and insufficiency of teachers (Lawson & Thompson, 1988). Misconceptions are a big impediment in meaningful learning. Especially, the permanent mistakes create great difficulties for the math education to reach its goals if they aren’t avoided on time. Traditional teaching techniques are the major reason in the occurrence of mistakes in terms of learning concepts. According to Marek, Cowan & Cavallo (2011) studies have showed that, as misconceptions are permanent and continuous, and at the same time they are not sufficient to make the student develop the right concepts, it is hard to relieve misconceptions by traditional teaching techniques.

According to Skemp (1976) misconceptions and errors may be a result of the mismatches that occur during the teaching and learning of mathematics. Skemp argues that learning mathematics involves two types of understanding, namely relational and instrumental understanding. To Skemp, the word understanding in mathematics is a ‘faux amis’. ‘Faux amis’ is a term used by the French to describe words which are the same, or very alike, but whose meanings are different (Skemp, 1976). The word understanding is a ‘faux amis’ in mathematics because it has two alternatives meanings attached to it. One can understand mathematics relationally or instructionally. According to Skemp (1976, p. 1), relational understanding means “knowing both what to do and why”. He argues that it is about how mathematical concepts are related to each other to forming mathematical logical structure. It can be called a rich understanding. Instrumental understanding on the other hand refers to understanding of “rules without reason” (Skemp, 1976, p. 1). It simply means not understanding at all. It has to do with asking rules and formulas blindly without understanding. According to Skemp (1976), it involves rote memorization of mathematical
concepts. According to Skemp, two types of mismatches are likely to occur as a result of relational and instrumental understandings. The first is a situation where learners who want to understand instrumentally are taught by a teacher who wants them to understand relationally. The second is a situation where the educator wants his learners to understand instrumentally but his learners want to understand relationally. In the first mismatch, learners will need some kind of rule for getting answers. In the second one the teacher can become a tool of destroying the understanding and reasoning skills of learners.

Skemp (1976) is against instrumental understanding because “it involves a multiplicity of rules rather than principles of more general application”. He argues for relational understanding because “it is more adaptable to new tasks, tasks are easier to remember and that relational schemas are organic in quality” (Skemp, 1976). However Skemp plays a devil’s advocate. He argues that within each context, instructional understanding is usually beneficial more especially if tasks which are learnt require a page of right answers. Also one can often find the right answers more quickly just because less knowledge is involved. Learners therefore need to be taught geometry relationally, and need to understand geometry concepts relationally in order to avoid errors and misconceptions.

2.2.8 Analysis of errors and misconceptions

According to Luneta (2008, p. 385) the concept of “error analysis in the teaching of physics and mathematics has received little attention in physics and mathematics teacher education curricula, let alone in university designed professionally development programmes for science and mathematics academic staff”. This little attention given to error analysis in the educator training background results in educators finding it difficult to identify and address, pedagogically, errors made by learners.

Error analysis has a long history in mathematics education. According to Howell, Fox, & Morehead (1993) error analysis is an assessment approach that allows the educator to determine whether learners are making consistent mistakes when performing basic computations. By pinpointing the pattern of and individual learners’ errors, the educator can then directly teach the correct procedure for solving the problem. Error analysis involves the evaluation of learners’ errors to determine their cause (Luneta, 2013). When errors are diagnosed one determines the learner’s areas of weaknesses, studies the specific errors the
learner is frequently making and attempts to explain why these errors are being made. It is important to know why learners are making certain errors and how to capitalize on the error to facilitate learning. It is critical to identify the cause of the errors because errors are a symptom of the difficulties learners are encountering during a learning experience (Luneta, 2008).

Error pattern analysis provides the educator an effective and efficient method for pinpointing specific problems learners are having with solving problems in geometry. By determining that your learner is consistently using an inaccurate procedure for solving geometry problems, the educator can then provide specific instruction and monitoring to assist the learner to use an effective procedure for solving specific types of computations. Additionally, the educator may discover through error pattern analysis that a learner does not have an accurate working knowledge of a major mathematical concept. In other words, specific types of error patterns can cue the educator that a learner not only uses an ineffective procedure to compute a problem, but that the learner also does not accurately understand an important math concept. More oftentimes than not, error pattern analysis is much more than a diagnostic tool for determining a learner's procedural effectiveness; often, it provides the educator a window for determining what type of knowledge a learner lacks.

Educators typically analyze learners' mathematical errors with the intent to improve instruction and correct misconceptions (Mastropieri & Scruggs, 2002). Evaluating learners' work to determine an appropriate instructional focus to correct errors is one of the main tenets of remedial or corrective education (Luneta, 2013). Identification and analysis of learners' errors has the potential to improve instructional planning and, ultimately, learner performance. It is important for both educators and learners to have the skills to analyze errors. Error patterns can reflect common misconceptions and wrongly applied strategies. Awareness of specific errors can help mathematics educators select appropriate remedial actions, as errors can indicate both knowledge and lack of knowledge.

The identification and analysis of misconceptions in learners’ work is a vital part of the process of moving towards a focus on learning rather than teaching (Eggleton & Moldavan, 2001). Educators need to predict the misconceptions which are likely to occur with particular pieces of work. They should plan questions and approaches which would expose such misconceptions if they occurred. Why, how and what would happen if type of questions
enable educators to probe their learners’ understanding of a topic (Gable & Cohen, 1990). Through error analysis educators can predict errors learners are likely to make in learning certain physics and mathematical concepts, explain why learners make these errors and as such necessitate remedial activities, analyse both computational skills and mathematical concepts and help learners resolve the misconceptions and errors understand and find out where the learners’ misconception lie (Gable & Cohen, 1990).

Error analysis is therefore critical in mathematics teaching because it enables educators to understand the root cause of learners’ misconceptions and how to address them. For example, Kimberle & Kembitzky (2009) examined the improvement of learners’ comprehension of geometric concepts through analytical writing about their own misconceptions using a reflective tool called an ERNIe (acronym for Error Analysis). The purpose of this study was to determine whether the ERNIe process could be used to correct geometric misconceptions, as well as how the accuracy at which the learners were able to analyze their misconceptions related to their ability to correct their misconceptions over the course of an entire academic year. The results of this study were that learners who accurately analyzed their own misconceptions through writing were significantly less likely to repeat them. The factor that most strongly influenced the level of accuracy at which a learner was able to analyze a misconception was prior and present mathematical achievement.

Another method that can be used to analyze learner errors is the Data Informed Practice Improvement Project (DIPIP). According to Brodie, Shalem, Sapire, & Manson (2010), the DIPIP is a professional development project that works with teachers to build and sustain professional learning communities in which educators engage with data from a range of sources and work together to better understand the nature of learners’ errors and how they might respond to them. The focus of the DIPIP project is on educators learning to engage with learner errors. DIPIP promotes the embracing of errors as a point of contact with learners’ thinking and as points of conversation, which can generate discussions about mathematical ideas. In this way learners’ thinking and mathematical knowledge are brought into contact with each other. In terms of the DIPIP project, a key theoretical consequence is that errors are a normal part of the learning process, for both oldtimers and newcomers (Smith, DiSessa, & Roschelle, 1993). The key point is that errors are reasonable, make sense to the person who makes the error and are part of gaining access to mathematics and developing it further. So errors make for points of engagement with current knowledge. This
notion of errors gives us a way to help educators to see learners as reasoning and reasonable thinkers and the practice of mathematics as reasoned and reasonable (Ball & Bass, 2003). If educators search for ways to understand why learners may have made errors, they may come to value learners’ thinking and find ways to engage their current knowledge in order to create new knowledge. Learning about learner errors in professional learning communities in the DIPIP project help educators come to a stronger understanding of data about learner errors, informed by the discussion in the community.

Learners’ errors do not necessarily arise from poor explanations by educators; they are a normal part of the learning process and can come about because a learner is struggling to reconcile two correct understandings. Furthermore, if we understand the source of the error we may be in a better position to correct it. The advantage of a professional learning community is that the focus comes from real classroom data, so educators are motivated to understand the error and understanding it is likely to help in similar situations in the future. Secondly, the educator is willing to acknowledge that he/she does not understand the error and his/her colleagues are able to explain it, given their knowledge of learners. A key element for the success of professional learning communities is to be able to admit to one’s own difficulties in understanding learning and to be able to learn from the knowledge of others. All members of the community need to be able to do this.

The DIPIP creates a collective engagement and inquiry into learners’ errors. Educators are willing to acknowledge weaknesses in their practices, to refrain from blaming learners for their errors, to critique their own teaching and to find ways to improve their practices. The community comes together with various voices and positions to build a stronger, collective understanding of errors in mathematics and the reasons for them.

One of the key principles of the DIPIP project is that in coming to understand learner needs, teachers can come to understand their own learning needs and how to challenge and improve their practice. They come to understand learner errors and the reasons for and reason behind these. A common focus is on understanding why the learners made the errors, what could they have been thinking or what were they doing. Teachers should start asking themselves these questions, as they take on the work of leading the community.

According to Marchand – Martella, Slocum & Martella (2004), educators should recognize the errors, prescribe an appropriate instructional focus, and implement an effective and efficient reteaching plan. The first step in this process, recognizing the errors, is completed through a systematic examination of learners’ mathematics work (Ashlock, 2002). Educators
analyze learners’ mathematical errors with the intent to improve instruction and correct misconceptions (Mastropieri & Scruggs, 2002). Evaluating learners’ work to determine an appropriate instructional focus to correct errors is one of the main tenets of remedial or corrective education for all learners, but especially for low performing students (Fuchs, Fuchs & Hamlet, 1994). Identification and analysis of learners’ geometric errors has the potential to improve instructional planning and, ultimately learners’ performance.

Educators should therefore have the skills to analyze learners’ errors. The following table shows steps that describe the process for completing error pattern analysis, (Howell, Fox, & Morehead, 1993):

Table 2.11: Steps that describe the process for completing error pattern analysis (Howell, Fox, & Morehead, 1993).

<table>
<thead>
<tr>
<th>Step</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Collect a sufficient number of learner computation samples for each type of problem you are interested in (at least 3 to 5 samples for each type of problem).</td>
</tr>
<tr>
<td>2</td>
<td>As the learner works the problems, encourage him/her to talk aloud about what they are doing. Do not cue the learner in any way.</td>
</tr>
<tr>
<td>3</td>
<td>Record all learner responses, both written and verbal.</td>
</tr>
<tr>
<td>4</td>
<td>Review the responses and look for patterns among common problem types</td>
</tr>
<tr>
<td>5</td>
<td>Also look for examples of &quot;exceptions&quot; to an apparent pattern (accurate &quot;exceptions&quot; may indicate that the student has partial understanding of the procedure or of a basic concept).</td>
</tr>
<tr>
<td>6</td>
<td>List in simple words the patterns you discover, then write beside each pattern why you think it is causing the student problems (e.g. if a learner fails to regroup double digit addition problems, it may indicate that he does not understand the concept of place value).</td>
</tr>
<tr>
<td>7</td>
<td>Interview the learner by asking her to explain how she solved the problem. Hearing what a learner was thinking can help you confirm suspected error patterns and how they are impacting your learner’s success.</td>
</tr>
</tbody>
</table>

2.2.9 Handling of errors and misconceptions

2.2.9.1 General handling

Educators and learners should have the ability and skills to handle errors. The most important issue is how to engage with errors without blaming learners. Educators can handle errors in different ways. According to Swan (2005) one way is to avoid errors, which “may arise from educator concerns about judging or shaming learners, or a fear that bringing errors into the
public realm will support a spread of errors among learners and create more obstacles and stumbling blocks”. This approach ignores the need for learners to gain access to appropriate mathematical knowledge.

A second way is to correct errors. This makes the appropriate knowledge available to learners but depending on how the errors are corrected, may not illuminate the criteria by which something is judged to be an error, thus still denying access to important mathematical knowledge.

A third possibility is to embrace errors as a point of contact with learners’ thinking and as points of conversation, which can generate discussions about mathematical ideas (Swan, 2005). In this way learners’ thinking and mathematical knowledge are brought into contact with each other. According to (Borasi, 1994), errors are reasonable, make sense to the person who makes the error and are part of gaining access to mathematics and developing it further. So errors make for points of engagement with current knowledge. This notion of errors gives us a way to help educators to see learners as reasoning and reasonable thinkers and the practice of mathematics as reasoned and reasonable (Ball & Bass, 2003). If educators search for ways to understand why learners may have made errors, they may come to value learners’ thinking and find ways to engage their current knowledge in order to create new knowledge.

Educators should understand learners’ errors, contemplate their causes and methodologically correct them. Educators have to accept students’ right to err, especially when learners face a new, unusual situation. A familiar action scheme cannot be immediately applied in an unusual situation, it has to be either adjusted (accommodation) or a new scheme has to be formed in order to solve a problem. Educators should try to understand learners’ errors, try to understand the way learners think, because learners “do not make errors in mathematics thoughtlessly; they either believe that what they are doing is correct, or are not at all sure what they are doing” (Booker, 1988). According to Booker (1988) “to err is human (errare humanum est) as the Latin saying has it”; however, the important thing is what conclusions are drawn from errors, how we learn while erring “(errando discimus)”. Educators should teach learners how to control themselves if they make errors, how to correct them, and teach them how to master errors. A systematic self-control (auto-control) teaching consists in teaching learners certain strategies, certain actions which increase their chances for a correct final solution. They do not guarantee that learners will not make errors or that they will
recognize errors in their solution and will be able to correct them, but they facilitate correctness control and error recognition (Polya, 1945).

Eggleton & Moldavan’s study (2001) showed that the instructional strategy of warning learners about errors did not significantly reduce the number of errors made. Vinner (1990) contends that learners’ misconceptions lead to logical inconsistencies when the instructor’s cognitive structure is imposed on the learners. Learners may fail to appreciate the implication of a counterexample because they do not perceive this stimulus as being related to the formula under consideration. Logical contradictions may not foster understanding, because learners do not expect mathematics to make sense (Carnoy, et al., 2008). Educators should allow learners to work through erroneous thinking as they would in a problem solving situation, that is, with minimal learning guidance. Educators may benefit from some examples of orchestrating classroom discourse using strategies that address error production.

### 2.2.9.2 Handling learners’ errors and misconceptions in terms of a theory.

Learners’ errors in mathematics learning are a world-wide phenomenon, and there is a long history for error analysis in mathematics education (Radatz, 1979). Due to the variety and significance of learners’ mathematical errors, it attracts a number of researchers’ interests, which leads to the formation of many theories about the nature of mathematical errors, their interpretation and the ways of overcoming them (Gagatsis & Kyriakides, 2000; Luo, 2004).

It is a fact is that our learners often make mistakes in geometry. Unless we can say why they make these mistakes, we are unable to do something about it. According to Olivier (1989), errors and misconceptions should be interpreted in terms of learning theories. According to Olivier (1989), a theory is like a lens through which one views the facts; it influences what one sees and what one does not see. Below is the discussion of three common learning theories (behaviorism and constructivism, and cognitivism), which can be used to interpret and understand geometry misconceptions and errors.

According to Ertmer & Newby (1993), behaviorism is based on observable changes in behaviour. Behaviorism assumes that learners learn what they are taught, or at least some subset of what they are taught, because it is assumed that knowledge can be transferred intact from one person to another (Jung & Orey, 2008). The learner is viewed as a passive recipient
of knowledge, an empty vessel to be filled, and a blank sheet on which the teacher can write. Behaviourists therefore, believe that knowledge is taken directly from experience, and that learners’ current knowledge is unnecessary to learning. According to Baine (1982), this theory sees learning as conditioning, whereby specific responses are linked with specific stimuli. The organization of learning must proceed from the simple to the complex; short sequences of small items of knowledge and exercise of these in turn through drill and practice. One learns by stockpiling, by accumulation of ideas.

There are criticisms that are levelled against behaviorism. Behaviorism does not take mental processes of learning into consideration (Jung & Orey, 2008). Behaviorism views learning as something that happens to a person, with the person being passive. We all know that every learner is active, both mentally and physically, when engaged in learning. Behaviourism does not account for all types of learning, since it disregards the activities of the mind. What goes on inside the mind of a person is of extreme importance in understanding the learning processes. Behaviourism does not explain some forms of learning such as the recognition of new language patterns by young children.

The early research on mathematics education viewed learners’ errors as problems that needed to be avoided, and misconceptions as something that needed to be replaced with accurate information (Even & Tirosh, 2008). This stems from a behaviorist view of learning. With behaviorism it is solely the experience that makes the associations that determine what is learned by the person (Skinner, 1938). For the behaviorist, learning simply requires making new associates from the environment. With this approach the learner is a passive recipient responding to environmental stimuli. The learner is viewed as a blank slate (i.e. tabula rasa) and behaviour is shaped through positive reinforcement or negative reinforcement. For education purposes the underlying assumption is that knowledge can be transferred intact from one person to another and any current knowledge a learner has is irrelevant to the learning process (Mamba, 2011).

From a behaviorist perspective, errors and misconceptions are not important, because it does not consider learners’ current concepts as relevant to learning. According to Olivier (1989), errors and misconceptions are seen rather like a faulty byte in a computer’s memory. If people don’t like what is there, it can simply be erased or written over, by telling the learner the correct view of the matter. According to behaviorists, the effects of incorrect rules of computation, as exhibited in faulty performance, can most readily be overcome by deliberate teaching of correct
rules. This means that teachers would best ignore the incorrect performances and set about as directly as possible teaching the rules for correct ones. Behavioural approaches to instruction tend to consider that errors, though an unavoidable element in learning, can be controlled and eventually eliminated through well-designed instruction (Baine, 1982).

The cognitive view of learning sees knowledge as given and absolute (Bigge & Shermis, 2004). Many of the information processing models of teaching and learning are based on the cognitive view of learning. Cognitivism stresses the acquisition of knowledge and internal mental structures (Jung & Orey, 2008). In other words, it focuses on how information is received, organized, stored, and retrieved by the mind. Cognitivists also place great emphasis on environmental conditions to facilitate learning (Cecil, 2008). However, contrary to behaviorism, the learner is characterized as being very active in the learning process (Ertmer & Newby, 1993). Therefore, environmental conditions are not enough to explain whole instructional situation. According to Jung & Orey (2008), the way that learners attend to, code, transform, rehearse, store and retrieve information and learners’ thoughts, beliefs, attitudes, and values are also key elements of learning process. Memory in cognitivism is prominent because cognitivists regard memory as the result of learning. The actual goal of instruction for behaviorism and cognitivism is often to communicate or transfer knowledge to students. According to Ertmer & Newby (1993), some of the characteristics of instructions in cognitivism are emphasis on the active involvement of the learner in the learning process, the use of hierarchical analyses to identify and illustrate prerequisite relationships, the emphasis on structuring, organizing, and sequencing information to facilitate optimal processing, the creation of learning environments that allow and encourage learners to make connections with previously learned material.

The cognitivists approach views misconceptions as crucially important to learning and teaching, because misconceptions form part of a learner’s conceptual structure that will interact with new concepts, and influence new learning, mostly in a negative way, because misconceptions generate errors. Misconceptions and errors therefore serve as the basis for improving teaching and learning in mathematics. According to cognitivists, errors can be controlled and eventually eliminated in ways that are similar to those that behaviorists suggest: through careful scheduling and review of new and difficult information (Cecil, 2008).

The constructivists’ theory of teaching and learning look at learning from the perspective of the learner (Li & Li, 2008). Because knowledge cannot be transferred ready-made, to support
the learner to construct his own knowledge, discussion, communication, reflection and negotiation are essential components of a constructivist approach to teaching. According to Mamba (2011) from a constructivist perspective, misconceptions are crucial to learning and teaching because misconceptions form part of learners’ conceptual structures that will interact with new concepts and influence new learning mostly in a negative way, because misconceptions generate errors. Errors and misconceptions are seen as the natural result of learners’ efforts to construct their own knowledge, and these misconceptions are intelligent constructions based on correct or incomplete (but not wrong) previous knowledge. Making errors should be regarded as part of the process of learning.

A constructivist perspective on learning (Smith, diSessa & Roschelle, 1993) assumes humans generate knowledge and meaning from an interaction between their experiences and their ideas. Here the current knowledge a learner has will help determine what the learner will learn. With constructivism it is the interaction between cognitive structures and experiences that makes the associations that determine what is learned by the person. An important concept in regards to constructivism is schemas. For the constructivist learning often requires assimilating into existing schemas. Misconceptions are important to this view because they occur when old information in schemas interacts with newly assimilated information (Derry, 1996). These misconceptions can twist and distort information generating errors. Because of the nature of learning, according to this view, misconceptions are considered inevitable and can be used as an important source of information about the learning process (Derry, 1996). This means educators should regard errors as a clue for uncovering what learners already know and how they have constructed such knowledge (Mamba, 2011). Constructivism emphasizes prior knowledge. Errors are expected in the early stages of learning and are considered to be part of the process. Learners usually recognize the current level of knowledge to be inadequate and learn by “transforming and refining that prior knowledge into more sophisticated forms” (Smith, diSessa, & Roschelle, 1993, p. 123).

This research embraces the constructivist learning theory as the theory to handle misconceptions and errors. The behaviourists and cognitive approaches are rejected. They are not learner centred and errors are viewed from a negative perspective. Errors and misconceptions are seen rather like a faulty byte in computer’s memory which can easily be removed like when you delete something from a computer. This is not true because errors are strongly held by learners and are part of their schema. The constructive perspective is
embraced because they regard making errors as part of the learning process. Misconceptions are considered inevitable and should be used as an important source of information about the learning process. Educators should regard errors as a clue for uncovering what learners already know and how they have constructed such knowledge.

2.2.10 Recognising mistakes: the better option

It is important for educators to understand how errors should be viewed and handled. The most important question to ask is: Are errors to be avoided, or are they to be recognized as part of the problem-solving process? An educator’s answer to this question may not be clear-cut. Some educators have become so accustomed to the common mistakes made by learners that their strategy for teaching precision is to warn learners about possible mistakes in advance. Despite their warnings, the same errors consistently appear on subsequent classworks and tests. This is a clear indication that no matter what strategies educators use in class, errors will never be avoided. According to Askew & William (1995, p. 13) it seems that to “teach in a way that avoid pupils creating any misconceptions … is not possible, and that we have to accept that pupils will make some generalisations that are not correct and many of these misconceptions will remain hidden unless the teacher makes specific efforts to uncover them”. This simply means that no matter how hard educators try to ensure that learners avoid making mistakes, they will not succeed because mistakes are part and parcel of the learning process.

Rather than try to tell the learners about a common errors to avoid, educators should require learners to confront these errors, using a learner’s incorrect answer to develop further understanding and reasoning through exploration. As learners share their responses to their errors, they will gain new knowledge. They may still make similar errors in the future, but they have now had the experience of analyzing the mistake to know why it is incorrect. When the learners make errors in practicing new skills, educators should encourage them to explore their errors. The approach should be to allow learners to value the work of correcting errors by accepting ownership of the common mistakes and the reasoning processes used to overcome them. According to Schoenfeld (1992) errors are part of the process of problem solving, which implies that both educators and learners need to be more tolerant of them. If no errors are made, then almost certainly no problem solving is taking place. Unfortunately, one tradition of schooling is that perfect performance is often exalted as an ideal. Errors are
seen as failures, as signs that the highest marks are not quite merited. Worse still, errors are sometimes ridiculed or taken as ridiculous. Mistakes and embarrassment often go hand in hand. Perfect performance may be a reasonable criterion for evaluating algorithmic performance, but it is incompatible with problem solving. Errors should therefore be encouraged for new learning to take place.

There is no doubt that some mathematical errors could be avoided by educator awareness, skilful choice of task and clarity of explanation. However, Swan (2005) suggests that, despite what they are taught, learners seem to make the same mathematical errors, and construct their own alternative meanings for mathematics, all over the world. This challenges notions about teaching to avoid learners developing such mistakes and misconceptions. Educators need to learn how to recognize common learner errors and misconceptions in mathematics, and to understand how these arise, how they can be prevented and how to remedy them (Mamba 2011). Cockburn (1999) and Koshy (2000) both reflect a growing view in the research evidence that mathematical errors can provide a useful insight for educators into a learner’s thinking and understanding, an effective mechanism for assessment for learning and, with sensitive handling, can enable learners to learn from mathematical mistakes viewing them as learning agents (Askew & William, 1995). The questions to be asked in terms of (TTA, 2003, p.7) are whether educators use learners’ mistakes and misunderstandings positively to provide an opportunity to improve understanding for all, and do they avoid causing embarrassment to pupils or making them afraid to make mistakes?

Current thinking and research is recommending a shift in how both educators and learners regard mathematical errors and misconceptions, moving from a let’s plan to avoid strategy to one which seeks to give greater status and value to learning from mistakes as a mechanism to assist further learning. It is self-evident that such a shift will necessitate adopting a “constructive attitude to their pupils’ mistakes” (Koshy, 2000, p. 173), as well as educators and learners recognizing that analysis and discussion of mistakes or misconceptions can be helpful to their mathematical development. Spooner (2002) suggests that placing learners in situations where they feel in control of identifying mathematical errors/miscceptions leads to greater openness on the part of the learners to explore and discuss their own misconceptions. Errors are therefore marvelous vehicles, when put to a maximum use, for clarifying misunderstandings and diagnosing weaknesses. This approach has an underlying
belief that learners’ mathematical understanding is more likely to be developed if children are given opportunities to:
  - explain their thinking;
  - compare their thinking with that of peers and educators.

To be effective in terms of long-term gains these opportunities need to be embedded within a school and classroom culture which accepts and promotes that learners can learn effectively from their peers and need encouragement to be brave to express their mathematical ideas. The most effective educators cultivate an ethos where learners do not mind making mistakes because errors are seen as part of the learning. In these cases learners are prepared to take risks with their answers (Ofsted, 2002). A significant feature of such approaches would be recognition by learners that learning often involves having to shift one’s thinking. Swan (2005) encourages a more radical shift in educators’ thinking, suggesting that far from trying to teach to avoid learners’ developing misconceptions the latter should be viewed as helpful, and possibly ‘necessary’, stages in children’s mathematical development. This suggests that a focus on how learners are taught mathematics, rather than on what mathematics they are taught, is needed.

**Implications for teaching approaches**

According to Mestre, (1989) misconceptions are a problem for two reasons. First they interfere with the learning when students use them to interpret new experiences. Second, learners are emotionally and intellectually attached to their misconceptions, because they have actively constructed them. Hence, learners give up their misconceptions, which can have such a harmful effect on learning, only with great reluctance. Hasan et. al. (1999) claim, “misconceptions are strongly held cognitive structures that are different from the accepted understanding in a field and that are presumed to interfere with the acquisition of new knowledge”. The treatment to overcome the misconception requires the elimination of the misconception followed by re-teaching, reconstructing and reinforcing the correct concept and skills.

The only way to avoid the formation of entrenched misconceptions is through discussion and interaction. A trouble shared, in mathematical discourse, may become a problem solved (Wood, 1988, p. 210). Anghileri (2000) refutes the notion that common errors or
misconceptions will be spread amongst children through discussion. Rather, she suggests that such activities will encourage children to review their thinking, leading to self-correction. The value in listening to explanations and the reasoning of others is viewed not only in the benefits to the restructuring of the specific and immediate mathematical idea, but also in the overall contribution to the development of individual mathematical thinking. This would suggest that the skills involved in using logic, reasoning, communication and problem solving – the very skills inherent in learners’ ability to use and apply mathematics – are actively developed by teaching beliefs and approaches which are deemed as connectionist (Anghileri, 2000).

Tanner & Jones (2000) suggest that restructuring thinking to accommodate new knowledge is not easy. This could be described as presenting the learners with uncomfortable learning as previously assimilated knowledge has to be revisited, reshaped and challenged. The learning process and environment needs to be of sufficient importance to the children in order for them to make the effort to restructure and change their thinking. Educators need to accept that just explaining the misconception is not enough. Learners will also need help in the restructuring process. Brodie, Shalem, Sapire, & Manson, (2010) suggest that the benefit to long-term learning is greater when children encountered misconceptions through their own work than when teachers choose to draw attention to potential errors/misconceptions in their introduction to topics. Swan (2005) believes that mistakes and misconceptions should be welcomed, made explicit, discussed and modified if long-term learning is to take place. He suggests that this is unlikely to happen unless the educator and the learner negotiate the social nature of the classroom and establish a classroom ethos based on trust, mutual support and value of individual viewpoints.

**Implications on planning lessons**

Learners will always make some generalizations that are not correct and many of these misconceptions remain hidden unless the teacher makes specific efforts to uncover them (Askew & William, 1995, p. 12). Effective teaching of mathematics should involve the planning to expose and discuss errors and misconceptions in such a way that learners are challenged to think, encouraged to ask questions and listen to explanations, and helped to reflect upon these experiences. This suggests that the more aware educators are of the common errors and possible misconceptions associated with a topic, the more effective will
be the planning to address and deal with learners’ potential difficulties. The role of questioning, dialogue and discussion is significant if learners are to shift their perspectives on only contributing if they think they have a correct answer, or the answer they believe is wanted by their teacher.

2.2.11 The confrontation of misconceptions and errors.

To neutralize the interference of misconceptions, instruction should confront learners with the disparity between their misconceptions and expert concepts. When the disparity becomes explicit, students will appreciate the advantages of the expert concepts and give up their misconceptions. Researchers who developed classroom approaches to misconceptions have often proposed rational competition between misconceptions and corresponding expert concepts (Champagne, Gunstone & Klopfer, 1985). This instruction first has students articulate their unconscious misconceptions and then establishes a framework for comparing the validity of the competing ideas (Champagne, Gunstone & Klopfer, 1985). Confrontation begins as an external, social interaction in the classroom, but for confrontation to succeed, the competition between misconceptions and expert concepts must be internalized by students. Confrontation and replacement are therefore inextricably linked: Successful instructional confrontation leads to learning by replacement.

For classroom instruction to be successful in confronting misconceptions, educators should present expert concepts in clear opposition to students’ faulty conceptions. This instruction should include demonstrations and activities that produce counterevidence and plausible conceptual alternatives to target misconceptions. The confrontation of ideas in the classroom is then internalized by students as a psychological process of competition that finally results in the replacement of the misconception. There are both strengths and weaknesses in this conceptualization of classroom instruction. Energetic classroom discussions, in which learners take positions, make sense of and explain problematic phenomena, and articulate justifications for their ideas are needed. Activities that learners are given should produce cognitive conflict and conducive to conceptual change.

In confronting misconceptions, educators need to be careful. According Champagne, Gunstone & Klopfer (1985) the instruction designed to confront learners’ misconceptions head-on is not the most promising pedagogy. It denies the validity of learners’ conceptions in
all contexts; it presumes that replacement is an adequate model of learning; and it seems destined to undercut learners' confidence in their own sense-making abilities. Rather than engaging learners in a process of examining and refining their conceptions, confrontation will be more likely drive learners underground. Targeting particular misconceptions for confrontation and replacement overemphasizes their individual importance relative to broader system-level issues. The goal of instruction should be not to exchange misconceptions for expert concepts but to provide the experiential basis for complex and gradual processes of conceptual change. Cognitive conflict is a state that leads not to the choice of an expert concept over an existing novice conception but to a more complex pattern of system-level changes that collectively engage many related knowledge elements.

### 2.2.12 Error feedback

According to Luneta (2013, p. 44), the management of learners’ work, that is correcting the work, giving sufficient feedback and reinforcing the learners’ efforts is critical to learning mathematics. Learners’ work should be marked and sufficient feedback is provided on time. Feedback should be accompanied by encouraging comments. Marking would include identification of learners’ misconceptions and associated errors. Good marking is essentially error analysis.

It is important for learners to get feedback on their misconceptions and errors. According to the literature corrective feedback should be varied (Allwright & Bailey, 1991). It should facilitate monitor use, i.e. the ability to self-correct resulting from learned knowledge of grammatical forms (Krashen, 1987). It should be appropriately pitched with effective support (Tomasello & Heron, 1989). It should emphasize content and communication of meaning (Holley & King, 1971). There are four dimensions of corrective feedback as suggested by various researchers.

- Types and features of corrective feedback include recasts (repetitions and expansions), clarification requests, and confirmation checks (Chaudron, 1988).
- Cognitive orientation which focuses on linguistic devices that allow learners to develop their explicit grammatical knowledge necessary to self-monitoring (Krashen, 1987).
Psychological corrective feedback which can be negative, positive and neutral (Vigil and Oller, 1976). Each must address the affective (i.e. appeal to emotional attitudes) as well as the cognitive nature of learning, but there must be balance between them (Edwards, 1995). The most effective in encouraging appropriate grammatical modifications in learners is the positive-affective and negative-cognitive combination (Vigil and Oller, 1976), i.e. when corrective feedback is accompanied by positive and encouraging tones of voice, gestures and facial expressions.

Implicit–explicit dichotomy. Explicit feedback is any feedback that overtly states that a learner’s output is not correct, with clear information about the state of the learner’s utterance. Implicit feedback consists of devices such as confirmation checks and requests for clarification, from which learners should infer that the form of their utterance is responsible for the teacher’s comprehension problems (Carroll & Swain, 1993 p. 361).

Giving learners feedback on their misconceptions and errors will go a long way in helping learners handle their misconceptions.

2.2.13 Replacement of misconceptions and errors

Because of their strength and flawed content, misconceptions interfere with learning expert concepts. According to Hiebert & Behr (1988) misconceptions interfere with proportional reasoning. The source of learners' difficulty in learning in mathematics has been attributed to the interference of misconceptions (Shaughnessy, 1985). Because misconceptions are so prevalent, learning mathematics must involve a shift away from misconceptions to expert concepts. This shift is often characterized as replacement: More adequate expert ideas must be developed and replace existing misconceptions. Learning involves both the acquisition of expert concepts and the dispelling of misconceptions.

Much misconceptions research has suggested that learning is a process of replacing misconceptions with appropriate expert knowledge. As dictated by constructivist thought, inadequate knowledge is replaced by more sophisticated levels of thinking. Therefore, any misconceptions would be flawed and consequently replaced. According to Smith, diSessa & Roschelle (1993) two main reasons exist to question the idea of replacement: first, “empirical evidence of the complexity of knowledge learning” (Smith, diSessa, & Roschelle, 1993, p. 125), and secondly, replacement conflicts with the constructivist concept of adapting prior
knowledge. The plausibility of replacement depends on very simple models of knowledge. Misconceptions (and expert conceptions) are taken to be unitary, independent, and therefore separable cognitive elements. Learning is a process of removing misconceptions from learners' cognitive structures and inserting appropriate expert concepts in their place. However, the relationship between particular conceptions and the cognitive structures that embed them are far more complicated than such unitary models suggest (Schoenfeld, Smith & Arcavi, 1993).

Replacement conflicts with the constructivist premise that learning is the process of adapting prior knowledge. According to Schoenfeld, Smith & Arcavi (1993), the critical question raised by replacement is: What prior knowledge is involved in the construction of the expert concepts that replace misconceptions? If we accept the mistaken character of misconceptions, they cannot serve as resources. The other possibility is that learners have some complementary pool of productive knowledge that can be brought into competition with misconceptions, but misconceptions researchers have not identified such resources within the novice understanding.

The simple addition of new expert knowledge and the deletion of faulty misconceptions-oversimplifies the changes involved in learning complex subject matter. By remaining mute on the processes and the specific conceptual resources involved in learning, replacement is similar to tabula rasa models of learning in asserting that any new acquisition is possible (Schoenfeld, Smith & Arcavi, 1993). According to Smith (1992), literal replacement itself cannot be a central cognitive mechanism nor does it even seem helpful as a guiding metaphor. Evidence that knowledge is reused in new contexts (that knowledge is often refined into more productive forms) and that misconceptions thought to be extinguished often reappear all suggest that learning processes are much more complex than replacement suggests (Schoenfeld, Smith & Arcavi, 1993).

To be successful in confronting misconceptions, educators should provide clear opposition to learners’ flawed conceptions (Derry, 1996). As learners demonstrate flawed thinking, expert ideas must be presented to counteract the misconception. Learners are then faced with a choice. They must make a rational choice whether or not to replace their possible misconception with the expert idea presented to them (Derry, 1996). Replacement of faulty misconceptions is an oversimplification of the complex processes involved in learning.
replacement itself cannot be a central cognitive mechanism, nor does it even seem helpful as
a guiding metaphor. Ideas which initially have appeared to be misconceptions have not in fact
been replaced, but seem to have been “refined into more productive forms” and often
reappear suggesting that learning processes are in fact much more complex than simple
replacement suggests (Smith, diSessa & Roschelle, 1993, p.153). Confrontation appears to
undermine learners’ thinking and confidence in their own ideas rather than engaging them in
examination and refinement of their conceptions (Smith, diSessa & Roschelle, 1993, p.154).

Smith, diSessa & Roschelle (1993, p.154) argue that “the goal of instruction should be not to
exchange misconceptions for expert concepts but to provide the experiential basis for
complex and gradual processes of conceptual change”. Cognitive conflict is
a state that leads
not to the choice of an expert concept over an existing novice conception but to a more
complex pattern of system-level changes that collectively engage many related knowledge
elements”. This view connects nicely with Vygotsky’s theory of zone of proximal
development. This is the realm between what a learner understands and does not understand
and where misconceptions are common place (Barke & Hazari, 2009). To navigate these
murky waters Vygotsky’s suggests learners need a guide to which they can collaborate. It is
only through this experience that a student’s understanding will grow. From the constructivist
perspective learners make sense of new information when it is incorporated in mental
constructs or schemas (Smith, diSessa & Roschelle, 1993). Often an educator’s job is to help
learners to organize their schemas and to find out how well these schemas are connected
(Donovan & Bransford, 2005). This is done by having the learner interact with where they
currently are and where they need to be. This interaction can come from educators, peers, or
even carefully diagnostic feedback (Reveles, Kelly & Durán, 2007). This interaction works
best if it is close to the boundary of what a learner knows and does not know, called the zone
of proximal development (ZPD). For Vygotsky (1979) learning is a social process that
required interacting with others. Vygotsky’s theory distinguishes two levels of development.
The first is the current level of the individual. The second is the level of potential
development. This potential development is the level that the student is capable of reaching
with the assistance of a teacher or peers (Heritage, 2010). To guide in this process it is
important to build scaffolding for the learner (Brooks & Brooks, 1993). An educator may ask
leading or probing questions to elaborate the knowledge the learner already possesses, or
providing feedback that assists the learner to take steps to move forward through the ZPD
(Heritage, 2010, p.8). As the learner becomes more competent in the subject area the
scaffolding is gradually reduced until the learner is able to function independently (Heritage, 2010).

An important entailment of constructivist theory is that misconceptions and the errors they produce cannot be removed or replaced through instruction (Smith, diSessa & Roschelle, 1993). Since they are constructions, they need to be restructured by learners (Hatano, 1996) into more appropriate conceptual structures. According to Hatano (1996), errors and misconceptions provide evidence of the fact that we do construct our own knowledge, precisely because they are not explicitly taught and yet they are constructed by so many learners. Because the misconception makes sense to the learner and is often the product of valid reasoning on the part of the learner, many researchers prefer the terminology of alternate conception, suggesting the appropriateness of the idea in relation to the learner’s different and not deficient conceptual structures.

2.2.14 How to get past errors, from an emotional perspective

Learners are emotional beings. Educators should be careful not to trigger the emotions of learners by referring to their mistakes as stupid. Referring errors as silly or stupid mistakes carries tremendous amount of emotional charge (Merrit, 2012). Learners may feel that they themselves are silly or stupid due to errors that they are committing. Judging a learner’s behaviour is detrimental to his/her self-esteem. According to Merritt (2012), judging, blaming, and shaming are counter-productive. It's best to unconditionally support the student regardless of how many problems he solves correctly. Holding the intention that the learner is smart, creative, and resourceful will go a long way toward minimizing emotional stressors. Praising the learner for trying, even if he/she makes a mistake, is very helpful in getting past the judgment that the student is somehow a better person if he gets the right answer.

It is important to slow down when doing calculations. Many learners do calculations too quickly and fail to slow down even when it is in our best interest. This is because learners want to prove to others that they are intelligent and they have been conditioned to believe that smart people think and act rapidly. Merritt (2012) suggests that slowing down, being willing to look at emotional factors, and changing our attitudes about how faster is better will go a long way toward reducing errors.
It is also important to strengthen the foundation. Rote memorization with no solid foundation will likely lead to a greater number of errors with a smaller than desired chance that the student will check and find the errors. Ferreting out the source of calculation errors may not always be easy but the process can be very straightforward and rewarding. Keeping the focus on finding and correcting the emotional, physiological, and foundational causes of the situation benefits everyone and keeps the learner’s self-esteem intact.

2.2.15 The role played by errors

In supporting the rationale of studying students’ misconceptions or preconceptions, Carpenter (1995) is convinced that wrong answers by learners can become a point of departure for rich discussion about mathematics pedagogy. Indeed, as Fang (2010) similarly notes, wrong answers can engender a cultural pedagogy of transforming errors in geometric proof assignments into resources for teaching and learning of logical thinking habits from early on. This is alluded to by Smith, diSessa & Roschelle (1993) who argue that misconceptions and the errors they produce are a normal part of the process of constructing knowledge and in fact may be a necessary step in the construction of certain ideas.

Olivier (1989) maintains that misconceptions play a very important role in learning and teaching, because misconceptions form part of the learners’ conceptual structure that will interact negatively with new concepts, which then lead to further misconceptions or alternative conceptions. Errors are seen as value sources of information about the learning process, providing clues that educators should take advantage of in order to uncover current learners’ knowledge and how they come to construct such knowledge. According to Olivier, learners’ erroneous thinking is an important part of learning process. Similarly, Nesher (1987) describes the learners’ errors and misconceptions as the learners’ “expertise, his contribution to the process of learning,” while she discusses the role of learners in a learning situation according to their contribution of expertise. She builds an argument on the contribution of performance errors to the process of learning, indicating that errors and misconceptions do not originate in a consistent conceptual framework based on earlier acquired knowledge but rather are usually an outgrowth of an already acquired system of concepts and beliefs wrongly applied to an extended domain. Any future instructional theory will have to change its perspective from condemning errors into one that seeks them. Also, a good instructional program should predict the types of errors and purposefully allow
them in the learning process. Educators should be aware of their learners’ possible errors and misconceptions and incorporate them into their instructional considerations since they cannot fully predict the effect of the learners’ earlier knowledge system in a new environment.

Several studies have been done that support the notion and point to the many positive roles misconceptions can play in learners’ conceptual understanding (Donovan & Bransford, 2005; Confrey, 1990). Fisher & Lipson (1986) agree with the former statement by arguing that learners’ errors can serve as anchor points for motivating learners and targeting instruction. Perso (1992, p.1) states that errors are “not to be understood simply as a failure of learners, but as the symptoms of the nature of the conceptions which underlie learners’ mathematical activity.”

Mistakes made in the class are actually catalysts for the learning that took place. Borasi (1994, p. 169) describes such errors as springboards for inquiry. The pedagogical approach to errors advocated in this article suggests that not only mathematicians and mathematics educators but mathematics learners as well could benefit from capitalizing on errors as springboards for inquiry by raising and pursuing questions such as: What are the causes and consequences of this error? In what circumstances could this result be considered correct? What would happen if we chose to accept this result?

Swan (2005) views misconceptions as conceptual lenses through which to view learners written and spoken words. Misconceptions have an important place in the mathematics classroom and the mathematics educator should provide opportunity to the learner to manifest his/her misconceptions, and then relate his/her subsequent instruction to these misconceptions. Swan (2005) argues that a ‘misconception’ is not a wrong thinking but is a concept in embryo or a local generalization that the learner has made. It may in fact be a natural stage of development. Nesher (1987, p. 39) states that “a good instructional program will have to predict types of errors and purposely allow for them in the process of learning”. If misconceptions are made explicit and used in learning, learners could restructure and construct new richer knowledge. Therefore, the mathematics educator “should provide opportunity to the student to manifest his misconceptions, and then relate his subsequent instruction to these misconceptions” (Nesher, 1987, p. 39).
Despite the fact that misconceptions lead to errors, Thabane (1994) argued that they “should be viewed as thoughtful efforts to make Mathematical sense”. Misconceptions and errors are legitimate attempts to understand mathematics. The understanding of learners’ misconceptions and errors can therefore help us to understand and remedy errors caused by learners. Oliver (1989) points out that errors and misconceptions are important parts of learning, since they form part of an individual’s conceptual structure that influence, mostly in a negative way, the learning of new concepts. According to Brodie, Shalem, Sapire & Manson (2010), educators should reflect on learners’ performance in ways that do not blame learners or themselves and which provide ways for them to work with learner errors in order to transform them. Errors are an important part of any practice, because they illuminate what mechanisms need to be put in place to give access to the practice. Errors point to the demands of the practice, while at the same time is the point of leverage for opening access to the practice. Errors help educators to see learners as reasoning and reasonable thinkers and the practice as reasoned and reasonable, and bring these two into a relationship. If educators search for ways to understand why learners may have made errors, they may come to value their thinking and find ways to work it into classroom conversations. Errors are also a key area of evaluation for educators. According to Smith, DiSessa, & Roschelle (1993), learner errors are a normal part of the learning process are reasonable and make sense to the learners. Everyone makes errors in mathematics, even good learners and educators, and they provide for points of engagement with current knowledge. Errors also provide a useful focus because educators orient towards errors in different ways. When educators teach learners that mistakes can be the beginning of the learning process, that they are to be used as stepping stones leading to success, educators are much more likely to engender learners (Brodie, Shalem, Sapire & Manson, 2010). Mistakes are like little lessons all in themselves. If learners face them as learning opportunities rather than reminders of their inabilities, then they are bound to be better mathematicians.

Awareness of specific errors can help mathematics educators select appropriate remedial actions, as errors can indicate both knowledge and lack of knowledge. From this perspective, systematic errors are particularly useful. For example, if a learner continually applies the same mistake, the error is easily identifiable, and appropriate action may be taken. Knowledge of the common mathematical errors and misconceptions of learners can provide educators with an insight into learners’ thinking and a focus for teaching and learning (Williams & Ryan, 2000). A social constructivist view of learning suggests that errors are
ripe for classroom consideration; via discussion, justification, persuasion and finally even change of mind, so that it is learners who reorganize their own conception (Cobb, Yackel & McClain, 2000; Tsamir & Tirosh, 2003).

Learner errors can serve as anchor points for motivating learners and targeting instruction. Errors are critical in fundamental mathematics teaching because they enable educators to understand the root cause of learners’ misconceptions and how to address them. According to Melis (2003), a proper usage of erroneous examples stimulates meta-cognition such as self-explanation, reflection, inquiry, critical thinking, diverse reasoning strategies and epistemological attitudes. More specifically, the task to search, explain, and correct errors in (somebody else’s) solution can foster self-explanation (Derry, 1999). Errors in worked examples can serve as a source for stimulating inquiry learning since the learner has to explore the potential problem solving space, think about alternatives, discover the structure, dependencies, and essence of a worked example (Melis, 2003). At the same time, worked example provides a skeleton for the learner’s exploration. Working on erroneous examples and other forms of using failures productively can help in practicing critical thinking. Handling of mistakes is a decisive part of meta-cognition (Melis, 2003). More specifically, finding an error in an erroneous example could be useful for learning because the student has to look at alternative (including erroneous) ways to solve a problem. However, the memorization of errors instead of correct steps has to be prevented. Similarly, error identification may help to develop self-monitoring for situations, where similar errors are made by the learner herself. The learner may train to reason (backwards) from a solution and to find inferences/rules that led to this solution.

Borasi (1994) acknowledges that “errors can be a powerful tool to diagnose learning difficulties and consequently direct remediation.” She encourages an increased awareness of individual differences and difficulties in learning mathematics, and the realization of the inefficiency of remediating errors by simply explaining the same topic over again or assigning additional practice exercises. Cherkas (1992) agrees when he states that “errors with even a strained logic have the potential to be understood and can be used to promote dynamic learning in the classroom.”

Teaching to avoid learners developing misconceptions appears to be unhelpful and could result in misconceptions being hidden from the educator and from learners themselves. This
implies that a shift in the mindset is needed for educators to move from planning mathematical lessons to avoid errors/misconceptions occurring, to actively planning lessons which will confront learners with carefully chosen examples that will allow for accommodation. Swan (2005) argues that misconceptions are a “natural stage of conceptual development” and, consequently, greater time in mathematical lessons should be given to encourage learners to make connections between aspects of mathematical learning and their own meanings. The time needed for reflection, examination of ideas and comparison with ideas of others challenges the present emphasis of the daily mathematics lessons.

Regardless of the time allocated to mathematical discussion or activity, the culture of the classroom has to be one in which learners are rewarded for having the courage to test out their mathematical ideas in order for errors and misconceptions to be aired, discussed and resolved. If getting the right answer, presenting the work in a neat way or completing a set of exercises in a given time is the aim of the activity, and then probing learners’ misunderstandings and misconceptions may prove difficult and counterproductive to effective mathematical learning.

Learners have the right to err. An error might teach a lot both to learners and educators if it evokes a reflection. Educators cannot be afraid of learners’ errors. Educators should try to understand learners’ errors; sometimes learners’ errors reveal more than a correct answer. Errors have to be corrected methodologically. Educators should try to make learners aware of their errors; and create such situations in which learners can discover their errors themselves. If learners do not correct their errors themselves, educators can use the help of other learners. An error analysis and correction with the participation of good learners can be educational for others, also for good learners (Freudenthal, 1988). It is important that learners correct their errors themselves. Educators should create a space for an error provocation. This simply means they should create such a situation in which learners’ errors will be revealed and an opportunity will be created to clarify them. It is significant for the error prevention to systematically teach students self-control (auto-control).

According to Swan (2005) educators should provoke misconceptions and errors, and use them as learning opportunities. Educators should actively encourage learners to make mistakes and learn from them”. Misconceptions and errors are not diseases that can be avoided by better teaching, neither are they caught through a casual encounter. They are reasoned, alternative ways of thinking. Research suggests that teaching approaches which
encourage the exploration of misconceptions through discussions result in deeper, longer term learning than approaches which try to avoid mistakes by explaining the right way to see things from the start (Swan, 2005). According to Mamba (2011) mistakes will not disappear because they have been pointed out to the learner. In fact making mistakes is a natural process of learning and must be considered as part of cognition. As a result Mamba (2011) therefore advocates that errors must be viewed positively

2.2.16 Learning from erroneous examples

According to Wellham (2011), the most powerful learning experiences often result from making mistakes. It is therefore important for learners to learn from their own errors. Tsovaltzi, Melis, McLaren, Meyer, Dietrich & Goguadae (2010) believe that learning from errors can help learners develop (or enhance) their critical thinking, error detection, and error awareness skills. It is therefore important for learners to learn error detection and correction skills. Learners can learn important error detection and correction skills by studying erroneous examples, something that is not possible with correct examples and difficult with unsupported problem solving. Erroneous examples may weaken learners’ incorrect strategies, as opposed to worked examples that strengthen correct strategies (Siegler, 2002). Additionally, similar to worked examples, erroneous examples do not ask a learner to perform as in problem solving, but provide a worked-out solution that includes one or more errors. They could, in effect, reduce extraneous cognitive load in comparison to problem solving while increasing germane cognitive load in the sense of creating cognitive conflict situations (Paas, Renkl & Sweller, 2003). Erroneous examples may, further, guide the learner toward a learning orientation rather than a performance orientation, especially in combination with help that increases learners’ understanding and, hence, their involvement in the learning process (Siegler, 2002). Learners may benefit from erroneous examples when they encounter them at the right time and in the right way. For example, rewarding the learner for error detection may lead to annotations of these errors in memory such that they will be avoided in subsequent retrieval. At the same time, a learner is unlikely to be demotivated by studying common errors in the domain, made by others, as when emphasising errors the student has made him/herself. Erroneous examples have motivational potential (Melis, 2005). Presenting erroneous examples to learners will give them the opportunity to find and react to errors in a way that will lead to deeper, more conceptual learning and better error-detection (i.e., metacognitive) skills. This, in turn, will help them improve their cognitive skills and will
promote transfer. According to Tsovaltzi, Melis, McLaren, Meyer, Dietrich & Goguadze (2010), the effect of erroneous examples depends on whether learners are supported in finding and correcting the error and on when and how they are introduced to the learners. Erroneous examples include instances of typical errors learners make, which address standard problems students face with rule-application, or errors that target common misconceptions and deal with more fundamental conceptual understanding in geometry. Erroneous examples consist of two phases: error detection and error correction.

In their studies with different levels of students, Tsovaltzi, Melis, McLaren, Meyer, Dietrich & Goguadze (2010) found that more advanced learners benefit from erroneous examples with help in terms of cognitive skills (including standard problem solving) in general, as opposed to erroneous examples without help or no use of erroneous examples at all. They also found that deep conceptual understanding is influenced by erroneous examples with additional error detection and correction help. Additionally, they also found that erroneous examples can also influence the metacognitive skill of error detection for highly competent students. Ohlsson (1996) argued that when the competency for finding errors is active, it functions as a self-correction mechanism that, given enough learning opportunities, can lead to a reduction of performance errors. The contribution of such help is also in line with the theoretical work by van Gog, Pass & van Merrienboerg (2004) who have advocated the use of erroneous examples in the context of worked examples as a way for promoting conceptual understanding.

2.2.17 Primary school geometry as a base for minimizing misconceptions and errors

De Villiers (1996) claims that the future of secondary school geometry is dependent on primary school geometry. Unfortunately, Atebe & Schäfer (2011) states that the geometry instruction which primary schools offer is inadequate in providing learners with the necessary thinking skills needed to operate at the level of axiomatic thinking required for most high school geometry. This is an unfortunate situation because the geometry taught in these first years should provide a sufficiently adequate foundation on which to base the later more advanced concepts of Euclidian geometry. The properly teaching of geometry in primary schools can lay a solid foundation which will help to minimize misconceptions and errors.
Most content about geometry in primary school education has largely focused on knowing terms, definitions and attributes of shapes, while neglecting opportunities to manipulate concrete materials (Copley, 2000). Learners tend to memorize mathematical facts such as names and attributes of shapes without understanding them. They learn geometry with misconceptions and carry these ideas throughout their adulthood. Lack of authentic experiences in the primary school, apart from inadequate teaching; have been identified as some of the possible reasons for learners’ misconceptions of geometry (Oberdorf & Taylor-Cox, 1999). Below is an example of a misconception as displayed by a primary school learner in geometry:

**Figure 2.10: A typical learner response on the identification of 2-D shapes and 3-D objects (DBE, 2011c).**

This is a clear sign that learners experience some challenges which needs to be addressed. The challenge is compounded by learners’ competencies in language, since the majority of learners are learning mathematics in English which is a second language to them.

Teaching geometry in primary school level has been described as the most challenging of all teaching (Egsgard, 1970). Piaget (1970) has categorized the cognitive development of children into 3 stages in Primary school:

- The pre-operational stage, in which the child learns through seeing and touching.
- The stage of concrete operations, during which children make comparisons and are able to make simple analyses.
- The formal operational stage, where children make constructions, generalize, and highlight specific properties.
Because Grade R learners are mostly at the pre-operational stage of development, then appropriate activities would be playing with, sorting and ordering solid Shapes such as spheres, cubes, prisms and pyramids, using the visual and haptic senses. Egsgard (1970) says that it is more important to sort and recognize similarities of solids in kindergarten, than to learn the names of the shapes. Once learners have examined the faces of the solids, they will progress to working with two-dimensional shapes. Finally, the “concepts will eventually grow from the experience the learners have with the edges and vertices of solids” (Egsgard, 1970, p. 481). Furthermore, the building of walls and other drawings will help learners to discover the properties of shapes. According to Lee & Das Gupta (1995), Vygotsky opposes Piaget’s view that the “learner’s cognitive development is a spontaneous, internal process”. Vygotsky argues for the cultural significance of concepts, language, and world view. Vygotsky’s theory declares that learning is social in origin. He believes that dialogue with educators is an important aspect of learning. He proposes that the role of the facilitator is to challenge the learner by working in the zone of proximal development. The task of the educator or peer when learners are solving problems is to extend learners beyond their comfort zone and in this way achieve a higher level of cognition. Bruner’s (1966) discovery learning principle has as its basis, on the premise that a discovery approach will encourage the development of thinking. The emphasis of the approach is on the importance of understanding and reasoning in the subject or concept being taught and learned (Anghileri, 1995). The role of the educator is to encourage learners to ‘discover’ relationships and connections between ideas and concepts. Bruner (1966, p. 103) argued that discovery learning ensures a high level of transfer. Biggs in Egsgard (1970) advocates a discovery approach, in groups, for the teaching of geometry in the early years, comprising of learners discovering for themselves, followed by discussion to find out about the learners’ thinking, and the solving of problems which occur in the learners’ physical environment.

Learning in this way will allow learners to practice thinking mathematically. This will help learners to be mathematically literate. To be mathematically literate implies “having the capacity to use mathematics and other techniques and considerations to make sense of authentic real world problems (DBE, 2012a, p. 2). Therefore, learners in the primary school should be assisted by carefully selected activities, to refine the topological ideas they have begun to develop, to make the transition from topological ideas to simple Euclidian ideas, and be exposed to activities which help them discover properties of space, and relationships in space” (Troutman & Lichtenberg, 1995, p. 422).
The use of free drawing (which should follow careful observation) should be encouraged instead of photocopied worksheets which will “interfere with learning to make discriminating observations.” (Troutman & Lichtenberg, 1995, p. 423) According to Teachers’ Lab (1997), the earliest shape activities in school should deal with identifying different figures. At a later, educators should use concrete apparatus and explore the relationships between shapes. According to Teachers’ Lab (1997) “Geometry suffers because we have the mistaken impression that it doesn’t become real, serious mathematics until it gets abstract and we deal with proof.” It is clear from the literature that well-formulated concrete activities early on in the learner’s schooling, will lead to an ability to understand abstraction in Geometry. This will indeed lay a solid foundation to minimize misconceptions and errors.

In conclusion, it is worth mentioning that the Department of Basic Education is well aware of the challenges in education. Annual National Assessments confirm that South African learners are performing well below par, and well below their own talents and abilities (DBE, 2012a). The DBE is well aware that potential of learners is not being realized, talents are not being developed and abilities are not being stretched. The department reveals that the reasons and causes are constantly researched and probed and the results inform specific strategies developed. This research is therefore worth carrying out as it will assist the department in addressing problems of this nature.

Educators, thus, need to help children to develop and build up the right schemata. Learning activities should enable children to assimilate and accommodate experiences into the existing schemata.
CHAPTER 3: RESEARCH METHODOLOGY

3.1 RESEARCH PARADIGM

Stanage (1987) traced ‘paradigm’ back to its Greek (paradeigma) and Latin origins (paradigma) meaning pattern, model or example. Huitt (2011) defined a paradigm as "the basic way of perceiving, thinking, valuing, and doing associated with a particular vision of reality...". The term paradigm in educational research mean a framework that determines the way knowledge is studied and interpreted and the motivation and goal of the research (Mackenzie & Knipe, 2006).

According to Gay & Airasian (2000), there are four main research paradigms in educational research, namely positivism, postpositivism, interpretivism, criticalism. Positivism is a paradigm for natural sciences (Cohen, Manion & Morrison, 2007). It is based on things that can be seen or proved rather than on speculation. Positivism focus on the study of observable behaviour in order to build scientific knowledge (Grix, 2004). The positivist paradigm emphasizes observation and reason as means of understanding human behaviour. In terms of this paradigm true knowledge is based on experience of senses and can be obtained by observation and experiment. According to Trochim (2006) positivistic thinkers adopt this scientific method as a means of knowledge generation; hence, it has to be understood within the framework of the principles and assumptions of science (Bryman, as cited in Grix, 2004).

According to Bryman, as cited in Grix (2004) postpositivism is, as the prefix indicates, a metatheoretical stance following positivism. There has been criticism of the positivist paradigm for applying the scientific method to research on human affairs (Weber, 2004). These opponents argued that uniform causal links that can be established in the study of natural science cannot be made in the world of the classroom where teachers and learners construct meaning (Trochim, 2006). According to Ernest (1994) these criticisms have given rise to postpositivism. Postpositivism is not a rejection of the scientific method, but rather a reformation of positivism to meet these critiques. It critiques and amends positivism. While positivists believe that the researcher and the researched person are independent of each other, postpositivists accept that theories, background, knowledge and values of the researcher can influence what is observed. Postpositivists believe that human knowledge is based not on unchallengeable, rock-solid foundations, but rather upon human conjunctures
Postpositivists hold that knowledge can be known only imperfectly and probabilistically. According to Philips, Nicholas & Burbules (2000), postpositivists believe that human knowledge is not based on unchallengeable, rock-solid foundations; it is conjectural. Postpositivism paradigm focuses on seeking subject understanding through textual reconstruction, without trapping on the certainty of objectivity (Healy & Hoyles, 2000). This research paradigm assumes that knowledge is never fixed and is always changing (Trochim, 2006). Reality can only be estimated. It is not fixed.

The critical paradigm promotes the notion of social justice in order to create the world which is “fairer, more equitable, more inclusive and more harmonious” (Taylor, 2008). In addition, according to Kincheloe & McLaren (2002), critical theory is concerned with the power and justice of several issues in society such as economy, race, gender and education. It considers the power of social politics and ideology which influence educational research (Fay as cited in Cohen, Manion & Morrison, 2007). Critical paradigm thus deals with inequalities in the social world. According to Kincheloe & McLaren (2002, p.100), It is grounded on the premises that the world is composed of people who dominate others and the dominated. The purpose is to expose social inequalities and try to reduce them in order to create the world which is fairer, more equitable, more inclusive and more harmonious (Dash, 1993). The critical paradigm stems from critical theory and the belief that research is conducted for “the emancipation of individuals and groups in an egalitarian society” (Cohen, Manion, and Morrison, 2007, p. 26). The critical educational researcher aims not only to understand or give an account of behaviours in societies but to change these behaviours. Critical theory originated from the criticism that educational research was too technical and concerned with only efficiency and rationality of design, neglecting social inequalities and issues of power (Gage, 1989). According to the critical theorists, researchers should be looking for the “political and economic foundations of our construction of knowledge, curriculum, and teaching” (Gage, 1989, p. 5). Schools play an explicit part in this construction of knowledge based on power in society. In other words, education serves the interests of those who have power, usually the rich males. Schools function to reproduce these inequalities and maintain the status quo. Educational research in the critical paradigm should challenge these reproductions of inequalities. People must challenge dominant discourses. Educational research and schools, like other social institutions, such as the media and the legislatures must be the scenes of the necessary struggles for power (Gage, 1989, p.5). Moreover this research has an agenda, to change the participants’ lives or the structures of the institution.
Interpretivist paradigms study individuals with their many characteristics, different human behaviours, opinions, and attitudes (Cohen, Manion & Morrison, 2007). The interpretivist paradigm can be also called the “anti positivist” paradigm because it was developed as a reaction to positivism. It emphasizes the ability of the individual to construct meaning (Ernest, 1994, p. 25). Interpretivists contend that only through the subjective interpretation of and intervention in reality can reality be fully understood (Bryman, 2001). The study of phenomena in their natural environment is a key to the interpretivist philosophy, together with the acknowledgement that scientists cannot avoid affecting those phenomena they study. Interpretivism’s main tenet is that research can never be objectively observed from the outside rather it must be observed from inside through the direct experience of the people (Olssen, 2006). Researchers in this paradigm seek to understand rather than explain. Knowledge is gained through personal experience. Therefore, its advantage is in finding a meaningful observation of objects. It gives opportunities to seek understanding and make sense of others’ perspectives which are shaped by the philosophy of social constructions (Taylor 2008). Through this paradigm, we can gain a fuller understanding of meanings, reasons, and insight human action (Bryman as cited in Grix, 2004).

This study was exploratory and interpretative in nature, thus data collection and analysis thereof was primarily determined in relation to the contextual setting and perspectives of the learners. In this research, the researcher therefore employed interpretivism paradigm. This is because interpretivism paradigm studies individuals with their many characteristics, different human behaviours, opinions, and attitudes. This paradigm gave the researcher the opportunity to analyse the misconceptions and resulting errors in the teaching and learning of geometry. The advantage of this paradigm is that it helps the researcher acquire knowledge by investigating the phenomena of the world and human in many ways. Its advantage is therefore in finding a meaningful observation of objects. It gives an opportunity to seek understanding and make sense of others’ perspectives which are shaped by the philosophy of social constructions. Through this paradigm, the researcher can gain a fuller understanding of meanings, reasons, and insight human action. Through this paradigm, I got an opportunity to analyse learners’ misconceptions and resulting errors. This gave me an opportunity to understand which misconceptions are held by learners, which errors are made by learners and some of the sources and causes of these misconceptions and resulting errors.
3.2 RESEARCH DESIGN

According to Mcmillan & Schumacher (1997), a research design is a plan and structure of the investigation used to obtain the evidence to answer the research question. This is alluded to by Creswell (2009, p.3) who defines research designs as plans, strategies and procedures for research comprising decisions from the underlying worldviews to the detailed methods of data collection and analysis. Luneta (2013, p. 66) defines the research design as a road map of how the research will be conducted. It includes the method to be used, the data to be collected, where, how and from whom the information will be collected as well as the circumstances under which the information will be collected.

The decision of using a specific research design is influenced by the: worldview assumptions of the researcher, personal experiences of the researcher, audiences of the study, nature of the research problem, research strategy, and methods of data collection analysis and interpretation (Creswell, 2009, p.3). Schulze (2003) views research design as the choice of research strategy and methods based on the researcher’s opinion on how solutions to the research problems may be obtained. The data of this research is contained within the perspectives of learners who were participants. The researcher engaged with them through interviews and through analyzing their written work. I therefore identified the phenomenological research design as the best means for this type of study. The intention of this research was to gather data regarding the perspectives of research participants about the phenomenon of misconceptions and resulting errors in the learning of geometry. I implemented the qualitative research method of phenomenology to allow for an exploration of the misconceptions and resulting errors. In phenomenological research design, the researcher develops an understanding of a subject’s or subjects’ “reality” (Leedy, 1997, p. 161). In essence, this approach investigates an individual’s or group’s perception of reality as he or she constructs it. The researcher becomes personally involved with the research participants and perspectives are shared Leedy (1997). The researcher therefore gains first-hand, holistic understanding of the phenomena.

Phenomenology was founded in the early years of the 20th century by the German philosopher Edmund Husserl, who hoped to return philosophy to concrete experience and to reveal the essential structures of consciousness (Crotty, 1996, p. 14). Phenomenology is from Greek word “phainómenon” which means "that which appears"; and “lógos” which means to
Phenomenology as developed by Edmund Husserl is the unbiased study of things as they appear so that an essential understanding (essence) of human consciousness and experience may be arrived at (Dowling, 2004). It is the philosophical study of the structures of subjective experience and consciousness. Phenomenology is the study of phenomena as experienced by man. The primary emphasis is on the phenomenon itself exactly as it reveals itself to the experiencing subject in all its concreteness and particularity (Giorgi, 1994). Phenomenologists therefore appear to be obsessed by the concrete, the minute and the private, essentially all the experiences that shape human behaviours, values and attitudes (Von Echartsberg 1998).

Phenomenology is both a philosophy and a methodological approach situated in the interpretive paradigm (Guba & Lincoln, 2005). Phenomenology seeks to examine phenomena from the perspective of first-hand accounts and through the lifeworld of people (Schwandt, 1997). Phenomenology is the art and science of drawing meaning and understanding from the way phenomena manifests in everyday experiences (Heidegger, 2008). Phenomenological methods are particularly effective at bringing to the fore the experiences and perceptions of individuals from their own perspectives, and therefore at challenging structural or normative assumptions (Husserl, 1970). For phenomenologists, the starting point of knowledge generation is human experience.

Phenomenology as a research method in education tries to "ward off any tendency toward constructing a predetermined set of fixed procedures, techniques and concepts that would rule-govern the research project" (van Manen, 1997, p. 29). At the heart of every phenomenological research endeavour is a deep questioning of an experience. Phenomenology is a school of thought that emphasizes a focus on people's subjective experiences and interpretations of the world. This simply means that the phenomenologist wants to understand how the world appears to others. Phenomenology is focused on the participants lived experiences. This type of research is interested in specific concrete experiences and how the participants perceive and feel about those experiences. According to Rossman & Rallis (1998) those engaged in phenomenological research focus on the in-depth of the meaning of a particular aspect of experience, assuming that through dialogue and reflection the quintessential meaning of the experience will be reviewed. Language is viewed as the primary symbol system through which meaning is both constructed and conveyed (Holstein & Gubrium, 1994).
The strengths of the phenomenological approach are that it provides a rich and complete description of human experiences and meanings (van Manen, 1997). Findings are allowed to emerge, rather than being imposed by an investigator. Careful techniques are used to keep descriptions as faithful as possible to the experiential raw data; this is accomplished by extreme care in moving step by step and in being ever mindful not to delete from, add to, change, or distort anything originally present in the initial “meaning units” of the participant transcripts. The investigator attempts to “bracket” presuppositions and biases to hold them in consciousness through all phases of the research and minimize their influence on the findings. According to Welman & Kruger (1999, p. 189) “the phenomenologists are concerned with understanding social and psychological phenomena from the perspectives of people involved”. A researcher applying phenomenology is concerned with the lived experiences of the people involved (Maypole & Davies, 2001). Another strength is that according to Wiersma (1995, pp. 211-212) phenomena are viewed in its entirety or holistically. Researchers do not impose their assumptions upon emerging data. The researcher’s role is to record what he/she collects from subjects’ in their natural environment. Reality exists as the subjects see it. The researcher is to record, fully, accurately and unbiasedly, reality as seen through the eyes of subjects.

This research is rooted in a journey that has sought to find meaning as to why learners hold misconceptions which result in errors. The phenomenological inquiry is particularly appropriate in this case because it seeks to address meanings and perspectives of research participants. The major concern of phenomenological analysis is to understand "how the everyday, inter-subjective world is constituted" (Schwandt, 2000) from the participants’ perspective. The basic philosophical assumption underlying phenomenological method has most often been illustrated by Husserl’s statements when he explained that "we can only know what we experience" (Husserl, 1970). Thus, any inquiry cannot engage in sciences of facts because there are not absolutely facts; we only can establish knowledge of essences. The essence is the central underlying meaning of the experience shared within the different lived experiences.

Phenomenologists take an experiential view toward understanding a phenomenon, highlighting human experience as not only valid, but of great importance to understanding human existence. Phenomenologists investigate people’s experiences of life events and the meanings these events have to them, and as such it is particularly relevant to the exploration
of learners’ misconceptions and errors in geometry. Phenomenology, with its emphasis on the importance of subjective experience, is the most appropriate method for understanding this phenomenon of learners’ misconceptions and errors. According to Streubert & Carpenter (1999, p. 56), topics “appropriate to phenomenological research method include those central to humans’ life experiences.” Learners’ misconceptions and errors in geometry are part of learners’ life experiences in a geometry class. I therefore chose phenomenological research because it involves both rich description of the life world or lived experience. I examined a phenomenon – the learning of mathematics learners in a geometry class with a view of understanding the type of misconceptions held and the resulting errors.

3.3 POPULATION AND SAMPLING

A population in research refers to those elements that make up the focus of the study that fit fixed criteria (LoBiondo-Wood & Haber, 2010). Schulze (2003, p. 33) defines a population as “the totality of persons, events, organization units, case records or other sampling units with which our problem is concerned”. A population therefore refers to the totality of objects or individuals regarding which inferences are to be made in a sampling study. It includes the group of people and individuals, items or units under investigation. The population of this research was all grade 11 mathematics learners.

Boyd (2001) describes a sample is a collection consisting of a part or subset of the objects or individuals of population which is selected for the purpose, representing the population sample obtained by collecting information only about some members of a population. A sample according to Gerrish & Lacey (2010) is a subset of a target population, normally defined by the sampling process. Sampling is the process of selecting participants from the population to take part in the research on the basis that they can provide detailed information that is relevant to the enquiry. According to Schulze (2003, p. 33), sampling means “taking any person of a population or universe representative of that population or universe”.

Creswell (2007) discusses the importance of selecting the appropriate candidates for interviews. He asserts that the researcher should utilize one of the various types of sampling strategies in order to obtain qualified candidates that will provide the most credible information to the study. Creswell (2007, p. 133) also emphasizes the importance of acquiring participants who will be willing to openly and honestly share information or “their story”.

113
The purpose of phenomenological research is not to describe the characteristics of a group of people who have had a particular experience but to describe the structure of the experience itself (Polkinghorne 1989, p. 48). Therefore the choice of participants does not depend on whether or not they form a representative sample but on whether or not they are able to “...generate a fund of possible relationships that can be used in determining the essential structure of the phenomena” (Polkinghorne 1989, p. 48). Both Stones (1988) and Polkinghorne (1989) give clear guidelines for selecting such research participants: the subject must have experienced the topic under research and must be able to provide full and sensitive descriptions.

According to Hycner (1999, p. 156) “the phenomenon dictates the method and the type of participants.” The researcher has chosen to use purposive sampling which, according to Parahoo (2006) is predominantly used in qualitative research. Purposive sampling involves the researcher selecting individuals who will have knowledge of the phenomena studied or deemed potential information rich cases (Mapp, 2008). Purposeful sampling was the means for selecting participants in the study. Learners who were chosen were able to function as informants by providing rich descriptions of their experiences. Participants were chosen so that a full range of richly varied descriptions could be obtained. According to Greig & Taylor, (1999) purposeful sampling involves looking for those who “have had experiences relating to the phenomenon to be researched”. The participants are selected with the sole purpose of assisting in understanding the phenomenon. They help develop a detailed understanding of the phenomenon. This always helps people to learn about the phenomenon and give voice to the silenced people. I therefore used purposeful sampling as a method of drawing a sample from my population because the learners were the best from the other learners who would supply me with the quality and relevant information needed by this research. The logic and power of purposeful sampling lies in selecting information-rich, in depth cases study. Information rich cases are those from which one can learn a great deal about issues of central importance to the purpose of the research (Patton 1990, p. 169). The sample was made up of 10 high performing learners, who excel in geometry and who displayed few misconceptions and errors in the tasks given, 10 average performing learners who displayed average misconceptions and errors in the written tasks given and 10 low performing learners who displayed lots of misconceptions and errors in the written tasks given. These learners were selected by their geometry teacher based on their performance in previous assessment tasks in geometry. A small sample of 30 learners was therefore used, so
that each description of misconceptions and resulting errors is examined in depth. Munhall (2007) notes that the advantages of a small sample size means that a good rapport can be built between researcher and subjects and may solicit more authentic responses.

3.4 DATA COLLECTION INSTRUMENTS AND PROCEDURES

The data collection strategies can be in the form of questionnaires, observation, interviews, documentary analysis or reflective journals. According to Luneta (2013), data collection is in essence the execution of the research plan. Polkinghorne (1989) suggested three steps in accomplishing a phenomenological study: the first is to gather information from participants who have experienced the phenomenon under investigation, the second is to analyze the data for the common elements that are essential to make the experience what it is, and the third is to write an accurate research report articulating the common essential themes of the phenomenon. These steps were followed. Furthermore, in phenomenological research, three sources can be employed to generate descriptions of the experiences under investigation. These are the researchers’ personal self-reflections on the incidents of the topics that they have experienced, other subjects in the study who describe their experiences of the phenomenon either orally or in written statements, as a response to interview questions; and in addition to the student interviews, data in the form of classroom assessments (Polkinghorne, 1989, p. 46).

Phenomenological research employs methods which allow the researcher to collect data surrounding the lived experiences of people (Polit & Beck, 2008). This is made permissible through a number of data collection strategies such as interviews, focus groups, action research and observation (Parahoo, 2006). In this research, data was collected from learners’ written work and interviews.

3.4.1 Learners’ written work

Learners’ written work was selected as the primary data gathering technique. In order to understand the grade 11’s learners’ misconceptions and resulting errors, it was necessary for me to gather in-depth information from learners’ written work. Learners completed given tasks on their own. The purposes for the learners to complete these tasks were three fold. Firstly, learners’ misconceptions and resulting errors would be identified. Secondly, the
strategies, which learners employ to solve geometric problems, could be ascertained. Thirdly, using the learner’s responses to the task, I would then be able to identify the learners to be interviewed. Learners’ written work helped to analyse past as well as present records of the participants involved in the study. Parahoo (2006), believe that a “person’s or group’s conscious and unconscious beliefs, attitudes, values, and ideas are often revealed in the documents they produce”. These learners’ tasks included a test and classwork which I personally set. The questions were well-suited for grade 11 learners and the content was selected using the Curriculum and Assessment Statement (CAPS) for mathematics Grade R-12. The tasks that were administered were based on cyclic quadrilaterals and tangent theorems.

The classwork consisted of 4 questions based on cognitive levels 1 and 2 as determined by the CAPS. Cognitive level 1 questions are knowledge questions based on straight recall of knowledge learnt, and questions which requires appropriate use of mathematical vocabulary (DBE, 2011a, p. 15). Cognitive level 2 questions are questions of routine procedures that require learners to perform well known procedures and simple applications and calculations which might involve many steps (DBE, 2011a, p. 15). Cognitive level 1 and 2 questions were therefore knowledge or computation category which comprised routine questions that require direct recall or application of the definition and properties of shapes, as well as simple manipulation or computation with answers obtained within one to two steps.

The test consisted of 5 questions based on cognitive levels 3 and 4 as determined by CAPS. Cognitive level 3 questions are questions involving complex procedures, problems involving complex calculations and/or higher order reasoning (DBE, 2011a, p. 15). There is often not an obvious route to the solution. These questions require conceptual understanding. Cognitive Level 4 questions are problem solving questions which might require the ability to break the problem down into its constituent parts (DBE, 2011a, p. 15). Cognitive level 4 and 5 questions were therefore questions in the Understanding category that do not just simply involve recalling the definition or the application of logarithmic laws, but require some understanding of the underlying concepts of logarithms. Learners had to decide not only what to simplify but how to simplify them. Learners were expected to develop their own techniques for solving these problems.
Participants for the study were selected on the basis of learners’ performance in written tasks. Below average learners were selected because they were the ones displaying misconceptions and resulting errors.

The test scripts and classworks were analysed for learners’ errors and misconceptions. The items were marked right or wrong, but comments were made alongside to indicate whether the student showed a misconception, a misinterpretation or a peculiar way of answering the question. The questions were scored correct if the learner gave the correct answer and followed the correct procedure. In the case where the answer was correct but the procedure wrong, I had to look for misconception or misinterpretation, which might have cropped up. Where the learner got the question wrong I looked at the process of getting the answer to see whether there was a misconception or misinterpretation.

3.4.2 Interviews

According to Kvale (1996), the interview is a rich site for constructing knowledge from a qualitative research perspective. Interviews are one of the powerful research tools used to understand participants. Van Teijlinjen & Ireland (2003) argues that an interview enables “interchange of views between two persons conversing about a theme of mutual interest”. Fontana & Frey (2000) contend that interviews are one of the most common and powerful ways in which we endeavour to understand each other. Interviews allow the researcher to enter into the inner thinking of each of the interviewees and to gain an understanding of their perspectives (Patton, 1990). Van Teijlinjen & Ireland (2003) state that when applying Husserlian phenomenology to research, the optimum way of collecting data is through one-to-one interviews. Charmaz (2006) posits that interviews are a powerful vehicle where discovery of people’s life-worlds can be realized by exploring metaphors context and meanings of unique experiences.

In order to understand human phenomena as lived and experienced, the researcher should first look into the individual point of view, i.e. the realization of subject consciousness perceived in the objects, which Giorgi & Giorgi (2003) pointed out as the major characteristics of a phenomenological method. The major data source for this inner perspective is interviewing. Patton (1990) stated the purpose of interviewing specifically as "to find out what is in and on someone else's mind", and that is exactly what the target of the phenomenological study focuses on, i.e. the perception of lived experience. Johnson &
Christensen (2004) state that interviewing; in the hands of a capable researcher, has the potential to generate “a wealth of valuable data”. In this research, the researcher was interested in aspects of geometric misconceptions and resulting errors, which are deeply buried in the minds of participants. According to Creswell (2008), researchers normally use either questionnaires or interviews in order to reach beyond the physical reach of the participants. In this study personal interviews were selected as the preferred mode of collecting data because they afforded the researcher the opportunity to ask questions. The advantage was that there was a room for clarifications and a personal rapport with the participants involved.

The interview conducted involved some of the selected learners who displayed some misconceptions and errors. The interviews were conducted after data analysis was done. The main aim was to understand why the learners held those misconceptions and the reasons for making errors when solving geometric problems. I was also interested in understanding the effects of these misconceptions and resulting errors on the attitude and feelings towards geometry, and mathematics as a whole. The learners who were interviewed were learners who had shown some misconceptions, and misinterpretations when answering some of the written questions from the geometry test and classworks. I gathered data about learners’ backgrounds (experience, education, etc.), their knowledge and beliefs about what geometry and mathematics is (i.e., content perspective), about how geometry is learned and should be taught, about how geometry curriculum should be organized, and lessons prepared and delivered. Furthermore, learners’ thinking and difficulties in geometry and learners’ knowledge, beliefs and approaches to those issues were in the focus of interviews.

As is characteristic of phenomenological interviews, the learners were “encouraged to share with the researcher the details of their experience” (Polkinghorne, 1989, p. 49). Probing questions were asked when necessary to gain the rich description needed for the study and to clarify meaning of participant’s statements. Learners in this study were asked what experiences they had that influenced how they view their own math skills and what learning strategies they developed while they were trying to learn basic concepts in the past. They were asked if their attitudes have changed and if so, in what way. Learners were also asked if there are different learning strategies that they have developed which help them to be successful. The researcher’s reflective notes of her observations of the interviews were also collected and added to the interview data.
The interviews were conducted under the following phases: the initial planning phase; the formulation of questions; the conducting of the interviews. In the initial planning phase, the researcher demarcated the area to be explored during the interview and used this as a guide when formulating the questions. The researcher then had face-to-face conversations with each learner participating in the dissertation research. The face-to-face interview format added the benefit of being able to clarify questions, ask further probing questions, and observe non-verbal communications. The purpose of the interview was explained and the terms of confidentiality addressed. The format of the interview was explained and participants were afforded an opportunity to ask questions for clarifications before the actual interview takes place. In the formulating of questions the researcher devised probing questions; ensured that the same questions were posed to all participants to ensure comparability of results. The researcher also ensured that questions were worded clearly and unambiguously; and these questions were a mix of open-ended and direct questions.

When conducting the actual interview, I demonstrated flexibility by shifting quickly between ideas, enhanced the participants’ self-esteem by making positive remarks to their responses. I did not dominate the interview sessions by talking excessively and unnecessarily. Instead the researcher afforded the participants ample time to air their views without any hindrance. During the interview, I used unstructured questions to facilitate the interview, and to ascertain the meaning and dimensions of each participant’s experience of their misconceptions and errors. According to Charmaz (2006) unstructured questions in interviews have a number of advantages. They are flexible; they allow the interviewer to probe respondents for more detail; they assist in clearing any misunderstandings; and they encourage cooperation and help establish rapport between the researcher and the respondents. Cohen, Manion & Morrison, (2007) are of the opinion that an unstructured interview allows the interviewer greater flexibility and freedom than a structured interview.

In each interview the conversation sought to draw out the personal story and meaning of each participant’s misconceptions and errors. Phenomenological researchers should help participants describe lived experience without leading the discussion (Streubert & Carpenter, 1999). Questions were directed to the participant’s experiences, feelings, beliefs and convictions about the learners’ misconceptions and resulting errors. According to Bentz & Shapiro (1998), Husserl called it bracketing when the inquiry is performed from the
perspective of the researcher. According to Carpenter (1995) bracketing, derived from mathematics, is a fundamental methodological principle of Husserlian phenomenology. The researcher’s preconceptions are held in abeyance to ensure researchers do not allow their assumptions to shape the data collection or impose their understanding and construction on the data (Polit & Beck, 2008). According to Crabtree & Miller (1992) the researcher “must bracket her/his own preconceptions and enter into the individual’s lifeworld and use the self as an experiencing interpreter”. Moustakas (1994, p. 85) points out that “Husserl called the freedom from suppositions the epoche, a Greek word meaning to stay away from or abstain”. The interview is reciprocal: both researcher and research subject are engaged in the dialogue. Kvale (1996, pp 1-2) remarks with regard to data capturing during the qualitative interview that it “is literally an interview, an interchange of views between two persons conversing about a theme of mutual interest,” where researcher attempts to “understand the world from the subjects' point of view, to unfold meaning of peoples' experiences”. At the root of phenomenology, “the intent is to understand the phenomena in their own terms — to provide a description of human experience as it is experienced by the person herself” (Bentz & Shapiro, 1998, p. 96) and allowing the essence to emerge. The maxim of Edmund Husserl was “back to things themselves” (Moustakas, 1994). By using bracketing in a study, the truth is arrived at based on participants’ descriptions rather than the sole interpretations of the researchers. The overall aim is to produce data and descriptions of the essence of the phenomenon that has not been adjusted, massaged, embellished or misinterpreted by participants or researchers alike. Mathematicians keep symbols, numbers, and operations within brackets; researchers attempt to keep biases inside brackets in the mind, thereby increasing the possibility for objectivity. Bracketing helps a researcher avoid making assumptions. Assumptions may simply not reflect reality; for example, some teachers disallow students to chew gum in class, based on the premise, the bias, that gum chewing distracts students from their work, and yet recent research suggests that gum chewing may actually enhance students’ ability to focus on their work (Steffenhagen, 2004).

During the interview process, I ensured that notes were taken. This is what Miles & Huberman (1984) called memoing. Memoing is the act of recording reflective notes about what the researcher (fieldworker, data coder and/or analyst) is learning from the data (Corbin & Strauss, 2008). This is the researcher’s field notes when he/she records what he/she hears, sees, experiences and thinks in the course of collecting and reflecting on the process. Field notes are a secondary data storage method in qualitative research. Because the human mind
tends to forget quickly, field notes by the researcher are crucial to retain data gathered (Lofland & Lofland, 1999). All participants were explained that notes will be made during the interview process and that they should not be distracted from their responses. According to Groenewald (2008), the human mind tends to forget much that has been experienced or observed at quite a rapid rate. Memoing should therefore be used to overcome this tendency. Memos are contributing substantially to the qualitative research process and its credibility. According to Birks, Chapman & Francis (2008) memoing helps to ‘transport the researcher from the concrete to the conceptual’.

The final stage in the interview process, once the data had been collected, involved the summaries of participants’ responses. Finally, the data was analyzed and interpreted according to the objectives of the research study.

3.5 DATA ANALYSIS PROCEDURES

Analysis involves breaking up the data into manageable themes, patterns, trends and relationships (Mouton, 2001, p. 108). This allows the researcher to organise the data into smaller sections, so that any obvious repetitions or errors may be easily noticed. The action of data analysis can be described as a process of bringing order, structure and interpretation to a massive amount of data in search of general statements about relationships and underlying themes that build grounded theory (Marshall and Rossman, 2006, p. 154). Burns (2000, p. 431) asserts that data analysis can be done by systematically arranging and presenting the data. Leedy & Ormrong (2001) state that the classification of the data collected entails unpacking the data with the intention of finding categories or themes of information. Marshall and Rossman (2006, p. 156) regard this categorizing process as the most fundamental operation in the analysis of qualitative data, because it requires the researcher to discover significant classes of things, persons and events, and the properties that differentiate them. These manageable themes are then written into different narratives. Gay & Airasian (2000, p. 25) state that in qualitative research, the researcher collects data from a small number of selected participants and makes use of a non-numerical, interpretive approach to provide narrative descriptions of the interviewees and how they see their contexts.

According to Luneta (2013), data analysis is a process of observing and extracting patterns in data, asking questions about those patterns. This involves constructing conjectures, collect data from selected participants on targeted topics. Luneta (2013) argues that data analysis can
be informed by the data that has been collected, and this is called inductive analyses. It is usually argued that inductive analysis generates theory as it is data led and the categories are derived from data. The analysis can also be informed by the themes, patterns and the categories that emanate from the literature, such analysis is called deductive analysis. Luneta (2013) argues that deductive analysis authenticates theory. The information collected must be analysed in order to arrive at the findings of the research.

Data analysis is often closely linked to the design of the study which is again closely linked to the methods one utilized to gather data. My data analysis involved deep and repeated reading of all data gathered during data collection. Phenomenologists communicate findings through detailed narratives exploring themes and patterns which emerged from data analysis and reduction. These themes and patterns are then placed within the context of virtually all instances of the phenomenon under study. When analyzing data, Cohen, Manion & Morrison (2007) argue that the researcher should strive to analyse and interpret the collected data in terms of the objectives of the research study; respect the anonymity of the participants, as information gathered should be treated in a highly confidential manner and limit their own biases and personal prejudices towards the study.

Phenomenological analysis has many benefits. The goal is to find common themes and broad patterns in data (Kvale, 1996). It involves a return to individual or corporate experience in order to obtain comprehensive descriptions about a phenomenon. The experiences and opinions of those interviewed during the data collection phase, is interpreted during phenomenological analysis (Moustakas, 1994, p. 13).

This study followed a deductive data analysis. In the process of data analysis I used Miles & Huberman’s (1994) guidelines for qualitative analysis. They define analysis as consisting of three concurrent flows of activity, namely data reduction, data display, and conclusion drawing and verification.
Data reduction refers to systematically displaying data so as to make it clearer. It is a continuous process throughout the analysis of the data. Miles & Huberman (1994, p. 10) refer to data reduction as, “the process of selecting, focusing, simplifying, abstracting, and transforming the data that appear in written-up field notes or transcriptions”. I read lots of books, articles and journals. Through data reduction I made summaries with a special focus on the research topic and questions.

In terms of data display, Miles & Huberman (1994, p. 11) suggest that it needs to move from the frequently used extended text to matrices, graphs, charts and networks. These forms of data displays are far more accessible and compact. The creation of data displays is also not separate from analysis, but forms part of the analysis process, externalising the thinking of the analyst. I therefore used diagrams and tables which displays the type of misconceptions and resulting errors displayed by learners.

The third stream of analysis activities as proposed by Miles & Huberman (1994) is conclusion drawing and verification. According to Miles & Huberman (1994), the researcher should be aware of regularities, patterns, possible configurations and logical causal flows throughout the research process. The pattern of my data analysis also followed a similar pattern as proposed by Ragpot (2013).
The raw data which I collected were organised in such a way that it was ready for analysis. This raw data were then analysed, according to smaller units of analysis or codes that was awarded units of meaning. The composition of codes was the first step in making the raw data more distinguishable according to semantic meaning-making. In each data set, the different codes were regrouped, where similar meaning was recognized and categories were formed. In the end each data set had a distinct number of categories which presented the raw data in the specific data set. The categories from each data set were then grouped together to form themes – the themes represent a specific aspect regarding students’ learning (the construct of the study). The themes served the purpose of unifying ideas that had been derived from the various categories. The themes were finally conceptually linked to from an overarching pattern which could be recognized across all the themes. These codes and categories are tabulated in chapter 4.

I analyzed learners’ written work and field notes made during interview to locate meaningful units which are small bits of text which are independently able to convey meaning. Phenomenologists search for themes and patterns, by logically linking these meaningful units. While analyzing data, I applied the process of phenomenological reduction. In keeping with the process of bracketing in phenomenological research, I placed all existing assumptions and biases in abeyance, so as not to influence data. I bracketed out the presuppositions to identify the data in pure form, uncontaminated by extraneous intrusions. Bracketing is a process of rendering oneself as non-influential and neutral as possible. In bracketing researchers aim to “bracket” their previous understandings, past knowledge, and assumptions about the phenomenon so as to focus on the phenomenon in its appearance.
(Sadala & Adorno, 2001). In fact, bracketing involves a process whereby “one simply refrains from positing altogether; one looks at the data with the attitude of relative openness” (Moran, 2000).

After identifying the themes, I applied what Sadala & Adorno (2001) calls “textural portrayal of each theme”. This is a critical phase of explicating the data, in that those statements that are seen to illuminate the researched phenomenon are extracted or ‘isolated’ (Hyten, 1999). The researcher is required to make amount of judgement calls while consciously bracketing her/his own presuppositions in order to avoid inappropriate subjective judgements. The list of units of relevant meaning extracted from each interview is carefully scrutinized and the clearly redundant units eliminated (Moustakas, 1994).

With the list of non-redundant units of meaning in hand, I then bracketed the presuppositions in order to remain true to the phenomenon. By rigorously examining the list of units of meaning I tried to elicit the essence of meaning of units within the holistic context. Clusters of themes are typically formed by grouping units of meaning together and. I then summarized notes made. I then looked for common themes and concluded the explicitation by writing a composite summary, which reflected the context or horizon from which the themes emerged. According to Sadala & Adorno (2001, p. 289) the researcher, at this point “transforms participants’ everyday expressions into expressions appropriate to the scientific discourse supporting the research”.

After the data analysis process, I interpreted and summarized the data collected. I tried to make sense out of what was uncovered and compile the data into sections or groups of information, also known as themes according to (Creswell, 2008). These themes were consistent phrases, expressions, or ideas that were common among research participants. This information is summarized in chapter 4.

3.6 ETHICAL CONSIDERATIONS

Ethics illustrate what is or is not correct to do, or what moral research procedure is involved. Neuman (2003, pp. 116-118) suggests that the researcher has a moral and professional obligation to be ethical, even when research subjects are unaware of or unconcerned about ethics. Neuman (2003, p. 124) further recommends that informed consent to participate in the study must be sought from the participants. That informed consent must contain the
following: A brief description of the purpose and procedure of the research, a statement of any risks or discomfort associated with participation, a guarantee of anonymity and the confidentiality of records, a statement that participation is completely voluntary and can be terminated at any time without penalty and offer to provide a summary of findings.

Research is viewed as a scientific human endeavour that is organized according to a range of protocols, methods, guidelines and legislation (Gerrish & Lacey, 2010). According to Koul (2008), research ethics is that domain of enquiry that identifies ethical challenges with a view to developing guidelines that safeguard against any harm and protects the rights of human subjects in research. Informed consent is the cornerstone of ethical research (Cassell & Young, 2002), and important aspect of this is the quality of information provided to potential subjects.

All participants were told of the aims and proposed outcomes of the research. The researcher provided all of the informants in the study with full information about the nature of the research proposed. This information included the role that they would be requested to take in the study and the possible benefits that the study might give to mathematical educators and learners. To get the consent of the participants, I provided all the participants with an information sheet detailing all information about the research process in a clear and concise manner. This information sheet contained information about the purpose and nature of the study, and guaranteed the privacy and confidentiality of all participants. Each participant was provided with an informed consent form, which they were asked to sign and date once they agreed to participate in the study.

Approval to conduct the study was obtained from the circuit manager, the principal, the learners’ parents and the SGB. To protect the identity of the participants, codes were used. The participants were informed of their right to withdraw whenever they wished. I was mindful that the interview could bring up painful and negative memories. I was therefore careful in choosing the words during the interview.
3.7 QUALITY ASSURANCE STRATEGIES

According to Koul (2008), the interpretivist paradigm seeks the understanding and meaning of people and situations which involve the subjectivity of the researchers. Therefore, the quality standards in this research paradigm are trustworthiness and authenticity.

❖ Trustworthiness

According to Guba & Lincoln (2005), many qualitative researchers agree that data trustworthiness, whether collected from direct observations, focus groups, or interviews, is evidenced by the following: transferability (via thick description), dependability (via outside reviewer), and confirmability (via data audit and credibility (via member checking). According to Merriam (1991), strategies that promote trustworthiness in a study include, triangulation, or multiple sources of data as evidence, member checks, or arranging for those who provided data to evaluate the conclusions, saturation, or continuous data collection to the point where more data add little to regularities that have already surfaced, peer review, or consultation with experts, audit trail, or the detailed record of data collection and rationale for important decisions and thick description, or providing rich detail of the context of the study.

❖ Credibility

Koul (2008) defines Credibility refers to the believability of the findings while Guba & Lincoln (2005) defines it as the confidence that can be placed in both the data and the analysis. Credibility is enhanced by evidence such as confirming evaluation of conclusions by research participants, convergence of multiple sources of evidence, control of unwanted influences, and theoretical fit. Maximum confidence in the believability of conclusions comes from support provided by participants’ agreement, analysis of multiple sources of data, others’ interpretations, and prediction based on relevant theoretical models (i.e., a predicted pattern matches an actual pattern).

Credibility means that the findings should be congruent with what is observed (Merriam, 1991). A study is credible when it presents faithful descriptions, and when readers or other researchers confronted with the experience can recognize it (Koul, 2008). Credibility of this research was achieved through notes made in the researcher’s journal and by giving sufficient
attention to bracketing. Bracketing (or suspending belief) makes it possible to focus on the respondents’ experience, while “allowing informants to construct and give meaning to their own reality” (Crotty, 1996, p. 19). In phenomenology, bracketing is a means of ensuring the researcher presents faithful descriptions of the experience. Every effort was made to stay faithful to participants’ words and descriptions throughout the analysis and development of the models, without changing the meaning or intent of descriptive passages. The credibility of this study lies in the faithful presentation of experience of learners’ misconceptions and errors as reflected in their written work and interviews. Checks were conducted with participants to verify that the findings and summaries were a true reflection and depiction of their experience, and that the portrait accurately reflected their summary errors and misconceptions. These findings were sent for review to colleagues and supervisors for continued confirmation of credibility. Its credibility is also ensured by the fact that my supervisor is an expert in respect of qualitative research methodology and the fact that I have read a lot on qualitative research methodology.

**Transferability**

Merriam (1991), defines transferability describes as the ability to apply the results of research in one context to another similar context. Similarly, Koul (2008) defines transferability refers to evidence supporting the generalization of findings to other contexts—across different participants, groups, situations, and so forth. According to Crotty (1996), transferability is enhanced by detailed descriptions that enable judgments about a “fit” with other contexts. Crotty (1996) further argues that comparison across cases (“cross-case comparisons”) or other units of analysis (classrooms, schools, etc.) that yield similar findings also increase transferability. Transferability means that a body of research’s findings can fit into other contexts outside the study situation and when readers regard findings as meaningful and find it applicable within their own contexts and experiences (Koul, 2008). It therefore refers to the extent to which a study invites readers to make connections between elements of the study and their own experiences (Howitt, 2007). The researcher furnished enough descriptive data in the study so that others may evaluate the applicability of data to other contexts and settings. The findings are transferable because the researcher provided a thorough description of the demographics of the participants and a rich description of the findings, including direct quotes of the participants.
Dependability

Dependability concerns itself with the responsibility of researchers to substantiate that every part of the research is transparent, methodical and clearly documented (Tobin & Begley, 2004). This also strengthens the confirmability of the proposed study with subjects understanding that the data produced is not exaggerated or fabricated by the researcher. Dependability concerns the stability of data over time (Guba & Lincoln, 2005). Dependability is enhanced by common qualitative strategies (audit trails, rich documentation, and triangulation). In interpretive research, data should be trackable and results should be consistent (Howitt, 2007). Authentic classworks and a test written by learners confirmed the dependability of this study.

Conformability

Conformability refers to objectivity (neutrality) and the control of researcher bias (Howitt, 2007). According to Guba & Lincoln (2005), bias in qualitative research is an ever-present concern, but unbiased interpretations are more likely once researcher self-reflection recognizes them overtly and factors them into the design by, for example, intentionally seeking potentially contradictory evidence predicted by alternatives (essentially different biases or worldviews). Conformability is enhanced by consistency with research findings that reach similar conclusions. Other evidence includes the consensus reached by peer review. According to Howitt (2007), conformability can be established by giving the readers clear track of data and interpretations. Conformability was achieved through data transcripts, analysis as well as thick description of findings. Its results were consisted with the data collected.

Authenticity

According to Bryman (2008) authenticity criterion is about relationships between others and researcher. Educative authenticity helps researchers to understand their role as educators as well as others who influence their professional practices (Crotty, 1996, p. 19). This criterion exposes the conversation between researchers and participants, and the situations and emotional compassion that arise during the study (Ellis & Bochner, 2000). Various viewpoints for measuring authenticity criteria as mention by Bryman (2008) are: ontological authenticity which helps members to understand their social milieu. Educative authenticity
helps members to appreciate others’ perspectives. Catalytic authenticity applies to provoke members to engage in action for circumstances change.
CHAPTER 4: DATA ANALYSIS

4.1 CATEGORIZATION OF MISCONCEPTIONS AND ERRORS

This chapter is an outcome of the research which I did whose aim was to determine the misconceptions and errors displayed by learners in the learning of geometry in grade 11. Learners were not passing mathematics and one of the problems was the poor performance in geometry. The problem was caused by misconceptions and errors displayed by learners in the learning of geometry. The analysis therefore sought to answer the following questions:

- What types of misconceptions and errors do learners have on Euclidean Geometry?
- What are the causes of these misconceptions and errors?

Qualitative analysis method was used in the analysis of data. The data was analyzed to uncover the kinds of misconceptions held by grade 11 learners and the resulting errors made because of these misconceptions when learning geometry. Data analysis was based on assessment done on learners’ classwork and test scripts. Assessment for learning has the purpose of continuously collecting information on a learner’s achievement that can be used to improve their learning (DBE, 2011a, p. 69).

I firstly collected data from learners’ classwork and test scripts in geometry. This was followed by an analysis of these scripts and a test. The analysis was therefore based on learners’ written classwork and test responses as given and marked by their teacher. These learners’ tasks were a rich source of valuable information as they revealed various misconceptions and resulting errors displayed by learners. I analyzed 30 learners’ classworks’ and test scripts based on geometry. I then coded the data for description to be used in the research report. In coding data, I followed Creswell’s (2002) data coding process which is diagrammatically illustrated below:
Figure 4.1: Coding process according to Creswell (2002).

I coded learners from L1 to L30 in terms of their test performance, starting from the lowest performer to the highest. The L refers to learner and the numbers 1 to 30 refers to the number of learners who participated in this research. This I did for the sole purpose of protecting the identity of learners as agreed in the consent forms.

I then categorized learners’ responses from C1 to C5. When solving geometry riders, you normally start by a statement and a reason. This category therefore focused on the statements written by learners and the reasons.
Table 4.1: Categorizing learners’ errors from C1 to C5.

<table>
<thead>
<tr>
<th>Category Number</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>Correct statement, correct reason/valid reasoning (Correct Answer)</td>
</tr>
<tr>
<td>C2</td>
<td>Incorrect statement, incorrect answer. Invalid reasoning (Wrong Answer)</td>
</tr>
<tr>
<td>C3</td>
<td>Correct statement, incorrect reason</td>
</tr>
<tr>
<td>C4</td>
<td>Incorrect statement, correct reason</td>
</tr>
<tr>
<td>C5</td>
<td>No attempt made</td>
</tr>
</tbody>
</table>

I also categorized learners’ responses in terms of the type of errors made by learners, namely slips, conceptual errors, procedural errors and order of operations errors.

Table 4.2: Categorizing learners’ errors in terms of slips, knowledge based errors, misapplication errors and order of operations errors.

<table>
<thead>
<tr>
<th>Category</th>
<th>Type of error</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Err0</td>
<td>No error</td>
<td>The learner made no error. Evidence of proof skills.</td>
</tr>
<tr>
<td>Err1</td>
<td>Slip</td>
<td>Blunder, minor error committed because the learner was in a hurry.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Caused by lack of concentration.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Wrong answers which learners can readily correct by themselves.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Due to carelessness, unlikely to be repeated.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Can be caught automatically upon reviewing one’s work</td>
</tr>
<tr>
<td>Err2</td>
<td>Conceptual error</td>
<td>Lack of knowledge of the concept</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Caused by unfamiliar with terminology</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Caused by an insufficient mastery of basic facts, concepts and skills.</td>
</tr>
<tr>
<td>Err3</td>
<td>Procedural error</td>
<td>Has knowledge concepts but misapplying a concept</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Learners have memorized concepts and properties without knowing when to apply them or why they're applying them when solving riders.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Learners know the concept and properties of figures but cannot apply it to the problem.</td>
</tr>
<tr>
<td>Err4</td>
<td>Order of operations</td>
<td>Lack of order when solving a rider which involves 2 or more steps.</td>
</tr>
<tr>
<td></td>
<td>error</td>
<td>Blindly apply procedures without really knowing what’s going on.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Problem in terms deductive reasoning.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Caused by incomplete schema</td>
</tr>
</tbody>
</table>

Lastly, learners’ responses were then categorized in terms of Van Hiele model of Geometric teaching. Van Hiele (1986) identified five levels of mental development in Geometry:

- Level 1 (visualization): Learners can name and identify common geometric shapes.
- Level 2 (analysis): learners can recognize a geometric shape based on its properties, but cannot recognize relationships between classes of figures.
Level 3 (abstraction): Learners identify class inclusion of shapes. Students can give definitions; they recognize how a definition identifies clearly a notion.

Level 4 (deduction): Learners understand the importance of proofs and they can construct geometric proofs.

Level 5 (rigor): learners understand the relations between geometrical concepts and they can see them in an abstract system.

Table 4.3: Categorizing learner’ responses in terms of Van Hiele Levels.

<table>
<thead>
<tr>
<th>Category number</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>VHL1</td>
<td>Van Hiele level 1</td>
</tr>
<tr>
<td>VHL2</td>
<td>Van Hiele level 2</td>
</tr>
<tr>
<td>VHL3</td>
<td>Van Hiele level 3</td>
</tr>
<tr>
<td>VHL4</td>
<td>Van Hiele level 4</td>
</tr>
<tr>
<td>VHL5</td>
<td>Van Hiele level 5</td>
</tr>
</tbody>
</table>

Errors were then summarized per question per learner in a tabular form in terms of the above categories (see tables 4.6 and 4.7). For example, if L1 is coded C2, Err2, and VHL1 for his or her classwork or test, it would mean the learner has given an incorrect statement and an incorrect answer, is showing a lack of the knowledge of a concept, has committed a conceptual error and is showing a sign of operating at van Hiele 1.

4.2 CLASSWORK AND TEST DISCUSSIONS

The format and style of questions for both the test and classwork followed the format and style required by the mathematics CAPS document for mathematics (DBE, 2011a). According to DBE (2011a), an assessment should include questions of all cognitive levels. These questions should include knowledge based questions, questions which require routine procedures, complex procedures and problem solving as summarized and explained in table 3.4 below:
Table 4.4: Cognitive levels, (DBE, 2011a).

<table>
<thead>
<tr>
<th>Cognitive levels</th>
<th>Description of skills to be demonstrated</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Knowledge</strong></td>
<td>Straight recall</td>
</tr>
<tr>
<td></td>
<td>Appropriate use of mathematical vocabulary</td>
</tr>
<tr>
<td></td>
<td>Proofs of prescribed theorems</td>
</tr>
<tr>
<td><strong>Routine procedures</strong></td>
<td>Perform well known procedures</td>
</tr>
<tr>
<td></td>
<td>Simple applications and calculations which might involve many steps</td>
</tr>
<tr>
<td><strong>Complex procedures</strong></td>
<td>Problems involving complex calculations and/or higher order reasoning</td>
</tr>
<tr>
<td></td>
<td>There is often not an obvious route to the solution</td>
</tr>
<tr>
<td></td>
<td>Require conceptual understanding</td>
</tr>
<tr>
<td><strong>Problem solving</strong></td>
<td>Higher order understanding and processes are often involved</td>
</tr>
<tr>
<td></td>
<td>Might require the ability to break the problem down into its constituent parts</td>
</tr>
</tbody>
</table>

4.2.1 Classwork

The classwork was based on cognitive levels 1 and 2 in terms of mathematics CAPS document (DBE, 2011a).

4.2.1.1 Question 1

This question consist of two questions, namely 1.1 and 1.2 both questions were on Cognitive levels 1 and 2 (refer to table 4.5). Two diagrams were given, namely a standard, familiar diagram 4.2 and a non-standard, disorientated diagram 4.7. In both diagrams, O is given as the centre of the circle. This question tested learners’ knowledge on the theorem which states that the angle which an arc subtends at the centre of the circle is twice the angle it subtends on the circumference of the circle. Learners were asked to determine the value of the variables.

![Figure 4.2: Question 1.1 (classwork).](image)
The correct answer for this question was:

\[ a = 2 \times 32^\circ \quad [\angle \text{ at centre} = 2 \times \angle \text{ at circum}] \]

\[ = 64^\circ \]

Of the 30 learners who were sampled, 17 of them managed to calculate the correct answer and state the correct reason. One-fifth of the learners calculated their answers as 16°. Theses learners stated the correct reason but they made the angle at the circumference to be twice the angle at centre. These learners display an inability to represent properties in values. An example of this error is a response by L1 below:

Figure 4.3: L1 response to 1.1 (classwork).

Instead of the value of a being twice 32°, the learner halved the angle at the circumference of the circle. This is the most common error made by learners. Learners seems to know the property that states that the angle at the centre of the circle is twice the angle it subtends at the circumference of the circle, but are confused when coming to the application of this property.

One-fifth of the learners gave their answers as 32°. Some of these learners gave their reason as \( \angle \text{ at centre} = 2 \times \angle \text{ at circum} \). This is surprising because while the reason is correct, these angles are not equal. The property states clearly that the angle at centre is not equal to the angle at circum, but it is twice that angle. Below is a typical example of learner 9 who responded in this way:

Figure 4.4: L9 response to 1.1 (classwork).

4 learners gave reasons such as alternate angles. This is wrong because there are no parallel lines in this figure.
In this question, 1 learner did not respond, 26 learners stated the correct reasons with only 17 calculated the correct answer. Learner 16 stated the correct statement and reason, but did not calculate the final answer. This is a typical example of a slip.

This is a disorientated figure of the theorem which states that the angle that an arc of a circle subtends at the centre of the circle is twice the angle it subtends at any point on the circumference of the circle. The maximum mark for this question was 5.

The correct answer for this question was:

$68° = 2b \ [\angle \text{at centre} = 2 \times \angle \text{at circum}]$

$b = 34°$

One-third of the learners managed to calculate the correct answer and stated the correct reason as displayed by L.21 below:
One-third of the learners stated incorrect reasons like cyclic quad and alternate angles as displayed by L8 below:

Figure 4.9: L8 response to 1.2 (classwork).

In this question, some learners stated the cyclic quad as a reason because the diagram had four vertices or points. The problem is that these learners fail to realize that not all four vertices are on the circumference of the circle. In the case of alternate angles, there are no given lines which are parallel.

Half of the learners calculated 136° as their answer. Like in question 1.1., they made the angle at the circumference to be twice the angle at centre. 2 learners calculated their answer as 68°, thereby implying that the angle at centre is equal to the angle at the circumference. 3 learners wrote the correct statement and reasons without simplifying as displayed in the below figure:

Figure 4.10: L16 response to 1.2 (classwork).

Learner 16, who made the mistake in 1.1., repeated the same mistake of not simplifying the final answer.
Summary learners’ performance in question 1 (See table 4.5)

Maximum: 5 marks. 11 learners got 0 marks. None got 1. 5 learners got 2 marks. 3 learners got 3. 1 learner got 4 marks. 10 learners got maximum marks. A third of the sampled learners got maximum marks for this question and almost a third got zero. Half of these learners failed to get half of the marks of this question. The performance of the learners in this question was therefore moderate.

4.2.1.2 Question 2

![Diagram](image)

**Figure 4.11: Question 2 (classwork).**

This question tested learners’ knowledge on the properties of the cyclic quadrilateral and parallel lines. It comprised of questions on Cognitive levels 1 and 2. In the diagram, learners were given Circle centre C with points A, B, and D on the circumference. AB//DC. AD is produced to E. \( \angle D_1 = 108^\circ \). Learners were to calculate the values of \( \angle A, \angle C_2, \angle C_1 \) and \( \angle B \). The maximum mark for this question is 9

**Question 2.1**

The correct answer for \( \angle A: \angle A = 108^\circ \) [corr \( \angle s, \ DC//AB \)]

Good work was seen in this question as many learners wrote correct statements and reasons. 29 learners calculated the correct answer and gave the correct reason. Only learner 2 gave a wrong reason, that of alternate angles.
Figure 4.12: L5 response to 2.1 (classwork).

This is wrong because $\angle A$ and $\angle EDC$ are corresponding angles.

Question 2.2

The answer for $\angle C_2$: $\angle C_2 = 2 \times 108^\circ = 216^\circ$  
$\angle$ at centre $= 2 \times \angle$ circum

One-third of the learners sampled calculated the correct answer and gave the correct reason as displayed by L14 below:

Figure 4.13: L14 response to 2.2 (classwork).

Half of the learners calculated the answer as $72^\circ$ and gave the reason as a cyclic quadrilateral (see figure 4.14 below). This is a misconception because C is at the centre of the circle and not the circumference of the circle, and therefore ABCD is not a cyclic quadrilateral. ABCD will only be a cyclic quadrilateral if it is given or through proof.

Figure 4.14: L1 response to 2.2 (classwork).

Learner 7 calculated the answer as $54^\circ$ and gave the reason as opposite angles of a cyclic quadrilateral (see figure 4.15 below). This is wrong because ABCD is not a cyclic quadrilateral. Even if it was a cyclic quadrilateral, it would still be wrong because $\angle C_2$ is outside the figure. Opposite angles should be inside a quadrilateral, hence they are called interior opposite angles.
The problem of learners not simplifying the statement was done by L12 as shown below:

L6 gave the reason of angles on a straight line as displayed in the figure below:

This is wrong because $\angle$ADC and $\angle$C₂ are not angles on straight line.

Question 2.3

Responses for $\angle$C₁

The correct answer for $\angle$C₁: $\angle$C₁ = 144° [rev or co-int. $\angle$s]

13 learners managed to calculate the correct answer and gave correct reasons (See figure 4.18 below).

As in previous questions, 3 learners did not simplify the statements as reflected by the answer given by L19
Figure 4.19: L19 response to 2.3 (classwork).
A quarter of learners sampled gave incorrect answers but correct reasons (see L16 answer below).

Figure 4.20: L16 response to 2.3 (classwork).

This is wrong because $\angle C_2$ is part of the revolution, and therefore $\angle C_1$ alone cannot be equal to 360°.

L13 wrote 360° - 180° instead of 360° - 108°. This was a slip.

Figure 4.21: L13 response to 2.3 (classwork).

About a quarter of the learners wrote that $\angle C_2 = 108°$. This implies that 108° = $\angle EDC$. This is wrong because there is no connection between these two angles and they are therefore not equal.

Question 2.4

The correct answer for $\angle B$: $\angle B = 36°$ [co-int $\angle s$, DC//AB]

All learners calculated the answer incorrectly. All the learners calculated the answer as 108° with some of them giving the reason as exterior angle of a cyclic quadrilateral. It seems the learners did not realize that ABCD is not a cyclic quadrilateral since only three vertices lie on a circle. Unfortunately the teacher marked these learners correctly. There was therefore a slip on the teacher who marked 108° as the correct value (see figure 4.22 below).
The question was therefore incorrectly marked by the educator. The correct answer is $36^\circ$.

**Summary learners’ performance in question 2 (see table 4.5)**

No learner got a maximum mark. One third of the learners got 7 marks. 2 learners got 6 marks. 2 got 5 marks. 3 learners got 4 marks. 12 learners got 2 marks. 1 learner got 1 mark. This question was also poorly answered because three quarter of the learners failed to get half of the mark.

4.2.1.3 Question 3

This question tested learners’ knowledge on the properties of the cyclic quadrilateral and the angle at centre in one diagram. Like questions 1 and 2, this question had questions based on cognitive levels 1 and 2. In the diagram O is the centre of the circle and learners were required to determine the values of the variables, giving reasons for your statements.

**The maximum mark for this question is 9**

Responses to a: correct answer: $114^\circ = 2a \quad [\angle \text{ at centre} = 2 \times \angle \text{ circum}]

\[a = 57^\circ\]
17 learners, which is more than half of the sampled learners managed to get the correct answer and stated the correct reason as reflected by L11 response below.

**Figure 4.24: L11 response to 3a (classwork).**

One third of the sampled learners gave incorrect answers and reasons. One of the incorrect reasons given by L1, L2 and L4 is radii.

**Figure 4.25: L1 response to 3a (classwork).**

L4 even wrote a wrong spelling of radii

**Figure 4.26: L4 response to 3a (classwork).**

This is a sign of learners’ inability to link properties in diagram. The radius in the diagram has no link to a. L8, L6 and L9 wrote cyclic quadrilaterals as their reason as shown in the below figure:

**Figure 4.27: L6 response to 3a (classwork).**
This is a wrong because learners had not yet calculated the value of c or b. This is a sign of learners randomly uses properties of figures randomly with no logic. This is a problem because the reason does not specify as to which of the three properties of a cyclic it refers to. Also there is no cyclic quad relationship between a as an unknown value and 114°.

L7 wrote his/her reason as supplementary angles. There is no way in which a and 114° can be supplementary as they are not co-interior angles, angles on a straight line or opposite angles on cyclic quadrilateral.

Figure 4.28: L7 response to 3a (classwork).

Responses to b: correct answer: b = 123° [∠ at centre = 2 x ∠ circum or opp ∠ s of a cyclic quad or ∠s on a str. line]

Two third of the learners gave the correct answer and reason. A quarter of learners gave incorrect answers and incorrect reasons.

One of the reasons given by learners for their statements is radii. Like in the previous question, there is no connection between the radii of the diagram and b. learners were therefore using this reason indiscriminately

Figure 4.29: L8 response to 3b (classwork).

L7 was the only learner who did not respond. The question was left blank.

Figure 4.30: L7 response to 3b (classwork).
Responses to c: the correct answer:

\[c = 57^\circ \text{ [ext } \angle \text{ of a cyclic quad or } \angle \text{s on str. Line]}\]

18 learners managed to calculate the correct answer and gave the correct reason.

For the other learners, it was a mixture of incorrect values and correct reasons as well as incorrect values and incorrect reasons as shown by the answer given by L3 below:

![L3 response to 3c](image)

L7, L4, L9 and L10 did not respond. All their answers were blank.

Figure 4.31: L3 response to 3c (classwork).

While exterior angle of a cyclic quad is part of the correct reasons, it does not match the given statement simply because the only angle which is equal to 123° is b. b and c are not in exterior angle of a triangle relationship with each other.

Figure 4.32: L10 response to 3c (classwork).

L6 wrote corr \( \angle \) s as a reason.

Figure 4.33: L6 response to 3c (classwork).

This is not correct since there are no parallel lines.
Summary learners’ performance in question 3 (see table 4.5)

Maximum mark is 9. 16 learners got maximum marks. 1 learner got 6 marks. 4 learners got 3 marks. 1 learner got 1 mark. 8 learners got 0 marks. Learners displayed an average performance as more than half of the sample got maximum marks, although a quarter of them scored no mark at all.

![Figure 4.34: Question 4 (classwork).]

4.2.1.4 Question 4

This question tested learners’ knowledge on the properties of the tangent. It had cognitive levels 1 and 2 questions. In the diagram, learners were given that TAN is a tangent and they were expected to calculate the value of unknown variables. The maximum mark of this question is 9.

Responses to a: the correct answer:

\[ a = 16^\circ \quad [\text{Tan perp to radius}] \]

Two-third of the learners answered this question correctly. They gave the correct answer and reason. Learners fared better on this question with many scoring full marks (see L15 response below)

![Figure 4.35: L15 response to 4a (classwork).]
A quarter of the learners sampled answered this question correctly but gave an incorrect reason. Some of the incorrect reasons were: $\angle$s of a triangle, $\angle$between tangent and chord, supplementary $\angle$s, and straight angles (see L16 response below).

Figure 4.36: L6 response to 4a (classwork).

Two learners got the wrong answer although they gave a correct reason. This is also an indication that reasons are indiscriminately used.

Responses to b: correct answer: $b = 74^\circ$ [Tan-chord theorem]

Two-thirds of the learners answered this question correctly. Both their answers and reasons were correct as shown by the answer written by L27.

Figure 4.37: L27 response to 4b (classwork).

One-fifth of the learners answered this question incorrectly. They also stated an incorrect reason. Their reason was than a tan is perpendicular to a radius (see the response by L7 below).

Figure 4.38: L7 response to 4b (classwork).

L4 and L5 gave incorrect answers and correct reason (refer to L4 response below). The learner is showing a sign of not understanding the concept of tan perp to a radius.
Figure 4.39: L4 response to 4b (classwork).

The fact that this reason is correct is a clear indication that the learner lacks a conceptual understanding of the tan-chord theorem.

L6 gave a correct answer but a wrong reason, confusing the tangent properties. Learners were guessing answers and reasons.

Figure 4.40: L6 response to 4b (classwork).

L2 did not respond to this question.

Figure 4.41: L2 response to 4b (classwork).

Responses to c: correct answer:

\[ C = 2 \times 74 = 148^\circ \quad \text{[\( \angle \) at centre} = 2 \times \angle \text{circum or Radii]} \]

One-third of the learners wrote the correct reason and answer (refer to L26 response below).

Figure 4.42: L26 response to 4c (classwork).

About half of the learners wrote the wrong answer but a correct reason, an example being L12 answer below:
Only L6 wrote an incorrect answer and incorrect reason. This is an indication that the learners have problems in terms of figure identification. The figure on this question has only three vertices on the circumference of a circle.

Learner 9 did not respond to this question, the question was left blank.

Some learners calculated their answers as 74° giving straight angles and exterior angle of a cyclic quadrilateral as their reasons. Angles on a straight line are supplementary not equal. This is a sign that the learner has not understood the application of angles on straight line which should have been mastered in the earlier grades.

In Question 4, there was an evidence of incorrect marking and awarding of marks as shown by the marking of L15

This learner was awarded 2 marks instead of 3 for the value of a.

In the case of L23, b was incorrectly marked even though it is correct.
Figure 4.46: L23 response to 4b (classwork).

Summary learners’ performance in question 4 (see table 4.5)

Maximum marks in this question were 9. 7 learners got maximum marks whereas 9 learners scored no mark at all. 2 learners got 8 marks. 9 learners got 6 marks. Generally, learners performed better on this question because more than half of them managed to score more than half of the marks from this question.
4.2.1.5 Total marks of learners’ performance in test and classwork

Table 4.5: Summary of learners’ marks per question.

<table>
<thead>
<tr>
<th>Learners</th>
<th>Test</th>
<th>Classwork</th>
<th>Max Mark</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Mark</td>
<td>Marks</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L3</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L4</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L5</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L6</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L7</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L8</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L9</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L10</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>L11</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>L12</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>L13</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>L14</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>L15</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>7</td>
<td>6</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>L16</td>
<td>5</td>
<td>12</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>L17</td>
<td>4</td>
<td>13</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>L18</td>
<td>5</td>
<td>13</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>9</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>L19</td>
<td>5</td>
<td>13</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>L20</td>
<td>4</td>
<td>13</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>9</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>L21</td>
<td>5</td>
<td>12</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>5</td>
<td>9</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>L22</td>
<td>5</td>
<td>13</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>4</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>L23</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>5</td>
<td>9</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>L24</td>
<td>3</td>
<td>13</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>L25</td>
<td>5</td>
<td>12</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>7</td>
<td>9</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>L26</td>
<td>5</td>
<td>13</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>L27</td>
<td>5</td>
<td>13</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>7</td>
<td>9</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>L28</td>
<td>5</td>
<td>13</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>L29</td>
<td>5</td>
<td>13</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>L30</td>
<td>5</td>
<td>13</td>
<td>12</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>
4.2.1.6 Analysis of learners’ classwork responses

Table 4.6: Analysis of learners’ classwork responses per question in terms of the outlined codes and categories.

<table>
<thead>
<tr>
<th>Learners</th>
<th>Question 1</th>
<th>Question 2</th>
<th>Question 3</th>
<th>Question 4</th>
<th>Van Hiele Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>C2, Err3</td>
<td>C3, Err3</td>
<td>C2, Err3</td>
<td>C2, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L2</td>
<td>C2, Err3</td>
<td>C3, Err3</td>
<td>C2, Err3</td>
<td>C2, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L3</td>
<td>C2, Err3</td>
<td>C2, Err3</td>
<td>C2, Err3</td>
<td>C2, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L4</td>
<td>C2, Err3</td>
<td>C2, Err3</td>
<td>C3, Err3</td>
<td>C2, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L5</td>
<td>C2, Err2</td>
<td>C2, Err3</td>
<td>C3, Err3</td>
<td>C2, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L6</td>
<td>C2, Err3</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L7</td>
<td>C2, Err2</td>
<td>C3, Err3</td>
<td>C2, Err3</td>
<td>C2, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L8</td>
<td>C2, Err3</td>
<td>C3, Err3</td>
<td>C1, Err3</td>
<td>C2, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L9</td>
<td>C2, Err3</td>
<td>C3, Err3</td>
<td>C2, Err3</td>
<td>C2, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L10</td>
<td>C2, Err3</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>C2, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L11</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>2</td>
</tr>
<tr>
<td>L12</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L13</td>
<td>C2, Err2</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L14</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L15</td>
<td>C3, Err3</td>
<td>C3, Err0</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>1</td>
</tr>
<tr>
<td>L16</td>
<td>C1, Err0</td>
<td>C3, Err1</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L17</td>
<td>C1, Err0</td>
<td>C3, Err0</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>2</td>
</tr>
<tr>
<td>L18</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L19</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L20</td>
<td>C3, Err3</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>1</td>
</tr>
<tr>
<td>L21</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>2</td>
</tr>
<tr>
<td>L22</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>2</td>
</tr>
<tr>
<td>L23</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>2</td>
</tr>
<tr>
<td>L24</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>2</td>
</tr>
<tr>
<td>L25</td>
<td>C1, Err1</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>2</td>
</tr>
<tr>
<td>L26</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>2</td>
</tr>
<tr>
<td>L27</td>
<td>C1, Err1</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>2</td>
</tr>
<tr>
<td>L28</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>2</td>
</tr>
<tr>
<td>L29</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>2</td>
</tr>
<tr>
<td>L30</td>
<td>C1, Err0</td>
<td>C3, Err3</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>2</td>
</tr>
</tbody>
</table>

Learners’ classwork shows lots of Err3, C1, and C3 errors. It means learners were able to give correct statements and reasons; were also giving incomplete answers and displayed lots of misapplication errors. This is an indication of poor competency in questions that requires logic. Table 4.5 clearly shows the first 16 learners are those that got lower marks. These are learners sampled from the average and below average group. This is an indication that misconceptions and errors were prevalent from this group.
4.2.2 Test

The test was based on cognitive levels 3 and 4 in terms of mathematics CAPS document (DBE, 2011a).

4.2.2.1 Question 1

Figure 4.47: Question 1 (test).

Question 1 consisted of cognitive level 3 and 4 questions. This question tested learners’ knowledge on angles on the same segment as well as the link with the prior knowledge of isosceles triangle. In the diagram, learners were given that A, B, C and D are four points on the circumference of a circle, with BC=DC and \( \angle B_2 = 70^\circ \).

Learners were required to prove that \( \angle A_1 = \angle A_2 \). This question is based on van Hiele level.....The maximum number of marks for this question was 5.

The correct proof:

\[
\angle D_2 = 70^\circ \quad [BC = DC]
\]

But \( \angle D_2 = \angle A_1 \) \[ \angle s \text{ on the same segment} \]

\[
\angle A_1 = 70^\circ
\]

\[
\angle A_2 = 70^\circ \quad [\angle s \text{ on the same segment}]
\]

\[
\angle A_1 = \angle A_2
\]

Two-third of the sampled learners provided the correct proof.
Two-third of the learners provided an incorrect proof. Some of these learners only provided a single step of the required proof with some even giving wrong reasons as witnessed by L4 answer below.

This clearly shows incompetence and lack of logic in terms of proof questions. In the figure, only $\angle B_2$ and $\angle D_2$ are isosceles angles. L5 provided the reason to the statement as exterior $\angle$ of triangle (see figure below).

This is not correct since cyclic quad ABCD has no exterior angle. L2 and L10 gave their reasons as ext $\angle$, without mentioning if it is the exterior angle of a cyclic quadrilateral or exterior angle of a triangle. It shows that these learners have not mastered the properties of exterior angle of a triangle and a cyclic quadrilateral.
Figure 4.51: L8 response to question 1(test).

This is wrong because there is no relationn of angles of a triangle between $\angle A_1$ and $\angle A_2$ or between $\angle A_2$ and the given $\angle B_2$ which is equal to 70°. This is a typical example of a learner who only memorized properties of angles without understanding.

Summary learners’ performance in question 4 (see table 4.5)

Maximum mark is 5. 15 learners got maximum marks. 2 learners got 4 marks. 3 learners got 3 marks. 1 learner got 2 marks. 6 learners got 1 mark each. 3 learners got 0 marks. Learners performed better in this question because half of the sampled scored maximum mark and two third got more than half of the maximum in this question.

4.2.2.2 Question 2

Figure 4.52: Question 2 (test).

Question 2 consisted of cognitive level 3 and 4 questions. This question tested learners’ knowledge on the properties of a diameter as well as the previous knowledge on the properties of parallel lines. In the diagram, learners were given that POQ is a diameter of the circle O, Chord SR is drawn parallel to PQ. OR and PR are drawn and that $\angle R_1 = 24^\circ$. Learners were required to calculate the sizes of $\angle O_1$, $\angle Q$ and $\angle S$

The maximum mark of this question is 13.
Solution FOR 2.1

\[ \angle R_1 = 24^\circ \quad \text{[Given]} \]
\[ \angle R_1 = \angle P_1 \quad \text{[Alt } \angle s, \text{ SR//PQ]} \]
\[ \angle P_1 = 24^\circ \]

But \[ \angle O_1 = 2 \angle P_1 \quad \text{[at centre = 2 x } \angle \text{ at circum or Radii]} \]
\[ = 2 \times 24^\circ \]
\[ = 48^\circ \]

17 learners provided the required answer with correct reasons. These learners showed signs of logic and proof competency in solving geometry questions (see L17 answer below).

Figure 4.53: L17 response to question 2.1 (test).

13 learners provided an incorrect proof. Most of them just attempted to give the value of \[ \angle O_1 \], and not the required proof as evidenced by L7 response below:

Figure 4.54: L7 response to question 2.1 (test).

This is a sign of learner showing lack of logic in solving geometry questions. The learner was guessing the answer and the reason after realizing that there are parallel lines. There is no corresponding relationship between \[ \angle O_1 \] and \[ R_1 \], the given angle equal to 24°.

Solution for 2.2

\[ \angle Q = \angle R_3 = 66^\circ \quad \text{[Radii]} \]

Half of the learners provided the correct solution with correct reasons (see L16 response below).
The other half provided an incorrect solution as evidenced by L6 below.

**Figure 4.56: L6 response to question 2.2 (test)**

These learners made unjustified conclusions based on wrong reasons, and it seems they were guessing the final answer and reasons. These learners show lack of deductive reasoning and logic in solving geometry questions. It is not surprising that they got wrong answers since most of them also got wrong answers for 2.1.

**Solution for 2.3**

\[ \angle S = 180^\circ - 66^\circ \quad \text{[Opp \, \angle s \, of \, a \, cyclic \, quad]} \]
\[ = 114^\circ \]

Half of the learners provided the correct solution, with L16 failing to simplify the final step. This was a slip because this learner provided the correct statement and reason.

**Figure 4.57: L16 response to question 2.3 (test).**

This is a common mistake done by learners as evidence in learners’ classwork scripts. Learners are always rushing to finish the test and it seems they don’t even check their answers after finishing writing. These learners failed to work out the final answer.

One third of the learners provided an incorrect solution (see L8 response below)
Learners who made mistakes gave reasons such as radii, corr $\angle s$, ext $\angle s$ of a cyclic quad, which are all wrong because there is no angle related to $\angle S$ with these reasons.

Figure 4.58: L8 response to question 2.3 (test).

Among the incorrect answers, most were based on wrong statements and wrong reasons.

4 learners did not attempt to answer this question (see L7 answer below):

Figure 4.59: L7 response to question 2.3 (test).

Summary of the whole question 2 (see table 4.5).

Maximum mark is 13. 11 learners got maximum marks. 3 learners got 12 marks. 1 learner got 10 marks. 4 learners got 5 marks. 1 learner got 4 marks. 2 learners got 1 mark each. 8 learners scored no mark at all. Learners’ performance on this question was moderate because half of the learners got 10 or more.

4.2.2.3 Question 3

Figure 4.60: Question 3 (test).
Question 3 consisted of cognitive level 3 and 4 questions. This question tested learners’ knowledge on the properties of a tangent and those of the cyclic quadrilaterals. In the diagram, learners were given TAN as a tangent to the circle at A and TN // CD. CE is produced to meet the tangent at T. $\angle C_1 = 30^\circ$ and $\angle E_2 = 51^\circ$. Learners were required to calculate the sizes of $\angle A_1$, $\angle T$, $\angle C_2$ and $\angle D$. The maximum mark of this question is 12.

**Solution for 3.1: $\angle A_1 = 30^\circ$ [Tan-chord theorem]**

28 learners provided the correct solution with correct reasons (see L9 answer below). Learners’ performance on this question was therefore excellent. This was a one statement and a reason question. Learners were able to use the tan-chord theorem reason to calculate the correct answer.

![Figure 4.61: L9 response to question 3.1 (test).](image)

The remaining two learners provided the correct answer but wrong reasons. L2 wrote the reason as radii while L20 wrote alternate angles.

![Figure 4.62: L20 response to question 3.1 (test).](image)

The **alternate angle reason is wrong because $\angle A_1$** is not in alternative relationship with $\angle C_1$, the given angle which is equal to $30^\circ$. The learner was guessing that since there are parallel lines, then these must be alternate angles.
Solution for 3.2:

\[ \angle T = 51^\circ - 30^\circ \quad \text{[Ext. } \angle \text{ of triangle]} \]

\[ = 21^\circ \]

16 learners provided correct solutions (L18 response below) while the remaining 14 provided incorrect solutions (L9 response below).

Figure 4.63: L18 response to question 3.2 (test).

Amongst the incorrect solutions, incorrect reasons which were not related to the question were given. Wrong reasons such as Iso triangle, Alt \( \angle s \), ext triangle, ext angles of a triangle, co-int \( \angle s \), were given. This is a sign of learners who were just guessing when giving reasons. They showed an inability of using properties in correct contexts.

Solution for 3.3

\[ \angle C_2 = 21^\circ \quad \text{[Alt } \angle \text{s, NT // CD]} \]

Learners’ responses to 3.3

Learners found this question challenging as evidenced by the number of questions that scored no marks. All learners except L30 provided incorrect solutions.

Figure 4.65: L9 response to question 3.3 (test).
The main reason was caused by the fact that in order to get the correct answer for 3.3 to have calculated the value of $\angle T$ correctly. While 16 learners correctly calculated the value of $\angle T$ correctly, they failed to see its connection with $\angle C_2$. Some learners like L9 above thought $\angle C_2$ and $\angle C_1$ are angles on the same segment. This is a misconception since the two angles are subtended by different arcs. They are also not equal because they are subtended by unequal chords.

Solution for 3.4

$$\angle A_2 = 180^\circ - 81^\circ \quad [\text{of a triangle}]$$

$$\angle D = 180^\circ - 99^\circ \quad [\text{opp } \angle \text{ of a cyclic quad.}]$$

Learners’ responses to question 3.4

9 learners provided full correct answer, 3 learners provided part of the answer whereas 17 learners provided incorrect answers. L4 did not respond to this question.

![L29 response to question 3.4 (test).](image)

![L11 response to question 3.4 (test).](image)

L15 partially gave the correct answer
The learner’s first statement is correct but failed to finish the question. This learner was on the right track of getting the correct answer.

L4 provided no response as shown below.

Summary of the whole question 3 (see table 4.5)
Maximum mark is 12. 1 learner got maximum mark. 8 learners scored 10 marks. 2 learners scored 7. 1 learner scored 6 marks. 6 learners scored 5 marks. 1 learner scored 4 marks. 10 learners scored 2 marks. 1 learner scored 1 mark. No learner scored a zero. Learners’ performance on this question was not good. More than half of the sampled learners failed to get half of the maximum mark of this question.

4.2.2.4 Question 4
Question 4 consisted of cognitive Level 4 questions. In the diagram, learners are given that \( AB = BC \). They were required to prove that \( AB \) is a tangent at \( A \) to the circle passing through \( A, E \) and \( D \). The maximum marks of this question were 5.

**Solution for question 4**

\[
\angle A_1 = \angle C \quad [AB = BC]
\]

But \( \angle C = \angle D \quad [\angle s \text{ on the same segment}] \)

\[
\angle A_1 = \angle D
\]

\( AB \) is a tangent at \( A \) to the circle passing through \( A, E \) and \( D \) \quad [Tan-chord theorem]

**Learners’ responses to question 4**

All learners except L30 failed to provide the required proof. L30 provided only half of the required proof. This is a clear sign that learners experiences challenges in terms of proof questions. A third of candidates scored no marks for this question (see L4 answer below).

**Figure 4.71: L4 response to question 4 (test).**

L30 provided a partially correct answer.

**Figure 4.72: L30 response to question 4 (test).**

**Summary of the whole question 4 (see table 4.5)**

Maximum mark is 5. 29 learners scored no mark only one learner scored 2 marks. Learners therefore performed poorly in this question.
4.2.2.5 Question 5

Figure 4.73: Question 1 (test).

Question 5 consisted of cognitive Level 4 questions. In the diagram, learners were given that PA and PB are tangents to the circle centre O. They were required to prove that $\angle ADB = 55^\circ$. The maximum mark for this question is 10.

Solution for 5.1

$\angle POA = 90^\circ$ [Tan perp. to radius]
$\angle POB = 90^\circ$ [Tan perp. to radius]
AOBP is a cyclic quad [Opp $\angle$s are supplementary]

Summary of learners’ responses to question 5.1

Like in the previous question, all learners except L30, who provided only one step of the solution, provided incorrect responses (refer to L6 incorrect response below).

Figure 4.74: L6 response to question 5.1 (test).

L30 provided a partially correct, only one step towards the answer, as displayed in the figure below.
Figure 4.75: L30 response to question 5.1 (test).

Solution for 5.2

\[ \angle O_1 = 110^\circ \]  \quad [\angle \text{ at centre} = 2 \times \angle \text{ at circum}]

But \( \angle O_1 + \angle P = 180^\circ \)  \quad [\text{Opp.} \\angle s \ \text{cyclic quad} \ AOBP]

\[ \angle P = 180^\circ - 110^\circ \]

\[ = 70^\circ \]

Learners’ responses to question 5.2

25 learners provided incorrect solutions (see L24 response below).

Figure 4.76: L24 incorrect response to question 5.2 (test).

L30 provided a correct solution. 4 learners did not attempt this question (refer to the learner response below).

Figure 4.77: L30 response to question 5.2 (test).

L4 did not attempt this question (refer to the below response)

Figure 4.78: L4 response to question 5.2 (test).
Summary of the whole question 5 (see table 4.5)

All learners got 0, except L30 who only got 6 marks. This question was therefore difficult for learners.

4.2.2.6 Analysis of learners’ test responses

Table 4.7: Analysis of learners’ test responses per question in terms of the outlined codes and categories

<table>
<thead>
<tr>
<th>Learners</th>
<th>Question 1</th>
<th>Question 2</th>
<th>Question 3</th>
<th>Question 4</th>
<th>Question 5</th>
<th>Van Hiele level</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>C2,Err4</td>
<td>C3,Err4</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C5,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L2</td>
<td>C2,Err4</td>
<td>C2,Err4</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L3</td>
<td>C3,Err4</td>
<td>C2,Err4</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C5,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L4</td>
<td>C3,Err4</td>
<td>C2,Err4</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C5,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L5</td>
<td>C3,Err4</td>
<td>C2,Err4</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L6</td>
<td>C3,Err4</td>
<td>C2,Err4</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L7</td>
<td>C3,Err3</td>
<td>C2,Err4</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L8</td>
<td>C2,Err4</td>
<td>C2,Err4</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L9</td>
<td>C3,Err4</td>
<td>C3,Err4</td>
<td>C3,Err4</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L10</td>
<td>C3,Err4</td>
<td>C2,Err4</td>
<td>C3,Err4</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L11</td>
<td>C3,Err4</td>
<td>C3,Err3</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L12</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L13</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L14</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L15</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L16</td>
<td>C1, Err0</td>
<td>C1,Err1</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L17</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L18</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C5,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L19</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L20</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L21</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L22</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L23</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L24</td>
<td>C3,Err4</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>1</td>
</tr>
<tr>
<td>L25</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L26</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L27</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L28</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L29</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>C3,Err3</td>
<td>C5,Err4</td>
<td>C2,Err4</td>
<td>2</td>
</tr>
<tr>
<td>L30</td>
<td>C1, Err0</td>
<td>C1, Err0</td>
<td>C3,Err4</td>
<td>C3,Err4</td>
<td>C1,Err4</td>
<td>2</td>
</tr>
</tbody>
</table>

Generally, learners did not perform well in the test. Learners have problems with questions from cognitive levels 3 and 4, that is questions that requires complex procedures and problem
solving respectively. Learners showed lots C3 errors (misapplication of concepts and properties) which made it difficult for them to attempt proof questions.

The above error analysis clearly reveals the types of errors made by learners and the van Hiele level at which they are operating. The analysis shows that all the three sampled groups made all the types of errors as categorized in tables 4.1 and 4.2. The only difference is the frequency of these errors. For example, slips were more frequent in the below average group. Orders of operations mistakes were more frequent in all the groups. This clearly shows the prevalence of misconceptions and errors as held by learners.

Most written responses provided by learners made no attempt to respond to the problem. The responses were inadequate and show very limited understanding of the problem. Some answers contain information that does not reflect the problem. Some responses were not totally complete in responding to all aspects of the problem. Most learners’ responses exhibit a moderate amount of reasoning. Responses therefore show some deficiencies in understanding aspects of geometric problems which made learners to commit errors.

The discussion of learners’ errors and misconceptions displayed by learners is fully discussed in chapter 5.
CHAPTER 5: DISCUSSION, RECOMMENDATIONS AND CONCLUSION

5.1 DISCUSSION

5.1.1 Introduction

The previous chapter analyzed learners’ scripts and the focus was on misconceptions and errors which were displayed by learners. A question by question analysis of the learners’ scripts was done on the 30 sampled scripts. The focus was on the nature and type of errors made by learners. In this last chapter a summary of the findings will be given first, followed by the conclusions drawn by the researcher. Thereafter the recommendations will be suggested.

The theoretical perspective that guided this research is the theory of constructivism. Constructivism is underpinned by the belief that the best way learners gain knowledge is by constructing knowledge themselves. In constructivism, the responsibility for learning lies with the learner. This has a direct bearing when analyzing learners’ misconceptions and errors. When analyzing learners’ scripts, we are able to understand how the learner is able to construct his/her knowledge, and the challenges the learner faces which might result in misconceptions that will eventually make the learners committing errors.

According to Mamba (2011) from a constructivist perspective, misconceptions and errors are crucial to learning and teaching because misconceptions form part of learners’ conceptual structures that will interact with new concepts and influence new learning mostly in a negative way, because misconceptions generate errors. Errors and misconceptions are seen as the natural result of learners’ efforts to construct their own knowledge, and these misconceptions are intelligent constructions based on correct or incomplete (but not wrong) previous knowledge. Making errors should be regarded as part of the process of learning. This means educators should regard errors as a clue for uncovering what learners already know and how they have constructed such knowledge (Borasi, 1994; Mamba, 2011). The constructive perspective is embraced because it regards making errors as part of the learning process. Misconceptions and errors are considered inevitable and should be used as an important source of information about the learning process.
This research was also guided by the van Hiele theory of geometry teaching. While analyzing learners’ errors and misconceptions, the van Hiele theory helped to understand the learners’ geometry competency levels. The conceptual framework of the van Hiele Model of learning geometry was built on a model consisting of five levels of thought development in geometry. The van Hieles were greatly concerned about the difficulties their learners encountered with secondary school geometry. The van Hiele model has three main components: insight, phases of learning, and thought levels (Usiskin, 1982). Insight exists when a person acts in a new situation adequately and with intention. The second component of the van Hiele model, the phases of learning, describes the phases through which learners’ progress in order to attain the next higher levels of thinking. Basically these phases constitute an outline for organizing instruction.

The third component of the van Hiele model grew out of the concern the van Hieles felt when their geometry learners repeatedly encountered difficulties with parts of the subject matter even after being given various explanations. Their joint interest in wanting to improve teaching outcomes led to the development of a theoretical model involving five levels of geometric thinking. According to the van Hieles, the learner, assisted by appropriate instructional experiences, passes through five levels, where the learner cannot achieve one level of thinking without having passed through the previous levels. It is clear that throughout those phases of learning, the teacher has various roles: planning tasks, directing students’ attention to geometric qualities of figures, introducing terminology and engaging students in discussions using these terms and encouraging explanations.

The findings and discussions of this research are also guided by the coding system which the researcher developed as displayed in chapter 4 (see tables 4.1, 4.2, 4.3 and 4.4). This coding system was inspired by various researchers like Luneta (2008); Luneta & Makonye, (2010); Eggleton & Moldavan, (2001); Gagatsis & Kyriakides, (2000); Luo, (2004); Even & Tirosh, (2008); Mamba, (2011); Tomita, (2008); Michael, (2001); Swan, (2001); Harper, (2010); Micawber, (2005); Riccomini, (2005); Green, Piel & Flowers, (2008) and many others who also undertook various studies in misconceptions and resulting errors in science and mathematics. These researchers inspired the coding system used in this research, which was eventually used for data analysis purposes.
The findings, conclusions and recommendations of this research arose in the background of the above explained theoretical assumptions, conceptual perspectives, and the coding system while answering the following main and sub research questions respectively: what types of misconceptions and errors do learners have on Euclidean Geometry and what are the causes of these misconceptions and errors?

The findings, conclusions and recommendations were also inspired by the aims of this research which were to establish the kind of misconceptions and errors learners display when learning Euclidean Geometry and identify the causes of these misconceptions and errors.

5.1.2 Research findings

The findings of this study provide answers to the main research question and its sub-questions. In chapter 2 I provided different definitions of both misconceptions and errors as defined by various researchers and scholars. In chapter 4, I provided different codes for errors made by learners as they solve geometry for data analysis purposes. This study revealed several misconceptions and errors made by learners. The errors varied with contexts and cognitive levels of the questions. Nesher (1987) pointed out that errors often lurk behind misconceptions. In this section I will outline and explain errors and misconceptions identified by the study. I will also attempt to provide explanations for why learners commit the identified errors.

The following major misconceptions and errors were identified:

- The misconception that for a line to be a tangent, there should always be a visible circle.

Learners displayed this misconception when answering test item question 4 (refer to figure 4.70). Learners were given the figure in question 4 to test their knowledge and competency of how to prove a tangent. Learners were required to prove that AB is a tangent at A to the circle passing through A, E and D. As indicated in chapter 4, all learners except L30 failed to provide the correct proof. Some of the following reasons were given as a response: impossible (see L4 response, figure 4.71), impossible the tangent does not pass through the circle, cannot prove AED is a triangle, wrong question, not possible AB will never be a
tangent, impossible the tangent does not pass through the circle and false just to mention the few.

This proved to be a difficult question for the learners judging by their responses and marks scored. It seems learners were taught that a tangent is a straight line that touches an outside part of a circle at one point only. AB was inside the circle. Learners’ responses and poor performance in this question is in consistent with the research by Hoffer (1981); Senk (1989) and Siyepu (2005) who all found that learning to write proofs in geometry “is one of the most difficult topics” for many high school learners. In South Africa, Siyepu’s (2005) study indicates that the majority of 11th-graders encounter difficulties with the proofs of circle theorems. This is the reason why learners struggle to reach van Hiele's fourth and fifth levels. Van Hiele’s Level 4 is deduction. In this level, learners are able to understand the importance of proofs and they can construct geometric proofs. Level 5 is rigor. At this level, learners are able to understand the relations between geometrical concepts and they can see them in an abstract system.

Learners’ performance also confirms Atebe’s (2008) findings that the first three levels identifying thinking are within the capacity of elementary school learners whilst the last two levels involve mathematical thinking typically needed in high school and tertiary courses but are difficult to achieve. This is a clear sign that learners will not be able to answer questions based on cognitive 3 and 4. Cognitive level 4 questions require learners to answer problems involving complex calculations and/or higher order reasoning. There is no obvious route to the solution and the question requires conceptual understanding. Cognitive level 4 questions involve problem solving type of questions. These questions require higher order understanding and processes, and the ability to break the problem down into its constituents’ parts (see table 4.4)

As a solution, learners need to be taught all the properties of a tangent. They should be taught that for a line to be a tangent, it must satisfy these properties, even if there is no visible circle. For example, in the below figure 5.1, AB may be a tangent if we can prove that $\angle B_1 = \angle D$ the reason will be that $\angle B_1$ will be the angle between tangent AB and chord AC while $\angle D$ will be the angle in the alternate segment.
Figure 5.1: An illustration of how to prove that a line is a tangent using the tan-chord theorem.

- The misconception that for quadrilateral to be cyclic, there should always be a visible circle.

Learners displayed this misconception when answering question 5.1 (See figure 4.73). This question tested their knowledge and competency of how to prove a cyclic quadrilateral. Learners were required to prove that AOBP is a cyclic quadrilateral.

There was no circle passing through AOBP. To make matters worse, P was drawn outside the circle. Like question 4, this proved both challenging and confusing for the learners judging by their responses and scores in this question. Of all the 30 sampled learners, including the high achievers, only L30, from the high achievers managed to score only 6 marks out of 10. The rest of the learners did not score even a single mark (see figures 4.74 and 4.75). The responses given by learners were almost similar to those given in question 4. Some of the responses given were “false, cannot prove, this question do not exist, impossible there is no circle on AOBP, no solution, not true P outside the circle, wrong question” to mention but few. It seems learners believe a quadrilateral will be cyclic only if there is a visible circle.

Learners need to be taught that four verices qualifies to be a cyclic qudrilateral even if there is no visble circle, as long as if it can be proved that there are angles on the same segment, there is an exterior angle which is equal to the interior opposite angle or if the interior opposite angles are supplementary. For example in the given diagram (figure 4.73), AOBP will be cyclic quadrilateral if it can be proved that $\angle A_1 + \angle B_1 = 180^\circ$ or angle $\angle O_1 + \angle P = 180^\circ$. 
The above misconceptions were caused by learners’ poor competency on proof. Learners lacked proof and problem solving ability skills. The interviews conducted revealed that learners were not exposed to proof skills. This is in consistent with the research conducted by Senk, (1989) and Siyepu, (2005) whose researches all found that learning to write proofs in geometry is one of the most difficult topics for many high school learners and that a large body of learners therefore lack understanding of the nature of proof. A study conducted by Senk (1989) in the United States of America, for instance, concludes that as much as 70% of high (secondary) school learners do not understand the proofs they study.

In Nigeria, testing learners’ ability to write proofs in geometry has been excluded from the examination syllabuses since about the late 1990s (WAEC, 2003, p.175) because it was challenging for both learners and educators. This was also done by the department of basic education in South Africa where geometry was made optional because it was difficult for both learners and educators.

In South Africa, Siyepu’s (2005) study indicates that the majority of 11th-graders encounter difficulties with the proofs of circle theorems. This is the reason why learners struggle to reach van Hiele's fourth and fifth levels. According to van Hiele (1986), learners need to master the first three levels for them to be able to master the art of proof. The learner is able to construct proof at level four.

Van Hiele (1986) identified five levels of mental development in Geometry:

- **Level 1 (visualization):** Learners can name and identify common geometric shapes.

- **Level 2 (analysis):** Learners can recognize a geometric shape based on its properties, but cannot recognize relationships between classes of figures.

- **Level 3 (abstraction):** Learners identify class inclusion of shapes. Students can give definitions; they recognize how a definition identifies clearly a notion.

- **Level 4 (deduction):** Learners understand the importance of proofs and they can construct geometric proofs.

- **Level 5 (rigor):** learners understand the relations between geometrical concepts and they can see them in an abstract system.
Van Hiele (1999) postulates that the first three levels identifying thinking are within the capacity of elementary school learners whilst the last two levels involve mathematical thinking typically needed in high school and tertiary courses. This is an indication that primary school geometry should serve as a basis for proof skills and competency in high schools. For learners to acquire proof skills and competencies, they should be thoroughly prepared at primary level.

Learners should be taught all the properties of a cyclic quadrilateral. They should be taught that a quadrilateral is cyclic if its opposite angles are supplementary, if the exterior angle is equal to the interior opposite angle and if the angles on the same segment are equal. This is true even if there is no cyclic quadrilateral. For example, in the below figure 5.2, ABCD will be a cyclic quadrilateral if it can be proved that $\angle B + \angle D_1 = 180^\circ$ [opp. $\angle$s of a cyclic quad] or if $\angle B = \angle D_2$ [ext = int. opp. $\angle$]. This is true if there is no visible circle.

Figure 5.2: An illustration of how to prove that a quadrilateral is cyclic.

- Lack of logical procedure when solving geometry problems

One of the flaws identified by this research is the lack of logic and correct procedure when proving geometry problems involving two or more steps. This was evident in both the classwork and test items, more especially when the answer required more than one steps. It was also evident in questions with sub-questions, where a learner had to get the correct answer in the first sub-question which he/she would use in the following questions. For example Question 2 in the classwork had 4 sub-questions. Learners had to use the answer for 2.1 to answer 2.2, answers for 2.1 and 2.2 to answer 2.3, answers for 2.1, 2.2, and 2.3 to answer 2.4. The maximum mark for this question was 9. Of the 30 sampled learners, no learner got a maximum mark. One third of the learners got 7 marks. 2 learners got 6 marks. 2 got 5 marks. 3 learners got 4 marks. 12 learners got 2 marks. 1 learner got 1 mark. This
question was also poorly answered because three quarter of the learners failed to get half of the maximum mark in this question (see table 4.5). There was a decreasing trend in terms of the marks obtained by learners from 2.1 to 2.4. This was caused by the fact that learners had to get the preceding answer before getting the next answer.

There was no logic or connections in most learners’ simplifications. Learners could not make logical connection between figures which was caused by low level of correct conceptual understanding of definitions or concepts. It was evident from the responses of these learners that many of them not only had a weak conceptual knowledge of geometry, but also lacked problem-solving ability in this learning area. This was caused by the learners’ lack of appropriate mental schemas.

This is inconsistent with researches done by de Villiers (1997), Cassim (2006) and Wu (2006) whose studies found that learners have problems with deductive reasoning in geometry. They found that most learners in geometry are unable to reason in a logical, coherent manner. Wu (2006, p. 13) suggests that learners should be taught to reason logically since “logical reasoning is the back-bone of problem solving”. This would be effective when educators in lower grades become aware of the global structure and coherence of mathematics (Cassim, 2006). Educators in lower grades should therefore be aware of the needs of advanced grades.

Learners need to improve both their conceptual and procedural fluency and understanding. Conceptual understanding is an integrated and functional grasp of mathematical ideas. Further, competency in this area is defined in terms of being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes. Procedural fluency is the knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately and efficiently. Mathematics curricula and classroom instruction must include learning outcomes that focus on both conceptual and procedural knowledge. Assessment must also test both types of knowledge. Correct answers are not always a safe indicator of understanding. Educators must examine more than answers to get a true picture of a learner’s conceptual and procedural understanding of mathematical concepts procedures.
The main challenge found by this study was therefore the lack conceptual and procedural understanding of geometric concepts (when and how concepts are to be used). Educators need to ensure that learners learn with understanding. Once learners have learned procedures without understanding, it can be difficult to get them to engage in activities to help learners understand the actual reasoning behind the procedure (Kilpatrick Swafford & Findell, 2001). Furthermore, when learners practice procedure without understanding, there is a danger that they will practice incorrect procedures. As a result, they make it more difficult for themselves to learn the correct procedures.

When a learner tries to learn a new concept without understanding, he/she is learning by rote because it is not linked to any previous knowledge. As a result, the new concept is not understood; and becomes isolated knowledge; therefore, it is difficult to remember. Such (rote) learning is the cause of many mistakes/misconceptions in mathematics as learners try to recall partially remembered and distorted rules (Olivier, 1989). Thus, it is important and necessary that teachers in primary and secondary schools help identify and eradicate such misconceptions because these mistakes will influence new learning, in a negative way, and generate errors. Another problem of learning without understanding is that learners separate what they learnt in school from what happens in the real world. Wu (2006) also echoes similar sentiments, where pure mathematics students learn abstract concepts before learning to apply them in the real world. Therefore, these learners will be limited in using what they learned at school to solve real-life problems. He adds that in mathematics literacy the learner looks at a number of applications and then extracts the abstractions, something that is common in business and engineering.

- **Learners know the concepts but apply them in incorrect situations to solve geometric problems.**

In this research I found that even though learners knew the names of geometric figures or shapes and some concepts, they could not apply their properties in correct contexts to simplify geometry concepts. This is a misapplication of concepts. This is a clear indication that learners are unable to use properties of the given shapes to simplify geometry problems. This kind of geometric reasoning displays the features of a learner who is operating at either the pre-recognition level or van Hiele level 1.
This was prevalent in both the test and classwork items. When facing a geometry problem, learners usually gave a reason unrelated to a problem. For example, they would give a reason of a corresponding angles or alternate angles even though there are no parallel lines. An example is L17, who when answering question 1.1 in the classwork question, gave a reason of alternate angle even though it was an angle at centre (see Figure 4.5). Most learners therefore display this type of error where they blindly apply concepts in incorrect situations to solve geometry problems. This simply shows that learners know the concepts but cannot apply them correctly.

These errors were exacerbated by lack of competency in concepts and procedures. This is inconsistent with the studies conducted by Cunningham & Roberts, (2010) and Kabaca, Karadag & Aktumen, (2011) who all confirmed that learners have difficulties in understanding geometry concepts. This is also alluded to by Siyepu (2005), whose study also reveals learners’ misconceptions about geometric concepts. This is also confirmed by studies done by Renne (2004) and Feza & Webb (2005) whose researches found that learners lack the appropriate vocabulary to express the distinguishing properties of a figure or compare shapes in an orderly manner. In the United States, Clements and Battista (1992) also found that learners are failing to learn basic geometric concepts, are underprepared for the study of more sophisticated geometric proofs. De Villiers (as cited in Feza & Webb, 2005, p.45) affirms by arguing that “success in geometry involves acquisition of the technical terminology”. Bloom (1956, p.63) asserts that “the most basic type of knowledge in any particular field is its terminology”. Oberdorf and Taylor-Cox (1999, p.340) “lack of exposure to proper vocabulary” is one of the reasons for learners’ misconceptions in geometry. For a success in learning Geometry the understanding of geometrical concepts is essential.

According to Ackoff (1989), a systems theorist and professor of organizational change, the content of the human mind can be classified into five categories: data which involves symbols, information which is data that are processed to be useful; provides answers to who, what, where, and when questions, knowledge which involves application of data and information; answers how questions, understanding which involves the appreciation of why and wisdom which is evaluated understanding.
According to Ackoff (1989), data is raw; data simply exists and has no significance beyond its existence (in and of itself); it does not have meaning of itself and represents a fact or statement of event without relation to other things.

Britton (2006) argues that information refers to data that has been given some meaning by way of relational connection. The meaning applied to the data may not necessarily be useful. Ackoff (1989) argues that knowledge is the appropriate collection of information, such that its intent is to be useful. Knowledge is structured and organized information that has developed inside of a cognitive system or is part of the cognitive heritage of an individual Britton (2006). Knowledge is embodied in humans as the capacity to understand, explain and negotiate concepts, actions and intentions.

Understanding is the process by which people take knowledge and synthesize new knowledge from the previously held knowledge. According to Ackoff (1989), understanding is cognitive and analytical. The difference between understanding and knowledge is the difference between learning and memorizing. People who have understanding can undertake useful actions because they can synthesize new knowledge, or in some cases, at least new information, from what is previously known (and understood). That is, understanding can build upon currently held information, knowledge and understanding itself.

According to Ackoff (1989) wisdom beckons to give us understanding about which there has previously been no understanding, and in doing so, goes far beyond understanding itself. Britton (2006) calls wisdom the essence of philosophical probing. Unlike the previous four levels, it asks questions to which there is no easily-achievable answer.

Data is a prerequisite for information and information is a prerequisite for knowledge. Knowledge itself is a prerequisite for understanding while understanding is a prerequisite for wisdom. The researcher’s argument in this case is that learners who misapply the concepts have data and information about concepts but lack knowledge. For them to be able to apply concepts correctly, they need knowledge, understanding and mathematical wisdom. Özerem, (2012), illustrates some of the concepts which are involved in understanding a geometrical fact.
According to Özerem (2012), in a learner learning process there are some key factors such as network, images, words, anecdotes, cases in point, formal principles and finally explanation structures that help in understanding a geometrical fact. In other words, for learners to understand geometric concepts and facts, which will eventually improve their proof skills, educators need to use lots of images, words and explanations, principles and anecdotes. This will make it easier for learners to understand rather than memorizing geometric concepts and facts.

The results of the study by Aydoğan (2007) show that geometrical concepts are mainly acquired by means of using figures. Different names should be assigned to concepts and prototypes. Assigning different names to the concepts, prototypes and the non-critical properties of the concepts have an important effect on geometrical thinking. Forming hierarchical relationships between different types of shapes is important for developing connections between shapes and their properties. De Villiers (1997) claimed that an advantage of hierarchical definition for a concept is that all theorems proved for that concept then automatically apply to its special cases.

- **Learners have problems with disorientated figure**

This research identified that learners have problems when solving geometry questions involving disorientated figures. This was evident when learners were answering question 1.2 in the classwork (see figure 4.6). The maximum mark for this question was 3. Only one third
of the learners got a maximum mark for the correct statement and reason. Learners who got an incorrect answer gave reasons such as a cyclic quadrilateral (see L1 response below)

![L1 response to 1.2 (classwork).](image)

Learners who mistook this diagram to be a cyclic quadrilateral just looked at the figure, and concluded that it has four vertices. They failed to realize that one of the four vertices is at the centre of the circle. Some learners gave their reasons as alternate angles (see figure 4.9). Some educators teach their learners that alternate angles are in a z position. This might have confused these learners. While the z position is correct, it was not given that the lines are parallel.

The results of this study reveal that learners are experiencing problems with disorientated figures. Learners find it difficult to identify and understand shapes when shapes are in a non-standard position. Learners perceive the figures differently when the figure’s orientation is changed. It is difficult for learners to recognize shapes in a non-standard orientation (upside down). This is inconsistent with the findings by Mayberry (1983, p.64) who report that some learners in her study “had difficulty in recognizing a square with a non-standard orientation”. This is alluded to by Clements and Battista (1992) whose research found that many learners in secondary education are able to recognize shapes only in some standard orientation. This finding is inconsistent with that of Prevost (1985) who reported that learners include irrelevant properties when orienting the figure. Drawing regular figures in teaching are likely to have affected students’ learning.

The above errors and failures confirms findings by de Villiers (1997), Siyepu (2005) and Roux (2003) that learners in South African have problems in understanding of features and properties of shapes – the very fundamentals of geometric understanding.

It is important for learners to be exposed to different shapes in different and non-standard shapes.
Learners often base their responses to a question on the visual appearance of a given diagram, resulting in learners making assumptions not directly related to the given diagram.

This was more evident when learners were answering question 2.4 (see figure 4.11). Most learners assumed that ABCD is a cyclic quadrilateral. Most learners who gave their reasons as cyclic quadrilateral based their response on the visual appearance of the diagram. They therefore incorrectly wrote 108° as their answer, the reason being exterior angle of a cyclic quadrilateral (see figure 4.22). The teacher marked this correctly even though it is wrong. This finding is inconsistent with the findings made by Cassim (2006). In the below proof (figure 5.5), the learner incorrectly identifies \( \angle Y_1 \), as the angle in the alternate segment in relation to \( \angle T_1 \). This is wrong since \( \angle Y_1 \) is not at the circumference of the circle. This learner based his/her response on the visual appearance of the diagram.

![Figure 5.5](image_url)

**Figure 5.5: Typical geometry problem involving tangent-chord theorem and a learner’s response to such a problem, Cassim (2006).**

Learners have problems with proof questions

The attempts at answers to Q4 and Q5 in the test item suggest that problems requiring formal deductive reasoning and proof are a challenge to learners. All the learners sampled, with the exception of L30 scored 0 (see table 4.5). Learners cannot reason deductively and present a logical, coherent argument. It was therefore not surprising that learners were committing errors because they had not developed deductive reasoning strategies. The analysis of learners’ work reveals poor performance in terms of questions which involves proof.
Many researchers also found that most learners had difficulties with constructing proofs (Healy & Hoyles, 2000; Lin, 2005; Weber, 2004). Moore (1994) identified seven major sources of the learners' difficulties in doing proofs: not knowing the definitions; having little intuitive understanding of the concepts; inadequate concept images for doing the proofs; being unable, or unwilling, to generate and use their own examples; not knowing how to use definitions to obtain the overall structure of proofs; being unable to understand and use mathematical language and notation; and not knowing how to begin proofs. Most of these difficulties reveal the importance of understanding the concepts of a domain in the process of writing proofs in that domain. For example, not knowing the definitions was often a reason for the learners' failure to produce a proof and a reason for having difficulty in learning the definitions were the abstractness of the concepts as seen by the student. As Clements and Battista (1992, p.421) put it, in the United States, elementary and middle school learners are “failing to learn basic geometric concepts and geometric problem solving; they are woefully underprepared for the study of more sophisticated geometric concepts and proofs”.

Proof is at the heart of mathematical thinking and deductive reasoning (Healy & Hoyles, 2000). Proofs let us distinguish between true results and results that seem plausible but are not generally true. The process of proving teaches us to reason logically. The careful formulation of arguments lets us see how individual mathematical results are related to broader mathematical ideas. Many subjects, from physics to politics, use mathematical models whose validity requires demonstration. Even qualitative arguments in these subjects use the techniques of logical proof. And the proof of a mathematical result lets us answer the question, “Why is this true?”

According to de Villiers, (2004), proof helps with explanation (providing insight into why it is true), with discovery (the discovery or invention of new results), with intellectual challenge (the self-realization/fulfillment derived from constructing a proof) and with systematization (the organization of various results into a deductive system of axioms, concepts and theorems)

According to de Villiers (2004), the main problem why learners experience proof questions is that proofs are given as finished products in textbooks. This does not challenge learners to think deductively.
The opposite angles of a cyclic quadrilateral are supplementary.

**Given:** Circle centre O with cyclic quad ABCD.

**RTP:** 1. $\hat{A} + \hat{C} = 180^\circ$
2. $\hat{ABC} + \hat{ADC} = 180^\circ$

**Construction:** Join OB and OD.

**Proof:**
1. $\hat{O}_1 = 2\hat{A}$
   and $\hat{O}_2 = 2\hat{C}$
   $\therefore \hat{O}_1 + \hat{O}_2 = 2\hat{A} + 2\hat{C}$
   But $\hat{O}_1 + \hat{O}_2 = 360^\circ$
   $\therefore 2\hat{A} + 2\hat{C} = 360^\circ$
   $\therefore \hat{A} + \hat{C} = 180^\circ$

2. Similarly, by joining AO and CO, we can prove that $\hat{ABC} + \hat{ADC} = 180^\circ$.

**Reason for use of theorem:** (Opp $\angle$s cyclic quad)

---

**Figure 5.6: A typical proof which is presented to Grade 11 learners’ as a finished product, Aird & Van Duyn, (2012).**

When one peruses through all mathematics textbooks, one observes that textbook authors have failed learners by not providing them with sufficient experiences to travel the path of the van Hiele model. Often, textbook authors provide learners with a finished product of the proof of a theorem only. Learners are given the theorems as a finished product. De Villiers, (1997, p. 45) refers to this product in mathematics, as the “the end-result of some mathematical activity which preceded it”. Learners are not afforded the opportunities to develop the geometry deductive and logical reasoning skills if teachers follow textbooks slavishly. The mathematical processes are not allowed to be developed fully within learners - although official policy documents, such as the syllabus encourages such processes to be nurtured and developed within learners. According MASA (1978, cited in de Villiers, 1997, p. 45), “The intrinsic value of mathematics is not only contained in the products of mathematical activity (i.e. polished concepts, definitions, structures, and axiomatic systems), but also, and especially, in the processes of mathematical.

The mathematics syllabi should be intended to reflect and increase emphasis on genuine mathematical activity as opposed to the mere assimilation of the finished products of such
activity. This should be reflected in the various sections on geometry”. According to de Villiers, (1997, p. 45), these good intentions cited above seem to have fallen on deaf ears. Teachers and many textbook authors continue providing learners with ready-made content, especially in geometry, which learners had to then “assimilate and regurgitate in tests and exams” (thereby confirming the assertion that geometry is useless outside the classroom.

By perpetuating the current product-driven approach to geometry, it will only increase learners’ negative attitude to the discipline. De Villiers (1997) argues from a theoretical vantage point that instead of teaching learners definitions of geometric concepts such as quadrilaterals, we should rather strive to develop students’ ability to define. Mathematicians such as (Blandford, 1908 and Freudenthal, 1973, as cited in de Villiers, 1998) are strong proponents allowing learners’ to be actively engaged in coming up with definitions for geometric concepts. Blandford, in de Villiers (1998) regards the method of giving learners’ readymade definitions as a “radically vicious method” (in de Villiers, 1997, p. 46), and by so doing one is robbing the learners’ of the most intellectually enriching activities. The evolving of the workable definition by the child’s own activity stimulated by appropriate questions is both interesting and highly educational (de Villiers, 1997, p. 46). Researchers like Ohtani (in de Villiers, 1998) have argued that the traditional method of providing learners with ready-made definitions by the teacher is an attempt by the teacher to exercise his control over learners, to avoid any dissension, and not having to deal with students’ ideas as well as to steer clear of any “hazardous” interaction with learners. This approach of providing learners with finished proofs limits learners’ competencies in proof. Educators should use complete proof from textbooks to help develop proof and deductive skills for their learners. They should design assessment activities that promote higher order thinking skills which will result in deductive and logical reasoning. This will help in solving problems involving riders in geometry.

- **Learners have problems with class inclusion, identification and linking of different properties of shape.**

Geometry shapes are class inclusions of shapes on their own. Data analysis of this research reveals that learners have a problem with identification and class inclusion of shapes in geometry. Identification of a shape simply means to identify the type of shape and its properties. Class inclusion means that learners are able to sort and classify different shapes
based on their appearances and properties. Within a single figure or diagram, learners should be able to identify different shapes and their properties. Learners who can’t identify and classify shapes have problems in solving riders and proof. Inability to identify and classify shapes results in misconceptions and errors.

For example learners who failed to score marks in question 2 in the test failed because of their inability to logically link different properties of figures. The diagram (see figure 4.5) had different shapes within it. There were parallel lines, radii, diameter, and angle at centre, triangles and a cyclic quadrilateral. The conundrum of answering this question was therefore to link alternate angles, corresponding angles, radii (identifying equal angles), diameter (identifying angle in the semi-circle), angle at centre, properties of a cyclic quadrilateral and the properties of a triangle.

![Figure 5.7: L3 response to question 2 (test).](image)

This typical response of L3 is almost the same as responses by other learners. It simply shows that the learner has no competency of linking properties within a figure.

These type misconceptions and errors are inconsistent with the findings of the research done by Feza & Webb (2005) whose research found that many learners have difficulties to perceive class inclusions of shapes, for example, they might think, that a square is not rectangle. This was also confirmed by studies done by de Villiers (1997), Siyepu, (2005) and Roux (2003) who all found that learners’ performance in South African high schools is poor when it comes to items involving understanding of features and properties of shapes – the very fundamentals of geometric understanding. According to Siyepu (2005), Secondary learners in South Africa cannot identify and name shapes like kite, rhombus, trapezium, parallelogram and triangle. Brombacher (2001) and Howie (2001) found that Learners can’t
recognize basic geometrical shapes. Learners can’t define correctly basic geometrical shapes. They don’t know the correct properties of the shapes.

Learners should be able to consider all shapes within a class. By focusing on a class of shapes, learners are able to think about what makes those shapes. The irrelevant features (e.g., size or orientation) fade into the background. Learners should begin to appreciate that a collection of shapes goes together because of properties. Ideas about an individual shape can now be generalized to all shapes that fit that class. If a shape belongs to a particular class, it has the corresponding properties of that class.

The research by van de Walle (2001) found that learners are able to develop the class inclusion of shapes if they are able to progress through the thought levels. Van de Walle (2001) and Fuys, Geddes & Tischler (1988) arranged van Hiele levels from 0 to 4, as compared to Usiskin (1982) who arranged van Hiele levels from 1 to 5.

According to van de Walle (2001), the objects of thought at level 0 are shapes and what they “look like.” learners recognize and name figures based on the global, visual characteristics of the figure. Learners operating at this level are able to make measurements and even talk about properties of shapes, but these properties are not thought about explicitly. It is the appearance of the shape that defines it for the student. A cyclic quadrilateral is a cyclic quadrilateral “because it looks like a cyclic quadrilateral.” Because appearance is dominant at this level, appearances can overpower properties of a shape. For example, a shape that has been rotated may not appear to be the same shape for a level 0 thinker. Learners at this level will sort and classify shapes based on their appearances. The products of thought at level 0 are classes or groupings of shapes that seem to be “alike.”

In terms of level 1, the objects of thought are classes of shapes rather than individual shapes. Learners at this level are able to consider all shapes within a class rather than a single shape. At this level, learners begin to appreciate that a collection of shapes goes together because of properties. Ideas about an individual shape can now be generalized to all shapes that fit that class. In defining a shape, level 1 thinkers are likely to list as many properties of a shape as they know. The products of thought at level 1 are the properties of shapes.
At Level 2, the objects of thought are the properties of shapes. As learners begin to be able to think about properties of geometric objects without the constraints of a particular object, they are able to develop relationships between and among these properties. Learners at level 2 will be able to follow and appreciate an informal deductive argument about shapes and their properties. “Proofs” may be more intuitive than rigorously deductive. However, there is an appreciation that a logical argument is compelling. An appreciation of the axiomatic structure of a formal deductive system, however, remains under the surface. The products of thought at level 2 are relationships among properties of geometric objects.

In terms of Level 3 the objects of thought at level 3 are relationships among properties of geometric objects. Learners are able to examine more than just the properties of shapes. Their earlier thinking has produced conjectures concerning relationships among properties. They begin to question whether conjectures are correct or true. As this analysis of the informal arguments takes place, the structure of a system complete with axioms, definitions, theorems, corollaries, and postulates begins to develop and can be appreciated as the necessary means of establishing geometric truth. At this level, learners begin to appreciate the need for a system of logic that rests on a minimum set of assumptions and from which other truths can be derived. The learner at this level is able to work with abstract statements about geometric properties and make conclusions based more on logic than intuition. The products of thought at level 3 are deductive axiomatic systems for geometry.

At Level 4, the objects of thought are deductive axiomatic systems for geometry. The products of thought at level 4 are comparisons and contrasts among different axiomatic systems of geometry. To promote class inclusion of shapes, van de Walle (2001) suggest various activities that might assist the educator. The van Hiele theory provides the thoughtful teacher with a framework within which to conduct geometric activities.

At Level 0 activities may involve lots of sorting, identifying, and describing of various shapes. The educator may use lots of physical models that can be manipulated by the students. This may include many different and varied examples of shapes so that irrelevant features do not become important. The educator may provide opportunities to build, make, draw, put together, and take apart shapes.
At Level 1, activities may focus more on properties of figures than on simple identification. Learners should be given activities to define, measure, observe, and change properties with the use of models. Activities should involve the use of problem-solving contexts in which properties of shapes are important components. Models used should permit the exploration of various properties of figures. Activities should involve the classification of figures based on properties of shapes as well as by names of shapes.

At Level 2, activities should continue include the use of models, with a focus on defining properties. Learners should make property lists, and discuss which properties are necessary and which are sufficient conditions for a specific shape or concept. Activities should also include the investigation of the converse of certain relationships for validity. For example, the converse of “If line is a diameter, it subtends the right angle on the circumference of the circle” is “If the angle subtended on the circumference of the circle is a right angle, then the line subtending that angle is a diameter.”

Knowledge of class inclusions of shapes is important in geometry because it enables the learners to reason about the relationships between different geometric shapes and their properties. According to van Hiele (1999), the ability to recognize and name shapes has been recognized as important for geometric conceptualization. Research evidence, however, indicates that many high school learners lack the ability to correctly identify, name, and classify many simple geometric shapes (Marchis, 2008; de Villiers, 1997; Siyepu, 2005; and Roux, 2003). The results of these researches suggest that the learners are not or rarely taught about class inclusion. Knowledge on class inclusion is essential, because it enables learners to establish family of shapes, with common properties. With this knowledge at their disposal learners will be able to do some proofs.

The difficulties that students experience relate to shape properties often stem from students having developed a concept image (a mental image of a shape) without a concept definition (a specified definition of the shape or its properties). Learners mostly they fail to establish relationships with other angles. When learners develop a concept image (a mental image of a shape) without a concept definition (a specified definition of the shape or its properties), they often fail to identify examples of shapes that are not identical to their own mental image of the shape or the shape prototype, i.e., the figure does not "look like" the shape. Although
characteristics such as orientation and proportions are irrelevant to the defining properties of a shape, they affect whether students recognize certain shapes.

**Incorrect marking of learners’ work by the educator**

There was an evidence of incorrect marking by the educator on the classwork. Question 2.4 was marked incorrectly. The teacher marked learners correctly even if their answers were wrong. This research was not concerned with educator’s misconceptions and errors. However this was a mistake that had a bearing on the learners’ test scores. The researcher had to communicate the error to the teacher. The teacher admitted to this error as a slip. For this reason the researcher adjusted the marks of learners who were wrongly awarded marks for incorrect answers. The learners’ scripts were not remarked for the purpose of evidence. Only the scores were adjusted.

There was also a wrong awarding of marks. This was also not the focus of this research, but it is important for this error to be pointed out because the wrong awarding of marks had an impact on the overall performance of learners.

In question 4 (classwork), there was an evidence of incorrect marking and awarding of marks L15 and 20 were incorrectly awarded 2 marks instead of 3 for the value of a

![Figure 5.8: L15 (incorrectly awarding of marks).](image1)

For learner L19, 2 marks should have been awarded for the statement \( a = 90° - 74° \)

![Figure 5.9: L19 (incorrectly awarding of marks).](image2)
For L23 and L24, b was incorrectly marked even though correct

![L23 incorrectly marked](image1)

**Figure 5.10: L23 (incorrectly marked).**

In the case of L29, the correct reason should have been awarded a mark when calculating the value of a. For the value of b, L23 was incorrectly awarded 2 marks instead of 3.

![L23 wrong marking and incorrect awarding of marks](image2)

**Figure 5.11: L23 (wrong marking and incorrect awarding of marks).**

The incorrect marking and wrong awarding of marks is an indication that teachers are not immune in terms of misconceptions and errors. This is inconsistent with the research done by Marek Cowan, & Cavallo (2011); Atebe & Schäfer (2010) and Hershey (2004) whose studies all found that misconceptions and errors may be communicated to learners by educators. According to Marek Cowan, & Cavallo (2011) there are some reasons why educators held misconceptions. The main reason is that educators have little abstract thinking ability about the concepts. In the case of Mathematics, and more especially geometry, an educator may teach Geometry badly because of the limited knowledge he/she has about it. According to Atebe & Schäfer (2010), many Mathematics educators are not up to par with the Mathematics syllabus. It is therefore not possible for our learners gain access to the good Mathematics education when our educators are not adequately equipped to provide that access. This may be a reason why our learners have misconceptions and errors. This is in consistent with the findings of numerous studies which found that educators operates at low levels of the van Hiele levels, which is a clear indication of their poor competency in geometry (Fuys, Geddes,
& Tischler, 1988; Mason & Schell, 1988; Sharp, 2001). This simply means that most of these educators harbor misconceptions and errors which will be eventually passed to their learners.

These types of errors call for educator development programmes in terms of geometry content. The research done by de Villiers (2004) found that the majority of black learners tend to be strongly disadvantaged in geometry due to the fact that many mathematics teachers in black township schools are uncomfortable with geometry. Many black teachers teaching mathematics are under-qualified or unqualified to teach the subject. Such teachers tend to feel that high school algebra and calculus are much easier topics to teach as they believe it can be taught algorithmically, whereas the solution of typical riders in our high school geometry often requires far more creative thinking, and is not only more difficult for themselves to accomplish, but also to teach. However, de Villiers (2004) also found that even well-qualified teachers (in all population groups) often have difficulty with geometry. The reason is that in most teacher education institutions in South Africa, such as universities and colleges, there is a heavy focus on calculus and algebra in the mathematics courses, with hardly any geometry being done. So many return to teach only with high school geometry as their highest qualification in geometry, and so perpetuate the cycle.

Educators’ misconceptions and errors are sometimes caused by lack of an understanding of the mathematical content. This was confirmed by Ma (2003) who characterized US elementary mathematics teachers as lacking a profound understanding of mathematics. Research indicate that much of the difficulty that learners experience with geometry is due to teachers’ lack of appropriate pedagogical content knowledge in these subject (van Hiele, 1986; Stoker, 2003; van der Sandt & Nieuwoudt, 2003; Feza & Webb, 2005; Mji & Makgato, 2006). In a study carried out in the North-West province in South Africa, for example, van der Sandt and Nieuwoudt (2003) report that grade 7 teachers and prospective teachers lacked the geometry content knowledge requisite for them to be successful teachers. In South Africa, the blame is put on apartheid education (Stoker, 2003; Mji & Makgato, 2006). Ma commented that unfortunately, low-quality school mathematics education and low-quality teacher knowledge of school mathematics reinforce each other. Mji & Makgato (2006) suggests that if we wish to efficiently help our learners develop their mathematical understanding, it makes sense to help the educators build their own understanding of these ideas and their relationships.
Research indicates that a teacher with good content knowledge of geometry coupled with a good teaching strategy in the subject would make learning much easier for the learners (van der Sandt & Nieuwoudt, 2003).

Ball & Bass (2003) argue that some educators do not possess an understanding of the principles underlying mathematical procedures adequate for teaching and that their knowledge of mathematics is not sufficiently connected. She advocates that subject matter knowledge should be a central focus of teacher education programs and that much more knowledge is needed about how teachers can be helped to increase and develop their understandings of mathematics in order to teach mathematics effectively.

Ma (2003) posits that “the quality of teacher subject matter knowledge directly affects student learning”. These teachers lacked the deep, connected understanding of mathematics, resulting in the imparting of limited exemplars of polygons to the teacher candidates. According to van der Sandt & Nieuwoudt (2003), a primary goal of a mathematics content course should be to provide educators with a strong foundation in mathematics. Lectures’ and workshop conductors of geometry courses need to explore students’ and educators’ concept images of shapes and expand their repertoire of concept images, as a means to build more and stronger connections between mathematical definitions and educators’ concept images. Mathematics educators need to provide learners with experiences that will enable them to develop that strong, connected foundational knowledge needed to learn mathematics.
Types of errors: slips, conceptual errors, procedural errors and order of operation errors.

Figure 5.12: Number of errors made by learners in terms of slips, knowledge-based, conceptual, procedural and order of operations.

Err0 shows that no error was made. Err1 represents slips, Err2 represents conceptual errors, Err3 represents procedural errors while Err4 represents order of operations errors.

The above bar graph clearly shows that most errors were Err4 errors (Figure 5.12). These errors were caused by lack of logic in attempting to answer the correct answer. These were prominent when learners attempted to answer proof questions.

There was an evidence of slips committed by learners. Slips were the lowest committed errors. For example L13 wrote 360° - 180° instead of 360° - 108° (see figure 4.21). This was a slip because learners were given that $\angle D_1 = 108^\circ$ not 180°. This was admitted by the learner during personal interviews, which were conducted with selected learners in order to get an explanation as to why they committed errors and misconceptions.

Lots of procedural errors were committed by learners in both the test and classwork (Err3). These errors were committed as a result of poor deductive reasoning and poor logic skills when answering geometry problems. Procedural errors are errors that occur when devising an effective course of action in an attempt to solve a problem. These errors can often result when
students are too quick to respond to a problem, consequently overlooking essential procedural steps when using a formula.

Figure 5.13: L1 response to question 1.1 (classwork).

This was a procedural error because the learner failed to make the angle at centre twice the angle at the circumference when calculating the value of a. instead, the learner made the angle at centre to be half the angle at circumference (see figure 4.2 for the diagram of this question).

Conceptual errors were also committed (Err2). Conceptual errors are errors that result from the misinterpretation of a mathematical rule, definition, or concept. These errors occur when a learner is unable to derive meaning from the definition or explanations provided, and; therefore, cannot adequately translate the concepts described and apply them to a given set of problems. Learners know the concepts but failed to use appropriately. Learners know alternate angles but use them in wrong contexts. For example L7 gave the reason of alternate angles even though it was angle at centre when answering question 1.1 in classwork (refer to figure 4.5).

Learners also committed misapplication errors when they misapplied concepts. Misapplication errors are classified as Err3 (table). Usually, when learners are faced with a geometry problem, they resort to blindly using reasons or concepts indiscriminately. This is caused by memorizing concepts which is caused by learning without understanding.

Figure 5.14: L8 response to question 2 (test).
This learner’s response is an example of misapplication error. \( \angle O_1 \) is not a co-interior to any angle in figure. \( \angle S \) does not correspond with any angle.

- **Errors made by learners as a result of incorrect statements and reasons**

![Graph showing number of errors](image)

**Figure 5.15: Number of errors made by learners in terms of statements and reasons.**

Category 1 indicates that learners were able to give the correct statement, correct reason and correct answer. Category 2 represents incorrect statements and incorrect answer. Category 3 represents that part of the answer is correct while the other part is incorrect. Category 4 represents a wrong answer which shows lack of logic in attempting to answer the question. Category 5 shows that no attempt was made to answer the question. From the bar graph above, it is clear that in the classwork, all learners attempted to answer all questions. It is only in the test that some learners did not attempt to answer some of the questions. This shows that the test was difficult as compared to the classwork. The bar graph also reveals that the most common problem was learners getting only part of the answer. This is an indication of learners experiencing problems with questions that requires multi-step answers.

- **Attitude (as revealed by the interview)**

The interview conducted reveals negative attitude and anxiety towards geometry. Learners felt geometry is very difficult. This was compounded by the fact that learners in grade 12 were not doing geometry since it was optional. This resulted in learners having a negative attitude towards geometry. Interviews with learners revealed that they are not comfortable
with geometry. Most of them expressed the fear that it will decrease their overall percentage in their grade 12 mathematics results. They felt it should be done by those learners who will pursue careers in engineering.

Such feelings are inconsistent with the studies conducted by Hlalele (2012), Tella (2007), Dörfler (2007), Rossnan (2006), Tsanwani (2009), Khatoon & Mahmood (2010), Leppavirta (2011), Newstead (2006), Perry (2004) and McAnallen (2010) whose studies found that learners have negative attitude towards mathematics. According to Oludipe (2009) negative feelings, negative attitude, anxiety and negative emotions resulted in misconceptions and errors which are thought to prevent some learners from reaching their academic potential. Tella (2007) argue that such feelings among learners’ limit their potential performance directly causing drop in the learner achievement. Such feelings result in learner anxiety. There are number of researches reporting anxiety as one of the major cause for learners’ underachievement and low performances at different levels of their educational life (Oludipe, 2009) and has been shown to affect learners’ ability to profit from instruction (Tella, 2007). Learners with low academic achievement have low negative attitude towards geometry. Statistically significant results revealed that all learners, especially those with high anxiety level, performed poorly and were less motivated to learn.

Because of errors and misconceptions, learners usually feel uneasy, upset, nervous, tense and panic. From the study conducted by Haris & Coy (2003), it is evident that feelings (affective) and worry (cognitive) related anxiety are sources of drop in learners’ achievement. Emotions can affect learning in a negative way. Huhtala (2000) confirm this in the following quote: When a learner experiences negative emotions, the learning process can be disabled.” Tella (2007) is of similar opinion as the following quote shows: “Students who are anxious, angry, or depressed don’t learn; people who are caught in these states do not take in information efficiently or deal with it well.”

According to Haris & Coy (2003) negative beliefs about mathematics are often manifested in misconceptions and errors. Tella (2007) identifies fear, sadness, anger, disgust and happiness as the basic emotions which affect learners’ performance in mathematics. Huhtala (2000) has studied weak learners’ relations to mathematics. Huhtala’s study reveal that learners’ experiences are that mathematics is unpleasant, terrifying, discouraging, and irritating. Some learners often feel that they are untalented and did not like mathematics lessons. Weak
learners have feelings like anxiety, fear, and disgust towards mathematics. Fear and disgust are seldom the emotions these learners have. Boredom is a negative experience. Some learners think about the negative sides of encountering mathematics, meaningless, avoidance, and estrangement.

There is a link between learners’ attitudes and their achievement in mathematics. According to Huhtala (2000) there is a relation between beliefs and learning of mathematics and that “students reorganize their beliefs about mathematics to resolve problems”. Another important element that affects learners’ behavior about mathematics is that of attitudes. Haris & Coy (2003) assert that attitudes are persons’ reactions to negative or positive emotions, with medium intensity, but with sufficient stability. Learners may express liking or disliking of mathematics because of emotions.

Van Hiele levels

The data analysis of this research reveals that most learners are operating at levels 1 and 2 (refer to figure 4.7).

Figure 5.16: Learners’ van Hiele’s levels.
Achieving the van Hiele level 3, 4 and 5 remains problematic for learners. This was evident in the marks that learners got in questions that involved proof or more than one step. The majority of the learners find it easier to calculate the value of unknown angles than proving equal angles, tangents; and cyclic quadrilateral. These learners are simply not ready for the study of deductive geometry problems. This is inconsistent to the study by Baffoe & Mereku (2010) which was an attempt to measure the van Hiele levels of geometric thought attained by Senior High School learners in Ghana. The results showed that 59% of the students attained Van Hiele level 1. Out of 59%, 11% reached level 2 and only 1% reached level 3 (refer to table 5.1, 5.2 and 5.3). De Villiers (1997) asserted that “unless we embark on a major revision of the primary school geometry curriculum along Van Hiele lines, it seems clear that no amount of effort at the secondary school will be successful”. These findings concur with those of the previous research studies Siyepu, (2005) and Atebe & Schäfer, (2008).

The findings of this research indicate that the majority of their learners were found to be operating at the pre-recognition level, and that a very small number of students operated at Van Hiele levels 2. This is in consistent with the research done by Mateya (2008) which found that out of the 50 students who participated in the study, 19 (38%) were at the precognition level, 11 (22%) at van Hiele level 1, 13 (26%) at van Hiele level 2 and 4 (8%) at van Hiele level 3. The results showed that the learners who participated in the study are functioning at a level of geometric thinking not fitting with their mathematics curriculum.

According to Usiskin (1982) van Hiele level 3 is a “guidepost” for the learner to be able to master the art of proof in geometry. At or above level 3 “… success in proof is likely, but below this level failure is just as likely”. Usiskin argues that results show that many learners are unsuccessful in geometry and the key factor is the lack of pre-requisite knowledge. Even after many years of geometry lessons, many learners are found to be “not yet versed in basic terminology or geometric ideas”. Van Hiele model can place students in levels by means of a simple test. Van Hiele level is a good descriptor of a concurrent performance in geometry and a reasonably good descriptor of the future performance.

Senk (1989) and Usiskin (1982) have indicated that many secondary learners are on van Hiele visual or analysis levels. In order for a learner to cope with the demands of an axiomatic system as required in secondary school, however, s/he needs to be on the van Hiele ordering level. Learners who have not received adequate experience on the visual and
analysis levels resort to memorization to cope with the demands of formal school geometry. It is in the primary school that the learners require experiences on the visual and analysis levels in preparation for activity on the van Hiele ordering level.

5.2 RECOMMENDATIONS

The following general recommendations are proposed to help deal with misconceptions and errors in order to improve the geometrical pedagogical practices and mathematical performance of learners.

❖ Educators should improve learners’ proof skills.

This recommendation is done because learners have performed poorly in proof questions. For example, question 4 and 5 in the test were based on proof questions. Question 4 had a maximum mark of 5. Only 1 learner got 2 marks, the rest scored zero. Question 5 had a maximum mark of 10. Only 1 learner got 6 marks, and the rest got 0. This is a clear indication that learners have problems with proof questions. The main aim of teaching geometry is to improve learners’ deductive reasoning through proof. This research presented the many reasons for teaching proof, amongst others as being to enable learners’ to develop thinking skills that are important within mathematics and transferable to other areas of inquiry.

Moore (1994) identified seven major sources of the learners' difficulties in doing proofs: not knowing the definitions; having little intuitive understanding of the concepts; inadequate concept images for doing the proofs; being unable, or unwilling, to generate and use their own examples; not knowing how to use definitions to obtain the overall structure of proofs; being unable to understand and use mathematical language and notation; and not knowing how to begin proofs. Most of these difficulties reveal the importance of understanding the concepts of a domain in the process of writing proofs in that domain. For example, not knowing the definitions was often a reason for the learners' failure to produce a proof and a reason for having difficulty in learning the definitions were the abstractness of the concepts as seen by the student.

The problem of learners failing in the questions involving proof is that most educators are still locked up in the traditional one-answer-one-method model of teaching mathematics.
Memorizing a proof without understanding the interconnectedness (the logical relationship) between one statement and the next can only serve to stifle understanding and to alienate students not merely from deductive geometric proof but also from mathematics in general. It is therefore recommended that educators should give learners more geometry questions and activities that involve proof. These questions and activities should be based on cognitive levels three and four as provided by the caps document which are to guide all assessment tasks (DBE, 2011a). These are complex procedures and problem solving. Complex procedures type of questions involve problems with complex calculations and/or higher order reasoning, where there is no obvious route to the solution, problems that involve making connections between different representations, and require conceptual understanding. Problem solving type questions involves questions that require higher order understanding and processes, and might require to break the problem into its constituencies questions

- **Learners’ conceptual understanding should be improved.**

This recommendation is given because learners are applying geometry properties in incorrect situations. They seem not to understand when and how a property should be used. This is caused by not understanding geometry concepts. Educators should therefore try to improve learners’ conceptual understanding. Conceptual understanding is described as ‘an integrated and functional grasp of mathematical ideas. Competency in conceptual understanding is being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes. To overcome misconceptions and errors, educators will need to help develop the learner’s conceptual understanding.

Learners should learn geometry with understanding because this will result in the efficient and meaningful learning of geometry. Studying geometry is an important component of learning mathematics because it allows students to analyse and interpret the world they live in as well as equip them with tools they can apply in other areas of mathematics. Therefore, students need to develop an understanding of geometric concepts as well as gaining adequate geometry related skills. Proper terminology should be used in context verbally when introducing new terms in geometry. Learners will become familiar with the written solutions and this helps them to develop and learn the necessary vocabulary. The educator can also insist on learners using concepts as much as possible and associate the new terms with
diagrams, representations and symbols so that students can easily connect them with the newly presented topic.

Learners’ should be helped to develop patience, persistence, and tolerance for lack of sufficient clarity while they are developing meaning for new concepts. Connections between concepts already learned and new concepts being introduced must be an integral part of the curriculum and instruction. Such connections must include not only real-world applications and relevance, but also assistance in building mathematical abstractions, so students can see how the results can be transferred from one context to another.

- **Educators should be empowered in terms of how geometric concepts are developed.**

This recommendation is made because of learners’ poor conceptual understanding of geometry. It is also made because research also found that educators themselves have weak conceptual understanding of geometry, which can then be transferred to their learners. Taylor & Vinjevold (1999) for example contend that teachers’ poor grasp of the knowledge structure and concepts of mathematics acts as a major inhibition to teaching and learning of this subject. They are of the opinion that strengthening mathematics teachers’ content and conceptual knowledge should be an essential component of any professional development programme.

According to de Villiers (2004), the problem why educators fail their learners in terms of conceptual knowledge is their use of traditional teaching methods. De Villiers, (2004) has argued that the traditional practice of simply telling definitions to learners is a method of moral persuasion with several social functions, amongst which are: to justify the educator’s control over the learners; to attain a degree of uniformity; to avoid having to deal with learners’ ideas; and to circumvent problematic interactions with learners. Just knowing the definition of a concept does not at all guarantee understanding of the concept.

According de Villiers (2004), it is essential to engage learners at some stage in the process of defining of geometric concepts in order to increase learners’ understanding of geometric definitions, and of the concepts to which they relate. Since defining concepts is not easy, it would appear to be unreasonable to expect learners to immediately come up with formal definitions on their own, unless they have been guided in a didactic fashion through some
examples of the process of defining which they can later use as models for their own attempts.

According to de Villiers, (2004), it is useful to distinguish between two different types of defining of concepts, namely, descriptive (a posteriori) and constructive (a priori) defining. With the descriptive (a posteriori) defining of a concept is meant here that the concept and its properties have already been known for some time and is defined only afterwards. In other words, the concept image of the concept is already well developed before a concept definition is formulated for it. A posteriori defining is usually accomplished by selecting an appropriate subset of the total set of properties of the concept from which all the other properties can be deduced. This subset then serves as the definition and the other remaining properties are then logically derived from it as theorems.

![Concept Image](image)

**Figure 5.17: An example of descriptive (a posteriori) defining of a concept, de Villiers, (2004).**

Constructive (a priori) defining takes place when a given definition of a concept is changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process. In other words, a new concept is defined ‘into being’, the further properties of which can then be experimentally or logically explored. In this case, the concept definition of the new concept precedes the later exploration and further development of its concept image.

![Concept Definition](image)

**Figure 5.18: An example of constructive (a priori) defining of a concept, de Villiers, (2004).**
Whereas the main purpose or function of a posteriori defining is that of the systematisation of existing knowledge, the main function of a priori defining is the production of new knowledge. Speaking from a historical perspective, as well as personal experience, de Villiers, (2004) argues that most concepts are usually defined in a descriptive manner although constructive defining has become a more prominent process in the last century.

The analyses of the junior secondary and senior secondary phase geometry content revealed that there is a strong link and coherence between the two syllabi contents. The two syllabi contain the same themes/topics. It further emerged that the importance of geometry is emphasized in the curriculum. This is illustrated by the inclusion of the theme/topic such as “constructions”. Constructions are fundamental to train students to gain basic geometrical skills that they can use to solve problems in the real-life world.

- **Educators should use relevant vocabulary.**

This recommendation is made because the research found that learners have problems with geometry concepts and properties. They blindly apply properties in incorrect context. This recommendation is also made because other studies found that, for many learners, the greatest barrier to learning mathematics is language (Henderson & Wellington, 1998). The problem is that like many other African countries, South Africa has developed a mathematics curricula and content upon western trends, and teaches mathematics mainly in English or Afrikaans. Geometry has got its own language and symbols. Most second language learners have difficulty with the transition from Level 1 to Level 2. Educators should ensure that learners are equipped with the relevant vocabulary in accordance with van Hiele teaching model. Educators should use relevant vocabulary to describe relevant geometric statements and their relationships. To do this a teacher can not only identify assumption, hypothesis for geometric statements but also explain and show the role of definitions, conjectures, theorems, proofs and counter examples in mathematical reasoning by using geometric examples to illustrate these concepts.

- **Learners should be exposed to different shapes and properties.**

This recommendation is made because learners have problems with the correct properties of shapes. They have problems in identifying shapes and their properties. This problem was also found by
studies done by researchers like de Villiers (1997), Siyepu, (2005) and Roux (2003) who found that learners’ performance in South African high schools is poor when it comes to items involving understanding of features and properties of shapes. Educators should therefore reconsider teaching basic shapes only through examples (Clements & Sarama, 2000). Learners need elaboration not just pictures. Educators need to help learners develop the language of attributes and description of shapes.

To succeed in geometry learning, it is very important to know different shapes and their properties. Learners get confused at recognizing the shapes. The reason for this is human perception. To eliminate this problem the educator should first make learners recognize the shapes then teach how to rotate the objects mentally to perceive them more clearly. The educator should continuously remind learners that rotation of an object does not change its shape. Educators should use manipulatives including pattern blocks, nets, cubes, miras, and geo-boards are fantastic resources to enhance the geometry unit allowing students to learn in a hands-on and exciting manner

Educators should include disorientated figures in their lessons.

This recommendation is made because it was found that learners are easily confused by disorientated shapes or figures. It is important for educators to rotate shapes when teaching learners. For example, the most common standard diagram for the angle at centre theorem, which is displayed in figure 4.2, has different variations, some of which are displayed below:

![Figure 5.19: Examples of disorientated figures](image)

When giving learners test and classworks, teachers should always change the positions of diagrams. This will help learners not to be distracted when solving geometry problems with disorientated figures.
Educators should also teach converse theorems. Learners should know and understand converse theorems.

- **Learners’ thinking and reasoning abilities should be improved.**

This research has identified inadequate thinking and reasoning abilities as some of the major obstacles in geometry. Language may be a source of this problem. It is therefore recommended that learners’ language competencies should be improved for them to be able think and reason deductively. The role of the educator is very crucial to overcome this problem.

- **Educators should improve learners’ problem solving skills.**

This research has identified learners’ problem solving skills as one of the contributors to learners’ poor performance in geometry. To improve learners’ problem solving skills, the guided problem solving teaching model as proposed by Brijlall & Maharaj in Mudaly, (2010) is recommended. To solve a geometry problem, learners will need to be able to reason from the inductive level to the deductive level and vice versa. All theorems in geometry have converses. It is therefore recommended that learners should be able to prove mathematical problems from an inductive level to deductive level or both as diagrammatically represented below:

![Guided problem solving teaching model](image)

Figure 5.20: A Guided problem solving teaching model [Brijlall & Maharaj in Mudaly, (2010)].

Educators should improve learners’ geometric strategic competence which is the ability to formulate mathematical problems, represent them and solve them.
Educators should also improve learners’ Adaptive reasoning which is the capacity to think logically about the relationships among concepts and situations.

When confronted with a geometry problem, learners should always try to first understand the problem by identifying the given information and specifying the unknowns in the problem. They should then formulate a plan for solving the problem. Prior knowledge or past experiences may assist the learners in this stage. Learners should then execute the plan developed. It is important for the learner to justify and check each step. Learners should always reflect on the solution. This includes verifying the result and consolidating what has been learned so that it can be generalized to the solutions of new problems.

- **Educators should focus on goal-oriented activities which promote higher order thinking.**

This recommendation is made because learners are struggling with cognitive levels 3 and 4 questions. It appears that the educator is struggling to handle the diversity of ability in the class and concentrate on the lower cognitive levels and often do not expose intelligent learners to the necessary challenge required. This is evident from the fact that many learners, including those from the intelligent group, struggled to do questions that required deeper insight.

From the analyses, it is very clear that many learners are dismally unprepared for the challenges of the Grade 11 and grade 12 geometry mathematics examinations. It is for this reason that a recommendations that educators should focus on goal-oriented activities which promote higher order thinking is made. When writing examinations, learners are going to be tested in all the cognitive levels. School based activities should therefore include questions from all the cognitive levels. This will ensure that learners are exposed to activities that promote higher order thinking.

- **Learners need to be guided to read, interpret, analyze and solve various types of mathematical problems.**

Learners should avoid memorization at all costs. Memorization results in learners misapplying concepts when solving geometry problems. Memorization is against the teaching promoted by CAPS. The National Curriculum Statement for Grades R - 12 aims to produce
learners that are able “to identify and solve problems and make decisions using critical and creative thinking; work effectively as individuals and with others as members of a team; organise and manage themselves and their activities responsibly and effectively; collect, analyse, organise and critically evaluate information; communicate effectively using visual, symbolic and/or language skills in various modes” (DBE, 2011a, p. 3). These aims cannot be attained through teachings that are teacher centred which promotes the memorization of concepts. Problem solving teaching methods which are learner centred and which promotes relational understanding should be used.

- **Geometric teaching strategies.**

This recommendation is made as a result of learners’ poor performance in geometry. It is also made based on the information given by learners in the interviews that their teacher is not that much confident when teaching geometry. They revealed that he seems to be having some challenges as he does not give them much time to ask questions or give feedback. This recommendation is also made because of studies done by researchers like Morar (2002, p. 274) who observed that in South Africa, mathematics education has been identified as a critical area for reform in schools because there are at present a large number of under-qualified and inexperienced teachers who lack both the subject knowledge and appropriate classroom teaching and management skills. Taylor & Vinjevold (1999) contended that teachers’ poor grasp of the knowledge structure of mathematics, science and geography, acts as a major inhibition to teaching and learning of these subjects.

The way in which mathematics is taught in schools has a bearing on the way learners perform in it. Therefore the methodology of teaching has a definite role to play in learners’ performance in mathematics.

Activities in class should be interesting and relevant to real-life in order to foster learning. Learners should be allowed to investigate and discover mathematical facts for themselves; thus investigative mathematics teaching is encouraged. If learners are not actively involved, the teaching and learning process becomes mundane, and learners will not respond to mathematical challenges in an appropriate manner, and the teacher would end up providing answers to his/her questions. Investigative teaching based on constructivism principles should replace the use of traditional chalk and talk methods.
The direct teaching of geometric definitions (the direct axiomatic-deductive approach) should be replaced by the reconstructive approach. In terms of the reconstructive, the content is not directly introduced to pupils (as finished products of mathematical activity), but that the content is newly reconstructed during teaching in a typical mathematical manner by the teacher and/or pupils.

The pedagogical advantages of employing a reconstructive approach are that its implementation accentuates the meaning of the content and allows the learners to become actively engaged in the construction of the content.

Thus to enhance learners understanding of geometric definitions, it is necessary to encourage learners’ to engage in activities that will afford them the opportunities to develop the requisite skills which the curriculum intends to develop within the learners. the provision of definitions by educators is un-constructive and out of sync with the current curriculum reform practices which encourages learners to communicate appropriately by using descriptions in words, graphs, symbols, tables and diagrams; use mathematical process skills to identify, pose and solve problems creatively and critically.

To increase learners’ understanding of geometric definitions then, it becomes incumbent on every educator to engage learners in the process of defining geometric concepts. Learners should not be provided with ready-made definitions of concepts. By providing learners with such ready-made definitions, a misconception may be created in the learners that there exists only one correct definition for each concept. Learners are thus denied the opportunity to search for alternative definitions, in cases where definitions are presented as items cast in stone. Defining concepts accurately in mathematics is certainly not an easy task, and is only developed after lots of experience and practice.

- **Educators should teach according to the van Hiele’s levels.**

This recommendation is made because most learners managed to answer questions that required one step and a reason answer. Questions which involve more than one step proved to be challenging for most learners. It clearly shows that learners are operating at van Hiele levels 1 and 2. According to the Van Hiele theory, the main reason for the failure of the traditional geometry curriculum at the high school is that the curriculum is presented at a lower level or at higher level than those of the learners; in other words learners cannot
The teaching and learning of geometry should involve more hands-on activities that will actively engage the students. This will enhance learners’ conceptual understanding of geometric concepts. When teaching about geometric concepts, teachers should ensure that students understand and know the properties of all geometric shapes. By knowing the properties of the geometric shapes, students will be able to establish class inclusion, which according to this study is sorely lacking. Learners can only recognize, describe and distinguish geometric shapes from each other by knowing their properties. When teaching about geometric shapes and concepts, teachers should ensure that the proper geometric terminologies are used by both the educators and learners.

- **Improve learner-educator contact time to increase learners’ content knowledge (time allocated for geometry). Educators should not wait until towards the end of the year.**

The empirical findings of this study have revealed that learners enter high school with weak numeracy and literacy skills. This is confirmed by the ANA results as discussed in chapter 1. These underdeveloped skills interfere with the teaching and learning at high school level because learners are unable to read, write and do simple mathematics problems. It is recommended that early literacy and numeracy skills should be closely monitored in the lower grades so that learners can fully develop these skills. Effective instruction in early years is crucial in ensuring learners are able to master reading, writing, and arithmetic. Early intervention is important for learners who struggle with numeracy at lower levels in order for them to become successful learners. Sufficient time should therefore be spent on teaching geometry.

Teaching according to the annual teaching plan, assessing learners in terms of the assessment guidelines, strict monitoring and moderation will go a long way in addressing this problem. The Mathematics head of department at school level, the deputy principal and the principal should ensure that mathematics teachers teach and learners learn. The time allocation for mathematics by the syllabus should be adhered to.
The biggest school-related factor that contributes to the low success rate of learners in mathematics is progression from one grade to another with inadequate content knowledge. Learners are being set up for failure by being allowed to progress to higher grades without mastering the content knowledge in lower grades. It cannot be expected of learners to be successful at Grade 12 level when they are entering this grade with weak foundation.

The study has established that learners’ insufficient content knowledge is the main school associated factor that contributes to their low confidence level of learning and low success in academic achievement. This is not only a problem for learners, but it is also difficult for teachers to teach learners who lack adequate background knowledge. Learners are unable to make meaningful connections to work they are supposed to already understand as a result of them not mastering the subject at the lower grades. Too much emphasis is placed on Grade 12 pass rates, and as a result not enough is being done to motivate learners in lower grades.

It is recommended that the mathematics teachers teaching in the lower grades should teach effectively. Educators need to ensure that learners are mastering the content knowledge in all grades by regularly assessing them.

- **ANA results should be improved.**

De Villiers (1996) claims that the future of secondary school geometry is dependent on primary school geometry. Unfortunately, Atebe & Schäfer (2011) states that the geometry instruction which primary schools offer is inadequate in providing learners with the necessary thinking skills needed to operate at the level of axiomatic thinking required for most high school Geometry. The improvement of the quality of basic education has been identified as the top priority of the South African Government on which the Department of Basic Education has to deliver. Within this context, the Annual National Assessment (ANA) is a critical measure for monitoring progress in learner achievement. Government has identified the ANA as a strategic tool for monitoring and improving the level and quality of basic education, with a special focus on the foundational skills of Literacy and Numeracy. Results in ANA tests and NSC exams are good barometers of the quality of education in a school. They give strong indicators but there’s far more to an excellent school.

This research has identified lower grades education as possible contributor to the learners’ misconceptions and errors. This was confirmed by the ANA results as of the school. This is even alluded to by the South African ANA results have found low levels of literacy and numeracy for South African learners. Without secure foundations of literacy and numeracy,
our learners will never obtain the high-level skills needed by a nation to address poverty and inequality for development and growth.

- **Focus on professional development of educators.**

The need for professional development of educators is necessitated by the fact that geometry had been optional and therefore most educators were not teaching it. This is a challenge for many mathematics educators. This is even necessitated by the fact that in South Africa, mathematics education has been identified as a critical area for reform in schools because there are at present a large number of under-qualified and inexperienced teachers who lack both the subject knowledge and appropriate classroom teaching and management skills. Educators’ poor grasp of the knowledge structure of mathematics acts as a major inhibition to teaching and learning of these subjects. Strengthening mathematics educators’ content knowledge should be an essential component of any professional development programme. Mathematics educators should have knowledge and understanding regarding the foundation of the teaching of this subject in both the intermediate and senior phases at schools. Most mathematics educators are not qualified to teach this subject. Mathematics educators therefore need to be targeted for in-service training to address the lack of subject knowledge. The lack of proper content knowledge in particular, leaves teachers poorly prepared to teach their learners

- **Learners with mathematical anxiety should be helped.**

This recommendation is done because literature indicates a positive correlation between anxiety and misconceptions displayed by learners in mathematics. Anxiety is a basic human emotion consisting of fear and uncertainty that typically appears when an individual perceives an event as being a threat to the ego or self-esteem (Sarason, 1988). When students develop an extreme fear of performing poorly on an examination, they experience test anxiety. Sarason (1988) argues that Test anxiety is a major factor contributing to a variety of negative outcomes including psychological distress, academic underachievement, academic failure, and insecurity. Many learners have the cognitive ability to do well on exams but may not do so because of high levels of test anxiety. Learners with mathematical problems usually develop a negative attitude which will lead to a mathematical anxiety resulting in errors and misconceptions. These learners develop
negative feelings which make them to be afraid to participate in lesson activities. These learners need special treatment in terms of individual and differentiation teaching. Special and individual care and support should be provided. Corrective feedback should always be available. They should always be supported emotionally. Making the learning of mathematics fun, meaningful and relevant goes a long way to inculcating positive attitudes towards the subject. Care and attention should be given to the design of the learning activities, to build confidence in and develop appreciation for the subject.

- The use of manipulatives.

A recommendation is made for the use of manipulatives because of the prevalence of various misconceptions and errors as identified by this research. The van Hiele’s theory strongly emphasizes the use of manipulatives in teaching geometry to facilitate the transition from one level to the next. Manipulatives are defined as physical or concrete objects that are used as teaching tools to engage students in the hands-on learning of Mathematics. When learners touch and manipulate concrete objects, they become more proficient in knowing positions or locations in space (for examples; above, horizontal) and structure (for example: number of parallel sides). Ultimately, hands-on study of geometric objects helps young children develop a strong intuitive grasp of geometric properties and relationships. The spatial orientation of learners should be developed and enhanced through the use of teaching aids and manipulatives in the class room. Learners must understand that geometric shapes are defined by their properties and not by their orientations in space. Educators need to provide learners with activities for discovering the properties of simple geometric shapes in different orientations.

5.3 CONCLUSION

The purpose of the study was to explore the misconceptions and resulting errors made by Grade 11 learners when solving problems on geometry problems at a High School in Bohlabela district in Mpumalanga province. This study aimed both to explore and explicate misconceptions and errors which learners had in geometry. Furthermore, it aimed to provide a rich and in-depth description of geometry errors and misconceptions. This study can alert teachers and curriculum developers to help learners overcome some of their misconceptions,
discussed in this research study. It can also aid these professionals in developing more mathematically proficient grade ten learners.

In pursuance of this aim, answers were sought to the major research question and its sub-question. The main research question was: What types of misconceptions and errors do learners have on Euclidean Geometry? It also seeks to answer the following sub-question: What are the causes of these misconceptions and errors?

In Chapter 1, the three main goals of this study were articulated as follows: To establish the kind of misconceptions and errors learners display when learning Euclidean Geometry and to identify the causes of these misconceptions and errors.

The significance of the study was highlighted in a number of ways. It was emphasized that it is important for mathematicians to study and understand learners’ misconceptions and errors as this has a potential of improving learners’ performance in mathematics. Teaching becomes irrelevant if it does not correct learners’ misconceptions and errors. Misconceptions are like bacteria which multiply rapidly and can be fatal if not treated early. This makes the study of misconceptions and errors extremely important in mathematics, more especially in geometry, considered to be one of the most difficult sections of mathematics. Analysis of errors provides teachers with insight regarding the learners’ procedural and conceptual misunderstandings. These errors can sometimes be more informative to teachers as errors often provide insight into learners’ misunderstandings about a particular mathematics concept or skill.

In Chapter two, the literature review and theoretical background of this research was discussed. It was indicated that this study is informed by the theory of constructivism because the national curriculum is based on the theory of constructivism. Different definitions of misconceptions and errors as defined by various researchers were explored and reviewed. The sources and causes of misconceptions and errors as explored by various researches were also reviewed. Various misconceptions and errors in geometry as identified by various researchers were also explored.

In Chapter three, the research methodology was discussed. Phenomenology was chosen as the research design. This is because Phenomenological methods are particularly effective at bringing to the fore the experiences and perceptions of individuals from their own perspectives. Negative feelings and negative attitude towards mathematics are attached to
errors and misconceptions. Negative feelings among learners limit learners’ potential performance, directly causing drop in the learner achievement. Anxiety, which develops as a result of misconceptions and errors, is one of the major causes for learners’ underachievement and low performances.

In Chapter four, learners’ responses were analyzed, coded and categorized as described in Section 4.1 (Table 10). Various tables were designed to help code and categorize misconceptions and errors.

The final chapter has certified the identified misconceptions and their associated errors as well as provided an explanation why learners made these errors. This research, through data analysis and interviews, has confirmed that like the results of studies done by Shannon (2002), misconceptions are a big impediment to meaningful learning. Mistakes create great difficulties for learners in mathematics and this spill into errors they make. Misconceptions and errors results in learners in viewing geometry as difficult and dull thereby generate a negative attitude towards it. Like the research done by Shaughnessy & Burger (1985), this research, through interviews, found that misconceptions and errors are associated with negative feelings. Those learners with high prevalence of misconceptions and errors expressed their frustrations and anger at not understanding geometry. This has been alluded to by Mestre (1989) whose research found that learners are emotionally and intellectually attached to their misconceptions, because they have actively constructed them. Misconceptions and errors results in the emotional disposition of a set of emotions like fear, anxiety, frustration, rage. According to Buxton (1981), this threatens both performance and participation in mathematics and this makes studies in misconceptions and errors of extreme importance.

However these research findings do not necessitate the view of mistakes, errors and misconceptions from a bad perspective. Misconceptions and errors should not be viewed from negative perspective. Errors are a permanent companion to human thought and action. Mistakes made in the class are actually catalysts for the learning that took place. Borasi (1994, p. 169) describes such errors as springboards for inquiry. Swan (2001) views misconceptions as “natural stage of conceptual development. Mistakes are necessary components of the learning process. Educators should understand learners’ errors, contemplate their causes and methodologically correct them. Misconceptions and errors serve
as the basis for improving teaching and learning in mathematics. There is therefore a need for professional development of educators in geometry, more especially on how to identify and handle errors and misconceptions. Teachers need to be supported not only with appropriate teaching materials, but adequate professional development opportunities for change to occur. This study supports the claim that the van Hiele theory is one of the best frameworks in exploring students’ geometric reasoning. It seems educators do not heed the levels at which concepts are to be introduced in the geometry life of a learner. It is unfair to blame teachers. Curriculum developers should ensure that information about the van Hiele theory is covered in the syllabus. Alternatively workshops should be arranged with experts on this theory invited to address teachers from grade 1 to grade 12. Educators need to understand the stages and levels at which concepts and figures are to be introduced. They should also be empowered in terms of identifying errors and misconceptions and how to handle them. This will indeed improve learners’ results in mathematics’

In conclusion, it should be noted that learners’ errors and misconceptions in learning mathematics cannot be avoided. It is therefore important for misconceptions and errors to be diagnosed, engaged and targeted in teaching and learning, in order to enhance learning and achievement. The most effective way for an educator is to identify learners’ skills and knowledge acquisition through adequate and effective evaluation of learners’ errors and misconceptions.
REFERENCES


Aydoğan, A (2007). The effect of dynamic geometry use together with open-ended explorations in sixth grade students’ performances in polygons and similarity and congruency of polygons. A thesis submitted to the graduate school of natural and applied sciences of Middle East Technical University in partial fulfillment of the requirements for the degree of Master of Science in secondary science and mathematics education. Middle East Technical University: Department of Education.


Britton, S. (2006). Are students able to transfer mathematical knowledge? University of Sydney: Australia


th, 2012, from https://www.google.com/m/search?dc=grlz-GHBB&client=ms-google-mmkt&channel=mf-bb-cht-all-611-


224


227


requirements for the degree of master in education (mathematics education) of Rhodes University.


Melis, E. (2003). Erroneous examples as a source of learning in mathematics. Germany: German Research Centre for Artificial Intelligence


Micawber, M. (2005). What is the difference between to make an error and to make a mistake?RetrievedMay,28th,2011,fromhtml:file://F:\What’s%20the%20%20difference%20between%20’to%20make%20%20error…


Van der Sandt, S., & Nieuwoudt, H.D. (2003). Grade 7 teachers’ and prospective teachers’ content knowledge of geometry. South Africa journal of education,


APPENDIX A: GEOMETRY CLASSWORK GRADE 11 TOTAL MARKS: 32

INSTRUCTIONS

❖ ANSWER ALL QUESTIONS.
❖ SHOW ALL YOUR WORKING AND GIVE REASONS FOR EACH STEP.
❖ NUMBER YOUR ANSWERS ACCORDING TO THE NUMBERING SYSTEM USED IN THIS QUESTION PAPER.

QUESTION 1

O is the centre of the circle in each diagram. Determine the value of the variables in each case.

1.1 (2)

1.2 (3)
QUESTION 2

Given: Circle centre C with points A, B, and D on the circumference. $AB \parallel DC$. AD is produced to E. $\angle D_1 = 108^\circ$. Calculate the size of:

2.1 $\angle A$  
2.2 $\angle C_2$  
2.3 $\angle C_1$  
2.4 $\angle B$

QUESTION 3

In the below diagram, determine the values of the variables, giving reasons for your statements. O is the centre of the circle.

In the below diagram, determine the values of the variables, giving reasons for your statements. O is the centre of the circle.

240
QUESTION 4

In the below diagram, determine the values of the variables, giving reasons for your statements. O is the centre of the circle.

TAN is a tangent.
APPENDIX B: MEMORANDUM CLASSWORK

QUESTION 1

1.1. \( \theta = 3 \times 32^\circ \) \[ \angle \text{at centre} = 2 \times \angle \text{at circum} \]
\[ = 64^\circ \]

1.2. \( b^\circ = 2b \) \[ \angle \text{at centre} = 2 \times \angle \text{at circum} \]
\[ b = 34^\circ \]

QUESTION 2

2.1. \( \angle A = 108^\circ \) \[ \text{Corr } \angle B, AB \parallel DC \]

2.2. \( \angle C_2 = 2 \times 108^\circ \) \[ \angle \text{at centre} = 2 \times \angle \text{at circum} \]
\[ = 216^\circ \]

2.3. \( \angle C_1 = 360^\circ - 216^\circ \) \[ \angle \text{in a point or rev} \]
\[ = 144^\circ \]

2.4. \( \angle B = 108^\circ \) \[ \text{Ext } \angle \text{of } \Delta \]

This Solution is Incorrect

Below is the correct solution

\( \angle B + 144^\circ = 180^\circ \) \[ \text{Co - int } \angle \text{s} \]

\[ \angle B = 36^\circ \]
**Question 3**

\[ 114^\circ = 2a \quad \checkmark \quad \text{[\angle \text{at centre} = 2 \times \angle \text{at circum}] \quad (2) }\]

\[ a = 57^\circ \quad \checkmark \]

\[ b + 57^\circ = 180^\circ \quad \checkmark \quad \text{[Opp \angle \text{s of cyclic quad]} \quad (2) }\]

\[ b = 123^\circ \]

\[ c = 57^\circ \quad \checkmark \quad \text{[\angle \text{ext \ of cyclic quad]} \quad (2) }\]

or

\[ c + 123^\circ = 180^\circ \quad \checkmark \quad \text{[\angle \text{on a str. line]} \quad (1) }\]

\[ c = 57^\circ \]

**Question 4**

\[ a = 90^\circ - 74^\circ \quad \checkmark \quad \text{[\sin 1 rad]} \quad (3) \]

\[ = 16^\circ \]

\[ b = 74^\circ \quad \checkmark \quad \text{[\tan - chord theorem]} \quad (3) \]

\[ c = 2 \times 74^\circ \quad \checkmark \quad \text{[\angle \text{at centre} = 3 \times \angle \text{at circum}]} \quad (3) \]

\[ = 148^\circ \]

**Total = 32 MARKS**
APPENDIX C: GEOMETRY TEST      GRADE 11                  TOTAL MARKS: 45

INSTRUCTIONS
 ANSWER ALL QUESTIONS
 YOUR WORK SHOULD BE NEAT AND LEGIBLE
 GIVE REASONS FOR YOUR STATEMENTS

QUESTION 1
Given: A, B, C and D are four points on the circumference of a circle, BC=DC and \( \angle B_2 = 70^\circ \).

Prove that \( \angle A_1 = \angle A_2 \)  \hspace{1cm} (5)

QUESTION 2
In the diagram, POQ is a diameter of the circle O. Chord SR is drawn parallel to PQ. OR and PR are drawn. \( \angle R_1 = 24^\circ \).

Calculate the size of
2.1 \( \angle O_1 \)  \hspace{1cm} (5)
2.2 \( \angle Q \)  \hspace{1cm} (5)
2.3 \( \angle S \)  \hspace{1cm} (3)
**QUESTION 3**

Given TAN is a tangent to the circle at A and TN is \( \parallel \) to CD. CE is produced to meet the tangent at T. \( \angle C_1 = 30° \) and \( \angle E_2 = 51° \).

Calculate the size of:

3.1 \( \angle A_1 \)  
3.2 \( \angle T \)  
3.3 \( \angle C_2 \)  
3.4 \( \angle D \)

**QUESTION 4**

Given: \( AB = BC \). Prove that AB is a tangent at A to the circle passing through A, E and D.

**QUESTION 5**

PA and PB are tangents to the circle centre O. \( \angle ADB = 55° \).
5.1 Prove that AOBP is a cyclic quadrilateral. (5)
5.2 Calculate the size of \( \angle P \). (5)
APPENDIX D: MEMORANDUM TEST

**MEMO**

**TEST**

**QUESTION 1**

\[ \angle A_2 = 70^\circ \quad \text{[base \(2\) of isosceles \(\Delta B_0 \), \(B_0 = C_0\)}\]

\[ \therefore \angle A_1 = 70^\circ \quad \text{[\(\angle 1\) on the same segment]}\]

but \[ \angle A_2 = 70^\circ \quad \text{[\(\angle A_2 = \angle B_2\), \(\angle 1\) on the same segment]}\]

\[ \therefore \angle A_1 = \angle A_2 \quad \text{(5)}\]

**QUESTION 2**

2.1. \[ \angle P_1 = 24^\circ \quad \text{[SR \parallel PQ, \(\angle 1\)}\]

\[ \therefore \angle Q = 2 \times 24^\circ \quad \text{[\(\angle 1\) on the same segment]}\]

\[ = 48^\circ \quad \text{(5)}\]

2.2. \[ \angle Q_2 = 132^\circ \quad \text{[\(\angle 1\) on a str. line]}\]

\[ \therefore Q = 132^\circ \quad \text{[\(\angle 1\) on the same segment]}\]

\[ = 66^\circ \quad \text{(5)}\]

2.3. \[ \angle S = 180^\circ - 66^\circ \quad \text{[opps. \ of cyclic quadr.]}\]

\[ = 114^\circ \quad \text{(3)}\]
QUESTION 3

3.1. \( \angle A_1 = 30^\circ \)  \( \text{[Sine-Chord theorem]} \)

3.2. \( \angle T = 51^\circ - 30^\circ \)  \( \text{[Ext. \(< 0^\circ \Delta]} \)

3.3. \( \angle C_2 = 21^\circ \)  \( \text{[Alt.}\angle \text{, } \text{N} \parallel \text{CD]} \)

3.4. \( \angle A_2 = 180^\circ - 81^\circ \)  \( \text{[Int. \( \text{of} \Delta]} \)

\[ = 99^\circ \]

\[ \therefore \angle A = 180^\circ - 99^\circ \text{[opp. \( \angle \) of cyclic quad]} \]

\[ = 81^\circ \]

QUESTION 4

\( \angle A_1 = \angle C \)  \( \text{[base \( \angle \) of Iso. } \Delta \text{, } AB = BC] \)

but \( \angle C = \angle D \)  \( \text{[Alt. \( \angle \) on the same segment]} \)

\[ \therefore \angle A_1 = \angle D \]

\[ \therefore \triangle AB \text{ is a Tangent Triangle, } AEB \text{ [Sine-Chord Theorem]} \]

QUESTION 5

5.1. \( \angle A_1 = 90^\circ \)  \( \text{[Sine 1 rad]} \)

\( \angle B_1 = 90^\circ \)  \( \text{[Sine 1 rad]} \)

\[ \angle A_1 + \angle B_1 = 180^\circ \]

\[ \therefore \triangle AOB \text{ is a cyclic quad [opp. \( \angle \) are supplementary]} \]

5.2. \( \angle \alpha = 2 \times 55^\circ = 110^\circ \)  \( \text{[\angle \text{ at the centre } = 2 \times \angle \text{ at the circumference}]} \)

\[ \angle \beta = 180^\circ - 110^\circ \]  \( \text{[\( \triangle AOB \text{ is a cyclic quad]} \)] \)

\[ = 70^\circ \]

TOTAL = 45 MARKS
APPENDIX E: LEARNERS’ QUESTIONS FOR INTERVIEWS

1. What challenges are you having with geometry as a school subject? Please explain.

2. What do you think are the factors that demotivate your learning of geometry? Please explain.

3. What factors do you believe contribute to the formation of misconceptions and the displaying of errors in your learning of geometry? Please explain.

4. What strategies do you suggest that should be employed by your teacher to enhance geometry teaching and learning? Please explain.

5. Is there anything about the way your geometry teacher presents his/her lessons that you think confuses you? Please explain.