Algebraic methods for hybrid logics

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Declaration

I hereby declare that this thesis entitled "*Algebraic methods for hybrid logics*" and the work presented in it has been produced as a result of my own research under the guidance and supervision of Prof W.E. Conradie and Prof C. van Alten, except where due reference has been made in the text. It is being submitted for the degree of Doctor of Philosophy at the University of Johannesburg. I confirm that this thesis has not been previously submitted for a degree or any other qualification at this university or any other institution.

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Abstract

Algebraic methods have been largely ignored within the field of hybrid logics. A main theme of this thesis is to illustrate the usefulness of algebraic methods in this field.

It is a well-known fact that certain properties of a logic correspond to properties of particular classes of algebras, and that we therefore can use these classes of algebras to answer questions about the logic. The first aim of this thesis is to identify a class of algebras corresponding to hybrid logics. In particular, we introduce *hybrid algebras* as algebraic semantics for the better known hybrid languages in the literature.

The second aim of this thesis is to use hybrid algebras to solve logical problems in the field of hybrid logic. Specifically, we will focus on proving general completeness results for some well-known hybrid logics with respect to hybrid algebras. Next, we study Sahlqvist theory for hybrid logics. We introduce syntactically defined classes of hybrid formulas that have first-order frame correspondents, which are preserved under taking Dedekind MacNeille completions of atomic hybrid algebras, and which are preserved under canonical extensions of permeated hybrid algebras. Finally, we investigate the finite model property (FMP) for several hybrid logics. In particular, we give analogues of Bull's theorem for the hybrid logics under consideration in this thesis. We also show that if certain syntactically defined classes of hybrid formulas are added to the normal modal logic **S4** as axioms, we obtain hybrid logics with the finite model property.

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Introduction

It is a well-known fact that most of the familiar logics have a (natural) algebraic semantics. More precisely, the algebraic semantics of classical modal logic takes the form of Boolean algebras with additional operators (BAO's). But why bring BAO's into the study of modal logic? There are two main reasons. First, when a logic has an algebraic counterpart, the powerful methods of modern algebra can be used in its investigation, and this can have a profound influence on the development of the theory of logic. Second, in many cases, the algebraic semantics of a modal logic turns out to be better-behaved than the frame-based semantics: for instance, it is possible to prove algebraic completeness for any normal modal logic, whereas no analogous results can be proved for frames (see [10]). Since hybrid logics extend modal logics, the following questions naturally arise: (i) Do hybrid logics also have an algebraic counterpart, and if so, what does the algebraic counterpart of hybrid logics look like? (ii) Can these algebras be used to solve logical problems in the field of hybrid logic, and if so, to what extent?

But why study hybrid logics to begin with? Although states form the foundation of the semantics of modal logics, the syntax of modal logic does not provide us with the necessary apparatus to refer to states directly — we evaluate modal formulas inside models at some state, and use the modalities to scan accessible states, but the syntax does not allow us to name these states or reason about state equality. For many reasons, this is a disadvantage. Hybrid languages correct this by enriching modal languages with *nominals*. Nominals are a second sort of atomic formula that range over singletons, thus acting as names for states in models. Additional syntactic apparatus which exploits the naming power of the nominals, like satisfaction operators or the global modality, are often added. This makes hybrid languages significantly more expressive than their modal counterparts, but what price do we pay? It turns out that because this apparatus is so straightforward, hybrid logics still remain well-behaved from a computational and mathematical point of view. For instance, the satisfiability problem for the minimal hybrid logic is known to be no more complex than the satisfiability problem for the minimal modal logic — both are PSPACE-complete (see [65]).

Although hybrid logics have been actively researched since the mid nineties, algebraic methods have been largely ignored within the field of hybrid logic. The only result we are aware of to this end is Litak's *algebraization* of a very expressive hybrid logic using algebras closely related to cylindrical algebras [61]. Despite this, many of the more familiar hybrid logics in the literature still lack algebraic semantics. Moreover, it is not known whether these hybrid logics are algebraizable in an appropriate sense. If so, what does this mean for us in terms of solving problems in the field of hybrid logic? What if these logics are not algebraizable? Does this mean that the new algebraic semantics is useless to us, or can it still be used to solve logical problems and therefore help us to better understand hybrid logics? These are the questions we will focus on in the rest of this thesis.

Addressing these questions requires techniques from modal logic, first-order logic, universal algebra and topology. So the first part of this thesis (see Chapter 1) consist of a thorough survey of the relevant literature. This survey aims to determine precisely the current state of knowledge on hybrid logics within certain parameters, and to gather some relevant techniques and results for modal logics. The techniques studied in this survey facilitate the formulation of strategies for addressing these shortcomings. The modal techniques and results do not always transfer in a straightforward way, and indeed, it is the cases in which they fail to do so that are of special interest to us.

We begin our contribution to the field of hybrid logic in Chapter 2 with the identification of plausible classes of algebras in which some of the more familiar hybrid languages can be interpreted. We provide two kinds of algebras. The first is Boolean algebras with additional operators in which the nominals are interpreted as constants. This class of algebras is just the standard algebraic semantics for modal logics with modal constants (nullary modalities), however, these algebras are not appropriately dual to the intended relational semantics of hybrid logics. We will refer to these algebras as *orthodox interpretations*. The second type of algebras, called *hybrid algebras*, consists of a BAO together with a non-empty subset of the atoms of the BAO called *designated atoms*. Of course this only takes us part of the way. We would also have to know how to evaluate a hybrid formula in this type of algebra. Hybrid formulas will be interpreted in hybrid algebras with nominals ranging over the set of designated atoms. Hybrid algebras are dual to the intended relational semantics of hybrid logics, and all main results in this thesis will be in terms of these algebraic structures. The orthodox interpretations will only be used as intermediate semantics in some of the proofs of our main results.

As in the case of modal logics, many of the properties of hybrid algebras correspond to properties of frames. These correspondences can be made exact in the form of *dualities*, which we investigate in Chapter 2. This may serve to prove new results, as well as reasserting known results. In particular, we will prove completeness for six different families of hybrid logics with respect to hybrid algebras, which reaffirms Ten Cate's completeness results in [72] through the dualities in Chapter 2.

The biggest and most interesting question now is: can our new hybrid algebras be used to solve logical problems in the field of hybrid logic? Often metalogical properties also end up having algebraic counterparts. For instance, as we mentioned before, in [10], we can find *algebraic completeness* results for normal modal logics. Furthermore, a central tool in proving completeness for modal logics is the notion of *canonicity*, which happens to have an equally important and interesting algebraic expression. In [20], Bull showed that all normal extensions of the modal logic **S4.3** have the finite model property using algebraic techniques. Since hybrid logics extend modal logics, one would expect that it is possible to obtain similar results for hybrid logics in terms of our hybrid algebras. Indeed, the remainder of this thesis is devoted to illustrating the usefulness of our hybrid algebras developed in Chapter 2 in the field of hybrid logic.

First, in Chapter 3, we turn our attention to proving *algebraic completeness* theorems for the better known hybrid logics. There is a standard method to prove that a logic is complete with respect to a class of algebras. This method can be said to have begun with Tarski's 1935 paper [70] (an English translation appears in [71]). In this paper, Tarski gives the precise relationship between Boolean algebras and propositional logic. The basic idea is to look at the set of formulas as an algebra with operations induced by the logical connectives. Tarski then observed that the logical equivalence relation is a congruence relation on the formula algebra, and therefore a quotient algebra could be built — the so called *Lindenbaum-Tarski algebra*. This method has come to be known as the *Lindenbaum-Tarski method*. Unfortunately, this method cannot be applied straightforwardly to prove the completeness of hybrid logics, mainly because of the two-sorted nature of the language and the restrictions on substitution in hybrid logics. So in order to obtain our completeness results, we need a different method. A key methodological tool in proving completeness is a construction simulating the process of taking generated submodels algebraically.

Chapter 4 is dedicated to developing a *hybrid* Sahlqvist theory. Some initial results of this kind can be found in [28] and [73]. We first extend the definition of inductive formulas in [55] to the hybrid language with satisfaction operators, and in addition, define two subclasses, called *skeletal* and *nominally skeletal* hybrid inductive formulas. Once we have this set up, we show that every hybrid inductive formula has a first-order frame correspondent, and that formulas of the subclasses are respectively preserved under canonical extensions and Dedekind-MacNeille completions of certain hybrid algebras. The latter is enough to ensure that these formulas axiomatize relationally complete logics. Our methods use a variation of the algorithm ALBA (**A**ckermann **L**emma **B**ased **A**lgorithm) developed in [35], which we formulate and call hybrid-ALBA.

Finally, in Chapter 5, we investigate the *finite model property* (FMP) for hybrid logics. In [7], it is shown that the finite model property is in general *not* transferred from a modal logic to its hybrid companion obtained by adding nominals to the modal logic. Is this always the case, or are there modal logics with the finite model property whose hybrid companion obtained by adding nominals to the modal logic also have the finite model property? In this thesis, we will show that the latter is true. In particular, we will prove an analogue of Robert Bull's famous result in [20] that all normal extensions of the modal logic **S4.3** have the finite model property for the hybrid logic obtained by adding nominals to this logic. As expected, the techniques used by Bull do not transfer in a straightforward way.

The question that naturally now arises is: to what extent does Bull's theorem hold when we add satisfaction operators or the global modality in addition to nominals to the modal logic **S4.3**? As we will see in Chapter 5, it is not clear if these hybrid companions of **S4.3** also have the finite model property. However, we show that the hybrid logic obtained by adding nominals and the global modality has the finite model property if we add an additional axiom enforcing *well-connectedness*, an algebraic property that plays a crucial role in Bull's proof in [20]. Unfortunately, if we add nominals and satisfaction operators, we have to settle for proving the finite model property for a specific fragment of the hybrid logic obtained.

Still on the topic of the finite model property, in [18], Bull characterized a class of axiomatic extensions of the normal modal logic **S4** with the finite model property. This result takes the form of a syntactic characterization of a class of formulas that may be added as axioms to **S4**, somewhat in the spirit of Sahlqvist's famous theorem in modal correspondence theory. In this thesis, we extend this result for hybrid languages: we expand the syntactic class given by Bull in [18] with hybrid formulas, and then show that if the formulas in these syntactic classes are added to the normal modal logic **S4**, then we obtain hybrid logics with the finite model property.

In short, the results provided in this thesis provide an algebraic semantics for hybrid logics, which we then apply in various settings, illustrating its usefulness.

Chapter 1

Preliminaries on modal and hybrid logics

In this chapter, we review some vital background knowledge. We focus our attention on the modal and hybrid logics that will be in the spotlight in the later chapters, and also on their relation to first-order logic. This chapter makes no original contribution, nor does it contain anything unusual or surprising, it simply aims to collect some relevant facts, and to fix some terminology and notation. The reader familiar with these logics might best skip over this chapter, only referring back to it should (s)he ever, in later sections, find (her)himself in doubt as to the meaning of some notation or term.

1.1 Modal logic

We obtain propositional modal languages by supplementing propositional logic with modal operators — in the most basic case, with the dual pair of unary operators $\Diamond$ and $\Box$. These operators can be interpreted in a number of diverse ways. First, $\Diamond\varphi$ can be read as 'it is possibly the case that φ'. Under this reading, $\Box\varphi$ means 'necessarily φ'. Second, in *epistemic logic*, modal languages are used to reason about knowledge. Instead of writing $\Box\varphi$ for 'the agent knows that φ', it is usual to write $K\varphi$. The language of tense (or temporal) logic enriches propositional logic with F and P, together with their duals G and H, and interprets $F\varphi$ as 'sometime in the future φ', $G\varphi$ as 'always in the future φ', $P\varphi$ as 'sometime in the past φ', and $H\varphi$ as 'always in the past φ'. We can list many more examples.

For this thesis, however, the appropriate perspective is to follow [10] and view modal languages as languages for talking about relational structures. In what follows, we fix some basic notions of modal logic. The reader is referred to any one of the references [10], [23] or [58] for very thorough and the most recent treatments of modal logic.

1.1.1 Syntax

For our purpose, we will only consider the *basic modal language*. So let PROP be a countably infinite set of propositional variables. Then we inductively define the set of basic modal formulas by the following rule:

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi.$$

We will make use of the standard abbreviations for disjunction, implication, bi-implication and the constant $\top$. In particular, $\Box\varphi$ is shorthand for $\neg\Diamond\neg\varphi$.

1.1.2 Semantics

The basic modal language can be interpreted over various structures. We will be concerned with Kripke frames, models, general frames, and Boolean algebras with operators.

Kripke frames and models

A (*Kripke*) *frame* for the basic modal language is a pair $\mathfrak{F} = (W, R)$ such that W is a non-empty set (called the *domain*) of objects called states or points, and R is a binary *accessibility relation* on W. A *model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ such that $\mathfrak{F}$ is a frame for the basic modal language and V is a map from PROP to the power set of W assigning to each propositional variable p in PROP a subset $V(p)$ of W. V is called a *valuation* . Given a model $\mathfrak{M} = (\mathfrak{F}, V)$, we say that $\mathfrak{M}$ is *based* on the frame $\mathfrak{F}$, or that $\mathfrak{F}$ is the frame *underlying* $\mathfrak{M}$.

Let $\mathfrak{M} = (W, V)$ be a model, and w a state in W. Then we inductively define the notion of a modal formula being *satisfied* (or *true*) in $\mathfrak{M}$ at w as follows:

$\mathfrak{M}, w \Vdash \bot$ never

$\mathfrak{M}, w \Vdash p$ iff $w \in V(p)$

$\mathfrak{M}, w \Vdash \neg\varphi$ iff $\mathfrak{M}, w \nVdash \varphi$

$\mathfrak{M}, w \Vdash \varphi \wedge \psi$ iff $\mathfrak{M}, w \Vdash \varphi$ and $\mathfrak{M}, w \Vdash \psi$

$\mathfrak{M}, w \Vdash \Diamond\varphi$ iff there exists v such that wRv and $\mathfrak{M}, v \Vdash \varphi$

A formula φ is *globally true* in a model $\mathfrak{M}$ (denoted $\mathfrak{M} \Vdash \varphi$), if it is true at all states in $\mathfrak{M}$. We say that φ is *satisfiable* in a model $\mathfrak{M}$, if it is true at some state in $\mathfrak{M}$. On the other hand, φ is *refutable* in a model if its negation is satisfiable. A set of formulas Γ is globally true in a model $\mathfrak{M}$, if every formula in Γ is globally true in $\mathfrak{M}$. A set of formulas Γ is satisfiable in a model $\mathfrak{M}$, if every formula in Γ is satisfiable in $\mathfrak{M}$.

A modal formula φ is *valid at a state w in a frame* $\mathfrak{F}$ (denoted $\mathfrak{F}, w \Vdash \varphi$), if it is true at w in every model $(\mathfrak{F}, V)$ based on $\mathfrak{F}$. We say that φ is *valid in a frame* $\mathfrak{F}$ (denoted $\mathfrak{F} \Vdash \varphi$), if φ is valid at all states in $\mathfrak{F}$. A formula φ is *valid on a class of frames* K (denoted $\mathsf{K} \Vdash \varphi$), if it is valid on all frames in K. A formula φ is *valid* (denoted $\Vdash \varphi$), if φ is valid on the class of all frames.

These concepts can be extended to sets of formulas in the obvious way. Specifically, a set Γ of modal formulas is valid on a frame $\mathfrak{F}$ (denoted $\mathfrak{F} \Vdash \Gamma$), if every formula in Γ is valid on $\mathfrak{F}$. A set of modal formulas Γ is valid on a class K of frames (denoted $\mathsf{K} \Vdash \Gamma$), if Γ is valid on every member of K.

General frames

A *general frame* is a structure $\mathfrak{g} = (W, R, A)$, where (W, R) is a frame, A is a non-empty collection of subsets of W (called *admissible subsets*) which is closed under finite intersection, relative complement, and under the operation $\langle R\rangle$ on the power set of W defined by

$$\langle R\rangle X := \{w \in W \mid \exists v \in X(wRv)\}.$$

Clearly, A is also closed under the dual operator $[R]$, defined as follows:

$$[R]X := \{w \in W \mid wRv \text{ implies } v \in X\}.$$

It is not difficult to see that the collection A is a Boolean algebra of subsets of W. Furthermore, by convention, Kripke frames may be identified with general frames for which the set of admissible subsets is the set of all subsets of its domain.

A valuation V on $\mathfrak{g}$ is called admissible for $\mathfrak{g}$, if for each propositional letter p, $V(p) \in A$. A *modal based on a general frame* is a pair $(\mathfrak{g}, V)$, where V is an admissible valuation for $\mathfrak{g}$. The notions of truth and validity of formulas are accordingly relativized with respect to general frames and models based on them.

Here are a few types of general frames that will be encountered further on. A general frame $\mathfrak{g} = (W, R, A)$ is said to be:

differentiated, if for every $w, v \in W$ such that $w \neq v$, there exists $a \in A$ such that $w \in a$ and $v \notin a$;

tight, if for all $u, v \in W$, it is the case that uRv iff $\forall a \in A(v \in a \Longrightarrow u \in \langle R\rangle a)$;

compact, if $\bigcap A_0 \neq \varnothing$ for every subset A_0 of A which has the finite intersection property[1];

refined, if it is differentiated and tight;

descriptive, if it is refined and compact;

discrete, if every singleton subset of W is in A.

Boolean algebras with operators

Algebraically, we interpret modal languages in Boolean algebras with additional operators, or BAOs.

Definition 1.1.1 (Boolean algebras with operators). A *Boolean algebra with operators* for the basic modal language is an algebra

$$\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, f)$$

such that $(A, \wedge, \vee, \neg, \bot, \top)$ is a Boolean algebra and f is an *operator*, i.e., a function from A to A, satisfying the following:

[1]A family of sets has the finite intersection property, if every finite subfamily has non-empty intersection.

(*normality*) $f(\bot) = \bot$, and

(*additivity*) $f(a \vee b) = f(a) \vee f(b)$.

Note that these equations correspond to the following modal formulas:

$$\begin{aligned} \Diamond\bot &\leftrightarrow \bot \\ \Diamond(p \vee q) &\leftrightarrow \Diamond p \vee \Diamond q. \end{aligned}$$

Both these formulas are modal validities, so the algebraic operators are well-named.

An operator f on a Boolean algebra is *monotonic*, if $a \leq b$ implies $f(a) \leq f(b)$. All operators are monotonic. To see this, assume $a \leq b$. Then $a \vee b = b$, so $f(a) \vee f(b) = f(a \vee b) = f(b)$, which means $f(a) \leq f(b)$. We say that operators have the property of *monotonicity*.

Having defined BAOs, how would we go about evaluating a formula in a BAO. Of course we would have to know how to evaluate the propositional variables in the formula first. This is done by a function assigning to each propositional variable an element of the BAO. Once we have this set up, we can extend such a map to terms and evaluate the formula's meaning under this assignment.

Definition 1.1.2. Let $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, f)$ be a BAO for the basic modal language. An *assignment* on $\mathbf{A}$ is a function v: $\mathsf{PROP} \to A$ associating an element of A with each propositional variable in PROP. Given such an assignment v, we calculate the *meaning* $\widetilde{v}(t)$ of a term t as follows:

$$\begin{aligned} \widetilde{v}(\bot) &= \bot \\ \widetilde{v}(p) &= v(p), \\ \widetilde{v}(\neg\psi) &= \neg\widetilde{v}(\psi), \\ \widetilde{v}(\psi_1 \wedge \psi_2) &= \widetilde{v}(\psi_1) \wedge \widetilde{v}(\psi_2), \text{ and} \\ \widetilde{v}(\Diamond\psi) &= f(\widetilde{v}(\psi)). \end{aligned}$$

We say that an equation $\varphi \approx \psi$ is *true* in a BAO $\mathbf{A}$ (denoted $\mathbf{A} \models \varphi \approx \psi$), if for all assignments θ, $\widetilde{\theta}(\varphi) = \widetilde{\theta}(\psi)$. A set E of equations is true in a BAO $\mathbf{A}$ (denoted $\mathbf{A} \models E$), if each equation in E is true in $\mathbf{A}$. An equation $\varphi \approx \psi$ is a *semantic consequence* of a set E of equations (denoted $E \models \varphi \approx \psi$), if for any BAO $\mathbf{A}$ such that $\mathbf{A} \models E$, $\mathbf{A} \models \varphi \approx \psi$.

1.1.3 The standard translation

Here we link modal logic with the wider logical world using what is called the *standard translation*. But first we define our *correspondence language* — the language we will translate modal formulas into.

Define $\mathcal{L}_0$ to be the first-order language with $=$, a binary relation symbol R, and individual variables $\mathsf{VAR} = \{x_0, x_1, \ldots\}$. Furthermore, let $\mathcal{L}_1$ be the extension of $\mathcal{L}_0$ with a set of unary predicates $\{P_0, P_1, \ldots\}$ corresponding to the propositional variables $p_0, p_1, \ldots$ in PROP.

Definition 1.1.3 (Standard translation). Let x be a first-order variable from VAR. The *standard translation* ST_x taking modal formulas to first-order formulas in $\mathcal{L}_1$ is inductively defined as follows:

$$\begin{aligned} ST_x(p) &= P(x), \\ ST_x(\bot) &= x \neq x, \\ ST_x(\neg\varphi) &= \neg ST_x(\varphi), \\ ST_x(\varphi \wedge \psi) &= ST_x(\varphi) \wedge ST_x(\psi), \text{ and} \\ ST_x(\Diamond\varphi) &= \exists y(xRy \wedge ST_y(\varphi)), \end{aligned}$$

where y is a variable that has not been used in the translation.

Models for the basic modal language can be viewed as models for the first-order language $\mathcal{L}_1$. Indeed, we have the following:

Proposition 1.1.4. *Let φ be a formula in the basic modal language. Then:*

(i) *For any model $\mathfrak{M}$ and all states of $\mathfrak{M}$, $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M} \models ST_x(\varphi)[x := w]$.*

(ii) *For all $\mathfrak{M}$, $\mathfrak{M} \Vdash \varphi$ iff $\mathfrak{M} \models \forall x ST_x(\varphi)$.*

1.1.4 Logics

Throughout this subsection we will work with a fixed countable infinite set of propositional variables.

Definition 1.1.5 (Normal modal logics). A *normal modal logic* Λ (in the basic modal language) is a set of modal formulas such that

(i) Λ contains all propositional tautologies,

(ii) Λ contains the axiom $\Box(p \to q) \to (\Box p \to \Box q)$ (known as the K-axiom) and $\Box p \leftrightarrow \neg\Diamond\neg p$ (*Dual*), and

(iii) Λ is closed under uniform substitution (if $\varphi \in \Lambda$, then all substitution instances of φ are also in Λ), modus ponens ($\varphi \to \psi, \varphi \in \Lambda$ implies $\psi \in \Lambda$), as well as necessitation ($\varphi \in \Lambda$ implies $\Box\varphi \in \Lambda$).

Given a logic Λ and a formula φ, if $\varphi \in \Lambda$, we say that φ is a Λ-*theorem* of Λ, and write $\vdash_\Lambda \varphi$; if not, we write $\nvdash_\Lambda \varphi$.

For any set of formulas Γ, there is a smallest normal modal logic containing it. We call this the normal modal logic *generated* or *axiomatized* by Γ. The normal logic generated by the empty set is called $\mathbf{K}$, in honour of Kripke, and it is the smallest (or minimal) normal modal logic. If Γ is a non-empty set of formula, we will denote the normal modal logic generated by Γ by $\mathbf{K} \oplus \Gamma$. We will also often refer to the formulas in Γ as *axioms* of the logic, and say that the logic was generated using the *rules of proof* substitution, modus ponens and generalization.

Here are some of the better known axioms, together with their traditional names, that we will encounter further on:

(4) $\Diamond\Diamond p \to \Diamond p$;

(T) $p \to \Diamond p$;

(B) $p \to \Box\Diamond p$;

(D) $\Box p \to \Diamond p$;

(.3) $\Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \wedge \Diamond(q \wedge \Diamond p)$.

There is also a tradition for denoting logics generated by such axioms: instead of writing $\mathbf{K} \oplus \{p \to \Diamond p\}, \mathbf{K} \oplus \{p \to \Box\Diamond p\}, \mathbf{K} \oplus \{p \to \Diamond p, \Diamond\Diamond p \to \Diamond p\}, \mathbf{K} \oplus \{p \to \Diamond p, \Diamond\Diamond p \to \Diamond p, \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \wedge \Diamond(q \wedge \Diamond p)\}$ and $\mathbf{K} \oplus \{p \to \Diamond p, \Diamond\Diamond p \to \Diamond p, p \to \Box\Diamond p\}$, we write **T**, **B**, **S4**, **S4.3** and **S5**.

Let $\Gamma \cup \{\varphi\}$ be a set of formulas. Then we say that φ is *deducible in* Λ *from* Γ, if $\vdash_\Lambda \varphi$ or there are formulas $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_\Lambda (\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$. If this is the case, we write $\Gamma \vdash_\Lambda \varphi$, if not, $\Gamma \nvdash_\Lambda \varphi$. A set of formulas Γ is Λ-*consistent*, if $\Gamma \nvdash_\Lambda \bot$, and Λ-*inconsistent* otherwise.

1.1.5 Completeness

Here we review two fundamental concepts linking the syntactic and semantic perspectives, namely, *soundness and completeness*.

Frame-theoretical perspective

Before we can give the precise definition of soundness and completeness, we need the following definition:

Definition 1.1.6 (**Local semantic consequence**). Let S be a class of frames, or models, or general frames, and let $\Gamma \cup \{\varphi\}$ be a set of formulas in the basic modal language. We then say that φ is a *local semantic consequence of* Γ *over* S (denote $\Gamma \Vdash_{\mathsf{S}} \varphi$), if for all structures $\mathfrak{G}$ in S and all states w in $\mathfrak{G}$, $\mathfrak{G}, w \Vdash \varphi$ whenever $\mathfrak{G}, w \Vdash \Gamma$.

Definition 1.1.7 (**Soundness and completeness**). Let S be a class of frames, or models, or general frames. A logic Λ is *sound* with respect to S, if for all formulas ψ and all structures $\mathfrak{G}$ in S, $\vdash_\Lambda \psi$ implies $\mathfrak{G} \Vdash \psi$. A logic Λ is *strongly complete* with respect to S, if for any set of formulas $\Gamma \cup \{\psi\}$, $\Gamma \Vdash_{\mathsf{S}} \psi$ implies $\Gamma \vdash_\Lambda \psi$. A logic Λ is *weakly complete* with respect to S, if for any formula ψ, $\mathsf{S} \Vdash \psi$ implies $\vdash_\Lambda \psi$.

Note that weak completeness is the special case of strong completeness in which Γ is empty, so strong completeness with respect to some class of structures implies weak completeness with respect to the same class of structures.

We often make use of the following result to prove completeness:

Proposition 1.1.8. *Let* S *be a class of frames, or models, or general frames. A logic* Λ *is strongly complete with respect to* S *iff every* Λ*-consistent set of formulas is satisfiable on some structure in* S*. A logic* Λ *is weakly complete with respect to* S *iff every* Λ*-consistent formula is satisfiable on some structure in* S*.*

We have the following general completeness results with respect to general frames:

Theorem 1.1.9. *Every normal modal logic is sound and strongly complete with respect to its class of general frames.*

In fact, this even holds if we restrict our attention to the class of descriptive general frames.

Theorem 1.1.10 ([47])**.** *Every normal modal logic is sound and strongly complete with respect to its class of descriptive general frames.*

Algebraic perspective

How do we prove completeness of modal logics algebraically? Obviously we have to show that any non-theorem of the modal logic can be refuted on some BAO. So the most important question is: how do we build a BAO refuting a non-theorem? It turns out that the Lindenbaum-Tarski algebra of a logic gives us a counter-example for any non-theorem. This algebra was first introduced by Tarski in [70] with an English translation appearing in [71]. The idea is to build an algebra on top of the formula algebra in such a way that the relation of logical equivalence between two modal formulas is a congruence relation[2]. So let us recall the definition of a formula algebra. As before, we only consider the basic modal language. In this exposition, we will be following [72].

Definition 1.1.11 (**Formula algebras**)**.** The *formula algebra* of the basic modal language over PROP is the algebra

$$\mathbf{Form}(\mathsf{PROP}) = (\mathit{Form}(\mathsf{PROP}), \cdot, +, -, 0, 1, f_\Diamond),$$

where $\mathit{Form}(\mathsf{PROP})$ is the set of basic modal formulas over PROP, $\varphi \cdot \psi := \varphi \wedge \psi, \varphi + \psi := \varphi \vee \psi, -\varphi := \neg\varphi$, and $f_\Diamond \varphi := \Diamond\varphi$.

Let Λ be a normal modal logic in the basic modal language. We define $\equiv_\Lambda$ as a binary relation between formulas by

$$\varphi \equiv_\Lambda \psi \text{ iff } \vdash_\Lambda \varphi \leftrightarrow \psi.$$

If $\varphi \equiv_\Lambda \psi$, we say that φ and ψ are *equivalent modulo* Λ.

Proposition 1.1.12. *Let Λ be a normal modal logic in the basic modal language. Then $\equiv_\Lambda$ is a congruence relation on* $\mathbf{Form}(\mathsf{PROP})$*.*

Given Proposition 1.1.12, we define the Lindenbaum-Tarski algebra of any normal modal logic Λ as the quotient algebra[3]

[2]Let $\mathbf{A}$ be an algebra. An equivalence relation $\sim$ on $\mathbf{A}$ is a *congruence*, if for every basic operation f of $\mathbf{A}$, $a_1 \sim b_1 \& \cdots \& a_n \sim b_n$ implies $f_{\mathbf{A}}(a_1, \ldots, a_n) \sim f_{\mathbf{A}}(b_1, \ldots, b_n)$.

[3]Let $\mathbf{A}$ be an algebra, and $\sim$ is a congruence relation on $\mathbf{A}$. The *quotient algebra* of $\mathbf{A}$ by $\sim$ is the algebra $\mathbf{A}/\sim$ whose carrier is the set $A/\sim = \{[a] \mid a \in a\}$ of equivalence classes of A under $\sim$, and whose operations are defined by

$$f_{\mathbf{A}/\sim}([a_1], \ldots [a_n]) = [f_{\mathbf{A}}(a_1, \ldots, a_n)].$$

of the formula algebra over the congruence relation $\equiv_\Lambda$:

Definition 1.1.13 (Lindenbaum-Tarski algebra). Let Λ be a normal modal logic in the basic modal language. The *Lindenbaum-Tarski algebra of* Λ *over* PROP is the structure

$$\mathcal{L}_\Lambda(\mathsf{PROP}) = (\mathit{Form}(\mathsf{PROP})/\equiv_\Lambda, \cdot, +, -, 0, 1, f_\Diamond),$$

where

$$\begin{aligned}
[\varphi]\cdot[\psi] &:= [\varphi \wedge \psi],\\
[\varphi]+[\psi] &:= [\varphi \vee \psi],\\
-[\varphi] &:= [\neg\varphi],\\
f_\Diamond[\varphi] &:= [\Diamond\varphi],\\
0 &:= [\bot], \text{ and}\\
1 &:= [\top].
\end{aligned}$$

Note that Proposition 1.1.12 tells us that the operations in Definition 1.1.13 are correct.

Theorem 1.1.14. *Let Λ be a modal logic in the basic modal language. Then*

$$\vdash_\Lambda \varphi \text{ iff } \mathcal{L}_\Lambda(\mathsf{PROP}) \models \varphi \approx \top.$$

Proof. For the completeness direction, suppose $\nvdash_\Lambda \varphi$. We have to find an assignment v such that $\mathcal{L}_\Lambda(\mathsf{PROP}), v \not\models \varphi \approx \top$. Note that $[\neg\varphi] \neq [\bot]$, for otherwise $\vdash_\Lambda \neg\varphi \leftrightarrow \bot$, which means that $\vdash_\Lambda \top \leftrightarrow \varphi$, and so $\vdash_\Lambda \varphi$, a contradiction. Now, let ι: $\mathsf{PROP} \to \mathit{Form}(\mathsf{PROP})/\equiv_\Lambda$ be defined by $\iota(p) = [p]$[4]. It can easily be verified by straightforward structural induction that $\widetilde{\iota}(\rho) = [\rho]$ for all formulas ρ that use variables from the set PROP. But then $\widetilde{\iota}(\varphi) = [\varphi] \neq [\top] = \widetilde{\iota}(\top)$, for otherwise $[\neg\varphi] = [\bot]$, which is a contradiction.

For the converse direction, let ψ be a theorem of Λ, and let ν be an arbitrary assignment on Λ. So ν assigns an equivalence class to each propositional variable. For each variable p in PROP choose a representing formula $\theta(p)$ in the equivalence class $\nu(p)$. Then $\nu(p) = [\theta(p)]$. We may therefore view θ as a function mapping propositional variables to formulas; in other words, θ is a substitution. Let $\theta(\rho)$ denote the effect of performing this substitution on ρ. Using structural induction on ρ, we can show that for any formula ρ, $\widetilde{\nu}(\rho) = [\theta(\rho)]$. Now, we know that Λ is closed under substitution, so $\theta(\psi)$ is a theorem, and so $\theta(\psi) \equiv_\Lambda \top$. Hence, $[\theta(\psi)] = [\top]$, which means that $\widetilde{\nu}(\psi) = [\top]$. $\square$

Let us now make sure that the Lindenbaum-Tarski algebra is an algebraic model of the right kind. But to give a precise formulation, we need the following definition:

Definition 1.1.15. Let Σ be a set of formulas in the basic modal language. We define V_Σ to be the class of Boolean algebras with operators in which the set $\Sigma^\approx = \{\sigma \approx \top \mid \sigma \in \Sigma\}$ is valid.

Theorem 1.1.16. *Let Λ be a normal modal logic in the basic modal language. Then we have* $\mathcal{L}_\Lambda(\mathsf{PROP}) \in \mathsf{V}_\Lambda$.

[4] Let $\mathbf{A}$ be an algebra and $\sim$ a congruence relation on $\mathbf{A}$. A function ι taking an element a of A to its equivalence class $[a]$ is called the *natural map* associated with the congruence.

Proof. Once we have shown that $\mathcal{L}_\Lambda(\mathsf{PROP})$ is a BAO, the result follows immediately from Theorem 1.1.14. First, that $\mathcal{L}_\Lambda(\mathsf{PROP})$ is a Boolean algebra is clear. So all we have to check is that $f_\diamond$ is normal and additive. Now, it is easy to check that $\vdash_\Lambda \diamond\bot \leftrightarrow \bot$. But then we have $[\diamond\bot] = [\bot]$, so $f_\diamond([\bot]) = [\diamond\bot] = [\bot]$. Likewise, it is not difficult to show that $\vdash_\Lambda \diamond(\varphi \vee \psi) \leftrightarrow (\diamond\varphi \vee \diamond\psi)$, which means that $[\diamond(\varphi \vee \psi)] = [\diamond\varphi \vee \diamond\psi]$. Hence, $f_\diamond([\varphi]+[\psi]) = f_\diamond([\varphi \vee \psi]) = [\diamond(\varphi\vee\psi)] = [\diamond\varphi \vee \diamond\psi] = [\diamond\varphi] + [\diamond\psi] = f_\diamond([\varphi]) + f_\diamond([\psi])$. $\square$

We are now ready to give the algebraic completeness theorem for modal logic.

Theorem 1.1.17. *Every normal modal logic $\mathbf{K} \oplus \Sigma$ is sound and complete with respect to the class of all BAOs which validate Σ. That is $\vdash_{\mathbf{K}\oplus\Sigma} \varphi$ iff $\mathsf{V}_\Sigma \models \varphi \approx \top$.*

Proof. The soundness direction is straightforward. The completeness direction follows from Theorems 1.1.14 and 1.1.16. $\square$

1.1.6 Frame definability

This part is mostly about using formulas to define classes of frames. In particular, we discuss a number of results concerning the relationship between classes of frames definable by modal formulas and frame classes definable by first-order formulas. But wait! What does it mean for a formula to define a class of frames?

Definition 1.1.18 (**Definability**)**.** Let φ be a modal formula, and K a class of frames. We say that φ *defines* K, if for all frames $\mathfrak{F}$, $\mathfrak{F} \in \mathsf{K}$ if and only if $\mathfrak{F} \Vdash \varphi$. Similarly, if Γ is a set of modal formulas, we say that Γ *defines* K, if for every $\mathfrak{F}$, $\mathfrak{F} \in \mathsf{K}$ if and only if $\mathfrak{F} \Vdash \Gamma$. A class of frames is *modally definable* if there is some set of modal formulas that defines it.

We will often say that a formula defines a *property*, if it defines the class of frames satisfying that property. For example, $p \to \diamond p$ defines the class of reflexive frames, or simply, $p \to \diamond p$ defines reflexivity. Similarly, $\diamond\diamond p \to \diamond p$ defines transitivity.

Since frames are just relational structures, we are free to define frame classes using non-modal languages. For instance, the class of reflexive frames is simply the class of all frames that make the first-order formula $\forall x(xRx)$ true, while the class of transitive frames make $\forall x\forall y\forall z(xRy \wedge yRz \to xRz)$ true. We say that these classes of frames are *elementary*. More precisely:

Definition 1.1.19 (**Elementary frame class**)**.** A frame class is *elementary*, if it is defined by a sentence of the first-order frame correspondence language $\mathcal{L}_0$.

We are interested in the relationship between modally definable frame classes and elementary frame classes. First, is it the case that all modally definable frame classes are elementary? The answer is *no* — the frame class defined by the Löb formula $\Box(\Box p \to p) \to \Box p$ is not elementary. On the other hand, are all elementary frame classes modally definable? The answer is again *no*, as we will soon see.

In what follows, we will review some model theoretic characterizations of the modally definable elementary frame classes, as well as some attempts at syntactic characterizations.

Model theoretic characterizations

One of the best-known results in modal logic is the *Goldblatt-Thomason Theorem*. This result allows us to characterize the elementary frame classes which are also modally definable in terms of the frame constructions generated subframes, disjoint unions, bounded morphic images, and ultrafilter extensions. So before we formally state the Goldblatt-Thomason Theorem, let us first recall the definitions of these frame constructions.

Definition 1.1.20 (Generated subframes). A frame $\mathfrak{G} = (W', S)$ is a *generated subframe* of the frame $\mathfrak{F} = (W, R)$ (written $\mathfrak{G} \rightarrowtail \mathfrak{F}$) if $W' \subseteq W$, $S = R \cap W' \times W'$, and the following condition holds:

$$w \in W', v \in W \text{ and } wRv \text{ implies } v \in W'.$$

Let X be a non-empty subset of W. The *subframe generated by* X (written $\mathfrak{F}_X$) is the smallest generated subframe of $\mathfrak{F}$ whose domain contains X. If X is a singleton $\{w\}$, we write $\mathfrak{F}_w$ for the subframe generated by w. If a frame $\mathfrak{F}$ is generated by a singleton $\{w\}$ we say it is *rooted* or *point-generated*. The point w is called the *root* of the frame.

Definition 1.1.21 (Disjoint unions). Let $\{\mathfrak{F}_i \mid i \in I\}$ be a collection of disjoint frames. Then their *disjoint union* is the structure

$$\biguplus_{i \in I} \mathfrak{F}_i = (W, R),$$

where W is the union of the domains W_i and R is the union of the relations R_i.

Definition 1.1.22 (Bounded morphisms). A mapping $f \colon \mathfrak{F} \to \mathfrak{G}$ is a *bounded morphism* from a frame $\mathfrak{F} = (W, R)$ to a frame $\mathfrak{G} = (W', S)$ if f satisfies the following conditions:

(*Forth*) wRv implies $f(w)Sf(v)$, and

(*Back*) $f(w)Sv'$ implies that wRv and $f(v) = v'$ for some $v \in W$.

We say that $\mathfrak{G}$ is a *bounded morphic image* of $\mathfrak{F}$, denoted $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$, if there is a surjective bounded morphism from $\mathfrak{F}$ onto $\mathfrak{G}$.

Definition 1.1.23 (Ultrafilter extensions). Let $\mathfrak{F} = (W, R)$ be a frame for the basic modal language. The ultrafilter extension $\mathfrak{ue}\mathfrak{F}$ of $\mathfrak{F}$ is defined as the frame $(Uf(W), R^{ue})$, where $Uf(W)$ is the set of ultrafilters[5] over W, and for $u, v \in Uf(W), uR^{ue}v$ iff for all $X \in v, \langle R \rangle X \in u$.

It is a well-known fact that the first three constructions preserve modal validity, while the fourth anti-preserves it (meaning that if a formula is valid on the ultrafilter extension of a frame, then it is also valid on the original frame). But this means that these constructions can be used to test for modal definability: if we can show that a class of frames is *not* closed under one of these constructions, then we can show that it is *not* modally definable. This is exactly what the Goldblatt-Thomason Theorem tells us. We will use the following terminology: a class of frames K *reflects* ultrafilter extensions, if $\mathfrak{ue}\mathfrak{F} \in \mathsf{K}$ implies $\mathfrak{F} \in \mathsf{K}$.

[5] We will define ultrafilters in Subsection 1.1.7.

Theorem 1.1.24 (**Goldblatt-Thomason Theorem** [50]). *An elementary frame class is modally definable iff it is closed under generated subframes, disjoint unions, bounded morphic images and reflects ultrafilter extensions.*

This tells us which elementary frame classes are modally definable, but which modally definable frame classes are elementary? This question was answered by Van Benthem in [76].

Theorem 1.1.25. *Let* K *be any modally definable frame class. Then the following statements are equivalent:*

(i) K *is elementary.*

(ii) K *is defined by a set of first-order sentences.*

(iii) K *is closed under elementary equivalence.*

(iv) K *is closed under ultrapowers*[6].

Syntactic characterizations

The results above do not tell us which modal formulas define an elementary frame class. The bad news is that the problem whether a given modal formula defines an elementary frame class is undecidable according to Chagrova in [22] and [24]. But if we are willing to be satisfied with *approximations*, all is not lost. Various large and interesting syntactically defined classes of formulas defining elementary frame classes are known. In what follows, we review the most famous among these classes.

The *Sahlqvist formulas* undoubtedly form the best known syntactically specified class of formulas defining elementary frame classes. They were first introduced by Sahlqvist in [64], in a slightly more restricted form than the definition we use today.

Definition 1.1.26. An occurrence of a propositional variable in a formula φ is *positive* (*negative*), if it is in the scope of an even (odd) number of negation signs. A formula φ is *positive* (*negative*) *in propositional variable* p, if all occurrences of p in φ are positive (negative). A formula is *positive* (*negative*), if it is positive (negative) in all propositional variables.

We will refer to the positivity or negativity of a formula in a propositional variable as its *polarity* in that propositional variable.

Definition 1.1.27. A *boxed atom* is a propositional variable, prefixed with finitely many (possibly no) boxes.

Definition 1.1.28 (**Sahlqvist antecedent**). A *Sahlqvist antecedent* is a formula built up from $\top, \bot$, boxed atoms and negative formulas, using $\wedge$, $\vee$ and diamonds.

In particular, note that any negative formula is a Sahlqvist antecedent.

[6]The reader unfamiliar with this construction is referred to any standard text on model theory, for example, [25] or [56].

Definition 1.1.29 (Sahlqvist implication). A *Sahlqvist implication* is a formula of the form $\varphi \to \psi$, where φ is a Sahlqvist antecedent and ψ is a positive formula.

Definition 1.1.30 (Sahlqvist formula). A *Sahlqvist formula* is a formula that is obtained from Sahlqvist implications by applying boxes, disjunctions and conjunctions.

Many well known formulas fall within the class of Sahlqvist formulas. Let us look at some examples.

Example 1.1.31. The following formulas are Sahlqvist:

(i) the formula $p \to \Diamond p$ defining reflexivity,

(ii) the formulas $\Diamond\Diamond p \to \Diamond p$ or $\Box p \to \Box\Box p$ defining transitivity,

(iii) the formula $p \to \Box\Diamond p$ defining symmetry,

(iv) the formula $\Box p \to \Diamond p$ defining seriality, and

(v) the Geach-formula $\Diamond\Box p \to \Box\Diamond p$ defining the Church-Rosser property.

It can be shown that the formula obtained by negating a Sahlqvist formula and importing the negation over all connectives is a Sahlqvist antecedent (see [27]). We then have the following proposition:

Proposition 1.1.32. *Every Sahlqvist formula is semantically equivalent*[7] *to a negated Sahlqvist antecedent, and hence to a Sahlqvist implication.*

Notice that Definition 1.1.30 differs from the usual definition of a Sahlqvist formula (see for instance [10]) in that disjunctions are only allowed between formulas that share no propositional variables. This would exclude a formula like $(\Diamond\Diamond p \to \Diamond p) \vee (\Diamond\Box p \to \Box\Diamond p)$, which would be admitted by Definition 1.1.30. In the light of Proposition 1.1.32, it is clear that this restriction on the occurrence of disjunctions is unnecessary as far as classes defining elementary frames are concerned. However, this requirement is indeed essential for the usual proof of elementarity to work (see once again [10]).

In [77], a natural syntactic generalization of the class of Sahlqvist formulas is given. Following [58], we will refer to this class as the class of Sahlqvist–van Benthem (SvB) formulas. It is defined as follows:

Definition 1.1.33 (Sahlqvist–van Benthem formula). A *Sahlqvist–van Benthem formula* is a formula in negation normal form[8] such that for every propositional variable p, either

(i) there is no positive occurrence of p in a subformula $\varphi \wedge \psi$ or $\Box\psi$ which is in the scope of a $\Diamond$, or

[7] Formulas φ and ψ are semantically equivalent, if they are true at exactly the same points in the same models.

[8] A formula is in *negation normal form* if it contains no occurrences of $\to$ and $\leftrightarrow$ and all negation signs occur only directly in front of propositional variables.

(ii) there is no negative occurrence of p in a subformula $\varphi \wedge \psi$ or $\Box\psi$ which is in the scope of a $\Diamond$.

Note that all Sahlqvist formulas, after being rewritten in negation normal form, are Sahlqvist–van Benthem. In particular, item (ii) of Definition 1.1.33 is satisfied with respect to every propositional variable in a Sahlqvist formula. The converse does not hold: the formula $\Diamond(p \wedge \Box\Diamond\neg p) \to (\Diamond\Box p \vee \Box\Box\neg p)$ is a Sahlqvist–van Benthem formula, but it is not Sahlqvist. However, it is not difficult to rewrite this formula as a Sahlqvist formula whilst maintaining semantic equivalence, namely as the formula $(\Diamond(p \wedge \Box\Diamond\neg p) \wedge \Box\Diamond\neg p \wedge \Diamond\Diamond p) \to \bot$. Had the polarity of p been reversed, this would not have worked — to obtain a Sahlqvist formula we would have had to switch the polarity. We have the following proposition (its proof can be found in [27]):

Proposition 1.1.34. *Every Sahlqvist–van Benthem formula is locally equivalent*[9] *to a Sahlqvist implication.*

1.1.7 Duality

In Subsection 1.1.2, we saw that the basic modal language can be interpreted in various structures. One naturally wonders how these structures are related to each other, and if it is possible to obtain one from the other. Here we review the relationship between these structures, starting with the connection between Kripke frames and BAOs.

Connections between modal algebras and Kripke frames were first studied explicitly by Lemmon in [60]. More precisely, he gave us a way to construct a BAO from a Kripke frame.

Definition 1.1.35 (Complex algebras). Let $\mathfrak{F} = (W, R)$ be a frame. The *complex algebra* of $\mathfrak{F}$ is the structure

$$\mathfrak{F}^+ = (\mathcal{P}(W), \cap, \cup, -, \varnothing, W, \langle R \rangle),$$

where $\cap$ is the intersection of two sets, $\cup$ the union of two sets, $-$ the complement of a set relative to W, and $\langle R \rangle$ is defined as on page 7.

Proposition 1.1.36. *Let $\mathfrak{F} = (W, R)$ be a Kripke frame for the basic modal language. Then $\mathfrak{F}^+$ is a Boolean algebra with operators.*

Before we go in the opposite direction, that is, from BAOs to Kripke frames, we recall the following definition:

Definition 1.1.37 (Filters). A *filter* of a Boolean algebra $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top)$ is a subset $F \subseteq A$ satisfying the following:

(i) $\top \in F$,

(ii) F is closed under taking meets, i.e., if $a, b \in F$, then $a \wedge b \in F$, and

(iii) F is upward closed, i.e., if $a \in F$ and $a \leq b$, then $b \in F$.

[9] Formulas φ and ψ are locally equivalent, if they are valid at exactly the same points in the same general frames.

A filter is *proper* if it does not contain $\perp$. An *ultrafilter* is a proper filter such that for every $a \in A$, either $a \in F$ or $\neg a \in F$. The collection of ultrafilters of $\mathbf{A}$ is denoted by $\mathit{Uf}\,\mathbf{A}$. An *ultrafilter over a set* S is an ultrafilter of the Boolean algebra of all subsets of S.

Example 1.1.38. Let $\mathbf{A}$ be a Boolean algebra, and let $a \in A$ such that $a \neq \perp$. Then the set $a\uparrow = \{b \in A \mid a \leq b\}$ is an ultrafilter (called the *principal ultrafilter generated by* A).

Example 1.1.39. Let $\mathbf{A} = (A, \wedge, \vee, \neg, \top, \perp)$ be a Boolean algebra, and let $C \subseteq A$. It is not difficult to see that $\{a \in A \mid \exists c_1 \cdots \exists c_n(c_1 \wedge \cdots \wedge c_n \leq a)\}$ is a filter containing C. We will denote this filter by F_C. This is the smallest filter containing C, and is sometimes referred to as the filter *generated by* C. If C has the finite meet property[10], then this filter is proper. To see this, assume that C has the finite meet property, and suppose $\perp \in F_C$. Then there are $c_1, \ldots, c_n \in C$ such that $c_1 \wedge \cdots \wedge c_n \leq \perp$, contradicting the fact that C has the finite meet property.

Let us gather some known properties of ultrafilters for future reference.

Proposition 1.1.40. *Let* $\mathbf{A}$ *be a Boolean algebra.*

(i) *For any ultrafilter* u *of* $\mathbf{A}$ *and* $a, b \in A$, $a \vee b \in u$ *iff* $a \in u$ *or* $b \in u$.

(ii) $\mathit{Uf}\,\mathbf{A}$ *coincides with the set of maximal proper filters of* $\mathbf{A}$.

Theorem 1.1.41 (Ultrafilter Theorem). *Let* $\mathbf{A}$ *be a Boolean algebra,* $a \in A$*, and* F *a filter of* $\mathbf{A}$ *that does not contain* a*. Then there is an ultrafilter extending* F *that does not contain* a*.*

A proof of this theorem can be found in [10].

Recall the following: given an operator f on a Boolean algebra $(A, \wedge, \vee, \neg, \perp, \top)$, we define the binary relation Q_f on the set of ultrafilters of the algebra by

$$uQ_fv \text{ iff } f(a) \in u \text{ for all } a \in v.$$

We also have the following alternative but equivalent definition of the relation Q_f:

$$uQ_fv \text{ iff } \neg f(\neg a) \in u \text{ implies } a \in v.$$

For the left-to-right direction, suppose uQ_fv, and let $\neg f(\neg a) \in u$. Then $f(\neg a) \notin u$. But as uQ_fv, $\neg a \notin v$, which means $a \in v$. Conversely, suppose $\neg f(\neg b) \in u$ implies $b \in v$, and assume $f(a) \notin u$. Then $\neg f(a) \in u$, and so $\neg f(\neg\neg a) \in u$. Hence, $\neg a \in v$, which means $a \notin v$.

Now, let $\mathbf{A} = (A, \cap, \vee, \neg, \perp, \top, f)$ be a BAO. The *ultrafilter frame* of $\mathbf{A}$, denoted $\mathbf{A}_+$, is the structure $(\mathit{Uf}\,\mathbf{A}, Q_f)$.

Let us now recall a number of operations that enable us to construct general frames out of Kripke frames or out of BAOs, and conversely.

[10] Let $(A, \wedge, \vee, \neg, \perp, \top)$ be a Boolean algebra. A subset D of A has the *finite meet property* if there is no finite subset $\{d_0, \ldots, d_n\}$ of D such that $d_0 \wedge \cdots \wedge d_n = \perp$.

Definition 1.1.42. Let $\mathfrak{g} = (W, R, A)$ be a general frame. The *underlying* (Kripke) frame of $\mathfrak{g}$ is the frame (W, R), denoted $\mathfrak{g}_\sharp$. In other words, the underlying frame is obtained by forgetting about the algebra of admissible sets. The structure $\mathfrak{g}^* = (A, \cap, \cup, -, \varnothing, W, \langle R\rangle)$ is called the *underlying Boolean algebra with operators* of $\mathfrak{g}$.

Conversely, the *full frame* of a Kripke frame $\mathfrak{F} = (W, R)$ is the general frame $\mathfrak{F}^\sharp = (W, R, \mathcal{P}(W))$. Finally, the *general ultrafilter frame* of a BAO $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, f)$ is defined as

$$\mathbf{A}_* = (\mathbf{A}_+, \widehat{A}),$$

where $\widehat{a} := \{u \in Uf\mathbf{A} \mid a \in u\}$ and $\widehat{A} := \{\widehat{a} \mid a \in A\}$.

Next, we devote our attention to the relationship between general frames and Boolean algebras with operators. But we will need the following lemma:

Lemma 1.1.43. *For any Boolean algebra* $\mathbf{A}$*,*

(i) $\widehat{\neg a} = -\widehat{a}$*,*

(ii) $\widehat{a \vee b} = \widehat{a} \cup \widehat{b}$*,*

(iii) $\widehat{a \wedge b} = \widehat{a} \cap \widehat{b}$*, and*

(iv) $\widehat{\Diamond a} = \langle Q_\Diamond\rangle \widehat{a}$*.*

Proof. (i) For the left-to-right inclusion, let $u \in \widehat{\neg a}$. Then $\neg a \in u$, and so $a \notin u$. Hence, by definition, $u \notin \widehat{a}$, which means $u \in -\widehat{a}$. Conversely, let $u \in -\widehat{a}$. Then we have $u \notin \widehat{a}$, and so $a \notin u$, which means $\neg a \in u$. Hence, $u \in \widehat{\neg a}$.

(ii) Let $u \in \widehat{a \vee b}$. Then $a \vee b \in u$, and so, by Proposition 1.1.40, $a \in u$ or $b \in u$. Hence, $u \in \widehat{a}$ or $u \in \widehat{b}$, and therefore $u \in \widehat{a} \cup \widehat{b}$. Conversely, assume $u \in \widehat{a} \cup \widehat{b}$. Then $u \in \widehat{a}$ or $u \in \widehat{b}$, which implies that $a \in u$ or $b \in u$. Thus, by Proposition 1.1.40, $a \vee b \in u$, and hence, $u \in \widehat{a \vee b}$.

(iii) Assume $u \in \widehat{a \wedge b}$. Then $a \wedge b \in u$, and so $\neg(a \wedge b) \notin u$. But this means that $\neg a \vee \neg b \notin u$, so, by Proposition 1.1.40, $\neg a \notin u$ and $\neg b \notin u$. Hence, $a \in u$ and $b \in u$, which implies $u \in \widehat{a}$ and $u \in \widehat{b}$, and therefore, $u \in \widehat{a} \cap \widehat{b}$. For the converse inclusion, let $u \in \widehat{a} \cap \widehat{b}$. Then $u \in \widehat{a}$ and $u \in \widehat{b}$, and so $a \in u$ and $b \in u$. But u is upward closed, which means $a \wedge b \in u$. Thus, $u \in \widehat{a \wedge b}$.

(iv) Let $u \in \langle Q_\Diamond\rangle \widehat{a}$. Then there is an ultrafilter $v \in \widehat{a}$ such that $uQ_\Diamond v$. Hence, $a \in v$, and so, by definition, $\Diamond a \in u$. This means that $u \in \widehat{\Diamond a}$. For the converse inclusion, take an arbitrary ultrafilter $u \in \widehat{\Diamond a}$. Then $\Diamond a \in u$. We have to find an ultrafilter v such that $uQ_\Diamond v$ and $v \in \widehat{a}$, or equivalently, $a \in v$. First, recall that we can reformulate the definition of $Q_\Diamond$ as follows:

$$uQ_\Diamond v \text{ iff } \neg\Diamond\neg b \in u \text{ implies } b \in v.$$

We claim that the set $B = \{b \in A \mid \neg\Diamond\neg b \in u\} \cup \{a\}$ has the finite meet property. For suppose not, then there are $c_1, c_2, \ldots, c_n \in B$ such that $c_1 \wedge c_2 \wedge \cdots c_n \wedge a = \bot$, and so $a \leq \neg c_1 \vee \neg c_2 \vee \cdots \neg c_n$. This means we have $\Diamond a \leq \Diamond\neg c_1 \vee \Diamond\neg c_2 \vee \cdots \vee \Diamond\neg c_n$. But $\Diamond a \in u$ and u is upward closed, so then $\Diamond\neg c_1 \vee \Diamond\neg c_2 \vee \cdots \vee \Diamond\neg c_n \in u$. Hence, $\Diamond\neg c_i \in u$ for some

$1 \leq i \leq n$, which means that $c_i \notin B$ for some $1 \leq i \leq n$, a contradiction. Now, since B has the finite meet property, we know from Example 1.1.39 that there is a proper filter v containing B. We can now use the Ultrafilter Theorem to extend this proper filter to an ultrafilter v^+. We have thus found an ultrafilter satisfying the desired properties. $\square$

Theorem 1.1.44. *Let* $\mathbf{A}$ *be a BAO, and* $\mathfrak{g} = (W, R, A)$ *a general frame. Then*

(i) $\mathbf{A}_*$ *is a descriptive two-sorted general frame,*

(ii) $(\mathbf{A}_*)^* \cong \mathbf{A}$, *and*

(iii) $(\mathbf{g}^*)_* \cong \mathfrak{g}$ *iff* $\mathfrak{g}$ *is descriptive.*

Proof. We will only prove items (i) and (ii). The proof of item (iii) can be found in [10].

(i) We first show that $\mathfrak{A}_*$ is a general frame. First, recall that $\bot \notin u$ for all ultrafilters u of $\mathfrak{A}$, and so, $\widehat{\bot} = \varnothing$. But $\bot \in A$, so $\widehat{\bot} \in \widehat{A}$. We can thus conclude that $\varnothing \in \widehat{A}$.

To show that $\widehat{A}$ is closed under $\cup$, let $\widehat{a}, \widehat{b} \in \widehat{A}$. Then $a, b \in A$. But A is closed under $\vee$, so $a \vee b \in A$. This means that $\widehat{a \vee b} \in \widehat{A}$, and so, since $\widehat{a} \cup \widehat{b} = \widehat{a \vee b}$ by Lemma 1.1.43, $\widehat{a} \cup \widehat{b} \in \widehat{A}$.

Next, let $\widehat{a} \in \widehat{A}$. Then $a \in A$, and so, since A is closed under $\neg$, $\neg a \in A$. Hence, $\widehat{\neg a} \in \widehat{A}$. But by Lemma 1.1.43, $\widehat{\neg a} = -\widehat{a}$, so $-\widehat{a} \in \widehat{A}$.

To show that $\widehat{A}$ is closed under $\langle Q_\diamond \rangle$, let $\widehat{a} \in \widehat{A}$. Then $a \in A$. But A is closed under $\diamond$, so $\diamond a \in A$. Hence, $\widehat{\diamond a} \in \widehat{A}$, and so, since $\widehat{\diamond a} = \langle Q_\diamond \rangle \widehat{a}$ by Lemma 1.1.43, $\langle Q_\diamond \rangle \widehat{a} \in \widehat{A}$.

To show that $\mathfrak{A}_*$ is descriptive, we have to prove that $\mathfrak{A}_*$ is differentiated, tight and compact. We first show that it is differentiated. The left-to-right direction is obvious. For the converse, assume u and v are different ultrafilters of $\mathfrak{A}$. Then we may assume without loss of generality that there is an a in A such that $a \in u$ but $a \in v$. Hence, $\widehat{a}$ is admissible, and, furthermore, $u \in \widehat{a}$ but $v \notin \widehat{a}$.

For the tightness, suppose $uQ_\diamond v$ does not hold for some ultrafilters u and v. Then there is some $a \in A$ such that $a \in v$ but $\diamond a \notin u$. Hence, $\widehat{a}$ is admissible, and $v \in \widehat{a}$ but $u \notin \widehat{\diamond a}$. Now, since $\langle Q_\diamond \rangle \widehat{a} = \widehat{\diamond a}$, $u \notin \langle Q_\diamond \rangle \widehat{a}$, as required. For the converse, assume there is an admissible set $\widehat{a}$ such that $v \in \widehat{a}$ but $u \notin \langle Q_\diamond \rangle \hat{a} = \widehat{\diamond a}$. This means that there is an $a \in A$ such that $a \in v$ but $\diamond a \notin u$. Hence, $uQ_\diamond v$ does not hold.

To show that $\mathfrak{A}_*$ is compact, let $\{\widehat{c} \mid c \in C \subseteq A\}$ be a non-empty collection of admissible sets with the finite intersection property. Then $\widehat{c_1} \cap \widehat{c_2} \cap \cdots \cap \widehat{c_n} \neq \varnothing$ for every finite subcollection $\{\widehat{c_1}, \widehat{c_2}, \ldots, \widehat{c_n}\}$. This means that there is an ultrafilter u that belongs to every $\widehat{c_i}$, $1 \leq i \leq n$. Hence, $c_i \in u$ for all c_i, $1 \leq i \leq n$. Thus, $c_1 \wedge c_2 \wedge \cdots \wedge c_n \in u$, which means that $c_1 \wedge c_2 \wedge \cdots \wedge c_n \neq \bot$. We can thus conclude that C has the finite meet property, and thus, there is a proper filter v such that $C \subseteq v$. By the Ultrafilter Theorem, v can be extended to an ultrafilter v^+. Hence, $C \subseteq v^+$, and so $v^+ \in \widehat{c}$ for all $c \in C$. We thus have that $v^+ \in \bigcap\{\widehat{c} \mid c \in C \subseteq A\}$, which means that $\bigcap\{\widehat{c} \mid c \in C \subseteq A\} \neq \varnothing$, as required.

(ii) Consider the map $h\colon A \to \widehat{A}$ defined by $h(a) = \widehat{a}$. We have to show that h is an isomorphism. First, h is clearly surjective from A onto $\widehat{A}$. Proving that h preserves all operations is also straightforward. To show that h is injective, suppose a and b are distinct elements of A. Then $a \not\leq b$ or $b \not\leq a$. Without loss of generality, we may assume the first. But if $a \not\leq b$, then b does not belong to the filter $a\uparrow$, so, by the Ultrafilter Theorem, there is some ultrafilter u such that $a\uparrow \subseteq u$ and $b \notin u$. Now, since $a \in a\uparrow$, $a \in u$. Hence, $u \in \widehat{a}$ but $u \notin \widehat{b}$, which means that $h(a) \neq h(b)$. $\square$

We now know how to construct algebras from frames and frames from algebras. Furthermore, we also know how to construct new frames from old frames and algebras from old algebras (for more on the operations on algebras, see [21]), but what is the relationship between the operations on frames and the operations on algebras? *Duality theory* studies these relationships. We will only review some of these relationships here, for an overview of the duality theory between BAOs and relational structures, the reader is referred to [46] and [49]. In our exposition, we will be following [10].

Theorem 1.1.45 below gives us a precise formulation of the links between Kripke frames and Boolean algebras with operators. Figure 1.1 shows the structures in Theorem 1.1.45 and the relationships between them.

$$\begin{tikzcd} \mathfrak{F}^+ \arrow[d, dash] & \mathfrak{G}^+ \arrow[l, two heads] \arrow[d, dash] \\ \mathfrak{F} \arrow[r, tail] & \mathfrak{G} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{F}^+ \arrow[d, dash] & \mathfrak{G}^+ \arrow[l, tail] \arrow[d, dash] \\ \mathfrak{F} \arrow[r, two heads] & \mathfrak{G} \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{A}_+ \arrow[d, dash] & \mathbf{B}_+ \arrow[l, two heads] \arrow[d, dash] \\ \mathbf{A} \arrow[r, tail] & \mathbf{B} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{A}_+ \arrow[d, dash] & \mathbf{B}_+ \arrow[l, tail] \arrow[d, dash] \\ \mathbf{A} \arrow[r, two heads] & \mathbf{B} \end{tikzcd}$$

Figure 1.1: The structures in Theorem 1.1.45 and their relationships.

Theorem 1.1.45. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be two Kripke frames, and $\mathbf{A}$ and $\mathbf{B}$ two Boolean algebras with operators.*

(i) *If $\mathfrak{F} \rightarrowtail \mathfrak{G}$, then $\mathfrak{G}^+ \twoheadrightarrow \mathfrak{F}^+$.*

(ii) *If $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$, then $\mathfrak{G}^+ \rightarrowtail \mathfrak{F}^+$.*

(iii) *If $\mathbf{A} \rightarrowtail \mathbf{B}$, then $\mathbf{B}_+ \twoheadrightarrow \mathbf{A}_+$.*

(iv) *If $\mathbf{A} \twoheadrightarrow \mathbf{B}$, then $\mathbf{B}_+ \rightarrowtail \mathbf{A}_+$.*

The proof of this theorem is similar to that of Theorem 1.1.48.

We can prove results analogous to the results in Theorem 1.1.45 for general frames and Boolean algebras with operators. But first, let us review the notions of disjoint unions and bounded morphisms in the setting of general frames.

Definition 1.1.46. For each $i \in I$, let $\mathfrak{g}_i = (W_i, R_i, A_i)$ be a general frame. Then their *disjoint union* is the structure

$$\biguplus_{i\in I} \mathfrak{g}_i = (W, R, A),$$

where W is the union of the domains W_i, R is the union of the relations R_i, and A consists of those subsets $a \subseteq \bigcup_{i\in I} W_i$ such that $a \cap W_i \in A_i$ for all $i \in I$.

Definition 1.1.47. Let $\mathfrak{g} = (W, R, A)$ and $\mathfrak{h} = (W', R', A')$ be two general frames. A map g: $W \to W'$ is a *bounded morphism between* $\mathfrak{g}$ *and* $\mathfrak{h}$, if g is a bounded morphism between the frames (W, R) and (W', R') such that $g^{-1}[a'] \in A$ for all $a' \in A'$. Such a bounded morphism g is called an *embedding*, if it is injective, and for all $a \in A$, there is an $a' \in A'$ such that $g[a] = g[W] \cap a'$. We say that $\mathfrak{h}$ is *embeddable* in $\mathfrak{g}$ (denoted $\mathfrak{h} \rightarrowtail \mathfrak{g}$), if there is an embedding from W' to W. The two-sorted general frame $\mathfrak{h}$ is called a *bounded morphic image* of $\mathfrak{g}$ (denoted $\mathfrak{g} \twoheadrightarrow \mathfrak{h}$), if there is a surjective bounded morphism g from W to W'. Finally, $\mathfrak{h}$ and $\mathfrak{g}$ are *isomorphic* (denoted $\mathfrak{h} \cong \mathfrak{g}$), if there is an bijective bounded morphism between W and W'.

Theorem 1.1.48. *Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be two general frames, and* $\mathbf{A}$ *and* $\mathbf{B}$ *two Boolean algebras with operators.*

(i) *If* $\mathfrak{g} \rightarrowtail \mathfrak{h}$*, then* $\mathfrak{h}^* \twoheadrightarrow \mathfrak{g}^*$*.*

(ii) *If* $\mathfrak{g} \twoheadrightarrow \mathfrak{h}$*, then* $\mathfrak{h}^* \rightarrowtail \mathfrak{g}^*$*.*

(iii) *If* $\mathbf{A} \rightarrowtail \mathbf{B}$*, then* $\mathbf{B}_* \twoheadrightarrow \mathbf{A}_*$*.*

(iv) *If* $\mathbf{A} \twoheadrightarrow \mathbf{B}$*, then* $\mathbf{B}_* \rightarrowtail \mathbf{A}_*$*.*

The proof of this theorem follows immediately from Propositions 1.1.51 and 1.1.52 below. But in order to prove these propositions, we need the following terminology:

Definition 1.1.49. Let $\mathfrak{g} = (W, R, A)$ and $\mathfrak{h} = (W', R', A')$ be two general frames. Given a map g: $W \to W'$, its *dual* g^*: $A' \to \mathcal{P}(W)$ is defined by

$$g^*(a') := g^{-1}[a'] \; (= \{w \in W \mid g(w) \in a'\}).$$

Definition 1.1.50. Let $\mathbf{A}$ and $\mathbf{B}$ be two BAOs, and let h be a map from A to B. Then its *dual* h_* is the map from $Uf\mathbf{B}$ to $\mathcal{P}(A)$ defined by

$$h_*(u') := h^{-1}[u'] \; (= \{a \in A \mid h(a) \in u'\}).$$

The proposition below asserts that the duals of bounded morphisms between general frames are nothing but homomorphisms between Boolean algebras with operators. The proof can be found in [10].

Proposition 1.1.51. *Let $\mathfrak{g} = (W, R, A)$ and $\mathfrak{h} = (W', R', A')$ be two general frames, $\mathfrak{g}^* = (A, \cap, \cup, -, \varnothing, W, \langle R \rangle)$ and $\mathfrak{h}^* = (A', \cap, \cup, -, \varnothing, W', \langle R' \rangle)$ their underlying BAOs, and g a map from W to W'.*

(i) *If g is a bounded morphism, g^* maps elements of A' to elements of A.*

(ii) *If g is a bounded morphism, then g^* is a homomorphism from $\mathfrak{h}^*$ to $\mathfrak{g}^*$.*

(iii) *If g is an embedding, then g^* is a surjective homomorphism.*

(iv) *If g is surjective, then g^* is injective.*

Going in the opposite direction, we find that the duals of homomorphisms between Boolean algebras with operators are bounded morphisms between general frames. We will only give proofs of the items not readily available in the literature. The proofs of the other items can be found in [10].

Proposition 1.1.52. *Let $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, \diamond)$ and $\mathbf{B} = (B, \wedge, \vee, \neg, \bot, \top, \diamond)$ be two BAOs, $\mathbf{A}_* = (\mathit{Uf}\,\mathbf{A}, Q_\diamond, \widehat{A})$ and $\mathbf{B}_* = (\mathit{Uf}\,\mathbf{B}, Q'_\diamond, \widehat{B})$ their general ultrafilter frames, and h a map from A to B.*

(i) *If h is a homomorphism, then h_* maps ultrafilters to ultrafilters.*

(ii) *If h is a homomorphism, then h_* is a bounded morphism from $\mathbf{B}_*$ to $\mathbf{A}_*$.*

(iii) *If h is a surjective homomorphism, then h_* is an embedding.*

(iv) *If h is an embedding, then h_* is a surjective.*

Proof. (i) We have to show that if $u' \in \mathit{Uf}\,\mathbf{B}$, then $h_*(u')$ is an ultrafilter of $\mathbf{A}$. First, since h is a homomorphism and u' is an ultrafilter of $\mathbf{B}$, $h(\top) = \top \in u'$. Hence, $\top \in h_*(u')$.

To show that $h_*(u')$ is closed under meets, assume $a, b \in h_*(u')$. Then $h(a)$ and $h(b)$ are in u'. But u' is closed under meets, so $h(a) \wedge h(b) \in u'$. Now, $h(a) \wedge h(b) = h(a \wedge b)$, which means that $h(a \wedge b) \in u'$. Hence, $a \wedge b \in h_*(u')$.

Next, let $a \in h_*(u')$ such that $a \leq b$. Then $h(a) \in u'$ and $a \wedge b = a$. From $a \wedge b = a$ we have that $h(a \wedge b) = h(a)$. But then $h(a) \wedge h(b) = h(a)$, which means that, $h(a) \leq h(b)$. Hence, since u' is upward closed, $h(b) \in u'$, and therefore, $b \in h_*(u')$.

To show that $h_*(u')$ is proper, assume $\bot \in h_*(u')$. Then $h(\bot) \in u'$. But $\bot = h(\bot)$, which means that $\bot \in u'$, contradicting the fact that u' is an ultrafilter.

Finally, we show that either $a \in h_*(u')$ or $\neg a \in h_*(u')$. Suppose $a \notin h_*(u')$. Then $h(a) \notin u'$, and so, since u' is an ultrafilter, $\neg h(a) \in u'$. But $h(\neg a) = \neg h(a)$, so $h(\neg a) \in u'$. Hence, $\neg a \in h_*(u')$. Similarly, if $\neg a \notin h_*(u')$, then $a \in h_*(u')$.

(ii) For the forth property, see [10]. For the back property, assume $h_*(u') Q_\diamond v$. We have to find an ultrafilter v' of $\mathbf{B}$ such that $h_*(v') = v$ and $u' Q'_\diamond v'$. So consider the sets $F_1 = \{a' \in A' \mid \Box a' \in u'\}$ and $F_2 = \{h(a) \mid a \in v\}$. It is not difficult to see that both F_1 and F_2 are closed under meets. Now, let $F = F_1 \cup F_2$. We will show that F has the finite meet property. So suppose F does not have the finite meet property. Using the fact that F_1 and F_2 are closed under meets, we then have the following cases:

Case 1: there is an $a' \in F_1$ such that $a' = \bot$. But then $\Box a' = \Box\bot \in u'$, which means $\Box h(\bot) = h(\Box\bot) \in u'$. Hence, $\Box\bot \in h_*(u')$, and so, by the definition of $Q_\Diamond$, $\bot \in v$, a contradiction.

Case 2: there is an $a \in v$ such that $h(a) = \bot$. But since $a \in v$, $\Diamond a \in h_*(u')$, which means that $h(\Diamond a) \in u'$. Hence, $\Diamond h(a) = \Diamond\bot = \bot \in u'$, a contradiction.

Case 3: there is an $a \in v$ and an $a' \in F_1$ such that $a' \wedge h(a) = \bot$. Then $a' \leq \neg h(a)$, and so, by the monotonicity of $\Box$, $\Box a' \leq \Box\neg h(a) = h(\Box\neg a)$. Now, $a' \in F_1$, so $\Box a' \in u'$. Hence, since u' is upward closed, $h(\Box\neg a) \in u'$. This means that $\Box\neg a \in h_*(u')$, and so, by the definition of $Q_\Diamond$, $\neg a \in v$. Therefore, $a \notin v$, which is a contradiction.

Now, by the Ultrafilter Theorem, there is an ultrafilter v' extending F. We just have to check that $h_*(v') = v$ and $u'Q'_\Diamond v'$. First, we show that $h_*(v') = v$, so for the right-to-left inclusion, assume $a \notin h_*(v')$. Then $h(a) \notin v'$, and so $h(a) \notin F$. But $F_1 \subseteq F$, so $a \notin F_1$, which means $a \notin v$. We thus have $v \subseteq h_*(v')$. Hence, by the maximality of v, $h_*(v') = v$. Finally, to show that $u'Q'_\Diamond v'$, assume $\Box\neg a' \in u'$. Then $a' \in F_1$, and so, since $F_1 \subseteq F \subseteq v'$, $a' \in v'$.

Finally, let $\widehat{a}$ be an admissible set of $\mathbf{A}_*$. Note that if we can show that $h_*^{-1}[\widehat{a}] = \widehat{h(a)}$, we are done since $h(a) \in B$. But this is indeed the case: $u' \in h_*^{-1}[\widehat{a}]$ iff $h_*(u') \in \widehat{a}$ iff $a \in h_*(u')$ iff $h(a) \in u'$ iff $u' \in \widehat{h(a)}$.

(iii) To show that h_* is injective, let u' and v' be two distinct ultrafilters of $\mathbf{B}$. Then we may assume without loss of generality that there is some $b \in B$ such that $b \in u'$ but $b \notin v'$. But we know that h is surjective, so there is some $a \in A$ such that $h(a) = b$. This means that $h(a) \in u'$ while $h(a) \notin v'$, and so, by definition, $a \in h_*(u')$ but $a \notin h_*(v')$. Hence, $h_*(u') \neq h_*(v')$.

Now, let $\widehat{b} \in \widehat{B}$. Then $b \in B$, and so, since h is surjective, there is some $a \in A$ such that $h(a) = b$. We claim that $h_*[\widehat{b}] = h_*[Uf\mathbf{B}] \cap \widehat{a}$. For the right-to-left inclusion, let $u \in h_*[Uf\mathbf{B}] \cap \widehat{a}$. Then there is some $u' \in Uf\mathbf{B}$ such that $u = h_*(u')$, and furthermore, $a \in u$. Hence, $a \in h_*(u')$, so $h(a) \in u'$. But since $h(a) = b$, $b \in u'$. We thus have that $u' \in \widehat{b}$, and therefore, since $u = h_*(u')$, $u \in h_*[\widehat{b}]$. For the other inclusion, let $u \in h_*[\widehat{b}]$. Then there is some $u' \in \widehat{b}$ such that $u = h_*(u')$. Since $u' \in \widehat{b}$, $b \in u'$, and so, $h(a) \in u'$. Hence, $a \in h_*(u')$, which means that $a \in u$. We thus have that $u \in \widehat{a}$. Now, since $\widehat{b} \subseteq Uf\mathbf{B}$ and $u = h_*(u')$, $u \in h_*[Uf\mathbf{B}]$, and thus, $u \in h_*[Uf\mathbf{B}] \cap \widehat{a}$.

(iv) For the proof of this, see [10]. □

1.1.8 Canonicity

There are two, closely related, notions of canonicity, namely, a frame-theoretical one and an algebraic one. In what follows, we give short overviews of both notions of canonicity.

The frame-theoretical perspective

Before we discuss the frame-theoretical perspective of canonicity, we first review the canonical model construction.

Definition 1.1.53 (**Λ-MCSs**)**.** Let Λ be a normal modal logic, and Γ a set of formulas. Γ is a *maximal Λ-consistent set* (Λ-MCS), if it is Λ-consistent and every proper superset of Γ is Λ-inconsistent.

A modal version of Lindenbaum's lemma holds, saying that every Λ-consistent set of formulas can be extended to a Λ-MCS.

Definition 1.1.54 (**Canonical models**)**.** The *canonical model* $\mathfrak{M}^\Lambda$ of a modal logic Λ is the triple $(W^\Lambda, R^\Lambda, V^\Lambda)$, where

(i) W^Λ is the set of all Λ-MCS's,

(ii) for all Λ-MCS's $u, v \in W^\Lambda$, $uR^\Lambda v$ iff $\psi \in v$ implies $\Diamond\psi \in u$ for all formulas ψ, and

(iii) for every propositional variable p, $V^\Lambda(p) = \{w \in W^\Lambda \mid p \in w\}$.

The pair $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda)$ is called the *canonical frame* for the logic Λ.

Suppose that we suspect that a normal modal logic is strongly complete with respect to some class of frames, how should we go about proving this? Actually, there is no precise strategy. Nonetheless, a very simple technique works in a large number of modal logics: simply show that the canonical frame for the logic belongs to the class of frames. We call such proofs *completeness–via–canonicity* proofs. The following completeness results can be proved using completeness–via-canonicity proofs:

Theorem 1.1.55. *The logic* **K4** *is strongly complete with respect to the class of transitive frames.* **T** *is strongly complete with respect to the class of reflexive frames.* **B** *is strongly complete with respect to the class of symmetric frames.* **S4** *is strongly complete with respect to the class of reflexive, transitive frames.* **S5** *is strongly complete with respect with respect to the class of frames whose relation is an equivalence relation.*

One naturally wonders what these logics have in common that makes them so nice. A remarkable answer to this question was given in [64], namely that the axioms $(4), (T)$ and (B) are all Sahlqvist. Since Sahlqvist formulas define elementary frame classes, these results hint at a link between frame definability and the properties of canonical frames. Let us recall some terminology to describe this important phenomenon.

Definition 1.1.56. A formula φ is *canonical*, if for any normal logic Λ, $\varphi \in \Lambda$ implies that φ is valid on the canonical frame for Λ. A normal modal logic Λ is *canonical*, if for all φ such that $\vdash_\Lambda \varphi$, φ is valid on the canonical frame for Λ.

Clearly, (4), (T) and (B) are all canonical formulas. Moreover, $\mathbf{K4}, \mathbf{T}, \mathbf{B}, \mathbf{S4}$ and **S5** are all canonical logics.

In general, we have the following:

Theorem 1.1.57. *Every canonical normal modal logic is sound and complete with respect to the class of frames they define.*

Theorem 1.1.57 tells us that proving completeness boils down to showing that the axioms of the logic are canonical.

The converse of Theorem 1.1.57 does not hold. An example of a complete logic that is not canonical is the logic obtained by adding the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$ to $\mathbf{K}$. That this logic is complete was proved by Fine in [38]. Goldblatt showed that the McKinsey axiom is not canonical in [48].

The algebraic perspective

Before we can give the algebraic version of the notion of canonicity, we need to review the canonical extension of a Boolean algebra with operators.

There are two approaches. One involves a concrete construction, namely, as the complex algebra of the ultrafilter frame of the original algebra:

Definition 1.1.58. Let $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, f)$ be a BAO. The complex algebra $((\mathbf{A})_+)^+$ of the ultrafilter frame $\mathbf{A}_+$ of $\mathbf{A}$ is called the *canonical extension* of $\mathbf{A}$.

The second approach is abstract: the canonical extension is not constructed, but axiomatically characterized as the unique completion of the original algebra in which the original algebra is *dense* and *compact*. But first we review these notions one by one.

A *complete* Boolean algebra is a Boolean algebra in which every subset has a supremum (least upper bound). A BAO $\mathbf{B}$ is a *completion* of a BAO $\mathbf{A}$, if $\mathbf{B}$ is complete and $\mathbf{A}$ is a subalgebra of $\mathbf{B}$.

Before we define the concept of density, we review some preliminary notions. Let $\mathbf{B}$ be a completion of a BAO $\mathbf{A}$. An element in B is *open* (*closed*), if it is the join (meet) in $\mathbf{B}$ of elements in $\mathbf{A}$. We will denote the collection of open and closed elements by $\mathbb{O}(\mathbf{B})$ and $\mathbb{K}(\mathbf{B})$, respectively. Elements that are both closed and open are called *clopen*.

Let $\mathbf{B}$ be a completion of a BAO $\mathbf{A}$. We say that $\mathbf{A}$ is *meet-dense* in $\mathbf{B}$ if $\mathbb{K}(\mathbf{B}) = B$, *join-dense* if $\mathbb{O}(\mathbf{B}) = B$, and *dense* if $\mathbb{K}(\mathbb{O}(\mathbf{B})) = \mathbb{O}(\mathbb{K}(\mathbf{B})) = B$. In other words, a completion of a Boolean algebra is dense, if every element in the completion is both a meet of open elements and a join of closed elements.

Next, we turn to the notion of compactness. Given a completion $\mathbf{B}$ of a BAO $\mathbf{A}$, we say that $\mathbf{A}$ is *compact* in $\mathbf{B}$, if for all sets $X \subseteq A$ and $Y \subseteq A$ of closed and open elements, respectively, whenever $\bigwedge X \leq \bigvee Y$, then there exist finite subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ such that $\bigwedge X_0 \leq \bigvee Y_0$.

We are now ready to give an abstract characterization of the canonical extension of a BAO with operators.

Definition 1.1.59 (Canonical extension). A *canonical extension* $\mathbf{B}$ of a BAO $\mathbf{A}$ is a completion of the BAO such that $\mathbf{A}$ is both dense and compact in $\mathbf{B}$.

It can be shown that the canonical extension of a BAO is unique up to isomorphism (for a proof, see for instance [43] and [78]). This justifies our speaking of 'the' canonical extension of a BAO $\mathbf{A}$, and from now on this algebra will be denoted by $\mathbf{A}^\delta$.

Now, recall that the normal modal logic $\mathbf{T}$ is strongly complete with respect to the class of reflexive frames. How can we prove this result algebraically?

It is a well-known fact that the canonical frame of a normal modal logic is actually isomorphic to the ultrafilter frame of its Lindenbaum-Tarski algebra. So in order to show that $\mathbf{T}$ is strongly complete with respect to the class of reflexive frames, we have to show that the ultrafilter frame $(\mathcal{L}_{\mathbf{T}}(\mathsf{PROP}))_+$ of $\mathcal{L}_{\mathbf{T}}(\mathsf{PROP})$ is reflexive, or algebraically, that the complex algebra $((\mathcal{L}_{\mathbf{T}}(\mathsf{PROP}))_+)^+$ belongs to the class V_T of Boolean algebras that validates $p \to \Diamond p$. Note that by Theorem 1.1.14, we already know that $\mathcal{L}_{\mathbf{T}}(\mathsf{PROP})$ belongs to V_T, so this example suggests that proving completeness actually boils down to answering the following question: *which classes of BAOs are closed under taking canonical extensions*? In fact, this gives us an algebraic grip on the notion of canonicity and motivates the following definitons:

Definition 1.1.60. Let C be a class of Boolean algebras with operators. We say that C is *canonical*, if it is closed under taking canonical extensions.

Definition 1.1.61. Let $\mathbf{A}$ be a BAO. An equation $\varphi \approx \psi$ is *canonical*, if $\mathbf{A} \models \varphi \approx \psi$ implies $\mathbf{A}^\delta \models \varphi \approx \psi$.

1.1.9 Persistence

Recall that every normal logic is strongly complete with respect to its class of descriptive general frames. Of course, we are interested in Kripke frames and not in general frames. However, Theorem 1.1.10 can be seen as an important first step towards proving Kripke completeness. The second step commonly involves persistence, a notion we define next.

Definition 1.1.62 (**Persistence**)**.** Let φ be a formula in the basic modal language, and let G be a class of general frames. Then φ is *persistent with respect to* G, if for every general frame $\mathfrak{g}$ in G, $\mathfrak{g} \Vdash \varphi$ implies $\mathfrak{g}_\sharp \Vdash \varphi$.

Persistence with respect to refined, descriptive and discrete general frames is called *r-persistence*, *d-persistence* and *di-persistence*, respectively.

An important result in modal logic is the following:

Theorem 1.1.63 ([64])**.** *Every Sahlqvist formula is d-persistent.*

If we put Theorems 1.1.10 and 1.1.63 together, we obtain the following Kripke completeness result for Sahlqvist formulas:

Corollary 1.1.64 ([64])**.** *Let Σ be a set of Sahlqvist formulas. Then $\mathbf{K} \oplus \Sigma$ is strongly complete with respect to the class of Kripke frames defined by Σ.*

Persistence also has an algebraic dimension. Moreover, persistence and canonicity are actually two sides of the same coin. This originates from the fact that the underlying Boolean algebra with operators $\mathfrak{g}^*$ of the general frame $\mathfrak{g}$ is a subalgebra of the complex algebra $(\mathfrak{g}_\sharp)^+$ of the underlying Kripke frame $\mathfrak{g}_\sharp$ of $\mathfrak{g}$. The notion of persistence then becomes: $\varphi \approx \psi$ is persistent, if $\mathfrak{g}^* \models \varphi \approx \psi$ implies $(\mathfrak{g}_\sharp)^+ \models \varphi \approx \psi$. But this is just canoniciy. More precisely, from the duality between BAOs and descriptive general frames, d-persistence and canonicity are really the same notions. The following important result thus follows from Theorem 1.1.63:

Theorem 1.1.65. *Every Sahlqvist formula is canonical.*

1.2 Hybrid logic

Hybrid languages extend the basic modal language with *nominals*. Syntactically, nominals act as a second sort of variables or atomic formulas, while, semantically, their interpretation is restricted to singleton sets. In other words, nominals are used to name states in a model, much like constants in first-order logic.

Hybrid logics have a long history, dating back to the fifties. It all began with Arthur Norman Prior's work in tense logics (see [63]). He devised a version of possible worlds semantics, and interestingly, this part of his work is already closely related to hybrid logic. Robert Bull, a student of Prior, pushed the idea of hybridization further in [19]. Since then, hybrid languages have been reinvented at several occasions. About fifteen years later in Sofia, Bulgaria, nominals were rediscovered by Solomon Passy, Tinko Tinchev and the late George Gargov in their investigation on Boolean modal logic and propositional dynamic logic (see [41]). The Sofia tradition in hybrid logics continued with the work of Valentin Goranko. In [40], Gargov and Goranko investigated the basic modal language extended first with nominals and the global modality, and then with the difference operator. Goranko was also the first to investigate the binder $\downarrow$ in the context of hybrid logic. In [51], he extended the basic modal language with the global modality and the binder $\downarrow$ with only a single state variable. In the nineties, Blackburn and Seligman [12] investigated a number of very expressive hybrid languages with various state binders, including $\downarrow$. The history of hybrid languages will not be discussed further here, but the reader is referred to [9] and [15] for expositions.

1.2.1 Syntax and relational semantics

In this section, we give the syntax and relational semantics of three hybrid languages, namely, $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. The first language extends the basic modal language with nominals, which we will denote with boldface letters $\mathbf{i}, \mathbf{j}, \mathbf{k}, \ldots$, possibly indexed. As already mentioned, nominals are variables evaluated to singleton subsets, thus serving as names for states. An example of a modal formula containing nominals is $\Diamond\mathbf{i}$, which says that the current state has a successor named by $\mathbf{i}$.

Besides nominals, the second language extends the basic modal language with *satisfaction operators*. Satisfaction operators enable us to express that a formula holds at a state named by a nominal. In other words, they allow us to jump freely to a state named by a nominal, bypassing the accessibility relation. An example of a formula containing a nominal and a satisfaction operator is $@_{\mathbf{i}}\Box p$, saying that all the successors of the state named by $\mathbf{i}$ satisfy p.

The last addition that we will consider is the modality E. Following [72], we will refer to this modality as the *global modality*. We say that $\mathsf{E}\varphi$ holds at a state, if there is at least one state in the model satisfying φ. The *dual* of E, denoted by A, expresses global truth: $\mathsf{A}\varphi$ holds at a state if φ is true at every state in the model. The global modality E and its dual A therefore behave syntactically just like an ordinary diamond-box pair, but we specify that the accessibility relation used to interpret them must always be the universal relation.

Note that satisfaction operators can be defined using the global modality, i.e $@_{\mathbf{i}}\varphi$ is the same as $\mathsf{E}(\mathbf{i} \wedge \varphi)$or, equivalently, $\mathsf{A}(\mathbf{i} \to \varphi)$.

Formally, let PROP and NOM be non-empty disjoint sets of propositional variables and *nominals*, respectively, then the syntax of the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ is defined as follows:

$$\begin{aligned}
\varphi &::= \perp \mid p \mid \mathbf{j} \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \\
\varphi &::= \perp \mid p \mid \mathbf{j} \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid @_{\mathbf{j}}\varphi \\
\varphi &::= \perp \mid p \mid \mathbf{j} \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \mathsf{E}\varphi
\end{aligned}$$

Here $p \in \mathsf{PROP}$ and $\mathbf{j} \in \mathsf{NOM}$.

Let us make a few conventions before we continue. Firs, we will make use of the abbreviation $\mathsf{A}\varphi$ for $\neg\mathsf{E}\neg\varphi$. Finally, from here on we will also assume that both PROP and NOM are countably infinite.

A hybrid formula is said to be *pure* , if its only atomic subformulas are nominals and $\perp$. A pure formula is therefore not allowed to contain propositional variables.

The definitions of a frame and model are unchanged. But although these definitions are the same as for modal logic, we want nominals to act as names, so we will insist that for all $\mathbf{j} \in \mathsf{NOM}$, $V(\mathbf{j})$ is a *singleton* subset of W.

Let $\mathfrak{M}$ be a model. Then the truth definition is extended with the following clauses:

$\mathfrak{M}, w \Vdash \mathbf{j}$ iff $V(\mathbf{j}) = \{w\}$,

$\mathfrak{M}, w \Vdash @_{\mathbf{j}}\varphi$ iff $\mathfrak{M}, v \Vdash \varphi$ where $V(\mathbf{j}) = \{v\}$, and

$\mathfrak{M}, w \Vdash \mathsf{E}\varphi$ iff there exists v such that $\mathfrak{M}, v \Vdash \varphi$.

Validity with respect to a frame or a class of frames is defined as for modal formulas.

Like modal languages, hybrid languages can also be interpreted in general frames. However, in the setting of hybrid logic, it seems more natural to consider general frames with two sorts of admissible sets, one for arbitrary formulas and one for nominals. We call these general frames *two-sorted general frames*, and they were first introduced by Ten Cate in [72].

Definition 1.2.1 (Two-sorted general frames). A *two-sorted general frame* is a structure $\mathfrak{g} = (W, R, A, B)$, where (W, R, A) is a general frame, $\varnothing \neq B \subseteq W$, and, for all $w \in B$, $\{w\} \in A$.

Given a two-sorted general frame $\mathfrak{g} = (W, R, A, B)$, a valuation V on $\mathfrak{g}$ is called admissible for $\mathfrak{g}$, if for each propositional variable p, $V(p) \in A$, and, for each nominal $\mathbf{i}$, $V(\mathbf{i}) \in \{\{w\} \mid w \in B\}$. A *model based on a two sorted general frame* is a pair $(\mathfrak{g}, V)$, where V is an admissible valuation for $\mathfrak{g}$. Truth in such a model is defined in the obvious way, that is, as if we were talking about the model (W, R, V). Truth and validity of formulas are defined as before.

The non-emptiness condition on B implies that A contains at least one singleton. There are general frames that do not contain any singleton admissible set. We might call such general frames atomless. Atomless general frames trivialize the notion of validity for hybrid logic, since they admit no hybrid valuations. In particular, the hybrid formula $\perp$ is valid on atomless frames, since, trivially, it holds under every hybrid valuation.

The following two-sorted general frames are of particular interest when it comes to the study of completeness theory for hybrid logics:

Definition 1.2.2. A two-sorted general frame $\mathfrak{g} = (W, R, A, B)$ is *descriptive*, if (W, R, A) is descriptive.

Definition 1.2.3. A two-sorted general frame $\mathfrak{g} = (W, R, A, B)$ is *strongly descriptive*, if it is descriptive and it satisfies the following conditions:

(i) for all $a \in A$, if $a \neq \varnothing$, then there is a $w \in B$ such that $w \in a$, and

(ii) for all $a \in A$ and $u \in B$, if $\{v \in a \mid uRv\} \neq \varnothing$, then there is a $w \in B$ such that $w \in a$ and uRw.

Definition 1.2.4. A two-sorted general frame $\mathfrak{g} = (W, R, A, B)$ is *discrete*, if $B = W$.

In other words, discrete two-sorted general frames are not really two-sorted since $B = W$. The admissible valuations for the nominals are already implicit in the underlying general frame. Hence, we may drop the "two sorted" and simply refer to these as discrete general frames.

1.2.2 Standard translation

The correspondence language for our hybrid languages is the first-order language $\mathcal{L}_1'$ extending $\mathcal{L}_1$ with a variable $y_{\mathbf{i}}$ for each nominal $\mathbf{i} \in \mathsf{NOM}$. The standard translation function ST_x is extended to $\mathcal{H}, \mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ as follows:

Definition 1.2.5 (**Standard translation**)**.** Let x be a first-order variable from VAR. The *standard translation* ST_x taking formulas of $\mathcal{H}, \mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ to first-order formulas in $\mathcal{L}_1'$ is inductively defined by the following clauses:

$$
\begin{aligned}
ST_x(p) &= P(x), \\
ST_x(\mathbf{i}) &= y_{\mathbf{i}} = x \\
ST_x(\bot) &= x \neq x, \\
ST_x(\neg\varphi) &= \neg ST_x(\varphi), \\
ST_x(\varphi \wedge \psi) &= ST_x(\varphi) \wedge ST_x(\psi), \\
ST_x(\Diamond\varphi) &= \exists y(xRy \wedge ST_y(\varphi)), \\
ST_x(@_{\mathbf{i}}\varphi) &= \exists y(y = y_{\mathbf{i}} \wedge ST_y(\varphi)), \text{ and} \\
ST_x(\mathsf{E}\varphi) &= \exists y ST_y(\varphi),
\end{aligned}
$$

where y is a variable that has not been used in the translation.

Proposition 1.2.6. *Let φ be any formula. Then:*

(i) *For any model $\mathfrak{M} = (W, R, V)$ and any state w in $\mathfrak{M}$, $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M} \models ST_x(\varphi)[x := w, y_{\mathbf{i}_1} := V(\mathbf{i}_1), \ldots, y_{\mathbf{i}_n} := V(\mathbf{i}_n)]$, where $y_{\mathbf{i}_1}, \ldots, y_{\mathbf{i}_n}$ are the free variables corresponding to the nominals $\mathbf{i}_1, \ldots, \mathbf{i}_n$, respectively.*

(ii) *For any model $\mathfrak{M} = (W, R, V)$, $\mathfrak{M} \Vdash \varphi$ iff $\mathfrak{M} \models \forall x \forall y_{\mathbf{i}_1} \cdots \forall y_{\mathbf{i}_n} ST_x(\varphi)$, where $y_{\mathbf{i}_1}, \ldots, y_{\mathbf{i}_n}$ are the variables corresponding to the nominals $\mathbf{i}_1, \ldots, \mathbf{i}_n$, respectively.*

Another, perhaps more natural, option is to translate nominals using individual constants, but for our purposes, we prefer to use variables.

1.2.3 Expressivity and frame definability

Some frame properties are not modally definable. For example, the class of irreflexive frames is not modally definable. To see this, consider the frames $\mathfrak{F} = (\{u, v\}, \{(u, v), (v, u)\})$ and $\mathfrak{G} = (\{u'\}, \{(u', u')\})$ in Figure 1.2. It is not difficult to see that $\mathfrak{G}$ is a bounded morphic image of $\mathfrak{F}$. However, $\mathfrak{F}$ is irreflexive, while $\mathfrak{G}$ is not. The class of irreflexive frames is thus not closed under taking bounded morphic images, and so we know from the Goldblatt-Thomason Theorem that the class of irreflexive frames is not modally definable.

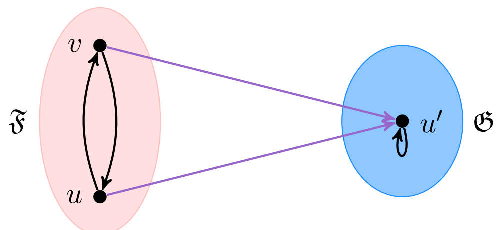

Figure 1.2: A bounded morphism

One of the main reasons why hybrid languages have gained popularity in the last decades is that many frame properties that are not definable in the basic modal language can be defined using pure formulas. For example:

(i) $\Diamond \mathbf{j}$ defines the class of frames in which the accessibility relation is the universal relation (universality);

(ii) $\mathbf{j} \to \neg\Diamond \mathbf{j}$ defines the class of frames in which the accessibility relation is irreflexive;

(iii) $\mathbf{j} \to \neg\Diamond\Diamond \mathbf{j}$ defines the class of frames in which the accessibility relation is asymmetric;

(iv) $\mathbf{j} \to \Box(\Diamond \mathbf{j} \to \mathbf{j})$ defines the class of frames in which the accessibility relation is antisymmetric;

(v) $\Diamond\Diamond \mathbf{j} \to \neg\Diamond \mathbf{j}$ defines the class of frames in which the accessibility relation is intransitive;

(vi) $@_{\mathbf{i}}(\neg \mathbf{j} \wedge \neg \mathbf{k}) \to @_{\mathbf{j}}\mathbf{k}$ defines the class of frames with at most two states;

(vii) $@_{\mathbf{j}}\Diamond \mathbf{i} \vee @_{\mathbf{j}}\mathbf{i} \vee @_{\mathbf{i}}\Diamond \mathbf{j}$ defines the class of frames in which the accessibility relation is trichotomous.

We thus see that hybrid languages extend the expressivity of modal languages, however, the preservation of the validity of hybrid formulas under frame operations draws boundaries to the expressivity of hybrid languages. In what follows, we will see that, unlike for modal formulas, validity of hybrid formulas is not always preserved under the frame operations discussed in Subsection 1.1.6.

Operations on frames and formulas they preserve

We first consider generated subframes. Let us start with the good news: taking generated subframes does preserve validity of $\mathcal{H}(@)$-formulas (see [72] for a proof of this). However, it is well known that the validity of modal formulas containing the global modality E is in general not preserved under taking generated subframes. To see this, consider the generated subframe $\mathfrak{F}_u = (\{u, v\}, \{(u, v), (v, u)\})$ of the frame $\mathfrak{F} = (\{u, v, w\}, \{(u, v), (v, u)\})$ in Figure 1.3. Clearly, $\mathfrak{F} \Vdash \mathsf{E}\neg\Diamond\top$ but $\mathfrak{F}_v \nVdash \mathsf{E}\neg\Diamond\top$.

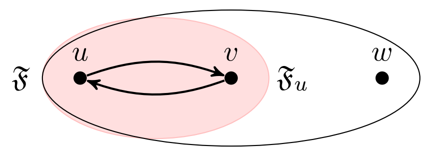

Figure 1.3: A frame with a generated subframe

Unlike validity of modal formulas, validity of hybrid formulas is in general not preserved under bounded morphisms. For example, consider the hybrid formula $\mathbf{i} \to \neg\Diamond\mathbf{i}$ which defines irreflexivity, and let $\mathfrak{F} = (\{u, v\}, \{(u, v), (v, u)\})$ and $\mathfrak{G} = (\{u'\}, \{(u', u')\})$ be the frames in Figure 1.2. It is straightforward to show that $\mathfrak{G}$ is a bounded morphic image of $\mathfrak{F}$. However, clearly $\mathfrak{F} \Vdash \mathbf{i} \to \neg\Diamond\mathbf{i}$, while $\mathfrak{G} \nVdash \mathbf{i} \to \neg\Diamond\mathbf{i}$.

The validity of hybrid formulas is also not preserved under disjoint unions of Kripke frames. For example, consider the formula $\mathbf{i}$ which defines the class of frames that contain exactly one element. Consider the frames $\mathfrak{F}_1 = (\{u\}, \{(u, u)\}\})$ and $\mathfrak{F}_2 = (\{v\}, \{(v, v)\}\})$, and let $\mathfrak{G} = (\{u, v\}, \{(u, u)(v, v)\})$ (see Figure 1.4). Then clearly

$$\mathfrak{G} = \mathfrak{F}_1 \uplus \mathfrak{F}_2.$$

However, both $\mathfrak{F}_1$ and $\mathfrak{F}_2$ contain exactly one element, while $\mathfrak{G}$ contains two elements.

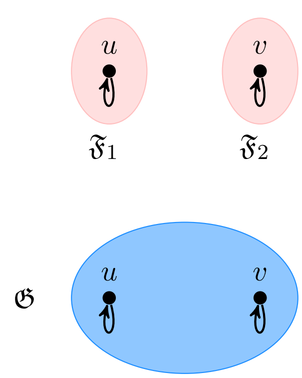

Figure 1.4: Frames and their disjoint union

Unlike the frame operations discussed above, ultrafilter extensions anti-preserve validity of hybrid formulas in exactly the same way as modal formulas. In fact, something stronger holds: validity of $\mathcal{H}(\mathsf{E})$-formulas is preserved under taking ultrafilter morphic images (for the proof of this, see [72] and [15]), and thus also of $\mathcal{H}$ and $\mathcal{H}(@)$.

Recall the following definitions:

Definition 1.2.7. Let $\mathfrak{F} = (W, R)$ and $\mathfrak{G} = (W', S)$ be two frames. A non-empty binary relation $Z \subseteq W \times W'$ is *called a bisimulation between* $\mathfrak{F}$ and $\mathfrak{G}$ if the following conditions are satisfied:

(*zig*) If wZw' and wRv, then there is a $v' \in W'$ such that vZv' and $w'Sv'$.

(*zag*) If wZw' and $w'Sv'$, then there is a $v \in W$ such that vZv' and wRv.

If $\mathfrak{F}$ and $\mathfrak{G}$ are linked by some bisumulation, we write $\mathfrak{F} \leftrightarrows \mathfrak{G}$.

Denote the domain and range of Z by $\mathrm{dom}(Z)$ and $\mathrm{rng}(Z)$, respectively. Recall that a bisimulation is *total*, if $\mathrm{dom}(Z) = W$ and $\mathrm{rng}(Z) = W'$.

Definition 1.2.8. Let Z be a bisimulation between frames $\mathfrak{F}$ and $\mathfrak{G}$, and let X be a subset of W'. We say that Z *respects* X if the following two conditions hold for all $v \in X$:

(i) There exists exactly one u such that uZv.

(ii) For all u, w, if uZv and uZw, then $w = v$.

Definition 1.2.9. A *bisimulation system* from $\mathfrak{F}$ to $\mathfrak{G}$ is a function Z that assigns to each finite subset $X \subseteq W'$ a total bisimulation $Z(X) \subseteq W \times W'$ respecting X.

It is a well-known fact that if a modal formula contains no propositional variables, then its validity on a frame is preserved under total bisimulations. In general, validity of pure hybrid formulas is not preserved under total bisimulations. However, Ten Cate showed in [72] that the validity of pure $\mathcal{H}(\mathsf{E})$-formulas is preserved under taking images of bisimulation systems, and hence also of $\mathcal{H}$ and $\mathcal{H}(@)$.

Frame definability

Although validity of hybrid formulas is not always preserved under taking generated subframes, bounded morphic images and disjoint unions, Goldblatt and Thomason's characterization of the modally definable elementary frame classes has hybrid analogues. Ten Cate was able to characterize the elementary frame classes definable in $\mathcal{H}, \mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, as well as the elementary frame classes definable by pure formulas of these languages. These characterizations can be found in [72] and [15]. We give a short summary of these results, beginning with the language $\mathcal{H}(@)$.

Theorem 1.2.10. *An elementary frame class* K *is definable by a set of* $\mathcal{H}(@)$*-formulas iff* K *is closed under ultrafilter morphic images and generated subframes.*

As a corollary, we have the following:

Corollary 1.2.11. *An elementary frame class* K *is definable by a set of* $\mathcal{H}(\mathsf{E})$*-formulas iff it is closed under ultrafilter morphic images.*

Gargov and Goranko gave a similar characterization of the $\mathcal{H}(\mathsf{E})$-definable elementary frame classes. Their proof was algebraic, and it is not clear how to generalize the proof to other hybrid languages. For more details, see [40].

Finally, we have the following result for the minimal hybrid language $\mathcal{H}$:

Theorem 1.2.12. *An elementary frame class* K *is definable by a set of* $\mathcal{H}$*-formulas iff* K *is closed under ultrafilter morphic images,* K *is closed under generated subframes, and, for any frame* $\mathfrak{F}$*, if every point generated subframe of* $\mathfrak{F}$ *is a proper generated subframe of a frame in* K*, then* $\mathfrak{F} \in \mathsf{K}$*.*

Frame definability by pure formulas

The question for a characterization of the frame classes definable by pure hybrid formulas was first asked in [40], and the following results have been obtained in [15]:

Theorem 1.2.13. *A frame class* K *is definable by a pure* $\mathcal{H}(@)$*-formula iff* K *is elementary, closed under generated subframes, and closed under images of bisimulation systems.*

Corollary 1.2.14. *A frame class* K *is definable by a pure* $\mathcal{H}(\mathsf{E})$*-formula iff it is elementary and closed under bisimulation systems.*

Theorem 1.2.15. *A frame class* K *is definable by a pure* $\mathcal{H}$*-formula iff* K *is elementary, closed under images of bisimulation systems, closed under generated subframes, and for any frame* $\mathfrak{F}$*, if every point generated subframe of* $\mathfrak{F}$ *is a proper generated subframe of a frame in* K*, then* $\mathfrak{F} \in \mathsf{K}$*.*

1.2.4 Logics

Here we give axiomatizations of minimal normal hybrid logics in the languages $\mathcal{H}, \mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, as well as compare some hybrid logics. We also give a short overview of some of the completeness results in the literature. The definitions of theorems, deducibility, consistency, inconsistency, soundness and completeness for modal logics remain unchanged for hybrid logics.

The axiomatizations

Different approaches have been taken in the literature in axiomatizing the hybrid logic of a frame class. The first complete axiomatization of a hybrid logic in the language $\mathcal{H}$ was given in [41]. This axiomatization is based on the notion of *necessity forms* and *possibility forms*. For a fixed symbol \$ not belonging to $\mathcal{H}$, necessity forms are inductively defined as follows:

(i) \$ is a necessity form.

(ii) If ψ is a necessity form and φ is a $\mathcal{H}$-formula, then $\varphi \to \psi$ is a necessity form.

(iii) If ψ is a necessity form, then $\Box\psi$ is also a necessity form.

Possibility forms are defined similarly, replacing implications by conjunctions and boxes by diamonds. Given a possibility form M and a formula φ, $M(\varphi)$ will denote the result of replacing the unique occurrence of \$ in M by φ. Similarly, given a necessity form L and a formula φ, $L(\varphi)$ will denote the result of replacing the unique occurrence of \$ in L by φ. Now, Gargov, Passy and Tinchev showed that any complete axiomatization of the basic modal language extended with the axiom scheme $M(\mathbf{i} \wedge p) \to L(\mathbf{i} \to p)$, for every possibility form $M(\$)$ and every necessity form $L(\$)$, completely axiomatizes the hybrid logic in the language $\mathcal{H}$ of the class of all frames. We will refer to this axiom as (Nom').

Similar axiomatizations of hybrid logics in the languages $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$, respectively, can be found in [40] and [10], respectively.

In [13], we can find axiomatizations of logics in the language $\mathcal{H}(@)$. These axiomatizations use the following 'non-orthodox' Burgess-Gabbay-style inference rules:

(*Name*) If $\vdash @_{\mathbf{j}}\varphi$, then $\vdash \varphi$ for $\mathbf{j}$ not occurring in φ.

(*BG*) If $\vdash @_{\mathbf{i}}\Diamond\mathbf{j} \to @_{\mathbf{j}}\varphi$, then $\vdash @_{\mathbf{i}}\Box\varphi$ for $\mathbf{j} \neq \mathbf{i}$ and $\mathbf{j}$ not occurring in φ.

These rules are called 'non-orthodox' rules because of their syntactic side-conditions, and cover only classes of frames definable by pure formulas. BG stands for Bounded Generalization. Because $\mathbf{j} \neq \mathbf{i}$ and it does not occur in φ, $@_{\mathbf{i}}\Diamond\mathbf{j}$ asserts the existence of a successor (arbitrarily named by $\mathbf{j}$) of the state named by $\mathbf{i}$. Accordingly, the antecedent condition of the rule can be read as follows: suppose we can prove that if the state named by $\mathbf{i}$ has an arbitrary successor named by $\mathbf{j}$, then φ holds at the state named by $\mathbf{j}$. But then, since the state named by $\mathbf{j}$ was an arbitrary successor of the state named by $\mathbf{i}$, the consequent condition of the rule tells us that φ must hold at all successors of the state named by $\mathbf{i}$. The rule (*Name*) tells us that if it is provable that φ holds at an arbitrary state named by $\mathbf{i}$ (the state is arbitrary because $\mathbf{i}$ does not occur in φ), then we can prove φ. These rules play a crucial role in the completeness proof regarding extensions with pure axioms, but more on this later.

In what follows, we give axiomatizations of two minimal normal hybrid logics in each of the hybrid languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$. The difference between these two axiomatizations lies each time in the addition of extra inference rules. The systems given here are based on those in [13], [10] and [72]. We begin with two minimal normal hybrid logics in the language $\mathcal{H}$. We will use the notation $\Diamond^n$ with $n \in \mathbb{N}$ to denote a string of n consecutive $\Diamond$'s. The notation $\Box^m$ is defined similarly.

Definition 1.2.16. The *minimal normal hybrid logic* $\mathbf{H}$ is the smallest set of $\mathcal{H}$-formulas containing all propositional tautologies, the axioms in Table 1.1, and which is closed under the inference rules in Table 1.1, except for (*Name*) and (*Paste*). $\mathbf{H}^+$ is defined similarly, closing in addition under (*Name*) and (*Paste*). If Σ is a set of $\mathcal{H}$-formulas, then $\mathbf{H} \oplus \Sigma$ and $\mathbf{H}^+ \oplus \Sigma$ are the normal hybrid logics *generated* by Σ.

It should be clear to the reader that all basic axioms are sound, and that all inference rules preserve validity with respect to any class of frames.

With *sorted substitution*, we mean replacing propositional variables by arbitrary formulas and nominals by nominals. Substituting nominals by arbitrary formulas does not preserve

Axioms:	
(*Taut*)	$\vdash \varphi$ for all classical propositional tautologies φ.
(*Dual*)	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$
(*K*)	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$
(*Nom*)	$\vdash \Diamond^n(\mathbf{i} \wedge p) \to \Box^m(\mathbf{i} \to p)$ for all $n, m \in \mathbb{N}$.
Rules of inference:	
(*Modus ponens*)	If $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$.
(*Sorted substitution*)	$\vdash \varphi'$ whenever $\vdash \varphi$, where φ' is obtained from φ by sorted substitution.
(*Nec*)	If $\vdash \varphi$, then $\vdash \Box\varphi$.
(*NameLite*)	If $\vdash \neg\mathbf{i}$, then $\vdash \bot$.
(*Name*)	If $\vdash \mathbf{i} \to \varphi$, then $\vdash \varphi$ for $\mathbf{i}$ not occurring in φ.
(*Paste*)	If $\vdash \Diamond^n(\mathbf{i} \wedge \Diamond(\mathbf{j} \wedge \varphi)) \to \psi$, the $\vdash \Diamond^n(\mathbf{i} \wedge \Diamond\varphi) \to \psi$ for $n \in \mathbb{N}$, $\mathbf{i} \neq \mathbf{j}$, and $\mathbf{j}$ not occurring in φ and ψ.

Table 1.1: Axioms and inference rules of $\mathbf{H}$ and $\mathbf{H}^+$

validity in general. To see this, consider the formula

$$\Diamond(\mathbf{j} \wedge p) \wedge \Diamond(\mathbf{j} \wedge q) \to \Diamond(p \wedge q).$$

It is not difficult to see that this formula is valid. However, if we substitute the ordinary propositional variable r for $\mathbf{j}$, the resulting formula can be falsified. Consider the model $\mathfrak{M} = (W, R, V)$, where $W = \{u, v, w\}, R = \{(u, v), (u, w)\}, V(p) = \{v\}, V(q) = \{w\}$ and $V(r) = \{v, w\}$ (see Figure 1.5). Clearly, $\mathfrak{M}, u \Vdash \Diamond(r \wedge p) \wedge \Diamond(r \wedge q)$ but $\mathfrak{M}, u \nVdash \Diamond(p \wedge q)$.

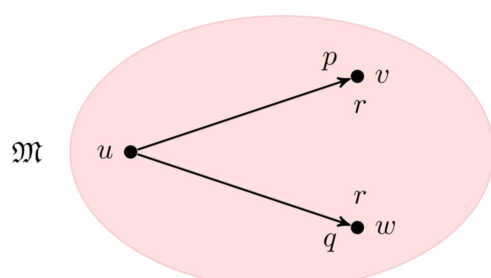

Figure 1.5: A model falsifying $\Diamond(r \wedge p) \wedge \Diamond(r \wedge q) \to \Diamond(p \wedge q)$.

The role of (*NameLite*) is to render logics that derive $\neg\mathbf{j}$ for some nominal $\mathbf{j}$, inconsistent, reflecting the fact that $\neg\mathbf{j}$ is not valid on any frame. As is not hard to see, without (*NameLite*), the logic $\mathbf{H} \oplus \{\neg\mathbf{j}\}$ would be consistent.

But why the additional rules? In order to answer this question, we have to get ahead of ourselves a bit. When pure formulas are added to the minimal hybrid logic $\mathbf{H}^+$ as axioms, then this logic is automatically complete with respect to the class of frames it defines. This result hinges on a rather simple observation. But first we recall the following: a model (W, R, V) is *named*, if for all states $w \in W$, there is some nominal $\mathbf{j} \in \mathsf{NOM}$ such that $V(\mathbf{j}) = \{w\}$. Now, it is very easy to show that if $(\mathfrak{F}, V)$ is a named model and φ is a pure formula, then $\mathfrak{F} \Vdash \varphi$ whenever $(\mathfrak{F}, V) \Vdash \psi$ for all pure instances ψ of φ. In other words, for named models and pure formulas the gap between truth in a model and validity in a frame is non-existent. So if we had a way of building named models, any pure formula would give rise to a strongly complete logic for the class of frames it defines. The idea is to build a named canonical model from the logic's MCSs and prove an Existence Lemma, once this is done, completeness will be immediate.

Now, given a consistent set of $\mathcal{H}$-formulas Σ, we can use the ordinary Lindenbaum's Lemma to extend it to a maximal consistent set of formulas Σ^+, and then build a canonical model out of the maximal consistent sets in the usual way. But we do not want to build just any model, we want a named model. In other words, each of the maximal consistent sets in our canonical model must contain a nominal. If this is the case, we will say that the maximal consistent set is *named*. However, nothing guarantees that Σ^+ will be named.

Suppose we overcame the first problem and succeeded in expanding a consistent set of formulas to a named maximal consistent set. Now, as we mentioned above, to build a named canonical model, only named maximal consistent sets should be used in the model construction. Unfortunately, nothing guarantees that there are enough maximal consistent sets to support an Existence Lemma. This is where the additional rules come in. The rule (*Name*) solves our first problem, while the rule (*Paste*) solves the second problem.

To get back to Gargov and Goranko's approach in [41], (*Nom*) and (*Paste*) can be replaced by the following:

(*Nom′*) $M(\mathbf{i} \wedge p) \to L(\mathbf{i} \to p)$, where $M(\$)$ is a possibility form and $L(\$)$ a necessity form.

(*Cov*) If $\vdash L(\neg \mathbf{i})$, then $\vdash L(\bot)$, where $L(\$)$ is a necessity form not containing the nominal $\mathbf{i}$.

It is not hard to see that (*Nom*) and (*Nom′*) are interderivable, as well as the rules (*Paste*) and (*Cov*). Moreover, (*NameLite*) can be seen as the simplest possible instance of (*Cov*).

Let us now move on to the axiomatizations of the minimal normal hybrid logics in the language $\mathcal{H}(@)$.

Definition 1.2.17. The *minimal normal hybrid logic* $\mathbf{H}(@)$ is the smallest set of $\mathcal{H}(@)$-formulas containing all propositional tautologies, the axioms in Table 1.2, and which is closed under the inference rules in Table 1.2, except for ($Name_@$) and ($BG_@$). $\mathbf{H}^+(@)$ is defined similarly, closing in addition under ($Name_@$) and ($BG_@$). If Σ is a set of $\mathcal{H}(@)$-formulas, then $\mathbf{H} \oplus \Sigma$ and $\mathbf{H}^+ \oplus \Sigma$ are the normal hybrid logics *generated* by Σ.

The axioms of the minimal hybrid logic $\mathbf{H}(@)$ can be divided into three categories. The axioms ($K_@$), (*Selfdual*) and (*Intro*) fall into the first category and establish the basic logic of the satisfaction operators. The inclusion of ($K_@$) should come as no surprise since the satisfaction operators are normal modal operators. The same for the axiom (*Selfdual*). The axiom (*Intro*) tells us how to put information under the scope of satisfaction operators. Not

Axioms:	
(*Taut*)	$\vdash \varphi$ for all classical propositional tautologies φ.
(*K*)	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$
(*Dual*)	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$
($K_@$)	$\vdash @_{\mathbf{j}}(p \to q) \to (@_{\mathbf{j}}p \to @_{\mathbf{j}}q)$ for all $\mathbf{j} \in \mathsf{NOM}$.
(*Selfdual*)	$\vdash \neg @_{\mathbf{j}}p \leftrightarrow @_{\mathbf{j}}\neg p$ for all $\mathbf{j} \in \mathsf{NOM}$.
(*Intro*)	$\vdash \mathbf{j} \wedge p \to @_{\mathbf{j}}p$ for all $\mathbf{j} \in \mathsf{NOM}$.
(*Ref*)	$\vdash @_{\mathbf{j}}\mathbf{j}$ for all $\mathbf{j} \in \mathsf{NOM}$.
(*Agree*)	$\vdash @_{\mathbf{i}}@_{\mathbf{j}}p \to @_{\mathbf{j}}p$ for all $\mathbf{i}, \mathbf{j} \in \mathsf{NOM}$.
(*Back*)	$\vdash \Diamond @_{\mathbf{j}}p \to @_{\mathbf{j}}p$ for all $\mathbf{j} \in \mathsf{NOM}$.
Rules of inference:	
(*Modus ponens*)	If $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$.
(*Sorted substitution*)	$\vdash \varphi'$ whenever $\vdash \varphi$, where φ' is obtained from φ by sorted substitution.
(*Nec*)	If $\vdash \varphi$, then $\vdash \Box\varphi$.
($Nec_@$)	If $\vdash \varphi$, then $\vdash @_{\mathbf{j}}\varphi$.
($Name_@$)	If $\vdash @_{\mathbf{j}}\varphi$, then $\vdash \varphi$ for $\mathbf{j}$ not occurring in φ.
($BG_@$)	If $\vdash @_{\mathbf{i}}\Diamond\mathbf{j} \wedge @_{\mathbf{j}}\varphi \to \psi$, then $\vdash @_{\mathbf{i}}\Diamond\varphi \to \psi$ for $\mathbf{j} \neq \mathbf{i}$ and $\mathbf{j}$ not occurring in φ and ψ.

Table 1.2: Axioms and inference rules of $\mathbf{H}(@)$ and $\mathbf{H}^+(@)$

only that, it also tells us how to obtain such information. To see this, replace p by $\neg p$, contrapose, and make use of (*Selfdual*), then we get

$$\mathbf{i} \wedge @_{\mathbf{i}}p \to p.$$

The second category includes the axioms (*Ref*) and (*Agree*). This category provide us with tools for naming states or reasoning about state equality. The third category includes the axiom (*Back*). This axiom tells us how $\Diamond$ and $@$ interact with each other.

Finally, the reason for the additional rules is the same as for the logic $\mathbf{H}^+$.

We conclude with axiomatizations of two minimal normal hybrid logics in the language $\mathcal{H}(\mathsf{E})$:

Definition 1.2.18. The *minimal normal hybrid logic* $\mathbf{H}(\mathsf{E})$ is the smallest set of $\mathcal{H}(\mathsf{E})$-formulas containing all propositional tautologies, the axioms in Table 1.3, and which is closed under the inference rules in Table 1.3, except for the ($Name_{\mathsf{E}}$), ($BG_{\mathsf{E}\Diamond}$) and (BG_{EE}) rules. $\mathbf{H}^+(\mathsf{E})$ is defined in the same way, closing in addition under ($Name_{\mathsf{E}}$), ($BG_{\mathsf{E}\Diamond}$) and (BG_{EE}). If Σ is a set of $\mathcal{H}(\mathsf{E})$-formulas, then $\mathbf{H}(\mathsf{E}) \oplus \Sigma$ and $\mathbf{H}^+(\mathsf{E}) \oplus \Sigma$ are the normal hybrid logics generated by Σ.

Axioms:	
($Taut$)	$\vdash \varphi$ for all classical propositional tautologies.
(K)	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$
($Dual$)	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$
(K_{A})	$\vdash \mathsf{A}(p \to q) \to (\mathsf{A}p \to \mathsf{A}q)$
($Dual_{\mathsf{A}}$)	$\vdash \mathsf{E}p \leftrightarrow \neg\mathsf{A}\neg p$
($Incl_{\mathbf{j}}$)	$\vdash \mathsf{E}\mathbf{j}$
(Nom_{E})	$\vdash \mathsf{E}(\mathbf{i} \wedge p) \to \mathsf{A}(\mathbf{i} \to p)$
($T\mathsf{E}$)	$\vdash p \to \mathsf{E}p$
($4\mathsf{E}$)	$\vdash \mathsf{EE}p \to \mathsf{E}p$
($B\mathsf{E}$)	$\vdash p \to \mathsf{AE}p$
($Incl_{\Diamond}$)	$\vdash \Diamond p \to \mathsf{E}p$
Rules of inference:	
($Modus\ ponens$)	If $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$.
($Sorted\ substitution$)	$\vdash \varphi'$ whenever $\vdash \varphi$, where φ' is obtained from φ by sorted substitution.
(Nec)	If $\vdash \varphi$, then $\vdash \Box\varphi$.
(Nec_{A})	If $\vdash \varphi$, then $\vdash \mathsf{A}\varphi$.
($Name_{\mathsf{E}}$)	If $\vdash \mathbf{i} \to \varphi$, then $\vdash \varphi$ for $\mathbf{i}$ not occurring in φ.
($BG_{\mathsf{E}\Diamond}$)	If $\vdash \mathsf{E}(\mathbf{i} \wedge \Diamond\mathbf{j}) \wedge \mathsf{E}(\mathbf{j} \wedge \varphi) \to \psi$, then $\vdash \mathsf{E}(\mathbf{i} \wedge \Diamond\varphi) \to \psi$ for $\mathbf{i} \neq \mathbf{j}$ and $\mathbf{j}$ not occurring in φ and ψ.
(BG_{EE})	If $\vdash \mathsf{E}(\mathbf{i} \wedge \mathsf{E}\mathbf{j}) \wedge \mathsf{E}(\mathbf{j} \wedge \varphi) \to \psi$, then $\vdash \mathsf{E}(\mathbf{i} \wedge \mathsf{E}\varphi) \to \psi$ for $\mathbf{i} \neq \mathbf{j}$ and $\mathbf{j}$ not occurring in φ and ψ.

Table 1.3: Axioms and inference rules of $\mathbf{H}(\mathsf{E})$ and $\mathbf{H}^{+}(\mathsf{E})$

First, note that the axioms ($T\mathsf{E}$), ($4\mathsf{E}$) and ($B\mathsf{E}$) determine that E is an $\mathbf{S}5$-modality. Second, notice that in addition to the BG rule for $\Diamond$, we also have a BG rule for E. This rule plays the same role as the BG rule for $\Diamond$.

Completeness

One of the most important results in modal logic is the Sahlqvist completeness theorem. This result says that the axiomatization obtained by adding a set of modal Sahlqvist formulas to the basic modal logic $\mathbf{K}$ is *complete* with respect to the class of frames definable by these Sahlqvist axioms. While this result covers many interesting frame classes, there are frame properties such as irreflexivity and asymmetry that cannot be defined by modal formulas. However, as we stated earlier, many of these properties are definable by pure formulas. What is more, as we mentioned earlier, when pure formulas are added to the basic hybrid logics with additional rules, they yield complete logics for the classes of frames they define. A second important reason why hybrid logics have become popular is therefore that there is a general completeness result for hybrid logics that applies to many frame classes not definable by modal Sahlqvist formulas. The following analogues of the Sahlqvist completeness theorem for modal logic can be obtained for hybrid logics:

Theorem 1.2.19. *If Σ is a set of pure $\mathcal{H}$-formulas, then $\mathbf{H}^+ \oplus \Sigma$ is strongly complete with respect to the class of frames defined by Σ. If Σ is a set of pure $\mathcal{H}(@)$-formulas, then $\mathbf{H}^+(@) \oplus \Sigma$ is strongly complete with respect to the class of frames defined by Σ. Finally, if Σ is a set of pure $\mathcal{H}(\mathsf{E})$-formulas, then $\mathbf{H}^+(\mathsf{E}) \oplus \Sigma$ is strongly complete with respect to the class of frames defined by Σ.*

For the hybrid language $\mathcal{H}$, the completeness of logics axiomatized by pure formulas was already proved in the eighties by Gargov, Passy and Tinchev in [41]. The case for $\mathcal{H}(\mathsf{E})$ was proved by Gargov and Goranko in [40], and adapted by Blackburn and Tzakova to $\mathcal{H}(@)$ in [14].

However, there are frame properties that are definable by modal Sahlqvist formulas but not by pure hybrid formulas. For example, no set of pure formulas defines the same class of frames as the modal Sahlqvist formula $\Diamond\Box p \to \Box\Diamond p$ (the Geach axiom). This was first proved by Gargov and Goranko in [40]. The following results are therefore interesting:

Theorem 1.2.20. *If Σ is a set of modal Sahlqvist formulas, then $\mathbf{H} \oplus \Sigma$ is strongly complete with respect to the class of frames defined by Σ. If Σ is a set of modal Sahlqvist formulas, then $\mathbf{H}(@) \oplus \Sigma$ is strongly complete with respect to the class of frames defined by Σ. Finally, if Σ is a set of modal Sahlqvist formulas, then $\mathbf{H}(\mathsf{E}) \oplus \Sigma$ is strongly complete with respect to the class of frames defined by Σ.*

The cases for $\mathcal{H}$ and $\mathcal{H}(@)$ were proved in [73], while the case for $\mathcal{H}(\mathsf{E})$ was first observed in [40].

To conclude, recall that every modal logic is sound and strongly complete with respect to the class of descriptive general frames (see [10]). In [72], Ten Cate obtained similar results for hybrid logics. In particular, Ten Cate proved that the axiomatizations without the additional inference rules are complete with respect to descriptive two-sorted general frames, whereas the axiomatizations with the additional inference rules are complete with respect to strongly descriptive two-sorted general frames and discrete general frames.

Logics with and without extra rules compared

One naturally wonders if the logics with and without the additional inference rules have the same theorems? Here we will answer this question.

Let us first consider the minimal hybrid logics $\mathbf{H}$ and $\mathbf{H}^+$. Since both $\mathbf{H}$ and $\mathbf{H}^+$ are sound and strongly complete with respect to the class of all frames, they both have the same theorems. This might lead one to believe that this is also the case when we extend these logics with a set Σ of axioms. However, this is not true. To see this, consider $\Sigma = \{\mathbf{j} \to \Box\bot\}$ and $\varphi = \Diamond\top$, and let $\mathfrak{g} = (W, R, A, B)$, where $W = \{u, v\}$, $R = \{(u, u)\}$, $A = \mathcal{P}(W)$ and $B = \{v\}$ (see Figure 1.6). Then $\mathfrak{g}$ validates the members of Σ, and furthermore, $\Diamond\top$ is satisfied at u. This means that $\Diamond\top$ is $\mathbf{H} \oplus \Sigma$-consistent, and so $\Box\bot \notin \mathbf{H} \oplus \Sigma$. On the other hand, if we apply the ($Name$) rule to $\mathbf{j} \to \Box\bot$, we get that $\mathbf{H}^+ \oplus \Sigma$ has $\Box\bot$ as a theorem. Hence, $\neg\varphi \in \mathbf{H}^+ \oplus \Sigma$, while $\neg\varphi \notin \mathbf{H} \oplus \Sigma$.

Similarly, if Σ is a set of $\mathcal{H}(@)$-formulas, then the logics $\mathbf{H}(@) \oplus \Sigma$ and $\mathbf{H}^+(@) \oplus \Sigma$ do not have the same theorems. Consider $\Sigma = \{@_{\mathbf{j}}\Box\bot\}$ and $\varphi = \Diamond\top$, and let $\mathfrak{g}$ be the two-sorted general frame in Figure 1.6. Clearly, $\mathfrak{g}$ validates the members of Σ, and furthermore, φ is satisfied at u. Hence, φ is $\mathbf{H}(@) \oplus \Sigma$-consistent, and so $\neg\varphi \notin \mathbf{H}(@) \oplus \Sigma$. However, $\mathbf{H}^+(@) \oplus \Sigma$ has $\neg\varphi$ as a theorem (apply ($Name_@$) to $@_{\mathbf{j}}\Box\bot$).

Finally, let Σ be a set of $\mathcal{H}(\mathsf{E})$-formulas. Then the same holds for the logics $\mathbf{H}(\mathsf{E}) \oplus \Sigma$ and $\mathbf{H}^+(\mathsf{E}) \oplus \Sigma$. To see this, let $\Sigma = \{\mathbf{j} \to \Box\bot\}$ and $\varphi = \Diamond\top$, and consider the two-sorted general frame in Figure 1.6.

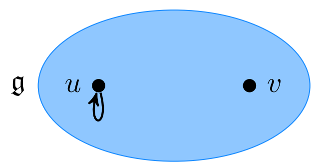

Figure 1.6: A two-sorted general frame satisfying the formula $\Diamond\top$.

1.2.5 A survey of results for hybrid logics

Hybrid logics offer an important advantage over modal logics, namely, increased expressive power. The natural question then is: How much do we gain by extending modal languages with nominals and operators like @ and E, and what price do we pay? We saw that boundaries are drawn when it comes to the preservation of validity of hybrid formulas under certain frame operations. However, in spite of this, Ten Cate was still able to obtain hybrid analogues of the Goldblatt-Thomason theorem. But how does the increased expressivity affect complexity, decidability, interpolation and Beth definability? Here we will give a short overview of known results in the literature to this end. We also give a short survey of some proof systems for hybrid logics. To conclude, we will look at some of the uses of hybrid logics.

Complexity

In this subsection, we will give a short review on the complexity of the satisfiability problem for some of the hybrid logics discussed so far.

For the basic modal logic $\mathbf{K}$, adding nominals does not increase the complexity: the complexity of the satisfiability problem for $\mathbf{H}$ is still in PSPACE, as was first shown by Schaerf [65].

It is not always the case that the complexity of the satisfiability problem remains unchanged when we add nominals. The satisfiability problem for the modal logic of the class of all symmetric frames is PSPACE-complete [26], however, the addition of just a single nominal, blows the complexity up to EXPTIME. For the proof of this, see [7].

The satisfiability problem of the logic $\mathbf{H}(@)$ on the class of all frames is PSPACE-complete. A proof of this can be found in [1] and [2]. In other words, since the satisfiability problem of the basic modal logic on the class of all frames is already PSPACE-complete, we can conclude that, for the class of all frames, the addition of nominals and the @ operator does not increase the complexity of the satisfiability problem. However, adding both nominals and the global modality E to the basic modal logic, raises the complexity to EXPTIME-complete (see for instance [69]).

From [59], we know that the satisfiability problem of the basic modal logic on the class of transitive frames is also PSPACE-complete. As for the class of all frames, adding nominals and the @ operator does not increase the complexity of the satisfiability problem. A proof of this can be found in [3]. From the results in [3], we can also conclude that the satisfiability problem of **H**(@) on linear orders is also PSPACE-complete.

Decidability

Here we ask the question of whether decidability is preserved when nominals, satisfaction operators or the global modality is added to the basic modal language. Gargov and Goranko were the first to ask this question explicitly (see [40]).

First, from the complexity results above, we know that the logics **H**, **H**(@) and **H**(E) are all decidable. We also know from [7] that **HB** is decidable. However, adding nominals can also result in logics that are undecidable and lack the finite model property. For example, in [7], Bezhanishvili and Ten Cate showed that the logic **KB23** is decidable and has the finite model property, while the hybrid logic **HB23** is undecidable and lacks the finite model property.

Interpolation

Here we turn our attention to the interpolation property for hybrid languages. Interpolation for hybrid languages were first investigated by Areces, Blackburn and Marx in [2]. They showed that $\mathcal{H}(@)$ does not have interpolation over propositional variables and nominals with respect to the class of all frames. Furthemore, Conradie proved in [29] that $\mathcal{H}(@)$ also lacks interpolation over propositional variables and nominals with respect to the class of **S5** frames. However, there is some good news: the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ have interpolation over propositional variables with respect to many frame classes, including the class of all frames (see [72] and [73]). In particular, as for the basic modal language, it can be shown that the languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ have interpolation over propositional variables relative to any elementary class of frames closed under bisimulation products and generated subframes.

Next, we consider interpolation over nominals. The languages $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$ all lack interpolation over nominals relative to the class of all frames. Consider the implication $\mathbf{i} \wedge \Diamond \mathbf{i} \to (\mathbf{j} \to \Diamond \mathbf{j})$. An interpolant for this implication has to express that the current state is related to itself without using any nominals. An easy bisimulation argument shows that this is not possible. Sadly, the bad news does not stop there: $\mathcal{H}(\mathsf{E})$ also lacks interpolation over nominals relative to any non-empty modally definable frame class. For the proof of this, see [72].

Finally, we know that the modal logics **K**, **GL**, **S5** and **Grz** have uniform interpolation (see [44] and [79]). It turns out that the corresponding $\mathcal{H}$-logics **H**, **HS5**, **HGL** and **HGrz** (with **HS5** we mean the $\mathcal{H}$-logic of the frame class defined by **S5**, and similar for other logics) also have uniform interpolation over propositional variables, as well as the $\mathcal{H}(@)$-logics $\mathbf{H}(@)$, $\mathbf{HS5}(@)$, $\mathbf{HGL}(@)$ and $\mathbf{HGrz}(@)$. For more details, see [72].

Beth definability

The Beth definability property for any logic normally follows from the interpolation property for propositional variables. In particular, using the results in Subsection 1.2.5, we can show that the language $\mathcal{H}(@)$ has the Beth definability property relative to any frame class.

In [72], Ten Cate showed that $\mathcal{H}(\mathsf{E})$ has the Beth definability property relative to any elementary frame class closed under generated subframes and bisimulation products.

Surprisingly, the hybrid language $\mathcal{H}$ does not have the Beth definability property relative to the class of all frames. For more details on this, see [7].

Proof systems

Here we give a short survey of proof systems for hybrid logics. First, in [66], we can find a sound and complete *sequent calculus* for hybrid logics developed from a sequent calculus for first-order logics by a series of transformations. This calculus is cut free, and it can be proved that the cut rule is admissible.

Seligman also proposed a *natural deduction* system in [67]. In this paper, he proved soundness and completeness, but he does not discuss whether the calculus is normalizing. In [17], Braüner introduced a natural deduction calculus for the hybrid language with the operators $@, \downarrow$ and $\forall$ and its sublanguages, and establishes normalization.

In [11], a *tableau calculus* for the hybrid language with the operators $@$ and $\downarrow$ is given. This calculus is @-based: to prove that a formula φ is unsatisfiable, we have to apply the rules to the formula $@_{\mathbf{i}}\varphi$ for a nominal $\mathbf{i}$ not in φ. If we can find a closed tableau (a tableau in which each branch contains a pair of formulas $@_{\mathbf{j}}\psi$ and $@_{\mathbf{j}}\neg\psi$, then φ is unsatisfiable.

We can also find an approach to hybrid tableaux in [74] that uses nominals both as part of the object language and as meta-logical labels.

Finally, *resolution calculi* for the hybrid language with $@$ and $\downarrow$ and its sublanguages were introduced in [4] and [5]. More recently, the calculus for $\mathcal{H}(@)$ was refined to include ordering and selection functions in [6].

Uses of hybrid logics

In [27], Conradie developed an algorithm called SQEMA, which computes first-order frame equivalents for modal formulas, by first transforming them into pure formulas in a reversive hybrid language. He showed that this algorithm subsumes the classes of Sahlqvist and inductive formulas, and that all formulas on which it succeeds are canonical, and hence axiomatize complete normal modal logics. Ian Hodkinson obtains modal axiomatizations of elementary frame classes via hybrid logics in [57]. It turns out that the proof of this is analogous to the proof of Sahlqvist's theorem.

Chapter 2

Hybrid algebras

It is a well-known fact that most of the familiar logical systems have a (natural) algebraic semantics. In this chapter, we develop algebraic semantics for the hybrid logics discussed in Chapter 1. But in order to do that, we must first know which kinds of algebras are relevant. A first step towards an answer comes from Ten Cate's work in [72]. He studied completeness theory for the axiomatizations of these languages. He proved, among other things, a general completeness result for the logics without the additional rules with respect to *descriptive two sorted general frames*, as well as completeness for the logics with the additional rules with respect to *strongly descriptive two sorted general frames*. This suggests using algebras corresponding to these two-sorted general frames. However, in the proof of his completeness result with respect to descriptive two-sorted general frames, he treat nominals as unary modalities. This suggests that there are two possible algebraic semantics for the hybrid languages discussed in Chapter 1. One involves interpreting the nominals as constants and then falling back on known results from algebraic logic. The second corresponds to two-sorted general frames where the nominals are seen as special variables ranging over a subset of the atoms of the algebra.

In this chapter, we will give formal definitions of both algebraic semantics for each of the hybrid languages discussed in Chapter 1, as well as define homomorphisms, embeddings and products for the second type of algebras and discuss the preservation of validity under these. As we already mentioned, when we work with the first type of algebras, we can fall back on known results from algebraic logic, so we will not look at these operations on the first type of algebras, but the reader is referred to [21] and [23] for more details on these constructions and the preservation of validity under these. Finally, we will also investigate the duality between the second type of algebras and two-sorted general frames.

2.1 Hybrid algebras for $\mathcal{H}$ and the operations on them

2.1.1 Hybrid algebras for $\mathcal{H}$

The first type of algebraic semantics for $\mathcal{H}$ is called an *orthodox interpretation*. We use the term "orthodox" since it is really the 'standard' algebraic semantics for modal logics with constants, however, for us it is 'non-standard' since it is not appropriately dual to the intended relational semantics of hybrid logic.

Definition 2.1.1 (**Orthodox interpretations**). An *orthodox interpretation* of $\mathcal{H}$ is a structure $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, \Diamond, \{s_\mathbf{i}\}_{\mathbf{i} \in \mathsf{NOM}})$, where $(A, \wedge, \vee, \neg, \bot, \top, \Diamond)$ is a BAO, and each $s_\mathbf{i}$ is the interpretation of the nominal $\mathbf{i}$ as a *constant* (i.e., nullary operation). Moreover, $\mathbf{A}$ is required to validate the inequality $\Diamond^n(s_\mathbf{i} \wedge a) \leq \Box^m(\neg s_\mathbf{i} \vee a)$ for all $\mathbf{i} \in \mathsf{NOM}$ and $n, m \in \mathbb{N}$.

Note that the rule (*sorted substitution*) is not generally sound on orthodox interpretations, in the sense that for an orthodox interpretation $\mathbf{A}$ and $\mathcal{H}$-formula ψ, it may happen that $\mathbf{A} \models \psi \approx \top$ but that $\mathbf{A} \not\models \psi' \approx \top$ for some sorted substitution instance ψ' of ψ. For example, consider the orthodox interpretation $\mathbf{A} = (\mathbf{2}, \Diamond, \{s_\mathbf{j}\}_{\mathbf{j} \in \mathsf{NOM}})$, where $\mathbf{2}$ is the two element Boolean algebra, $\Diamond 0 = 0$, $\Diamond 1 = 1$, $s_\mathbf{j} = 0$, and $s_\mathbf{i} = 1$ for $\mathbf{i} \neq \mathbf{j}$. Then $\mathbf{A} \models \Diamond\mathbf{i} \approx \top$ but $\mathbf{A} \not\models \Diamond\mathbf{j} \approx \top$. However, this will not be a concern to us, as in the ensuing we will always require that an orthodox interpretation validates (all theorems of a) logic $\mathbf{H} \oplus \Sigma$, which is by definition closed under sorted substitution already.

Let us now turn to the second type of algebras for $\mathcal{H}$. We will consider these algebras to be the standard algebras for the language $\mathcal{H}$.

Definition 2.1.2 (**Hybrid algebras**). A *hybrid algebra* is a pair $\mathfrak{A} = (\mathbf{A}, X_A)$, where $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, \Diamond)$ is a BAO containing at least one atom, and X_A is a non-empty subset of the set $At\mathbf{A}$ of atoms of $\mathbf{A}$.

We will often refer to X_A as a *set of designated atoms* of the algebra. We also make the following convention: $\Box a := \neg\Diamond\neg a$. Finally, we will denote the class of hybrid algebras by HA.

A hybrid algebra $\mathfrak{A} = (A, X_A)$ is said to be *complete* if $\mathbf{A}$ is a complete BAO, and it is *atomic* if $\mathbf{A}$ is atomic and $X_A = At\mathbf{A}$.

We now give a few examples of hybrid algebras.

Example 2.1.3. The structure $\mathfrak{A} = (\mathbf{2}, \Diamond, \{1\})$, where $\mathbf{2}$ is the two element Boolean algebra, $\Diamond 0 = 0$, and $\Diamond 1 = 1$, is clearly a hybrid algebra.

Example 2.1.4. Let A be the set of all binary strings of length 4, and let

$$\mathbf{A} = (A, \cdot, +, -, 0000, 1111, \Diamond),$$

where $\cdot, +$ and $-$ are defined bitwise by Boolean multiplication, addition and complementation[1], respectively, and $\Diamond$ maps an element to itself. Then the structure $\mathfrak{A} = (\mathbf{A}, X_A)$, where $X_A = \{0001, 0010, 0100, 1000\}$ is a hybrid algebra.

[1] *Boolean addition* is defined by $0+1 = 1, 1+1 = 1$ and $0+0 = 0$, *Boolean multiplication* by $0 \cdot 1 = 0, 1 \cdot 1 = 1$ and $0 \cdot 0 = 0$, and *Boolean complementation* by $-0 = 1$ and $-1 = 0$.

Example 2.1.5. Let A be the collection of finite and co-finite subsets of the set of integers $\mathbb{Z}$, and let $\mathbf{A} = (A, \cap, \cup, -, \varnothing, \mathbb{Z}, f)$, where

$$f(a) = \begin{cases} \{n-1 \mid n \in a\} & \text{if } a \text{ is finite} \\ \mathbb{Z} & \text{if } a \text{ is co-finite.} \end{cases}$$

Then the structure $\mathfrak{A} = (\mathbf{A}, X_A)$, where $X_A = \{\{2k\} \mid k \in \mathbb{Z}\}$, is a hybrid algebra. First, since the set of even integers is not empty, $X_A \neq \varnothing$. Second, the fact that $(A, \cap, \cup, -, \varnothing, \mathbb{Z})$ is a Boolean algebra follows from basic Set Theory. So all we have to check is that f is a normal modal operator. Clearly, $f(\varnothing) = \varnothing$. To show that f is additive, we consider the following cases:

Case 1: a and b are finite. In this case we also know that $a \cup b$ is finite. So

$$\begin{aligned} m \in f(a \cup b) &\iff \exists n(m = n-1 \wedge n \in a \cup b) \\ &\iff \exists n(m = n-1 \wedge (n \in a \vee n \in b)) \\ &\iff \exists y((m = n-1 \wedge n \in a) \vee (m = n-1 \wedge n \in b)) \\ &\iff \exists n(m = n-1 \wedge n \in a) \vee \exists n(m = n-1 \wedge n \in b) \\ &\iff m \in f(a) \vee m \in f(b) \\ &\iff m \in f(a) \cup f(b). \end{aligned}$$

Hence, $f(a \cup b) = f(a) \cup f(b)$, as required.

Case 2: a is finite and b is co-finite, or vice versa. Here $a \cup b$ is co-finite, so $f(a \cup b) = \mathbb{Z}$. But we also know that one of a and b is co-finite, which means that $f(a) \cup (b) = \mathbb{Z}$.

Case 3: a and b are co-finite. In this case, we also have that $a \cup b$ is co-finite, which means that $f(a \cup b) = \mathbb{Z}$. But a and b are also co-finite, so $f(a) \cup f(b) = \mathbb{Z} \cup \mathbb{Z} = \mathbb{Z}$.

Example 2.1.6. Let $\mathfrak{g} = (W, R, A, B)$ be a two-sorted general frame. Then the structure $\mathfrak{g}^* = (A, \cap, \cup, -, \varnothing, W, \langle R \rangle, X_B)$, where $X_B = \{\{w\} \mid w \in B\}$, is a hybrid algebra. This algebra is called the *underlying hybrid algebra of* $\mathfrak{g}$.

Example 2.1.7. Consider the set $\mathbb{A} = \{X_1 \cup X_2 \cup X_3 \mid X_i \in \mathbb{X}_i, i = 1, 2, 3\}$, where $\mathbb{X}_1$ contains all finite (possibly empty) subsets of natural numbers, $\mathbb{X}_2$ contains $\varnothing$ and all sets of the form $\{x \mid n \leq x \leq \omega\}$ for all $n \in \omega$, and $\mathbb{X}_3 = \{\varnothing, \{\omega + 1\}\}$. Then

$$\mathfrak{A} = (\mathbb{A}, \cap, \cup, -, \varnothing, \omega + 1 \cup \{\omega + 1\}, f, X_{\mathbb{A}}),$$

where

$$f(A) = \begin{cases} \{x \in \omega + 1 \cup \{\omega + 1\} \mid \min(A) < x\} & \text{if } A \neq \varnothing \\ \varnothing & \text{otherwise} \end{cases}$$

and $X_{\mathbb{A}} = \{\{x\} \mid x \in \mathbb{N}\} \cup \{\{\omega + 1\}\}$, is a hybrid algebra. First, it is not difficult to see that $\mathbb{A}$ is closed under these operations. Using basic set theory we can show that the structure

$(\mathbb{A}, \cap, \cup, -, \varnothing, \omega+1 \cup \{\omega+1\})$ is a Boolean algebra. So we just have to check that f is a normal modal operator. Clearly, $f(\varnothing) = \varnothing$, and finally, f is additive:

$$\begin{aligned} x \in f(A \cup B) &\iff x > \min(A \cup B) \\ &\iff x > \min(\min(A), \min(B)) \\ &\iff x > \min(A) \text{ or } x > \min(B) \\ &\iff x \in f(A) \text{ or } x \in f(B) \\ &\iff x \in f(A) \cup f(B) \end{aligned}$$

Note that the BAO $(\mathbb{A}, \cap, \cup, -, \varnothing, \omega+1 \cup \{\omega+1\}, f)$ is actually the underlying algebra of the strongly descriptive general frame in Example 3.3 of [28].

Having defined hybrid algebras, the question now is how do we interpret $\mathcal{H}$-terms in a hybrid algebra. $\mathcal{H}$-terms are interpreted in hybrid algebras $\mathfrak{A} = (\mathbf{A}, X_A)$ in the usual way but subject to the constraint that nominals range over X_A, while the propositional variables range over all elements of the algebra, as usual. Let us now give a more precise definition:

Definition 2.1.8. An *assignment* on $\mathfrak{A} = (\mathbf{A}, X_A)$ is a function $v\colon \mathsf{PROP} \cup \mathsf{NOM} \to A$ associating an element of A with each propositional variable in PROP and an atom of X_A with each nominal in NOM. Given such an assignment v, we calculate the *meaning* $\widetilde{v}(t)$ of a term t as follows:

$$\begin{aligned} \widetilde{v}(\bot) &= \bot \\ \widetilde{v}(p) &= v(p), \\ \widetilde{v}(\mathbf{j}) &= v(\mathbf{j}), \\ \widetilde{v}(\neg\psi) &= \neg\widetilde{v}(\psi), \\ \widetilde{v}(\psi_1 \wedge \psi_2) &= \widetilde{v}(\psi_1) \wedge \widetilde{v}(\psi_2), \text{ and} \\ \widetilde{v}(\Diamond\psi) &= \Diamond\widetilde{v}(\psi). \end{aligned}$$

An equation $\varphi \approx \psi$ is *true* in a hybrid algebra $\mathfrak{A}$ (denoted $\mathfrak{A} \models \varphi \approx \psi$), if for all assignments θ, $\widetilde{\theta}(\varphi) = \widetilde{\theta}(\psi)$. A set E of equations is true in a hybrid algebra $\mathfrak{A}$ (denoted $\mathfrak{A} \models E$), if each equation in E is true in $\mathfrak{A}$. An equation $\varphi \approx \psi$ is a *semantic consequence* of a set E of equations (denoted $E \models \varphi \approx \psi$), if for any hybrid algebra $\mathfrak{A}$ such that $\mathfrak{A} \models E$, $\mathfrak{A} \models \varphi \approx \psi$.

2.1.2 Operations on hybrid algebras

There are several important methods of constructing new algebras from old ones. Three of the most fundamental are the formation of subalgebras, homomorphic images and products. In the next few sections, we define these constructions for hybrid algebras, and discuss to what extent these operations preserve validity of $\mathcal{H}$-formulas.

Homomorphisms between hybrid algebras

Definition 2.1.9 (Homomorphisms). Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two hybrid algebras. A map $h\colon \mathfrak{A} \to \mathfrak{B}$ is a *homomorphism between* $\mathfrak{A}$ *and* $\mathfrak{B}$, if h is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ and h maps elements of X_A to elements of X_B. We say that h is a *surjective homomorphism*, if it is surjective from A onto B, and furthermore, h is surjective from X_A onto X_B. $\mathfrak{B}$ is a *homomorphic image* of $\mathfrak{A}$ (denoted $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$), if there is a surjective homomorphism h from A to B. We say that $\mathfrak{A}$ is *embeddable* in $\mathfrak{B}$ (denoted $\mathfrak{A} \rightarrowtail \mathfrak{B}$), if there is an injective homomorphism h from A to B. If a homomorphism is both surjective and injective, then it is called an *isomorphism*. Finally, $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorhic*, if there is an isomorphism h between A and B.

We will usually just talk about a homomorphism instead of a homomorphism between hybrid algebras $\mathfrak{A}$ and $\mathfrak{B}$, as it will normally be clear from the context what type of homomorphism we are working with.

Example 2.1.10. Consider the set $C = \{1, 2, 3, 4\}$, and let $\mathbf{B} = (\mathcal{P}(C), \cap, \cup, -\varnothing, C, f)$, where $f(X) = X$. It is not difficult to see that $\mathfrak{B} = (\mathbf{B}, X_B)$, where $X_B = \{\{1\}, \{2\}, \{3\}, \{4\}\}$, is a hybrid algebra. Now, consider the algebra $\mathfrak{A}$ in Example 2.1.4, and define the map $h\colon \mathfrak{A} \to \mathfrak{B}$ by

$$h(a_1a_2a_3a_4) = \{i \in \{1, 2, 3, 4\} \mid a_i = 1\}.$$

We leave it to the reader to check that h is a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{B}$, $h(X_A) \subseteq X_B$, and furthermore, that h is surjective from X_A onto X_B.

The validity of hybrid formulas is preserved under homomorphic images.

Proposition 2.1.11. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two hybrid algebras. If $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$, then $\mathfrak{B} \models \varphi \approx \psi$ whenever $\mathfrak{A} \models \varphi \approx \psi$.*

Proof. Assume $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$, and suppose $\mathfrak{A} \models \varphi \approx \psi$. First, $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$ means there is some surjective homomorphism h from A onto B, and, furthermore, h is surjective from X_A onto X_B. $\mathfrak{A} \models \varphi \approx \psi$ implies that $\varphi^{\mathfrak{A}}(a_1, a_2, \ldots, a_n, x_1, x_2, \ldots, x_m) = \psi^{\mathfrak{A}}(a_1, a_2, \ldots, a_n, x_1, x_2, \ldots, x_m)$ for all $a_1, \ldots, a_n \in A$ and $x_1, \ldots x_m \in X_A$. Now, let $b_1, \ldots, b_n \in B$ and $y_1, \ldots, y_m \in X_B$. We know that h is surjective from A onto B and from X_A onto X_B, so there are $a'_1, \ldots, a'_n \in A$ and $x'_1, \ldots, x'_m \in X_A$ such that

$$\varphi^{\mathfrak{B}}(b_1, b_2, \ldots, b_n, y_1, y_2, \ldots, y_m) = \varphi^{\mathfrak{B}}(h(a'_1), h(a'_2), \ldots, h(a'_n), h(x'_1), h(x'_2), \ldots, h(x'_m))$$

But h is a homomorphism, so

$$\varphi^{\mathfrak{B}}(h(a'_1), h(a'_2), \ldots, h(a'_n), h(x'_1), h(x'_2), \ldots, h(x'_m)) = h(\varphi^{\mathfrak{A}}(a'_1, a'_2, \ldots, a'_n, x'_1, x'_2, \ldots, x'_m)).$$

Hence, since $\varphi^{\mathfrak{A}}(a_1, a_2, \ldots, a_n, x_1, x_2, \ldots, x_m) = \psi^{\mathfrak{A}}(a_1, a_2, \ldots, a_n, x_1, x_2, \ldots, x_m)$ for all elements $a_1, \ldots, a_n \in A$ and atoms $x_1, \ldots x_m \in X_A$,

$$h(\varphi^{\mathfrak{A}}(a'_1, a'_2, \ldots, a'_n, x'_1, x'_2, \ldots, x'_m)) = h(\psi^{\mathfrak{A}}(a'_1, a'_2, \ldots, a'_n, x'_1, x'_2, \ldots, x'_m)),$$

and so, since h is a homomorphism,

$$h(\psi^{\mathfrak{A}}(a_1', a_2', \ldots, a_n', x_1', x_2', \ldots, x_m')) = \psi^{\mathfrak{B}}(h(a_1'), h(a_2'), \ldots, h(a_n'), h(x_1'), h(x_2'), \ldots, h(x_m')).$$

Therefore,

$$\psi^{\mathfrak{B}}(h(a_1'), h(a_2'), \ldots, h(a_n'), h(x_1'), h(x_2'), \ldots, h(x_m')) = \psi^{\mathfrak{B}}(b_1, b_2, \ldots, b_n, y_1, y_2, \ldots, y_m),$$

which means that

$$\varphi^{\mathfrak{B}}(b_1, b_2, \ldots, b_n, y_1, y_2, \ldots, y_m) = \psi^{\mathfrak{B}}(b_1, b_2, \ldots, b_n, y_1, y_2, \ldots, y_m).$$

We have thus shown that $\mathfrak{B} \models \varphi \approx \psi$. □

As expected, the validity of $\mathcal{H}$-formulas is transferred from superalgebras to subalgebras.

Proposition 2.1.12. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two hybrid algebras. If $\mathfrak{A} \rightarrowtail \mathfrak{B}$, then $\mathfrak{A} \models \varphi \approx \psi$ whenever $\mathfrak{B} \models \varphi \approx \psi$.*

Proof. Assume $\mathfrak{A} \rightarrowtail \mathfrak{B}$. Then there is an injective homomorphism h from A to B. Now, suppose $\mathfrak{A} \not\models \varphi \approx \psi$. We then have

$$\varphi^{\mathfrak{A}}(a_1, a_2, \ldots, a_n, x_1, x_2, \ldots, x_m) \neq \psi^{\mathfrak{A}}(a_1, a_2, \ldots, a_n, x_1, x_2, \ldots, x_m)$$

for some $a_1, a_2, \ldots, a_n \in A$ and $x_1, x_2, \ldots, x_m \in X$. But h is injective, so

$$h(\varphi^{\mathfrak{A}}(a_1, a_2, \ldots, a_n, x_1, x_2, \ldots, x_m)) \neq h(\psi^{\mathfrak{A}}(a_1, a_2, \ldots, a_n, x_1, x_2, \ldots, x_m)),$$

and hence, since h is a homomorphism,

$$\varphi^{\mathfrak{A}}(h(a_1), \ldots, h(a_n), h(x_1), \ldots, h(x_m)) \neq \psi^{\mathfrak{A}}(h(a_1), \ldots, h(a_n), h(x_1), \ldots, h(x_m)).$$

Therefore, since h maps elements of X_A to elements of X_B, we have $\mathfrak{B} \not\models \varphi \approx \psi$. □

Products of hybrid algebras

Taking the product of hybrid algebras gives us a means of combining several hybrid algebras into one.

Definition 2.1.13 (Products of hybrid algebras). Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two hybrid algebras. The product $\mathfrak{A} \times \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ is given by $(\mathbf{A} \times \mathbf{B}, X_{A \times B})$, where $\mathbf{A} \times \mathbf{B}$ is defined in the usual way and $X_{A \times B} = \{(x, \perp^{\mathfrak{B}}) \mid x \in X_A\} \cup \{(\perp^{\mathfrak{A}}, y) \mid y \in X_B\}$.

Validity is not generally preserved under products of hybrid algebras. Consider the hybrid algebra $\mathfrak{A} = (\mathbf{2}, \Diamond, \{1\})$ in Example 2.1.3. Then $\mathfrak{A} \models \Diamond\mathbf{i} \approx \top$ but $\mathfrak{A}^2 \not\models \Diamond\mathbf{i} \approx \top$. We can easily fix this by simply adding $\perp$ to the sets of designated atoms.

Definition 2.1.14 (Grounded hybrid algebras). A *grounded hybrid algebra* (GHA) is just like a hybrid algebra $\mathfrak{A} = (\mathbf{A}, X_A)$, except that the bottom element of the algebra is also included in the set X_A.

$\mathcal{H}$-terms and equations are interpreted in grounded hybrid algebras as they are in hybrid algebras with nominals ranging over the elements in the designated set of atoms and $\bot$. Given a hybrid algebra $\mathfrak{B} = (\mathbf{B}, X_B)$, the *associated grounded hybrid algebra* is the structure $\mathfrak{B}_0 = (\mathbf{B}, X_B \cup \{\bot\})$.

Although validity is not preserved under taking products of hybrid algebras, we can show that if each of the associated grounded hybrid algebras of two hybrid algebras validates a $\mathcal{H}$-formula, then the product of the original algebras does too. But first, we need the following definition:

Definition 2.1.15. Let $\mathfrak{A}_1 = (\mathbf{A}_1, X_{A_1})$ and $\mathfrak{A}_2 = (\mathbf{A}_2, X_{A_2})$ be two hybrid algebras. The *projection map* on the ith coordinate of $A_1 \times A_2$ is the map $\pi_i\colon A_1 \times A_2 \to A_i$ defined by $\pi_i(a_1, a_2) = a_i$.

Proposition 2.1.16. *If $\mathfrak{A}$ and $\mathfrak{B}$ are hybrid algebras such that $\mathfrak{A}_0 \models \varphi \approx \psi$ and $\mathfrak{B}_0 \models \varphi \approx \psi$, then $\mathfrak{A} \times \mathfrak{B} \models \varphi \approx \psi$.*

Proof. We prove the contrapositive, so assume $\mathfrak{A} \times \mathfrak{B} \not\models \varphi \approx \psi$. Then there is an assignment $\nu\colon \mathsf{PROP} \cup \mathsf{NOM} \to \mathfrak{A} \times \mathfrak{B}$ such that $\nu(\varphi) \neq \nu(\psi)$. But then $\pi_1(\nu(\varphi)) \neq \pi_1(\nu(\psi))$ or $\pi_2(\nu(\varphi)) \neq \pi_2(\nu(\psi))$. If $\pi_1(\nu(\varphi)) \neq \pi_1(\nu(\psi))$, consider the assignment $\iota\colon \mathsf{PROP} \cup \mathsf{NOM} \to \mathfrak{A}_0$ defined by $\iota(p) = \pi_1(\nu(p))$ and $\iota(\mathbf{j}) = \pi_1(\nu(\mathbf{j}))$. We work with $\mathfrak{A}_0$ rather than $\mathfrak{A}$ since some of the atoms in $X_{A \times B}$ are of the form $(\bot, y), y \in X_B$, which means that $\iota(\mathbf{j})$ might be the bottom element. Using structural induction, we can show that $\iota(\gamma) = \pi_1(\nu(\gamma))$ for all $\mathcal{H}$-formulas γ. Hence,

$$\iota(\varphi) = \pi_1(\nu(\pi)) \neq \pi_1(\nu(\psi)) = \iota(\psi),$$

and so $\mathfrak{A}_0 \not\models \varphi \approx \psi$. The case where $\pi_2(\nu(\varphi)) \neq \pi_2(\nu(\psi))$ is similar. $\square$

We also have the following result for any hybrid algebra and its associated grounded hybrid algebra.

Proposition 2.1.17. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a hybrid algebra. If $\mathfrak{A}_0 \models \varphi \approx \psi$, then $\mathfrak{A} \models \varphi \approx \psi$.*

Proof. To see this, we simply note that any assignment in $\mathfrak{A}$ is also an assignment in $\mathfrak{A}_0$. $\square$

The converse of the claim in Proposition 2.1.17 does not hold. To see this, consider the hybrid algebra $\mathfrak{A} = (\mathbf{2}, \Diamond, \{1\})$ in Example 2.1.3 again. Clearly, $\mathfrak{A} \models \Diamond\mathbf{i} \approx \top$, while $\mathfrak{A}_0 \not\models \Diamond\mathbf{i} \approx \top$ under any assignment mapping $\mathbf{i}$ to 0.

Sometimes we would like to take the product of two algebras $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$, where either X_A or X_B is empty. But since this contradicts our definition of a hybrid algebra, we will refer to these algebras as *degenerate hybrid algebras*.

Definition 2.1.18 (Degenerate hybrid algebras). A *degenerate hybrid algebra* is a pair $\mathfrak{A} = (\mathbf{A}, X_A)$, where $\mathbf{A}$ is a BAO and $X_A = \varnothing$. Given a degenerate hybrid algebra $\mathfrak{A} = (\mathbf{A}, X_A)$, the *associated grounded degenerate hybrid algebra* is the structure $\mathfrak{A}_0 = (\mathbf{A}, \{\bot\})$.

We then have the following useful preservation result:

Proposition 2.1.19. *Let $\mathfrak{A}$ be a degenerate hybrid algebra, and $\mathfrak{B}$ a hybrid algebra. If $\mathfrak{A}_0 \models \varphi \approx \psi$ and $\mathfrak{B} \models \varphi \approx \psi$, then $\mathfrak{A} \times \mathfrak{B} \models \varphi \approx \psi$.*

Proof. Assume $\mathfrak{A} \times \mathfrak{B} \not\models \varphi \approx \psi$. Then there is an assignment $\nu\colon \mathsf{PROP} \cup \mathsf{NOM} \to \mathfrak{A} \times \mathfrak{B}$ such that $\nu(\varphi) \neq \nu(\psi)$. But then $\pi_1(\nu(\varphi)) \neq \pi_1(\nu(\psi))$ or $\pi_2(\nu(\varphi)) \neq \pi_2(\nu(\psi))$. If $\pi_1(\nu(\varphi)) \neq \pi_1(\nu(\psi))$, consider the assignment $\iota\colon \mathsf{PROP} \cup \mathsf{NOM} \to \mathfrak{A}_0$ defined by $\iota(p) = \pi_1(\nu(p))$ and $\iota(\mathbf{j}) = \pi_1(\nu(\mathbf{j}))$. As before, it is not difficult to show using structural induction on γ that $\iota(\gamma) = \pi_1(\nu(\gamma))$. Hence,

$$\iota(\varphi) = \pi_1(\nu(\pi)) \neq \pi_1(\nu(\psi)) = \iota(\psi),$$

and so $\mathfrak{A}_0 \not\models \varphi \approx \psi$. The case where $\pi_2(\nu(\varphi)) \neq \pi_2(\nu(\psi))$ is similar, however, note that since all atoms in $X_{A\times B}$ are of the form $(\bot, b)$, $b \in X_B$, we do not need to work with $\mathfrak{B}_0$. $\square$

2.1.3 Permeated hybrid algebras

As stated before, in [72] Ten Cate showed that $\mathbf{H}^+\Sigma$ is sound and complete with respect to strongly descriptive two-sorted general frames. As we will soon show, algebraically, these correspond to permeated hybrid algebras.

Definition 2.1.20 (Permeated hybrid algebras). A *permeated hybrid algebra* is a hybrid algebra $\mathfrak{A} = (\mathbf{A}, X_A)$ satisfying the following additional conditions:

(i) for each $\bot \neq a \in A$, there is an atom $x \in X_A$ such that $x \leq a$, and

(ii) for all $x \in X_A$ and $a \in A$, if $x \leq \Diamond a$, then there exists a $y \in X_A$ such that $y \leq a$ and $x \leq \Diamond y$.

Note, in particular, that the first condition implies that $\mathbf{A}$ must be atomic. However, the first condition does not necessarily hold when $\mathbf{A}$ is atomic. To see why, note that X_A can be a proper subset of the atoms of $\mathbf{A}$. Finally, we will denote the class of permeated hybrid algebras by PHA.

We now give some examples of permeated hybrid algebras.

Example 2.1.21. It is not difficult to see that the hybrid algebra in Example 2.1.3 is permeated.

Example 2.1.22. The hybrid algebra in Example 2.1.5 is *not* permeated. To see this, note that the first condition of Definition 2.1.20 is violated since $X_A \neq At\mathfrak{A}$.

Example 2.1.23. The hybrid algebra in Example 2.1.7 is permeated. The first condition clearly holds since any non-empty set in $\mathbb{A}$ contains a natural number or $\omega + 1$, and all the singletons of natural numbers and $\{\omega + 1\}$ are designated atoms. For the second condition, let $\{m\} \in X_{\mathbb{A}}$ and $A \in \mathbb{A}$, and assume $\{m\} \subseteq f(A)$. Then $m \in f(A)$, and so $m > \min(A)$. But we know that $\{\min(A)\} \subseteq A$. Furthermore,

$$f(\{\min(A)\}) = \{x \in \omega + 1 \cup \{\omega + 1\} \mid \min(A) < x\}.$$

So $m \in f(\{\min(A)\})$, which means that $\{m\} \subseteq f(\{\min(A)\})$.

Later we will need the fact that the product of two permeated hybrid algebras is also permeated. We now show that this is indeed the case.

Proposition 2.1.24. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two permeated hybrid algebras. Then $\mathfrak{A} \times \mathfrak{B}$ is also permeated.*

Proof. For the first condition, let $(a, b) \in A \times B$ such that $(a, b) \neq (\bot, \bot)$. Then $a \neq \bot$ or $b \neq \bot$. Assume $a \neq \bot$. Now, since $\mathfrak{A}$ is permeated, there is some $x \in X_A$ such that $x \leq a$. So we know that $(x, \bot) \leq (a, b)$. Furthermore, $x \in X_A$, hence $(x, \bot) \in X_{A\times B}$ by definition. Similarly for $b \neq \bot$. For the second condition, we have two cases:

Case 1: $(\bot, y) \in X_{A\times B}$ such that $(\bot, y) \leq \Diamond(a, b)$. Then $(\bot, y) \leq (\Diamond a, \Diamond b)$, and so, $y \leq \Diamond b$. But we know that $\mathfrak{B}$ is permeated, so there is a $y' \in X_B$ such that $y' \leq b$ and $y \leq \Diamond y'$. Hence, $(\bot, y') \leq (a, b)$ and $(\bot, y) \leq (\bot, \Diamond y') = (\Diamond\bot, \Diamond y') = \Diamond(\bot, y')$. Furthermore, since $y' \in X_B$, $(\bot, y') \in X_{A\times B}$.

Case 2: $(x, \bot) \in X_{A\times B}$ such that $(x, \bot) \leq \Diamond(a, b)$. Similar to Case 1. $\square$

2.1.4 Duality between two-sorted general frames and hybrid algebras

Recall that the fundamental operations on frames, namely, taking generated subframes, bounded morphic images and disjoint unions, correspond very naturally to those on algebras, namely, taking homomorphic images, subalgebras and products. In this section, we wish to show that this is also the case for two-sorted general frames and hybrid algebras.

Theorems 2.1.35 and 2.1.36 at the end of this section give a concise formulation of the basic connections between hybrid algebras and two-sorted general frames. The proof of Theorem 2.1.35 follows immediately from Propositions 2.1.30 and 2.1.34 below. First, however, we adapt familiar notions from the setting of general frames to the setting of two-sorted general frames.

Definition 2.1.25. For each $i \in I$, let $\mathfrak{g}_i = (W_i, R_i, A_i, B_i)$ be a two-sorted general frame. Then their *disjoint union* is the structure

$$\biguplus_{i\in I} \mathfrak{g}_i = (W, R, A, B),$$

where W is the union of the domains W_i, R is the union of the relations R_i, A consists of those subsets $a \subseteq \bigcup_{i\in I} W_i$ such that $a \cap W_i \in A_i$ for all $i \in I$, and B is the union of the sets B_i.

Definition 2.1.26. Let $\mathfrak{g} = (W, R, A, B)$ and $\mathfrak{h} = (W', R', A', B')$ be two-sorted general frames. We say that a map $g\colon W \to W'$ is a *bounded morphism between* $\mathfrak{g}$ *and* $\mathfrak{h}$, if g is a bounded morphism between the general frames (W, R, A) and (W', R', A') such that $g^{-1}[\{w'\}] \in \{\{v\} \mid v \in B\}$ for all $\{w'\} \in \{\{v'\} \mid v' \in B'\}$. A map g is an *embedding*, if it is an embedding between the general frames (W, R, A) and (W', R', A'), and for all $w \in B$, there is a $w' \in B'$ such that $g[\{w\}] = g[W] \cap \{w'\}$. We say that $\mathfrak{h}$ is *embeddable* in $\mathfrak{g}$ (denoted $\mathfrak{h} \rightarrowtail \mathfrak{g}$), if there is an embedding from W' to W. The two-sorted general frame $\mathfrak{h}$ is called a *bounded morphic image* of $\mathfrak{g}$ (denoted $\mathfrak{g} \twoheadrightarrow \mathfrak{h}$), if there is a surjective bounded morphism g from W to W'. Finally, $\mathfrak{h}$ and $\mathfrak{g}$ are *isomorphic* (denoted $\mathfrak{h} \cong \mathfrak{g}$), if there is an bijective bounded morphism between W and W'.

Although validity on Kripke frames is not preserved under bounded morphic images, we do have some good news: validity is preserved under bounded morphic images between two-sorted general frames. To prove this, we need to show (as usual) that model satisfaction is invariant under bounded morphisms. A map g is a bounded morphism between models $\mathfrak{M} = (\mathfrak{g}, V)$ and $\mathfrak{M}' = (\mathfrak{h}, V')$, if g is a bounded morphism between the two-sorted general frames $\mathfrak{g}$ and $\mathfrak{h}$, and w and $g(w)$ satisfy the same propositional variables and nominals.

Proposition 2.1.27. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two models such that g is a bounded morphism between $\mathfrak{M}$ and $\mathfrak{M}'$. Then for each formula ψ and each state w in $\mathfrak{M}$, $\mathfrak{M}, w \Vdash \psi$ iff $\mathfrak{M}', g(w) \Vdash \psi$.*

Proof. The proof is by structural induction on ψ. □

Proposition 2.1.28. *Let $\mathfrak{g} = (W, R, A, B)$ and $\mathfrak{h} = (W', R', A', B')$ be two-sorted general frames such that $\mathfrak{g} \twoheadrightarrow \mathfrak{h}$, and let ψ be a hybrid formula. Then $\mathfrak{h} \Vdash \psi$ if $\mathfrak{g} \Vdash \psi$.*

Proof. Let $g\colon W \to W'$ be a surjective bounded morphism between $\mathfrak{g}$ and $\mathfrak{h}$. We will prove the contrapositive, so assume $\mathfrak{h} \nVdash \psi$. Then there is some state $w' \in W'$ and an admissible valuation V' on $\mathfrak{h}$ such that $(\mathfrak{h}, V'), w' \nVdash \psi$. Now, define the valuation V on $\mathfrak{g}$ such that for each $p \in \mathsf{PROP}$ and each $\mathbf{i} \in \mathsf{NOM}$, $V(p) = \{w \in W \mid g(w) \in V'(p)\}$ and $V(\mathbf{i}) = \{w \in W \mid g(w) \in V'(\mathbf{i})\}$. We have to make sure that V is admissible. To prove this, we first show that $g^{-1}[V'(\mathbf{i})] = V(\mathbf{i})$: $v \in g^{-1}[V'(\mathbf{i})]$ iff $g(v) \in V'(\mathbf{i})$ iff $v \in V(\mathbf{i})$. Now, since $V'(\mathbf{i}) \in \{\{v'\} \mid v' \in B'\}$, by Definition 2.1.26, $g^{-1}[V'(\mathbf{i})] \in \{\{v\} \mid v \in B\}$. This means V evaluates nominals to elements in B. Similarly, $g^{-1}[V'(p)] = V(p)$, and so, since $V'(p) \in A'$, $V(p) \in A$. We also want to show that g is a bounded morphism between the models $(\mathfrak{g}, V)$ and $(\mathfrak{h}, V')$. But g is a surjective bounded morphism between the frames $\mathfrak{g}$ and $\mathfrak{h}$, so we just have to check that for each v, v and $g(v)$ satisfy the same variables. So let $v \in W$ and $\mathbf{i} \in \mathsf{NOM}$. Then

$$\begin{aligned}(\mathfrak{g}, V), v \Vdash \mathbf{i} &\text{ iff } v \in V(\mathbf{i})\\ &\text{ iff } g(v) \in V'(\mathbf{i})\\ &\text{ iff } (\mathfrak{h}, V'), g(v) \Vdash \mathbf{i}.\end{aligned}$$

Similarly for the propositional variables. Now, we know that g is surjective, there is a $w \in W$ such that $g(w) = w'$, and hence, since g is a bounded morphism between the models $(\mathfrak{g}, V)$ and $(\mathfrak{h}, V')$, by Proposition 2.1.27, $(\mathfrak{g}, V), w \nVdash \psi$. □

The dual of a map between two-sorted general frames is defined in the same way as that of a map between general frames.

Definition 2.1.29. Let $\mathfrak{g} = (W, R, A, B)$ and $\mathfrak{h} = (W', R', A', B')$ be two two-sorted general frames. Given a map $g\colon W \to W'$, its *dual* $g^*\colon A' \to \mathcal{P}(W)$ is defined by

$$g^*(a') := g^{-1}[a'] \; (= \{w \in W \mid g(w) \in a'\}).$$

As for general frames and Boolean algebras with operators, the duals of bounded morphisms between two-sorted general frames are simply homomorphisms between hybrid algebras.

Proposition 2.1.30. *Let* $\mathfrak{g} = (W, R, A, B)$ *and* $\mathfrak{h} = (W', R', A', B')$ *be two-sorted general frames,* $\mathfrak{g}^* = (A, \cap, \cup, -, \varnothing, W, \langle R\rangle, X_B)$ *and* $\mathfrak{h}^* = (A', \cap, \cup, -, \varnothing, W', \langle R'\rangle, X_{B'})$ *their underlying hybrid algebras, and* g *a map from* W *to* W'*.*

(i) *If* g *is a bounded morphism,* g^* *maps elements of* A' *to elements of* A*.*

(ii) *If* g *is a bounded morphism, then* g^* *is a homomorphism from* $\mathfrak{h}^*$ *to* $\mathfrak{g}^*$*.*

(iii) *If* g *is an embedding, then* g^* *is a surjective homomorphism.*

(iv) *If* g *is surjective, then* g^* *is injective.*

Proof. The proofs of items (i) and (iv) are the same as that of items (i) and (iv) in Proposition 1.1.51. So we only have to check items (ii) and (iii).

(ii) The fact that g^* respects the operations is proved in the same way as in the proof of Proposition 1.1.51, so we will only show that g^* maps elements of $X_{B'}$ to elements of X_B. Let $\{w'\} \in X_{B'}$. Then $g^{-1}[\{w'\}] \in X_B$ by Definition 2.1.26. But $g^{-1}[\{w'\}] = g^*(\{w'\})$, so $g^*(\{w'\}) \in X_B$.

(iii) Assume g is an embedding. The fact that g^* is a homomorphism follows from (ii). The proof that g^* is surjective is the same as in the proof of Proposition 1.1.51. We only show that g^* is surjective from $X_{B'}$ onto X_B, so let $\{w\} \in X_B$. Then $w \in B$, and so, by Definition 2.1.26, there is $w' \in B'$ such that $g[\{w\}] = g[W] \cap \{w'\}$. We claim that this $\{w'\}$ has the desired properties. First, we have $g^*(g[\{w\}]) = g^*(g[W] \cap \{w'\})$, and so, since g^* is a Boolean homomorphism, $g^*(g[\{w\}]) = g^*(g[W]) \cap g^*(\{w'\})$. Note that since $g^*(\{w'\}) \subseteq W$, if we can show that $g^*(g[\{w\}]) = \{w\}$ and $g^*(g[\{w\}]) = W$, we would be done. So we will now show that $g^*(g[b]) = b$ for all subsets b of W. Let $u \in b$. Then $g(u) \in g[b]$, and so $u \in g^*(g[b])$. Hence, $b \subseteq g^*(g[b])$. For the converse inclusion, assume $u \in g^*(g[b])$. Then $g(u) \in g[b]$, which means that there is some $v \in b$ such that $g(u) = g(v)$. But since g is injective, $u = v$. Hence, $u \in b$, and therefore, $g^*(g[b]) \subseteq b$. □

Before we go in the opposite direction, that is, from hybrid algebras to two-sorted general frames, we have to extend the notion of a general ultrafilter frame of a BAO to a two-sorted general ultrafilter frame of a hybrid algebra. But first we show that if a is an atom, then $\widehat{a}$ contains only the principal ultrafilter generated by a.

Lemma 2.1.31. *For any Boolean algebra* $\mathbf{A}$*, if* a *is an atom of* $\mathbf{A}$*, then* $\widehat{a} = \{a\uparrow\}$*.*

Proof. First, we know from Example 1.1.38 that $\widehat{a}$ is an ultrafilter, and furthermore, $a \in a\uparrow$, so $\uparrow a \in \widehat{a}$. To show that $\widehat{a} = \{a\uparrow\}$, assume $\widehat{a}$ contains at least two ultrafilters, say w and w', such that $w \neq w'$. This means that there is an element $b \neq \perp$ that belongs to one, say w, but not w'. Hence, $a \in w'$, but $b \notin w'$, and so, by the definition of an ultrafilter, $a \not\leq b$. This means that $a \leq \neg b$. But $a \in w$, so $\neg b \in w$, which is a contradiction. □

We are now ready to extend the definition of general ultrafilter frames of a BAO to two-sorted general ultrafilter frames of hybrid algebras.

Definition 2.1.32. Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a hybrid algebra. Then the *two-sorted general ultrafilter frame* is defined as

$$\mathfrak{A}_* = (\mathit{Uf}\mathfrak{A}, Q_\Diamond, \widehat{A}, X_A\uparrow),$$

where $\widehat{A} := \{\widehat{a} \mid a \in A\}$ and $X_A \uparrow = \{x\uparrow \mid x \in X_A\}$.

We will also make use of the notation $\widehat{X_A}$ to denote the set $\{\widehat{x} \mid x \in X_A\}$.

We are almost ready to go in the opposite direction, we just need one more definition:

Definition 2.1.33. Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two hybrid algebras, and let h be a map from A to B. Then its *dual* h_* is the map from $\mathit{Uf}\mathfrak{B}$ to $\mathcal{P}(A)$ defined by

$$h_*(u') := h^{-1}[u'] \; (= \{a \in A \mid h(a) \in u'\}).$$

As expected the duals of homomorphisms between hybrid algebras are bounded morphisms between two-sorted general frames.

Proposition 2.1.34. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two hybrid algebras, $\mathfrak{A}_* = (\mathit{Uf}\mathfrak{A}, Q_\Diamond, \widehat{A}, X_A\uparrow)$ and $\mathfrak{B}_* = (\mathit{Uf}\mathfrak{B}, Q'_\Diamond, \widehat{B}, X_B\uparrow)$ their general ultrafilter frames, and h a map from A to B.*

(i) *If h is a homomorphism, then h_* maps ultrafilters to ultrafilters.*

(ii) *If h is a homomorphism, then h_* is a bounded morphism from $\mathfrak{B}_*$ to $\mathfrak{A}_*$.*

(iii) *If h is a surjective homomorphism, then h_* is an embedding.*

(iv) *If h is an embedding, then h_* is a surjective.*

Proof. The proofs of items (i) and (iv) are exactly the same as that of items (i) and (iv) in Proposition 1.1.52. The proofs of items (ii) and (iii) need a bit more work.

(ii) The proofs of the forth and back properties are the same as in the proof of Proposition 1.1.52. We can also show that $h_*^{-1}[\widehat{a}]$ is an admissible set of $\mathfrak{B}_*$ in the same way as in the proof of Proposition 1.1.52. So all we have to check is that $h_*^{-1}[\widehat{x}] \in \widehat{X_B}$ for all $\widehat{x} \in \widehat{X_A}$. Let $\widehat{x} \in \widehat{X_A}$. Now, h maps elements of X_A to elements of X_B, so if we can show that $h_*^{-1}[\widehat{x}] = \widehat{h(x)}$, we are done. But this is indeed the case: $u' \in h_*^{-1}[\widehat{x}]$ iff $h_*(u') \in \widehat{x}$ iff $x \in h_*(u')$ iff $h(x) \in u'$ iff $u' \in \widehat{h(x)}$.

(iii) Assume that h is a surjective homomorphism. For the proof that h_* is injective, see the proof of Proposition 1.1.52. In a similar way as in the proof of Proposition 1.1.52, we can show that for all $\widehat{b} \in \widehat{B}$, there is a $\widehat{a} \in \widehat{A}$ such that $h_*[\widehat{b}] = h_*[\mathit{Uf}\mathfrak{B}] \cap \widehat{a}$. Finally, let $\widehat{y} \in \widehat{X_B}$. Then $y \in X_B$. But h is surjective from X_A onto X_B, so there is some $x \in X_A$ such that $h(x) = y$. Since $x \in X_A$, $\widehat{x} \in \widehat{X_A}$. We also claim that $h_*[\widehat{y}] = h_*[\mathit{Uf}\mathfrak{B}] \cap \widehat{x}$. For the left-to-right inclusion, let $v \in h_*[\widehat{y}]$. Then $h_*(y\uparrow) = v$, so $v \in h_*[\mathit{Uf}\mathfrak{B}]$. Now, since $h(x) = y$ and $y \in y\uparrow$, $x \in h_*(y\uparrow)$, and therefore, $x \in v$, which means that $v \in \widehat{x}$. Thus, $v \in h_*[\mathit{Uf}\mathfrak{B}] \cap \widehat{x}$. Conversely, let $v \in h_*[\mathit{Uf}\mathfrak{B}] \cap \widehat{x}$. Then there is some $u' \in \mathit{Uf}\mathfrak{B}$ such that $v = h_*(u')$, and furthermore $v = x\uparrow$. Hence, $x \in h_*(u')$, and so $h(x) \in u'$. But then $y \in u'$, which gives $u' \in \widehat{y} = \{y\uparrow\}$. This implies that $u' = y\uparrow$, and therefore, $v \in h_*[\widehat{y}]$. $\square$

Theorem 2.1.35. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two two-sorted general frames, and $\mathfrak{A}$ and $\mathfrak{B}$ two hybrid algebras.*

(i) *If $\mathfrak{g} \rightarrowtail \mathfrak{h}$, then $\mathfrak{h}^* \twoheadrightarrow \mathfrak{g}^*$.*

(ii) *If $\mathfrak{g} \twoheadrightarrow \mathfrak{h}$, then $\mathfrak{h}^* \rightarrowtail \mathfrak{g}^*$.*

(iii) *If $\mathfrak{A} \rightarrowtail \mathfrak{B}$, then $\mathfrak{B}_* \twoheadrightarrow \mathfrak{A}_*$.*

(iv) *If $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$, then $\mathfrak{B}_* \rightarrowtail \mathfrak{A}_*$.*

The next theorem states a connection between disjoint unions of two-sorted general frames and products of hybrid algebras.

Theorem 2.1.36. *For each $i \in I$, let $\mathfrak{g}_i = (W_i, R_i, A_i, B_i)$ be a two-sorted general frame. Then*

$$\left(\biguplus_{i\in I} \mathfrak{g}_i\right)^* \cong \prod_{i\in I} \mathfrak{g}_i^*.$$

Proof. Let $(W, R, A, B) = \biguplus_{i\in I} \mathfrak{g}_i$, and consider the map h from A to $\prod_{i\in I} A_i$ defined by

$$h(a)(i) = a \cap W_i.$$

We have to show that h is an isomorphism. We first show that h is surjective from A onto $\prod_{i\in I} A_i$, so let $\overline{a} \in \prod_{i\in I} A_i$, where $\overline{a}(i) = a_i$ for each $i \in I$. We have to find an $a \in A$ such that $h(a)(j) = a_j$ for each $j \in I$. We claim that $\bigcup_{i\in I} a_i$ satisfies the required conditions. First, to see that $\bigcup_{i\in I} a_i \in A$, let j be any index in I. Then

$$\bigcup_{i\in I} a_i \cap W_j = \bigcup_{i\in I} (a_i \cap W_j) = a_j.$$

But $a_j \in A_j$, so, by the definition of the disjoint union of two-sorted general frames, $\bigcup_{i\in I} a_i \in A$. Furthermore, for each $j \in I$,

$$h\left(\bigcup_{i\in I} a_i\right)(j) = \bigcup_{i\in I} a_i \cap W_j = a_j.$$

To show that h is surjective from X_B onto $X_{\prod_{i\in I} B_i}$, let $\overline{x}$ be an element of $X_{\prod_{i\in I} B_i}$, where $\overline{x}(i) = x_i$ for each $i \in I$. Then $x_j \in X_{B_j}$ for some $j \in I$ and $x_i = \varnothing$ for $i \neq j$. We have to find an $x \in X_B$ such that $h(x)(j) = x_j$ and $h(x)(i) = \varnothing$ for $i \neq j$. We claim that x_j satisfies these conditions. First, since $x_j \in X_{B_j}$ and B is the union of the B_i's, $x_j \in X_B$. Furthermore, $h(x_j)(j) = x_j \cap W_j = x_j$ and $h(x_j)(i) = x_j \cap W_i = \varnothing$ for $i \neq j$.

We also have to show that h maps designated atoms to designated atoms, so let $x \in X_B$. Then $x \in X_{B_j}$ for some $j \in I$, while $x \notin X_{B_i}$ for $i \neq j$. Hence, $h(x)(j) = x \cap W_j = x$, while $h(x)(i) = x \cap W_i = \varnothing$ for $i \neq j$. This means that $h(x) \in X_{\prod_{i\in I} B_i}$.

Now, to show that h is injective, let $a, b \in A$ such that $a \neq b$. We may then assume without loss of generality that there is some $w \in W$ such that $w \in a$ but $w \notin b$. But $w \in W$

implies that $w \in W_j$ for some $j \in I$. Now, since $w \in a$, $w \in a \cap W_j = h(a)(j)$. On the other hand, since $w \notin b$, $w \notin b \cap W_j = h(b)(j)$. Hence, $h(a)(j) \neq h(b)(j)$, which means that $h(a) \neq h(b)$.

The fact that h is a Boolean homomorphism follows from basic set theory. Finally, we show that h respects $\langle R \rangle$. Assume $w \in h(\langle R \rangle a)(i)$. Then $w \in \langle R \rangle a \cap W_i$, and so $w \in \langle R \rangle a$ and $w \in W_i$. But $w \in \langle R \rangle a$ implies that there is some $v \in a$ such that wRv, and hence, since $w \in W_i$, $v \in W_i$ and $wR_i v$. Therefore, $v \in a \cap W_i$, i.e., $v \in h(a)(i)$. We can thus conclude that $w \in \langle R_i \rangle h(a)(i)$. Conversely, let $w \in \langle R_i \rangle h(a)(i)$. Then there is some $v \in h(a)(i)$ such that $wR_i v$. But then $v \in a \cap W_i$ and wRv, so $w \in \langle R \rangle$. Now, since $w \in W_i$, $w \in \langle R \rangle a \cap W_i$, which means $w \in h(\langle R \rangle a)(i)$. $\square$

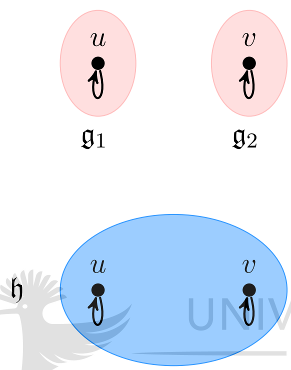

Figure 2.1: Two-sorted general frames and their disjoint union

It is not difficult to see that the validity of hybrid formulas is also not preserved under taking disjoint unions of two-sorted general frames. Again, consider the formula $\mathbf{i}$ which defines the class of frames that contain exactly one element. Let $\mathfrak{g}_1 = (\{u\}, \{(u,u)\}, \{\varnothing, \{u\}\}, \{u\})$, $\mathfrak{g}_2 = (\{v\}, \{(v,v)\}, \{\varnothing, \{v\}\}, \{v\})$, and $\mathfrak{h} = (\{u,v\}, \{(u,u),(v,v)\}, \{\varnothing, \{u\}, \{v\}, \{u,v\}\}, \{u,v\})$ (see Figure 2.1). Then

$$\mathfrak{h} = \mathfrak{g}_1 \uplus \mathfrak{g}_2.$$

However, both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ contain exactly one element, while $\mathfrak{h}$ contains two elements. This corresponds to the fact that the validity of hybrid formulas is not generally preserved under taking products of hybrid algebras. Now, recall that even though the validity of hybrid formulas is not preserved under taking products of hybrid algebras, it is the case that if each of the associated grounded hybrid algebras of these hybrid algebras validates an $\mathcal{H}$-formula, then the product of the original algebras does too. We can prove a similar result for two-sorted general frames. But first we make the following definition:

Definition 2.1.37. A *liberal valuation* for a (Kripke) frame $\mathfrak{F} = (W, R)$ is a map $V\colon \mathsf{PROP} \cup \mathsf{NOM} \to \mathcal{P}(W)$ that assigns to each propositional variable a subset of W and to each nominal a singleton or the empty set. A *liberal model based on a frame* $\mathfrak{F}$ is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where V a liberal valuation.

A formula φ is *groundedly valid at a state* w *in a frame* $\mathfrak{F}$ (denoted $\mathfrak{F}, w \Vdash_0 \varphi$), if it is true at w in every liberal model $(\mathfrak{F}, V)$ based on $\mathfrak{F}$. We say that φ is *groundedly valid in a* *frame* $\mathfrak{F}$ (denoted $\mathfrak{F} \Vdash_0 \varphi$) if φ is groundedly valid at all states in every liberal model $(\mathfrak{F}, V)$ based on $\mathfrak{F}$. A formula φ is *groundedly valid on a class of frames* K (denoted $\mathsf{K} \Vdash_0 \varphi$) if it is groundedly valid on all frames in K. Finally, φ is *groundedly valid* (denoted $\Vdash_0 \varphi$) if φ is groundedly valid on the class of all frames.

Given a two-sorted general frame $\mathfrak{g} = (W, R, A, B)$, we define a liberal valuation for $\mathfrak{g}$ in the same way as for a Kripke frame. A liberal valuation is called *admissible* for $\mathfrak{g}$, if for each propositional variable p, $V(p) \in A$, and, for each nominal $\mathbf{i}$, $V(\mathbf{i}) \in \{\{w\} \mid w \in B\} \cup \{\varnothing\}$. A *liberal model based on a two sorted general frame* is a pair $(\mathfrak{g}, V)$, where V is a liberal admissible valuation for $\mathfrak{g}$. Truth and validity of formulas are defined as before.

The following proposition shows that model satisfaction on liberal models is invariant under disjoint unions.

Proposition 2.1.38. *For each $i \in I$, let $\mathfrak{M}_i$ be a liberal model. If each nominal is evaluated to $\varnothing$ in all but one $\mathfrak{M}_i$, then, for each hybrid formula ψ, each $i \in I$, and each state w in $\mathfrak{M}_i$, $\mathfrak{M}_i, w \Vdash \psi$ iff $\biguplus_{j \in I} \mathfrak{M}_j, w \Vdash \psi$.*

Proof. The proof is by structural induction on ψ. $\square$

Now, as for the product of hybrid algebras, we can show that if each two-sorted general frame of a family of two-sorted general frames groundedly validates an $\mathcal{H}$-formula, then the disjoint union of these frames validates this formula.

Theorem 2.1.39. *For each $i \in I$, let $\mathfrak{g}_i = (W_i, R_i, A_i, B_i)$ be a two-sorted general frame. Then $\biguplus_{i \in I} \mathfrak{g}_i \Vdash \varphi$ whenever $\mathfrak{g}_i \Vdash_0 \varphi$ for each $i \in I$.*

Proof. We prove the contrapositive, so assume $\biguplus \mathfrak{g}_i \nVdash \varphi$. Then there is an admissible valuation V and a state $w \in \biguplus_{i \in I} W_i$ such that $(\biguplus \mathfrak{g}_i, V), w \nVdash \varphi$. Now, for each $i \in I$, each $p \in \mathsf{PROP}$, and each $\mathbf{j} \in \mathsf{NOM}$, define $V_i(p) = V(p) \cap W_i$ and $V_i(\mathbf{j}) = V(\mathbf{j}) \cap W_i$. Then $V_j(\mathbf{j}) = V(\mathbf{j})$ for some $j \in I$, while $V_i(\mathbf{j}) = \varnothing$ for $i \neq j$. We have to make sure $V_i(p)$ an $V_i(\mathbf{j})$ are admissible. But we know that $V(p)$ and $V(\mathbf{i})$ are admissible, so, by the definition of a disjoint union of two-sorted general frames, $V(p) \cap W_i \in A_i$ and $V(\mathbf{j}) \cap W_i \in A_i$ for all $i \in I$. Hence, for all $i \in I$, $V_i(p) \in A_i$ and $V_i(\mathbf{j}) \in A_i$. To use Proposition 2.1.38, have to show that $\biguplus_{i \in I} V_i(p) = V(p)$ and $\biguplus_{i \in I} V_i(\mathbf{j}) = V(\mathbf{j})$. First,

$$\begin{aligned}
\biguplus_{i \in I} V_i(\mathbf{j}) &= \biguplus_{i \in I} (V(\mathbf{j}) \cap W_i) \\
&= V(\mathbf{j}) \cap \biguplus_{i \in I} W_i \\
&= V(\mathbf{j}) \cap W \\
&= V(\mathbf{j}).
\end{aligned}$$

Similarly, $\biguplus_{i \in I} V_i(p) = V(p)$. Now, since $w \in \biguplus W_i$, $w \in W_k$ for some $k \in I$, and so, by Proposition 2.1.38, $(\mathfrak{g}_k, V_k), w \nVdash \psi$. But since V_k is liberal, $\mathfrak{g}_k \nVdash_0 \psi$. $\square$

Recall that Boolean algebras with operators and descriptive general frames are duals. The theorem that follows asserts that this is also the case for hybrid algebras and descriptive two-sorted general frames.

Theorem 2.1.40. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a hybrid algebra, and $\mathfrak{g} = (W, R, A, B)$ a two-sorted general frame. Then*

(i) $\mathfrak{A}_*$ *is a descriptive two-sorted general frame,*

(ii) $(\mathfrak{A}_*)^* \cong \mathfrak{A}$, *and*

(iii) $(\mathfrak{g}^*)_* \cong \mathfrak{g}$ *iff* $\mathfrak{g}$ *is descriptive.*

Proof. (i) The proof of this item is the same as that of item (i) in Theorem 1.1.44, all we have to check is that $\mathfrak{A}_*$ is two-sorted. Well, by definition $X_A \neq \varnothing$, so $X_A \uparrow \neq \varnothing$. Now, let $x \uparrow \in X_A \uparrow$. Then $x \in X_A$. But $X_A \subseteq A$, so $x \in A$. Hence, $\widehat{x} \in \widehat{A}$.

(ii) Here we just have to make sure that the map map $h\colon A \to \widehat{A}$ defined by $h(a) = \widehat{a}$ maps elements of X_A to elements of $\widehat{X_A}$ and that it is surjective from X_A onto $\widehat{X_A}$. But this is clearly true. (iii) The left-to-right direction follows from (i). The proof of the other direction is the same as in Theorem 1.1.44. We just have to make sure that $g^{-1}[\widehat{x}] \in \{\{w\} \mid w \in B\}$ for all $\widehat{x} \in \widehat{X_B}$, and that for all $s \in B$, there is some $y \uparrow$ such that $g[\{s\}] = g[W] \cap \{y \uparrow\}$. For the first condition, let $\widehat{x} \in \widehat{X_B}$. Then $x \in X_B$. We now claim that $\widehat{x} = g[x]$, and so, by the injectivity of g, $g^{-1}[\widehat{x}] = x$, which means that $g^{-1}[\widehat{x}] \in \{\{w\} \mid w \in B\}$ since $x = \{s\}$ for some $s \in B$. To see this, let $u \in \widehat{x}$ Then $x \in u$. But we know that $u = U_t$ for some $t \in W$, so $x \in U_t$. Hence, $t \in x$, so $g(t) \in g[x]$. Therefore, $U_t \in g[x]$, which gives $u \in g[x]$. Conversely, let $u \in g[x]$. Then there is some $t \in W$ such that $u = g(t)$ and $t \in x$. Hence, $u = t$, and so, since $t \in x$, $x \in u$, which means $u \in \widehat{x}$.

For the second condition, let $x = \{s\}$. We claim that $\widehat{x}$ satisfies this condition. For the left-to-right inclusion, let $u \in g[x]$. Then $u = g(s)$, which means $x \in U_s = g(s) = u$, so $u \in g[W]$ and $u \in \widehat{x}$. Hence, $u \in g[W] \cap \widehat{x}$. Conversely, let $u \in g[W] \cap \widehat{x}$. We then have $u \in g[W]$ and $u \in \widehat{x}$. But $u \in g[W]$ implies there is some $t \in W$ such that $g(t) = u$, and so, since $x \in u$, $x \in g(t) = U_t$, which gives $t \in x$. Therefore, $u \in g[x]$. But we know that $\widehat{x} = \{x \uparrow\}$, so we are done. □

Recall that the BAO of the hybrid algebra in Example 2.1.7 is the underlying algebra of the general frame in Example 3.3 of [28]. This general frame is strongly descriptive, and in Example 2.1.23 we showed that the hybrid algebra in Example 2.1.7 is permeated. In general, permeated hybrid algebras correspond to strongly descriptive two-sorted general frames.

Theorem 2.1.41. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a permeated hybrid algebra, and $\mathfrak{g} = (W, R, A, B)$ a two-sorted general frame. Then*

(i) $\mathfrak{A}_*$ *is a strongly descriptive two-sorted general frame,*

(ii) $(\mathfrak{A}_*)^* \cong \mathfrak{A}$, *and*

(iii) $(\mathfrak{g}^*)_* \cong \mathfrak{g}$ *iff* $\mathfrak{g}$ *is strongly descriptive.*

Proof. The proof of this theorem is the same as that of Theorem 2.1.40, all we have to check is that $\mathfrak{A}_*$ is strongly descriptive. Now, to prove the first condition of strongly descriptiveness, let $\widehat{a}$ be a non-empty admissible set of $\mathfrak{A}_*$. Since $\widehat{a}$ is non-empty, there is an ultrafilter u such that $a \in u$. But $\bot \notin u$, so $a \neq \bot$. Hence, since $\mathfrak{A}$ is permeated, there is a designated atom $x \in X_A$ such that $x \leq a$. By Lemma 2.1.31, $\widehat{x} = \{x\uparrow\}$. We just now have to check that $x\uparrow \in \widehat{a}$. We know that $x \in x\uparrow$. But $x \leq a$, and so, since $x\uparrow$ is upward closed, $a \in x\uparrow$. Hence, $x\uparrow \in \widehat{a}$, as required.

For the second condition, let $\widehat{a}$ be an admissible set of $\mathfrak{A}_*$, and let $x\uparrow \in X_A\uparrow$. Assume $\{u \in \widehat{a} \mid x\uparrow Q_\Diamond u\} \neq \varnothing$. Now, we know that $\widehat{x} = \{x\uparrow\}$. Since $\{u \in \widehat{a} \mid x\uparrow Q_\Diamond u\} \neq \varnothing$, there is an ultrafilter $w \in \{u \in \widehat{a} \mid x\uparrow Q_\Diamond u\}$. Hence, $w \in \widehat{a}$ and $x\uparrow Q_\Diamond w$. From $w \in \widehat{a}$ it follows that $a \in w$. This means that $\Diamond a \in x\uparrow$. We now claim that $x \leq \Diamond a$. For the sake of a contradiction, assume this is not the case. Since x is an atom, $x \leq \neg\Diamond a$. But $x \in x\uparrow$ and $x\uparrow$ is upward closed, so $\neg\Diamond a \in v$, which is a contradiction. Now, since $\mathfrak{A}$ is permeated, there is an atom $y \in X_A$ such that $y \leq a$ and $x \leq \Diamond y$. By Lemma 2.1.31, $\widehat{y} = \{y\uparrow\}$. But $y \in y\uparrow$ and $y \leq a$, so, since $y\uparrow$ is upward closed, $a \in y\uparrow$. Hence, $y\uparrow \in \widehat{a}$. To show that $x\uparrow Q_\Diamond y\uparrow$, let $b \in y\uparrow$. Then $y \leq b$. We thus now have that $x \leq \Diamond y \leq \Diamond b$, and so, since $x \in x\uparrow$ and $x\uparrow$ is upward closed, $\Diamond b \in x\uparrow$. Hence, by definition, $x\uparrow Q_\Diamond y\uparrow$. □

To conclude this subsection, we will give an algebraic construction that we will often use in the later chapters, and then briefly discuss what it is that we are doing frame-theoretically to illustrate how we use the duality between hybrid algebras and two-sorted general frames. This construction makes crucial use of the adjoint of $\Box$, so let us first recall some relevant preliminaries on adjoints and residuals. These will also play a crucial role in Chapter 4. In what follows, $\mathbf{A}$ and $\mathbf{B}$ are two complete BAOs. For the proofs of the propositions that follow, see [36].

Definition 2.1.42 (Adjoint pair). The monotone maps $f\colon \mathbf{A} \to \mathbf{B}$ and $g\colon \mathbf{B} \to \mathbf{A}$ form an *adjoint pair* (denoted $f \dashv g$), if for all a in $\mathbf{A}$ and b in $\mathbf{B}$,

$$f(a) \leq b \text{ iff } a \leq g(b).$$

If $f \dashv g$, f is called the *left adjoint* of g, while g is called the *right adjoint* of f.

An important property of adjoint pairs is that if a map is completely join-preserving (meet-preserving), then we can compute its right (left) adjoint pointwise from the map itself and the order relation on the BAO.

Proposition 2.1.43. *For monotone maps $f\colon A \to B$ and $g\colon B \to A$ such that $f \dashv g$,*

(i) $f(a) = \bigwedge\{b \in B \mid a \leq g(b)\}$, *and*

(ii) $g(b) = \bigvee\{a \in A \mid f(a) \leq b\}$.

Proposition 2.1.44. *For any map $f\colon A \to B$,*

(i) *f is completely join-preserving iff it has a right adjoint, and*

(ii) *f is completely meet preserving iff it has a left adjoint.*

Definition 2.1.45 (**Residual pair**)**.** Let $f\colon A^n \to A$ and $g\colon A^n \to A$ be n-ary maps. We say that f and g form a *residual pair* in the ith coordinate (denoted $f \dashv_i g$), if for all $a_1, \ldots, a_n, b \in A$,

$$f(a_1, \ldots, a_i, \ldots, a_n) \leq b \text{ iff } a_i \leq g(a_1, \ldots, b, \ldots, a_n).$$

If $f \dashv_i g$, then f is called the *left residual* of g in the ith coordinate, and g the *right residual* of f in the ith coordinate.

Proposition 2.1.46. *For $f\colon A^n \to A$ and $g\colon A^n \to A$ such that $f \dashv_i g$,*

(i) $f(a_1, \ldots, a_i, \ldots, a_n) = \bigwedge\{b \in A \mid a_i \leq g(a_1, \ldots, b, \ldots, a_n)\}$;

(ii) $g(a_1, \ldots, a_i, \ldots, a_n) = \bigvee\{b \in A \mid f(a_1, \ldots, b, \ldots, a_n) \leq a_i\}$.

Proposition 2.1.47. *For any n-ary map $f\colon A^n \to A$,*

(i) *f is completely join-preserving in the i-th coordinate iff it has a right residual in that same coordinate, and*

(ii) *f is completely meet-preserving in the i-th coordinate iff it has a left residual in that same coordinate.*

Example 2.1.48. Let $\mathbf{A}$ be a BAO. We know that in $\mathbf{A}^\delta$, the operations $\Diamond$ and $\Box$ are completely join- and meet-preserving, respectively, and therefore have right and left adjoints, respectively. We will denote the right adjoint of $\Diamond$ by $\Box^{-1}$, and the left adjoint of $\Box$ by $\Diamond^{-1}$.

Finally, before we give the construction, we need to extend the definition of the canonical extension of a BAO to the setting of hybrid algebras.

Definition 2.1.49. Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a hybrid algebra. The *canonical extension* of $\mathfrak{A}$ is the hybrid algebra $\mathfrak{A}^\delta = (\mathbf{A}^\delta, X_{A^\delta})$, where $\mathbf{A}^\delta$ is the canonical extension of $\mathbf{A}$ and X_{A^δ} is the set of all atoms of $\mathbf{A}^\delta$.

Note that the existence and uniqueness of the canonical extension of a hybrid algebra follows from the existence and uniqueness of the canonical extension of a BAO.

Now, given a hybrid algebra $\mathfrak{A} = (\mathbf{A}, X_A)$, consider its canonical extension $\mathfrak{A}^\delta$. Then choose some element d in $\mathfrak{A}^\delta$, and denote it by d_0. Suppose d_n is already defined, then define $d_{n+1} := \Diamond^{-1} d_n$. Let

$$D := \bigvee_{n \in \mathbb{N}} d_n,$$

and let

$$\mathbf{A}_D = (A_D, \wedge^D, \vee^D, \neg^D, \bot^D, \top^D, \Diamond^D),$$

where $A_D = \{a \wedge D \mid a \in A\}$, $\wedge^D$ and $\vee^D$ are the restriction of $\wedge$ and $\vee$ to A_D, and

$$\neg^D a = \neg a \wedge D \qquad \Diamond^D a = \Diamond a \wedge D$$
$$\bot^D = \bot \qquad \top^D = D.$$

Finally, let $\mathfrak{A}_D = (\mathbf{A}_D, X_D)$, where $X_D = \{x \in X \mid x \leq D\}$. It can then be shown that A_D is closed under these operations (see for instance 3.1.3), and hence that $\mathbf{A}_D$ is an algebra.

Let us now take a step back and look at what it is that we have done from a frame theoretical point of view. We will use the following terminology: a *full frame* of a frame $\mathfrak{F} = (W, R)$ is the two-sorted general frame $\mathfrak{F}^\sharp = (\mathfrak{F}, \mathcal{P}(W), W)$. Since the canonical extension of $\mathbf{A}$ is isomorphic to the complex algebra of the ultrafilter frame of $\mathfrak{A}$, it is not difficult to see that the canonical extension of $\mathfrak{A}$ is also isomorphic to the underlying hybrid algebra of the full frame of the ultrafilter frame of $\mathfrak{A}$. So taking the canonical extension enables us to work on a two-sorted general frame. We can therefore think of d as a subset of ultrafilters of $\mathfrak{A}$. We then closed under $\Diamond^{-1}$, so in a sense we are imitating the process of taking generated subframes on algebras. The element D can thus be thought of as the domain of the generated subframe. Next we defined the carrier of the algebra $\mathfrak{A}_D$ as the set of the meets of the elements in A with D. But we know that $\mathfrak{A}$ is a subalgebra of its canonical extension, so $\mathfrak{A}$ is isomorphic to a subalgebra of the complex algebra of the ultrafilter frame of $\mathfrak{A}$. This means that we actually just intersected admissible subsets of the full frame of the ultrafilter frame of $\mathfrak{A}$ with the domain of our generated subframe. Furthermore, we can think of X_D as a set of singletons all contained in the domain of the new subframe. We therefore get a new two-sorted general frame whose underlying frame is our new generated subframe, its set of admissible subsets consists of the intersections of the sets in the algebra isomorphic to $\mathfrak{A}$ with the domain of our new generated subframe, and its second sort of admissible sets consists of the ultrafilters u such that $\{u\}$ belong to the algebra isomorphic to $\mathfrak{A}$ and is contained in the domain of our new frame.

Now, in Chapter 5, we will prove that the algebra $\mathfrak{A}_D$ is a homomorphic image of $\mathfrak{A}$, so, by Theorem 2.1.35, we know that the new two-sorted general frame can be embedded into the two-sorted general frame whose underlying frame is the ultrafilter frame of $\mathfrak{A}$, its set of admissible sets is the algebra isomorphic to $\mathfrak{A}$, and its second sort of admissible sets consists of the ultrafilters the singletons of which belong to the designated set of atoms of the algebra isomorphic to $\mathfrak{A}$.

2.2 Hybrid algebras for $\mathcal{H}(@)$ and the operations on them

2.2.1 Hybrid algebras for $\mathcal{H}(@)$

As for the language $\mathcal{H}$, there are two possible algebraic semantics for $\mathcal{H}(@)$. Again the first involves interpreting nominals as constants, and as for $\mathcal{H}$, we will refer to these algebras as orthodox interpretations.

Definition 2.2.1 (Orthodox interpretations). An *orthodox interpretation* of $\mathcal{H}(@)$ is an algebra $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, \Diamond, @, \{s_\mathbf{i}\}_{\mathbf{i} \in \mathsf{NOM}})$, where $(A, \wedge, \vee, \neg, \bot, \top, \Diamond)$ is a BAO, @ is a binary operator, each $s_\mathbf{i}$ is the interpretation of the nominal $\mathbf{i}$ as a *constant* (i.e., a nullary operation), and **A** is required to validate the following for all $\mathbf{i}, \mathbf{j} \in \mathsf{NOM}$:

$$
\begin{array}{cc}
@_{s_{\mathbf{i}}}(\neg a \vee b) \leq \neg @_{s_{\mathbf{i}}} a \vee @_{s_{\mathbf{i}}} b & \neg @_{s_{\mathbf{i}}} a = @_{s_{\mathbf{i}}} \neg a \\
@_{s_{\mathbf{i}}} @_{s_{\mathbf{j}}} a \leq @_{s_{\mathbf{j}}} a & @_{s_{\mathbf{i}}} s_{\mathbf{i}} = \top \\
s_{\mathbf{i}} \wedge a \leq @_{s_{\mathbf{i}}} a & \Diamond @_{s_{\mathbf{i}}} a \leq @_{s_{\mathbf{i}}} a
\end{array}
$$

So exactly how does the @ operator behave in an orthodox interpretation of $\mathcal{H}(@)$? Proposition 2.2.3 gives us an answer. But first we need to show that @ preserves finite meets and joins in its second coordinate, as well as that it is monotone in its second coordinate in orthodox interpretations.

Lemma 2.2.2. *Let **A** be an orthodox interpretation of $\mathcal{H}(@)$, and let $a, b \in A$ and $s_{\mathbf{i}}$ the constant interpretation of $\mathbf{i}$. Then*

(i) $@_{s_{\mathbf{i}}} \top = \top$,

(ii) $@_{s_{\mathbf{i}}}(a \vee b) = @_{s_{\mathbf{i}}} a \vee @_{s_{\mathbf{i}}} b$,

(iii) $@_{s_{\mathbf{i}}}(a \wedge b) = @_{s_{\mathbf{i}}} a \wedge @_{s_{\mathbf{i}}} b$, *and*

(iv) *if $a \leq b$, then $@_{s_{\mathbf{i}}} a \leq @_{s_{\mathbf{i}}} b$.*

Proof. (i) First, we have $\neg a \vee a = \top$, so $@_{s_{\mathbf{i}}}(\neg a \vee a) = @_{s_{\mathbf{i}}} \top$. But then $\neg @_{s_{\mathbf{i}}} a \vee @_{s_{\mathbf{i}}} a = @_{s_{\mathbf{i}}} \top$. Hence, $@_{s_{\mathbf{i}}} \top = \top$.

(ii) First, note that $@_{s_{\mathbf{i}}}(a \vee b) = @_{s_{\mathbf{i}}}(\neg\neg a \vee b)$. But $@_{s_{\mathbf{i}}}(\neg\neg a \vee b) \leq \neg @_{s_{\mathbf{i}}} \neg a \vee @_{s_{\mathbf{i}}} b$, and so, since $\neg @_{s_{\mathbf{i}}} \neg a = @_{s_{\mathbf{i}}} \neg\neg a = @_{s_{\mathbf{i}}} a$, $@_{s_{\mathbf{i}}}(a \vee b) \leq @_{s_{\mathbf{i}}} a \vee @_{s_{\mathbf{i}}} b$. Conversely, first note that $a \leq a \vee b$, so $\neg a \vee (a \vee b) = \top$. Hence, $@_{s_{\mathbf{i}}}(\neg a \vee (a \vee b)) = @_{s_{\mathbf{i}}} \top$, which means $\neg @_{s_{\mathbf{i}}} a \vee @_{s_{\mathbf{i}}}(a \vee b) = \top$, and therefore $@_{s_{\mathbf{i}}} a \leq @_{s_{\mathbf{i}}}(a \vee b)$. Similarly, we can show that $@_{s_{\mathbf{i}}} b \leq @_{s_{\mathbf{i}}}(a \vee b)$. We thus have $@_{s_{\mathbf{i}}} a \vee @_{s_{\mathbf{i}}} b \leq @_{s_{\mathbf{i}}}(a \vee b)$.

(iii) From (ii) we have $@_{s_{\mathbf{i}}}(\neg a \vee \neg b) = @_{s_{\mathbf{i}}} \neg a \vee @_{s_{\mathbf{i}}} \neg b$, which means $\neg @_{s_{\mathbf{i}}}(\neg a \vee \neg b) = \neg(@_{s_{\mathbf{i}}} \neg a \vee @_{s_{\mathbf{i}}} \neg b)$. Hence, $@_{s_{\mathbf{i}}}(a \wedge b) = \neg @_{s_{\mathbf{i}}} \neg a \wedge \neg @_{s_{\mathbf{i}}} \neg b = @_{s_{\mathbf{i}}} a \wedge @_{s_{\mathbf{i}}} b$.

(iv) Assume $a \leq b$. Then $a \vee b = b$, and so $@_{s_{\mathbf{i}}}(a \vee b) = @_{s_{\mathbf{i}}} b$. Hence, by (ii), $@_{s_{\mathbf{i}}} a \vee @_{s_{\mathbf{i}}} b = @_{s_{\mathbf{i}}} b$, which means that $@_{s_{\mathbf{i}}} a \leq @_{s_{\mathbf{i}}} b$. $\square$

Proposition 2.2.3. *Let **A** be an orthodox interpretation of $\mathcal{H}(@)$, and let a be an element of A and $s_{\mathbf{i}}$ the constant interpretation of $\mathbf{i}$. Then $@_{s_{\mathbf{i}}} a = \top$ iff $s_{\mathbf{i}} \leq a$ and $@_{s_{\mathbf{i}}} a = \bot$ iff $s_{\mathbf{i}} \leq \neg a$.*

Proof. First, assume $@_{s_{\mathbf{i}}} a = \top$. Then $\neg @_{s_{\mathbf{i}}} a = \bot$, and so $@_{s_{\mathbf{i}}} \neg a = \bot$. But $s_{\mathbf{i}} \wedge \neg a \leq @_{s_{\mathbf{i}}} \neg a$, so $s_{\mathbf{i}} \wedge \neg a \leq \bot$. Hence, $s_{\mathbf{i}} \wedge \neg a = \bot$, which means that $s_{\mathbf{i}} \leq a$.

For the converse, assume $s_{\mathbf{i}} \leq a$. Then $s_{\mathbf{i}} \vee a = a$, and so $@_{s_{\mathbf{i}}}(s_{\mathbf{i}} \vee a) = @_{s_{\mathbf{i}}} a$. Hence, by Lemma 2.2.2, $@_{s_{\mathbf{i}}} s_{\mathbf{i}} \vee @_{s_{\mathbf{i}}} a = @_{s_{\mathbf{i}}} a$. Now, since $@_{s_{\mathbf{i}}} s_{\mathbf{i}} = \top$, it follows that $@_{s_{\mathbf{i}}} a = \top$.

Next, assume $@_{s_{\mathbf{i}}} a = \bot$. Then we have $s_{\mathbf{i}} \wedge a \leq \bot$, which gives $s_{\mathbf{i}} \wedge a = \bot$. But then $s_{\mathbf{i}} \leq \neg a$, as required.

Conversely, suppose $s_{\mathbf{i}} \leq \neg a$. By the monotonicity of @, $@_{s_{\mathbf{i}}} s_{\mathbf{i}} \leq @_{s_{\mathbf{i}}} \neg a$, and so, since $@_{s_{\mathbf{i}}} s_{\mathbf{i}} = \top$, $@_{s_{\mathbf{i}}} \neg a = \top$. Hence, $\neg @_{s_{\mathbf{i}}} a = \top$, and so $@_{s_{\mathbf{i}}} a = \bot$. $\square$

As for the language $\mathcal{H}$, the second type of algebraic semantics for $\mathcal{H}(@)$ dually corresponds to two-sorted general frames where the nominals are seen as special variables ranging over a subset of the atoms of the algebra.

Definition 2.2.4 (**Hybrid @-algebras**). A *hybrid @-algebra* is a pair $\mathfrak{A} = (\mathbf{A}, X_A)$, where $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, \Diamond, @)$ such that $(A, \wedge, \vee, \neg, \bot, \top, \Diamond)$ is a BAO containing at least one atom, X_A is non-empty subset of atoms of $\mathbf{A}$, @ is a binary operator whose first coordinate ranges over X_A and the second coordinate over all elements of the algebra, and for all $a, b \in A$ and all $x, y \in X_A$ the following holds:

($K@$) $@_x(\neg a \vee b) \leq \neg @_x a \vee @_x b$,

(*self-dual*) $\neg @_x a = @_x \neg a$,

(*agree*) $@_x @_y a \leq @_y a$,

(*ref*) $@_x x = \top$,

(*introduction*) $x \wedge a \leq @_x a$, and

(*back*) $\Diamond @_x a \leq @_x a$.

We will denote the class of hybrid @-algebras by $\mathsf{H@A}$.

Proposition 2.2.5 below tells us that the @ operator behaves correctly in hybrid @-algebras. More precisely, it characterizes hybrid @-algebras.

Proposition 2.2.5. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$, where $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, \Diamond, @)$, $(A, \wedge, \vee, \neg, \bot, \top, \Diamond)$ is a BAO, X_A is non-empty subset of atoms of $\mathbf{A}$, and @ is a binary operator whose first coordinate ranges over X_A and the second coordinate over all elements of the algebra. Then $\mathfrak{A}$ is a hybrid @-algebra iff for all $x \in X_A$ and $a \in A$, $@_x a = \top$ iff $x \leq a$ and $@_x a = \bot$ iff $x \nleq a$.*

Proof. The proof of the left-to-right direction is similar to that of Proposition 2.2.3. For the converse direction, we have to show that @ satisfies ($K@$), (*self-dual*), (*agree*), (*ref*), (*introduction*), and (*back*).

($K@$) Let $x \in X_A$ and $a, b \in A$, and assume $x \leq \neg a \vee b$. Then $@_x(\neg a \vee b) = \top$. But since x is an atom, $x \leq \neg a$ or $x \leq b$. Hence, $x \nleq a$ or $x \leq b$, and so $@_x a = \bot$ or $@_x b = \top$. We then have $\neg @_x a = \top$ or $@_x b = \top$, which means that $\neg @_x a \vee @_x b = \top$. Therefore, $@_x(\neg a \vee b) \leq \neg @_x a \vee @_x b$, as required. On the other hand, if $x \nleq \neg a \vee b$, $@_x(\neg a \vee b) = \bot$, which gives $@_x(\neg a \vee b) \leq \neg @_x a \vee @_x b$.

(*self-dual*) Let $x \in X_A$ and $b \in A$, and assume $x \leq b$. Then $@_x b = \top$, and so $\neg @_x b = \bot$. But $x \leq b$ implies $x \nleq \neg b$, which means that $@_x \neg b = \bot$. Hence, $\neg @_x b = @_x \neg b$. On the other hand, if $x \nleq b$, then $@_x b = \bot$, which gives $\neg @_x b = \top$. But we know from $x \nleq b$ that $x \leq \neg b$, so $@_x \neg b = \top$.

(*agree*) Let $x, y \in X_A$ and $b \in A$, and assume $y \leq b$. Then $@_y b = \top$, and so $@_x @_y b \leq @_y b$. On the other hand, if $y \nleq b$, $@_y b = \bot$, which means that $@_x @_y b = @_x \bot = \bot$. Hence, $@_x @_y b \leq @_y b$.

(*ref*) We have $x \leq x$ for all $x \in X_A$, so $@_x x = \top$.

(*introduction*) Let $x \in X$ and $b \in A$, and assume $x \leq b$. Then $@_x b = \top$, and so $x \wedge b \leq @_x b$. If $x \nleq b$, $x \wedge b = \bot$ and $@_x b = \bot$, which gives $x \wedge b \leq @_x b$.

(*back*) Let $x \in X_A$ and $b \in A$. If $x \leq b$, then $@_x b = \top$, which means that $\Diamond @_x b \leq @_x b$. On the other hand, assume $x \nleq b$. Then we have $@_x b = \bot$, and so we get $\Diamond @_x b = \Diamond \bot = \bot$. Therefore, $\Diamond @_x b \leq @_x b$. $\square$

Note that if $\mathbf{A}$ is an orthodox interpretation for $\mathcal{H}(@)$, then it is not necessarily the case that $@_{s_\mathbf{i}} a = \bot$ iff $s_\mathbf{i} \nleq a$. To see this, consider the orthodox interpretation $\mathbf{A} = (\mathbf{2}, \Diamond, @, \{s_\mathbf{j}\}_{\mathbf{j} \in \mathsf{NOM}})$, where $\mathbf{2}$ is the two element Boolean algebra, $\Diamond 0 = 0$, $\Diamond 1 = 1$, $s_\mathbf{j} = 0$, and $s_\mathbf{i} = 0$ for $\mathbf{i} \neq \mathbf{j}$. Then $@_{s_\mathbf{j}} \neg s_\mathbf{j} = \neg @_{s_\mathbf{j}} s_\mathbf{j} = \neg 1 = 0$ but $s_\mathbf{j} = 0 \leq 1 = \neg s_\mathbf{j}$. This further motivates our choice to work with hybrid algebras instead of orthodox interpretations.

Let us now give a few examples of hybrid @-algebras.

Example 2.2.6. The structure $\mathfrak{A} = (\mathbf{2}, \Diamond, @, \{1\})$, where $\mathbf{2}$ and $\Diamond$ are defined as in Example 2.1.3, $@_1 0 = 0$, and $@_1 1 = 1$, is clearly a hybrid @-algebra.

Example 2.2.7. The structure $\mathfrak{A} = (A, \cap, \cup, -, \varnothing, \mathbb{Z}, f, @, X_A)$, where $(A, \cap, \cup, -, \varnothing, \mathbb{Z}, f)$ and X_A are defined as in Example 2.1.5, and

$$@_{\{x\}} X = \begin{cases} \mathbb{Z} & \text{if } x \in X_A \\ \varnothing & \text{otherwise} \end{cases}$$

is a hybrid @-algebra. This is immediate from Proposition 2.2.5.

Example 2.2.8. The structure $\mathfrak{A} = (\mathbb{A}, \cap, \cup, -, \varnothing, \omega + 1 \cup \{\omega + 1\}, f, @, X_\mathbb{A})$, where $\mathbb{A}$, f and $X_\mathbb{A}$ are defined as in Example 2.1.7, and for all $\{x\} \in X_\mathbb{A}$ and $A \in \mathbb{A}$,

$$@_{\{x\}} A = \begin{cases} \omega + 1 \cup \{\omega + 1\} & \text{if } \{x\} \subseteq A \\ \varnothing & \text{otherwise,} \end{cases}$$

is a hybrid @-algebra.

Example 2.2.9. Let $\mathfrak{g} = (W, R, A, B)$ be a two-sorted general frame. Then the structure

$$\mathfrak{g}^* = (A, \cap, \cup, -, \varnothing, W, \langle R \rangle, @, X_B),$$

where $X_B = \{\{w\} \mid w \in B\}$, and for $\{w\} \in X_B$,

$$@_{\{w\}} a = \begin{cases} W \text{ if } \{w\} \subseteq a \\ \varnothing \text{ otherwise,} \end{cases}$$

is a hybrid @-algebra. This algebra is called the *underlying hybrid @-algebra* of $\mathfrak{g}$.

In general, Proposition 2.2.5 tells us that any hybrid algebra can be turned into a hybrid @-algebra by adding an operation @ defined as in the above examples to the hybrid algebra.

$\mathcal{H}(@)$-terms are interpreted in hybrid @-algebras $\mathfrak{A} = (\mathbf{A}, X_A)$ in the usual way, but subject to the constraint that nominals range over X_A, while the propositional variables range over all elements of the algebra. Formally:

Definition 2.2.10. Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a hybrid @-algebra. An *assignment* on $\mathfrak{A}$ is a map $v\colon \mathsf{PROP} \cup \mathsf{NOM} \to A$ associating an element of A with each propositional variable in PROP and an atom of X_A with each nominal in NOM. Given such an assignment v, we calculate the *meaning* $\widetilde{v}(t)$ of a term t as follows:

$$\begin{aligned}
\widetilde{v}(\bot) &= \bot \\
\widetilde{v}(p) &= v(p), \\
\widetilde{v}(\mathbf{j}) &= v(\mathbf{j}), \\
\widetilde{v}(\neg\psi) &= \neg\widetilde{v}(\psi), \\
\widetilde{v}(\psi_1 \wedge \psi_2) &= \widetilde{v}(\psi_1) \wedge \widetilde{v}(\psi_2), \\
\widetilde{v}(\Diamond\psi) &= \Diamond\widetilde{v}(\psi), \text{ and} \\
\widetilde{v}(@_{\mathbf{j}}\psi) &= @_{\widetilde{v}(\mathbf{j})}\widetilde{v}(\psi).
\end{aligned}$$

An equation $\varphi \approx \psi$ is *true* in a hybrid @-algebra $\mathfrak{A}$ (denoted $\mathfrak{A} \models \varphi \approx \psi$), if for all assignments θ, $\widetilde{\theta}(\varphi) = \widetilde{\theta}(\psi)$. A set E of equations is true in a hybrid @-algebra $\mathfrak{A}$ (denoted $\mathfrak{A} \models E$), if each equation in E is true in $\mathfrak{A}$. An equation $\varphi \approx \psi$ is a *semantic consequence* of a set E of equations (denoted $E \models \varphi \approx \psi$), if for any hybrid @-algebra $\mathfrak{A}$ such that $\mathfrak{A} \models E$, $\mathfrak{A} \models \varphi \approx \psi$.

2.2.2 Operations on hybrid @-algebras

Homomorphisms between hybrid @-algebras

Definition 2.2.11 (Homomorphisms). Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two hybrid @-algebras. A map $h\colon \mathfrak{A} \to \mathfrak{B}$ is called a *homomorphism between* $\mathfrak{A}$ *and* $\mathfrak{B}$ if h is a homomorphism between $(A, \wedge, \vee, \neg, \bot, \top, \Diamond)$ and $(B, \wedge, \vee, \neg, \bot, \top, \Diamond)$, h maps elements of X_A to elements of X_B, and for all $x \in X_A$ and $a \in A$,

$$h(@_x a) = @_{h(x)} h(a).$$

We say that h is a *surjective homomorphism*, if h is surjective from A onto B, and furthermore, h is surjective from X_A onto X_B. $\mathfrak{B}$ is a *homomorphic image* of $\mathfrak{A}$ (denoted $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$), if there is a surjective homomorphism h from A onto B. We say that $\mathfrak{A}$ is *embeddable* in $\mathfrak{B}$ (denoted $\mathfrak{A} \rightarrowtail \mathfrak{B}$), if there is an injective homomorphism h from A to B. If a homomorphism is both surjective and injective, then it is called an *isomorphism*. Finally, $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorhic*, if there is an isomorphism h between A and B.

The validity of $\mathcal{H}(@)$-formulas is also preserved under homomorphic images of hybrid @-algebras.

Proposition 2.2.12. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two hybrid @-algebras. If $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$, then $\mathfrak{B} \models \varphi \approx \psi$ whenever $\mathfrak{A} \models \varphi \approx \psi$.*

The proof of this is similar to that of Proposition 2.1.11.

In a similar way as in Proposition 2.1.12, we can show that the validity of $\mathcal{H}(@)$-formulas is transferred from superalgebras to subalgebras.

Proposition 2.2.13. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two hybrid @-algebras. If $\mathfrak{A} \rightarrowtail \mathfrak{B}$, then $\mathfrak{A} \models \varphi \approx \psi$ whenever $\mathfrak{B} \models \varphi \approx \psi$.*

Products of hybrid @-algebras

Unlike for hybrid algebras, we cannot form the product of hybrid @-algebras. To see why, consider the hybrid @-algebras $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$, and let

$$\mathfrak{A} \times \mathfrak{B} = (\mathbf{A} \times \mathbf{B}, X_{A \times B}),$$

where $\mathbf{A} \times \mathbf{B}$ is defined as usual and $X_{A\times B}$ is defined as in Definition 2.1.13. But then $@_{(\perp,x)}(a, b) = (@_\perp a, @_x b)$, which is undefined since $@_\perp a$ is not defined in $\mathfrak{A}$. Note that if $\perp$ was in X_A, we would not have had this problem. So can we form the product of grounded hybrid @-algebras (a grounded hybrid @-algebra is just like a hybrid @-algebra $\mathfrak{A} = (\mathbf{A}, X_A)$, except that $\perp$ is also included in the set X_A)? The answer is *no*. The reason for this is that grounded hybrid @-algebras are actually not defined. To see this, note that $@_\perp \perp = \top$ by (*ref*). But we also have

$$@_\perp \perp = @_\perp (a \wedge \neg a) = @_\perp a \wedge @_\perp \neg a = @_\perp a \wedge \neg @_\perp a = \top \wedge \perp = \perp,$$

which is a contradiction.

2.2.3 Permeated hybrid @-algebras

Algebraically, $\mathbf{H}^+(@)\Sigma$ is characterized by classes of *permeated hybrid @-algebras* (see Subsection 3.5). A *permeated hybrid @-algebra* (PH(@)A) is a hybrid @-algebra satisfying the same conditions as in Definition 2.1.20. The hybrid @-algebras in Examples 2.2.6 and 2.2.8 are both permeated.

2.2.4 Duality between two-sorted general frames and hybrid @-algebras

Theorem 2.2.14 below give a concise formulation of the basic relationships between hybrid @-algebras and two-sorted general frames. The proof of Theorem 2.2.14 follows immediately from Propositions 2.2.15 and 2.2.16. The definition of a two-sorted general ultrafilter frame of a hybrid @-algebra is unchanged.

Theorem 2.2.14. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two two-sorted general frames, and $\mathfrak{A}$ and $\mathfrak{B}$ two hybrid @-algebras.*

(i) *If $\mathfrak{g} \rightarrowtail \mathfrak{h}$, then $\mathfrak{h}^* \twoheadrightarrow \mathfrak{g}^*$.*

(ii) *If $\mathfrak{g} \twoheadrightarrow \mathfrak{h}$, then $\mathfrak{h}^* \rightarrowtail \mathfrak{g}^*$.*

(iii) *If $\mathfrak{A} \rightarrowtail \mathfrak{B}$, then $\mathfrak{B}_* \twoheadrightarrow \mathfrak{A}_*$.*

(iv) *If $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$, then $\mathfrak{B}_* \rightarrowtail \mathfrak{A}_*$.*

Recall that if $\mathfrak{g} = (W, R, A, B)$ and $\mathfrak{h} = (W', R', A', B')$ are two two-sorted general frames, and g is a map between W and W', then its *dual* $g^*\colon A' \to \mathcal{P}(W)$ is defined as

$$g^*(a') := g^{-1}[a'] \; (= \{w \in W \mid g(w) \in a'\}).$$

Proposition 2.2.15. *Let $\mathfrak{g} = (W, R, A, B)$ and $\mathfrak{h} = (W', R', A', B')$ be two-sorted general frames, $\mathfrak{g}^* = (A, \cap, \cup, -, \varnothing, W, \langle R\rangle, @, X_B)$ and $\mathfrak{h}^* = (A', \cap, \cup, -, \varnothing, W', \langle R'\rangle, @, X_{B'})$ their underlying hybrid @-algebras, and g a map from W to W'.*

(i) *If g is a bounded morphism between $\mathfrak{g}$ and $\mathfrak{g}'$, g^* maps elements of A' to elements of A.*

(ii) *If g is a bounded morphism, then g^* is a homomorphism between $\mathfrak{h}^*$ to $\mathfrak{g}^*$.*

(iii) *If g is an embedding, then g^* is a surjective homomorphism.*

(iv) *If g is surjective, then g^* is injective.*

Proof. The proof of this is the same as that of Proposition 2.1.30, all we have to check is that g^* respects @ to complete the proof of item (ii). So let $\{w'\} \in X_{B'}$ and $a' \in A'$, and consider the following two cases:

Case 1: $\{w'\} \subseteq a'$. Then $g^*(@_{\{w'\}}a') = g^*(W') = W$. We now claim that $g^*(\{w'\}) \subseteq g^*(a')$. To see this, let $w \in g^*(\{w'\})$. Then we have $g(w) = w'$, and so, since $\{w'\} \subseteq a'$, $g(w) \in a'$. Hence, $w \in g^*(a')$, which means $g^*(\{w'\}) \subseteq g^*(a')$. But then $@_{g^*(\{w'\})}g^*(a') = W$ by Proposition 2.2.5, and therefore $g^*(@_{\{w'\}}a') = @_{g^*(\{w'\})}g^*(a')$.

Case 2: $\{w'\} \nsubseteq a'$. Here $g^*(@_{\{w'\}}a') = g^*(\varnothing) = \varnothing$. We now claim that $g^*(\{w'\}) \nsubseteq g^*(a')$. Let $w \in g^*(\{w'\})$. Then $g(w) = w'$, and so, since $\{w'\} \subseteq -a'$, $g(w) \in -a'$. Hence, $w \in g^*(-a')$. But g^* is a Boolean homomorphism, so $w \in -g^*(a')$, which means $g^*(\{w'\}) \nsubseteq g^*(a')$. We thus have that $@_{g^*(\{w'\})}g^*(a') = \varnothing$ by Proposition 2.2.5, and therefore $g^*(@_{\{w'\}}a') = @_{g^*(\{w'\})}g^*(a')$. □

If $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ are two hybrid @-algebras, and h is a map from A to B, then its *dual* h_* is the map from $Uf\mathfrak{B}$ to $\mathcal{P}(A)$ defined as

$$h_*(u') := h^{-1}[u'] \; (= \{a \in A \mid h(a) \in u'\}).$$

Proposition 2.2.16. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two hybrid @-algebras, $\mathfrak{A}_* = (Uf\mathfrak{A}, Q_\Diamond, \widehat{A}, X_A\uparrow)$ and $\mathfrak{B}_* = (Uf\mathfrak{B}, Q'_\Diamond, \widehat{B}, X_B\uparrow)$ their two-sorted general ultrafilter frames, and h a map from A to B.*

(i) *If h is a homomorphism, then h_* maps ultrafilters to ultrafilters.*

(ii) *If h is a homomorphism, then h_* is a bounded morphism from $\mathfrak{B}_*$ to $\mathfrak{A}_*$.*

(iii) *If h is a surjective homomorphism, then h_* is an embedding.*

(iv) *If h is an embedding, then h_* is a surjective.*

Proof. The proof of this proposition is the same as that of Proposition 2.1.34. $\square$

As expected, descriptive two-sorted general frames and hybrid @-algebras are also duals.

Theorem 2.2.17. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a hybrid @-algebra, and $\mathfrak{g} = (W, R, A, B)$ a two-sorted general frame. Then*

(i) *$\mathfrak{A}_*$ is a descriptive two-sorted general frame,*

(ii) *$(\mathfrak{A}_*)^* \cong \mathfrak{A}$, and*

(iii) *$(\mathfrak{g}^*)_* \cong \mathfrak{g}$ iff $\mathfrak{g}$ is descriptive.*

Proof. The proof of this is the same as that of Proposition 2.1.40. The only item that needs a bit more work is item (ii), so recall the map $h\colon A \to \widehat{A}$ defined by $h(a) = \widehat{a}$. We have to make sure that h also respects @. So let $x \in X_A$ and $a \in A$, and consider the following two cases:

Case 1: $x \leq a$. In this case, by Proposition 2.2.5, $h(@_x a) = h(\top) = Uf\mathfrak{A}$. To show that $@_{h(x)}h(a) = Uf\mathfrak{A}$, we prove that $\widehat{x} \subseteq \widehat{a}$. So let $u \in \widehat{x}$. Then $x \in u$, and so, since $x \leq a$ and u is upward closed, $a \in u$. Hence, $u \in \widehat{a}$. We therefore have $@_{h(x)}h(a) = @_{\widehat{x}}\widehat{a} = Uf\mathfrak{A}$, which gives $h(@_x a) = @_{h(x)}h(a)$.

Case 2: $x \nleq a$. Here we have $h(@_x a) = h(\perp) = \varnothing$. We claim that $\widehat{x} \nsubseteq \widehat{a}$. To see this, let $u \in \widehat{x}$. Then $x \in u$. But x is an atom and $x \nleq a$, so $x \leq \neg a$. Since u is upward closed, $\neg a \in u$. Hence, $a \notin u$, and so, $u \notin \widehat{a}$. Now, $@_{h(x)}h(a) = @_{\widehat{x}}\widehat{a} = \varnothing$, and therefore, $h(@_x a) = @_{h(x)}h(a)$. $\square$

Likewise, permeated hybrid @-algebras and strongly descriptive two-sorted general frames are each others duals.

Theorem 2.2.18. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a permeated hybrid @-algebra, and $\mathfrak{g} = (W, R, A, B)$ a two-sorted general frame. Then*

(i) *$\mathfrak{A}_*$ is a strongly descriptive two-sorted general frame,*

(ii) *$(\mathfrak{A}_*)^* \cong \mathfrak{A}$, and*

(iii) *$(\mathfrak{g}^*)_* \cong \mathfrak{g}$ iff $\mathfrak{g}$ is strongly descriptive.*

2.3 Hybrid algebras for $\mathcal{H}(\mathsf{E})$ and the operations on them

2.3.1 Hybrid algebras for $\mathcal{H}(\mathsf{E})$

As in the previous sections, we begin with the definition of an orthodox interpretation of $\mathcal{H}(\mathsf{E})$. But first, we define the dual operator A of E: $\mathsf{A}a := \neg\mathsf{E}\neg a$.

Definition 2.3.1 (**Orthodox interpretations**). An *orthodox interpretation* of $\mathcal{H}(\mathsf{E})$ is an algebra $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, \Diamond, \mathsf{E}, \{s_{\mathbf{i}}\}_{\mathbf{i}\in\mathsf{NOM}})$, where $(A, \wedge, \vee, \neg, \bot, \top, \Diamond)$ is a BAO, E is a unary operation, each $s_{\mathbf{i}}$ is the interpretation of the nominal $\mathbf{i}$ as a *constant* (i.e., a nullary operation), and $\mathbf{A}$ is required to validate the following for all $\mathbf{i} \in \mathsf{NOM}$:

($K\mathsf{E}$) $\mathsf{A}(\neg a \vee b) \leq \neg \mathsf{A}a \vee \mathsf{A}b$,

($\mathit{refl}_{\mathsf{E}}$) $a \leq \mathsf{E}a$,

($\mathit{trans}_{\mathsf{E}}$) $\mathsf{EE}a \leq \mathsf{E}a$,

($\mathit{sym}_{\mathsf{E}}$) $a \leq \mathsf{AE}a$,

(incl) $\mathsf{E}s_{\mathbf{i}} = \top$,

($\mathit{nom}_{\mathsf{E}}$) $\mathsf{E}(s_{\mathbf{i}} \wedge a) \leq \mathsf{A}(\neg s_{\mathbf{i}} \vee a)$, and

($\mathit{incl}_{\Diamond}$) $\Diamond a \leq \mathsf{E}a$.

Definition 2.3.2 (**Hybrid E-algebras**). A *hybrid* E*-algebra* is a pair $\mathfrak{A} = (\mathbf{A}, X_A)$, where $\mathbf{A} = (A, \vee, \wedge, \neg, \bot, \top, \Diamond, \mathsf{E})$ such that $(A, \vee, \wedge, \neg, \bot, \top, \Diamond)$ is a BAO containing at least one atom, X_A is a non-empty subset of the set of atoms of $\mathbf{A}$, and for all $a \in A$,

$$\mathsf{E}a = \begin{cases} \top & \text{if } a > \bot \\ \bot & \text{otherwise.} \end{cases}$$

Note that the algebra $\mathbf{A} = (A, \vee, \wedge, \neg, \bot, \top, \Diamond, \mathsf{E})$ is defined in the same way as Goranko and Passy's definition of a model algebra with the additional operator E in [52]. Finally, we will denote the class of hybrid E-algebras by HEA.

Proposition 2.3.3. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a hybrid E-algebra. For all $a \in A$,*

$$\mathsf{A}a = \begin{cases} \top & \text{if } a = \top \\ \bot & \text{otherwise.} \end{cases}$$

Proof. Let $a \in A$ such that $a = \top$. Then we have $\neg a = \bot$, which means that $\mathsf{E}\neg a = \bot$. Hence, $\neg\mathsf{E}\neg a = \top$, and so, since $\mathsf{A}a = \neg\mathsf{E}\neg a$, $\mathsf{A}a = \top$. On the other hand, if $a \neq \top$, then $\neg a \neq \bot$, which means $\mathsf{E}\neg a = \top$. Therefore, $\neg\mathsf{E}\neg a = \bot$, and so $\mathsf{A}a = \bot$. □

Proposition 2.3.4. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a hybrid E-algebra. For all $a, b \in A$ and $x \in X$, the following holds:*

($K\mathsf{E}$) $\mathsf{A}(\neg a \vee b) \leq \neg \mathsf{A}a \vee \mathsf{A}b$,

($\mathit{refl}_{\mathsf{E}}$) $a \leq \mathsf{E}a$,

($\mathit{trans}_{\mathsf{E}}$) $\mathsf{EE}a \leq \mathsf{E}a$,

($\mathit{sym}_{\mathsf{E}}$) $a \leq \mathsf{AE}a$,

$(incl)$ $\mathsf{E}x = \top$,

(nom_{E}) $\mathsf{E}(x \wedge a) \leq \mathsf{A}(\neg x \vee a)$, *and*

$(incl_{\Diamond})$ $\Diamond a \leq \mathsf{E}a$.

Proof. $(K\mathsf{E})$ First, if $a = \top$ and $b = \top$, then $\neg \mathsf{A}a \vee \mathsf{A}b = \bot \vee \top = \top$. Hence, $\mathsf{A}(\neg a \vee b) \leq \neg \mathsf{A}a \vee \mathsf{A}b$, as required. If we have $a = \top$ and $b \neq \top$, then $\neg \mathsf{A}a \vee \mathsf{A}b = \bot \vee \bot = \bot$. But we also then have $\mathsf{A}(\neg a \vee b) = \mathsf{A}(\bot \vee b) = \mathsf{A}b = \bot$, so again $\mathsf{A}(\neg a \vee b) \leq \neg \mathsf{A}a \vee \mathsf{A}b$. If $a \neq \top$ and $b = \top$, then $\neg \mathsf{A}a \vee \mathsf{A}b = \top \vee \top = \top$, which gives $\mathsf{A}(\neg a \vee b) \leq \neg \mathsf{A}a \vee \mathsf{A}b$. Finally, if $a \neq \top$ and $b \neq \top$, then $\neg \mathsf{A}a \vee \mathsf{A}b = \top \vee \bot = \top$, and so $\mathsf{A}(\neg a \vee b) \leq \neg \mathsf{A}a \vee \mathsf{A}b$.

$(refl_{\mathsf{E}})$ First, assume $a > \bot$. Then $\mathsf{E}a = \top$, and so we have $a \leq \mathsf{E}a$. On the other hand, if $a = \bot$, then $\mathsf{E}a = \bot$, which clearly means that $a \leq \mathsf{E}a$.

$(trans_{\mathsf{E}})$ Assume $a > \bot$. Then we have $\mathsf{EE}a = \mathsf{E}\top = \top$ and $\mathsf{E}a = \top$. For $a = \bot$, we have $\mathsf{EE}a = \mathsf{E}\bot = \bot$ and $\mathsf{E}a = \bot$.

(sym_{E}) If $a > \bot$, $\mathsf{AE}a = \mathsf{A}\top = \top$, so clearly, $a \leq \mathsf{AE}a$. On the other hand, if $a = \bot$, $\mathsf{AE}a = \mathsf{A}\bot = \bot$, which means that $a \leq \mathsf{AE}a$.

$(incl)$ Since x is an atom, $x > \bot$, so, by definition, $\mathsf{E}x = \top$.

(nom_{E}) Assume $x \wedge a > \bot$. Then $\mathsf{E}(x \wedge a) = \top$ by definition. But since x is an atom, we have $x \leq a$, which means that $\neg x \vee a = \top$. Hence, $\mathsf{A}(\neg x \vee a) = \top$. Now, assume $x \wedge a = \bot$. Then $\mathsf{E}(x \wedge a) = \bot$, which clearly gives $\mathsf{E}(x \wedge a) \leq \mathsf{A}(\neg x \vee a)$.

$(incl_{\Diamond})$ If $a > \bot$, $\mathsf{E}a = \top$, so $\Diamond a \leq \mathsf{E}a$. On the other hand, if $a = \bot$, $\Diamond a = \bot$ and $\mathsf{E}a = \bot$, which means $\Diamond a \leq \mathsf{E}a$. □

Example 2.3.5. The structure $\mathfrak{A} = (\mathbf{2}, \Diamond, \mathsf{E}, \{1\})$, where $\mathbf{2}$ and $\Diamond$ are defined as in Example 2.1.3, $\mathsf{E}0 = 0$, and $\mathsf{E}1 = 1$, is clearly a hybrid E-algebra.

Example 2.3.6. The structure $\mathfrak{A} = (\mathbb{A}, \cap, \cup, -, \varnothing, \omega + 1 \cup \{\omega + 1\}, f, \mathsf{E}, X_A)$, where $\mathbb{A}$, f and X_A are defined as in Example 2.1.7, and

$$\mathsf{E}A = \begin{cases} \omega + 1 \cup \{\omega + 1\} & \text{if } A \neq \varnothing \\ \varnothing & \text{otherwise,} \end{cases}$$

is a hybrid E-algebra.

In general, we can turn any hybrid algebra into a hybrid E-algebra by simply adding an operation E defined as in Definition 2.3.2 to the hybrid algebra.

We also interpret $\mathcal{H}(\mathsf{E})$-terms in hybrid E-algebras $\mathfrak{A} = (\mathbf{A}, X_A)$ in the usual way but subject to the constraint that nominals range over X_A, while the propositional variables range over all elements of the algebra.

Definition 2.3.7. An *assignment* on $\mathfrak{A} = (\mathbf{A}, X_A)$ is a function $v\colon \mathsf{PROP} \cup \mathsf{NOM} \to A$ associating an element of A with each propositional variable in PROP and an atom of X_A with each nominal in NOM. Given such an assignment v, we calculate the *meaning* $\widetilde{v}(t)$ of a term t as follows:

$$\begin{aligned}
\widetilde{v}(\bot) &= \bot \\
\widetilde{v}(p) &= v(p), \\
\widetilde{v}(\mathbf{j}) &= v(\mathbf{j}), \\
\widetilde{v}(\neg\psi) &= \neg\widetilde{v}(\psi), \\
\widetilde{v}(\psi_1 \wedge \psi_2) &= \widetilde{v}(\psi_1) \wedge \widetilde{v}(\psi_2), \\
\widetilde{v}(\Diamond\psi) &= \Diamond\widetilde{v}(\psi), \text{ and} \\
\widetilde{v}(\mathsf{E}\psi) &= \mathsf{E}\widetilde{v}(\psi).
\end{aligned}$$

An equation $\varphi \approx \psi$ is *true* in a hybrid E-algebra $\mathfrak{A}$ (denoted $\mathfrak{A} \models \varphi \approx \psi$), if for all assignments θ, $\widetilde{\theta}(\varphi) = \widetilde{\theta}(\psi)$. A set E of equations is true in a hybrid E-algebra $\mathfrak{A}$ (denoted $\mathfrak{A} \models E$), if each equation in E is true in $\mathfrak{A}$. An equation $\varphi \approx \psi$ is a *semantic consequence* of a set E of equations (denoted $E \models \varphi \approx \psi$), if for any E-algebra $\mathfrak{A}$ such that $\mathfrak{A} \models E$, $\mathfrak{A} \models \varphi \approx \psi$.

2.3.2 Operations on hybrid E-algebras

Homomorphisms between hybrid E-algebras

Definition 2.3.8 (**Homomorphisms**)**.** Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two hybrid E-algebras. A map h: $\mathfrak{A} \to \mathfrak{B}$ is called a *homomorphism between* $\mathfrak{A}$ *and* $\mathfrak{B}$ if h is a homomorphism between $(A, \wedge, \vee, \neg, \bot, \top, \Diamond)$ and $(B, \wedge, \vee, \neg, \bot, \top, \Diamond)$, h maps elements of X_A to elements of X_B, and for all $a \in A$,

$$h(\mathsf{E}a) = \mathsf{E}h(a).$$

We say that h is a *surjective homomorphism*, if h is surjective from A onto B, and furthermore, h is surjective from X_A onto X_B. $\mathfrak{B}$ is a *homomorphic image* of $\mathfrak{A}$ (denoted $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$), if there is a surjective homomorphism h from A onto B. We say that $\mathfrak{A}$ is *embeddable* in $\mathfrak{B}$ (denoted $\mathfrak{A} \rightarrowtail \mathfrak{B}$), if there is an injective homomorphism h from A to B. If a homomorphism is both surjective and injective, then it is called an *isomorphism*. Finally, $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorhic*, if there is an isomorphism h between A and B.

The validity of $\mathcal{H}(\mathsf{E})$-formulas is also preserved under homomorphic images.

Proposition 2.3.9. *Let* $\mathfrak{A} = (\mathbf{A}, X_A)$ *and* $\mathfrak{B} = (\mathbf{B}, X_B)$ *be two hybrid* E*-algebras. If* $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$*, then* $\mathfrak{B} \models \varphi \approx \psi$ *whenever* $\mathfrak{A} \models \varphi \approx \psi$*.*

The proof of this is the same as that of Proposition 2.1.11.

As in Proposition 2.1.12, we can also show that the validity of $\mathcal{H}(\mathsf{E})$-formulas is transferred from superalgebras to subalgebras.

Proposition 2.3.10. *Let* $\mathfrak{A} = (\mathbf{A}, X_A)$ *and* $\mathfrak{B} = (\mathbf{B}, X_B)$ *be two hybrid* E*-algebras. If* $\mathfrak{A} \rightarrowtail \mathfrak{B}$*, then* $\mathfrak{A} \models \varphi \approx \psi$ *whenever* $\mathfrak{B} \models \varphi \approx \psi$*.*

Products of hybrid E-algebras

As in the case for hybrid @-algebras, we cannot form the product of hybrid E-algebras. To see why, consider the hybrid E-algebras $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$, and let $\mathfrak{A} \times \mathfrak{B} = (\mathbf{A} \times \mathbf{B}, X_{A \times B})$, where $\mathbf{A} \times \mathbf{B}$ is defined as usual and $X_{A \times B}$ is defined as in Definition 2.1.13. Now, note that $(\top, \bot) \neq (\bot, \bot)$ but $\mathsf{E}(\top, \bot) = (\mathsf{E}\top, \mathsf{E}\bot) = (\top, \bot) \neq (\top, \top)$. This means that $\mathfrak{A} \times \mathfrak{B}$ is therefore not a hybrid E-algebra.

2.3.3 Permeated hybrid E-algebras

A *permeated hybrid* E-*algebra* (PH(E)A) is a hybrid E-algebra satisfying the same conditions as in Definition 2.1.20. The class of permeated hybrid E-algebras will be denoted by PHEA.

The hybrid E-algebra in Example 2.3.5 is clearly permeated. Also recall that the hybrid algebra in Example 2.1.7 is permeated (see Example 2.1.23), so the hybrid E-algebra in Example 2.3.6 is also permeated.

2.3.4 Duality between two-sorted general frames and hybrid E-algebras

Consider the two-sorted general frames $\mathfrak{g} = (W, R, A, B)$ and $\mathfrak{h} = (W', R', A', B)$, where $W = \{u, v\}, R = \{(u, v), (v, u)\}, A = \{\varnothing, \{u\}, \{v\}, \{u, v\}\}, B = \{u, v\}$, $W' = \{u, v, w\}$, $R = \{(u, v), (v, u)\}, A = \{\varnothing, \{u\}, \{v\}, \{w\}, \{u, v\}, \{u, w\}, \{v, w\}, \{u, v, w\}\}$, and $B' = \{u, v\}$ (see Figure 2.2). Now, it is easy to see that $\mathfrak{g}$ can be embedded into $\mathfrak{h}$: take the map $g\colon W \to W'$ to be the identity map. But then $\mathfrak{h} \Vdash \mathsf{E}\neg\Diamond\top$, while $\mathfrak{g} \nVdash \mathsf{E}\neg\Diamond\top$. However, we know from Proposition 2.3.9 that the validity of $\mathcal{H}(\mathsf{E})$-formulas is preserved under taking homomorphic images. It turns out that the dual g^* (see below) does not preserve E, and is therefore not a homomorphism between $\mathfrak{g}^* = (A, \cap, \cup, -, \varnothing, W, \langle R \rangle, \mathsf{E}, X_B)$ and $\mathfrak{h}^* = (A', \cap, \cup, -, \varnothing, W', \langle R' \rangle, \mathsf{E}, X_{B'})$. So unlike for hybrid algebras and hybrid @-algebras, embeddings between two-sorted general frames and homomorphic images between hybrid E-algebras are not each others duals. However, all is not lost: if there is an isomorphism g between two two-sorted general frames, then we can show that g^* respects E. In other words, isomorphisms between two-sorted general frames correspond to homomorphic images. The other link is the same as before, i.e. bounded morphic images correspond to embeddings between hybrid E-algebras. Theorem 2.3.11 below give a concise formulation of the basic links between hybrid E-algebras and two-sorted general frames. The proof of Theorem 2.3.11 follows immediately from Propositions 2.3.12 and 2.3.13.

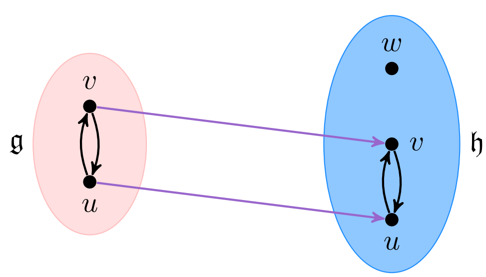

Figure 2.2: An embedding between two-sorted general frames

Theorem 2.3.11. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two two-sorted general frames, and $\mathfrak{A}$ and $\mathfrak{B}$ two hybrid* E*-algebras.*

(i) *If $\mathfrak{g} \twoheadrightarrow \mathfrak{h}$, then $\mathfrak{h}^* \rightarrowtail \mathfrak{g}^*$.*

(ii) *If $\mathfrak{g} \cong \mathfrak{h}$, then $\mathfrak{h}^* \cong \mathfrak{g}^*$.*

(iii) *If $\mathfrak{A} \rightarrowtail \mathfrak{B}$, then $\mathfrak{B}_* \twoheadrightarrow \mathfrak{A}_*$.*

(iv) *If $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$, then $\mathfrak{B}_* \cong \mathfrak{A}_*$.*

Recall that if $\mathfrak{g} = (W, R, A, B)$ and $\mathfrak{h} = (W', R', A', B')$ are two two-sorted general frames, and g is a map between W and W', then its *dual* $g^*\colon A' \to \mathcal{P}(W)$ is defined as

$$g^*(a') := g^{-1}[a'] \ (= \{w \in W \mid g(w) \in a'\}).$$

Proposition 2.3.12. *Let $\mathfrak{g} = (W, R, A, B)$ and $\mathfrak{h} = (W', R', A', B')$ be two-sorted general frames, $\mathfrak{g}^* = (A, \cap, \cup, -, \varnothing, W, \langle R\rangle, \mathsf{E}, X_B)$ and $\mathfrak{h}^* = (A', \cap, \cup, -, \varnothing, W', \langle R'\rangle, \mathsf{E}, X_{B'})$ their underlying hybrid* E*-algebras, and g a map from W to W'.*

(i) *If g is a bounded morphism between $\mathfrak{g}$ and $\mathfrak{g}'$, g^* maps elements of A' to elements of A.*

(ii) *If g is a surjective bounded morphism, then g^* is an injective homomorphism between $\mathfrak{h}^*$ to $\mathfrak{g}^*$.*

(iii) *If g is a bijective bounded morphism, then g^* is a bijective homomorphism between $\mathfrak{h}^*$ to $\mathfrak{g}^*$.*

Proof. (i) The proof of this is the same as that of item (i) of Proposition 1.1.51.

(ii) We only show that g^* respects E as the rest of the proof is the same as that of items (ii) and (iv) of Proposition 1.1.51.

Case 1: $a' = \varnothing$. Then $g^*(\mathsf{E}a') = g^*(\varnothing) = \varnothing$ and $\mathsf{E}g^*(\varnothing) = \mathsf{E}\varnothing = \varnothing$.

Case 2: $a' \neq \varnothing$. Here $g^*(\mathsf{E}a') = g^*(W') = W$. We now claim that $g^*(a') \neq \varnothing$. We know that from $a' \neq \varnothing$, there is some $w' \in W'$ such that $w' \in a'$. But since g is surjective, there is some $w \in W$ such that $g(w) = w'$. Hence, $g(w) \in a'$, which means that $w \in g^*(a')$. We therefore have that $\mathsf{E}g^*(a') = W$, and so $g^*(\mathsf{E}a') = \mathsf{E}g^*(a')$.

(iii) First, using the injectivity of g, we can show that g^* is surjective in the same way as in the proof of item (iii) of Proposition 1.1.51. The surjectivity of g is used to prove that g^* is injective in the same way as in the proof of item (iv) of Proposition 1.1.51. We also use the surjectivity of g to prove that g^* respects E. The fact that g^* respects the other operations is proved in the same way as in the proof of item (ii) of Proposition 1.1.51. □

If $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ are two hybrid E-algebras, and h is a map from A to B, then its *dual* h_* is the map from *Uf*$\mathfrak{B}$ to $\mathcal{P}(A)$ defined as

$$h_*(u') := h^{-1}[u'] \ (= \{a \in A \mid h(a) \in u'\}).$$

Proposition 2.3.13. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ and $\mathfrak{B} = (\mathbf{B}, X_B)$ be two hybrid* E*-algebras, $\mathfrak{A}_* = (Uf\mathfrak{A}, Q_\diamond, \widehat{A}, X_A\uparrow)$ and $\mathfrak{B}_* = (Uf\mathfrak{B}, Q'_\diamond, \widehat{B}, X_B\uparrow)$ their two-sorted general ultrafilter frames, and h a map from A to B.*

(i) *If h is a homomorphism, then h_* maps ultrafilters to ultrafilters.*

(ii) *If h is a homomorphism, then h_* is a bounded morphism from $(Uf\mathfrak{B}, Q'_\diamond)$ to $(Uf\mathfrak{A}, Q_\diamond)$.*

(iii) *If h is a surjective homomorphism, then h_* is bijective.*

(iv) *If h is an embedding, then h_* is a surjective.*

Proof. Since the definition of a bounded morphism does not involve the global modality, all items, except (iii), has been proved in Proposition 1.1.52. So let us move on to (iii). The injectivity of h_* is also proved in the same way as in item (iii) of Proposition 1.1.52. For the surjectivity, let u be an ultrafilter of $\mathfrak{A}$, and define

$$F := \{h(a) \mid a \in u\}.$$

Now, $h_*(F)$ might not be defined, the reason being that F might not be upward closed, and therefore not an ultrafilter, while h_* is only defined on ultrafilters. So let

$$F' := \{a' \mid \exists a \in u(h(a) \leq a')\}.$$

In the same way as in the proof of Proposition 5.52 in [10], we can show that F' is a filter. To show that F' is proper, suppose $\bot \in F'$. Then $h(a) = \bot$ for some $a \in u$. Hence, $\mathsf{E}h(\bot) = h(\mathsf{E}\bot) = \bot$, which means that $a = \bot$, for if not $\mathsf{E}a = \top$, and so $h(\mathsf{E}a) = h(\top) = \top$, a contradiction. But then $\bot \in u$, contradicting the fact that u is an ultrafilter. Now, by the Ultrafilter Theorem, F' can be extended to an ultrafilter u'. The fact that $h_*(u') = u$ can be proved in the same way as in the proof of Proposition 5.52 in [10]. □

As for the cases of hybrid algebras and hybrid @-algebras, hybrid E-algebras and permeated hybrid E-algebras correspond to descriptive two-sorted general frames and strongly descriptive two-sorted general frames, respectively.

Theorem 2.3.14. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a hybrid* E*-algebra, and $\mathfrak{g} = (W, R, A, B)$ a two-sorted general frame. Then*

(i) *$\mathfrak{A}_*$ is a descriptive two-sorted general frame,*

(ii) *$(\mathfrak{A}_*)^* \cong \mathfrak{A}$, and*

(iii) *$(\mathfrak{g}^*)_* \cong \mathfrak{g}$ iff $\mathfrak{g}$ is descriptive.*

Proof. The proofs of items (i) and (iii) are same as that of items (i) and (iii) of Theorem 2.1.40. For item (ii), let $h\colon A \to \widehat{A}$be defined by $h(a) = \widehat{a}$. We have to show that h is an isomorphism. But we only show that h respects E since the rest of the proof is the same as that of item (ii) of Proposition 2.1.40. So let $a \in A$, and consider the following cases:

Case 1: $a = \perp$. Then $h(\mathsf{E}a) = h(\perp) = \widehat{\perp} = \varnothing = \mathsf{E}\varnothing = \mathsf{E}\widehat{\perp} = \mathsf{E}h(a)$.

Case 2: $a \neq \perp$. In this case $h(\mathsf{E}a) = h(\top) = Uf\mathfrak{A}$. But we also have that $h(a) \neq \varnothing$. To see this, recall that $a \uparrow\in \widehat{a}$, so $a \uparrow\in h(a)$. Hence, $\mathsf{E}h(a) = Uf\mathfrak{A}$, which means that $h(\mathsf{E}a) = \mathsf{E}h(a)$. $\square$

Theorem 2.3.15. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a permeated hybrid E-algebra, and $\mathfrak{g} = (W, R, A, B)$ a two-sorted general frame. Then*

(i) $\mathfrak{A}_*$ *is a strongly descriptive two-sorted general frame,*

(ii) $(\mathfrak{A}_*)^* \cong \mathfrak{A}$, *and*

(iii) $(\mathfrak{g}^*)_* \cong \mathfrak{g}$ *iff* $\mathfrak{g}$ *is strongly descriptive.*

Chapter 3

Completeness with respect to hybrid algebras

Now that we know which kinds of algebras correspond to the hybrid logics in Chapter 1, our next goal is to put these algebras to the test and see what uses they have in terms of solving logical problems in the field of hybrid logic. In this chapter, we focus on proving *algebraic completeness* for each of the hybrid logics defined in Chapter 1. The general pattern is as follows: the axiomatizations without the additional 'non-orthodox' rules are complete with respect to the class of hybrid algebras, whereas the axiomatizations with the additional 'non-orthodox' rules are complete with respect to the class of permeated hybrid algebras. Combining the results in this chapter with our duality results in Chapter 2, we see that the axiomatizations without the additional 'non-orthodox' rules are complete with respect to the class of descriptive two-sorted general frames, while the axiomatizations with the additional 'non-orthodox' rules are complete with respect to the class of strongly descriptive two-sorted general frames, thus reaffirming Ten Cate's completeness results in [72].

3.1 Algebraic completeness of $\mathbf{H} \oplus \Sigma$

Recall that in Subsection 1.1.4 we showed that modal logics are complete with respect Boolean algebras with operators by making use of the Lindenbaum-Tarski method. One naturally wonders if the same method can be applied to show that hybrid logics are complete with respect to hybrid algebras. The answer is *no*. For this to work, the equivalence classes $[\mathbf{i}]$, $\mathbf{i} \in \mathsf{NOM}$, must be atoms of the Lindenbaum-Tarski algebra. Actually, the sorted substitution rule tells us that it is enough for only one such equivalence class to be an atom. However, for all $\mathbf{j} \in \mathsf{NOM}$, $[\mathbf{j}]$ is not an atom of the Lindenbaum-Tarski algebra. To see this, note for instance that $\vdash_{\mathbf{H}\oplus\Sigma} \mathbf{j} \wedge \mathbf{j} \leftrightarrow \mathbf{j}$, which means that $\mathbf{j} \wedge \mathbf{j} \in [\mathbf{j}]$ for all $\mathbf{j} \in \mathsf{NOM}$. This implies that for all $\mathbf{j} \in \mathsf{NOM}$, $[\mathbf{j}]$ is not a singleton. We must therefore find a different method of proving completeness with respect to hybrid algebras.

This does not mean that we have to forget about the Lindenbaum-Tarski algebra completely. In our approach we temporarily interpret the nominals as modal constants and work with the orthodox Lindenbaum-Tarski-algebra of $\mathbf{H} \oplus \Sigma$ over PROP. But since the interpretations of the nominals in the Lindenbaum-Tarski algebra are not atoms, the Lindenbaum-Tarski algebra requires a bit of sculpting to change it into a hybrid algebra of the right kind. In order to do this, we consider the canonical extension of the Lindenbaum-Tarski algebra. The canonical extension provides us with two useful tools, namely, atomicity and the existence of $\Diamond^{-1}$. Using the atomicity of the canonical extension, we get an atom in the canonical extension. We then construct an algebra validating the axioms and refuting a non-theorem in the same way as on page 63. Recall that closing under $\Diamond^{-1}$ lets us simulate taking generated subframes algebraically. To be more precise, since we will be working with an algebra and an assignment, we are actually simulating the process of taking generated submodels. It can then be shown that the interpretation of a nominal in this algebra is either bottom or an atom. Dropping the constant interpretations, we break the proof up into cases depending on whether all the interpretations of the nominals are atoms or not. If all the interpretations of the nominals are atoms, we can just declare all the constant interpretations of the nominals as designated atoms, and we are done. On the other hand, for the cases where at least one constant interpretation is $\perp$ but not all, or all the constant interpretations are $\perp$, we will form a product of algebras that is a hybrid algebra validating the axioms and refuting a non-theorem. It is here where the preservation of the validity of a hybrid formula under taking the product of two hybrid algebras whose associated grounded hybrid algebras validates the formula comes in.

So let us get to work. We first state and prove the main theorem, and consequently prove the lemmas needed in it.

Theorem 3.1.1. *For any set Σ of $\mathcal{H}$-formulas, the logic $\mathbf{H} \oplus \Sigma$ is sound and complete with respect to the class of all hybrid algebras which validate Σ. That is to say, $\vdash_{\mathbf{H}\oplus\Sigma} \varphi$ iff $\models_{\mathsf{HA}(\Sigma)} \varphi \approx \top$.*

Proof. It is straightforward to check the "soundness" direction of the above. For the "completeness" direction, we prove the contrapositive. So suppose $\nvdash_{\mathbf{H}\oplus\Sigma} \varphi$. We need to find a hybrid algebra $\mathfrak{A}$ and an assignment v such that $\mathfrak{A}, v \not\models \varphi \approx \top$. For the purpose of this proof, we will temporarily treat the nominals as modal constants and work with orthodox interpretations of $\mathcal{H}$.

Now, consider the orthodox Lindenbaum-Tarski algebra of $\mathbf{H} \oplus \Sigma$ over PROP. For the sake of brevity, we will denote this algebra simply by $\mathbf{A}$. Note that $[\neg\varphi] > \perp$ in $\mathbf{A}$, for suppose not, then $\neg\varphi \equiv \perp$, which means that $\vdash_{\mathbf{H}\oplus\Sigma} \neg\varphi \leftrightarrow \perp$. Hence, $\vdash_{\mathbf{H}\oplus\Sigma} \varphi \leftrightarrow \top$, and so $\vdash_{\mathbf{H}\oplus\Sigma} \top \to \varphi$. But $\vdash_{\mathbf{H}\oplus\Sigma} \top$, so, by (*Modus Ponens*), $\vdash_{\mathbf{H}\oplus\Sigma} \varphi$, which is a contradiction.

The fact that $\mathbf{A}$ validates precisely the theorems of $\mathbf{H} \oplus \Sigma$ is proved in the usual way. To see that $\mathbf{A} \not\models \varphi \approx \top$, let ν be the map $\nu\colon \mathsf{PROP} \to A$ defined by $\nu(p) = [p]$. It can easily be verified by straightforward structural induction that $\widetilde{\nu}(\psi) = [\psi]$ for all formulas ψ that use variables from the set PROP. But then $\widetilde{\nu}(\varphi) = [\varphi] \neq [\top] = \widetilde{\nu}(\top)$, for otherwise, $[\varphi] = [\top]$, which means that $[\neg\varphi] = [\perp]$, a contradiction.

Next, consider the orthodox canonical extension $\mathbf{A}^\delta$ of $\mathbf{A}$. First, note that since all axioms of $\mathbf{H}$ are Sahlqvist under the orthodox interpretation, it follows from the canonicity of Sahlqvist equations that $\mathbf{A}^\delta \models \mathbf{H}^\approx$ (see Theorem 1.1.65). However, the validity of the equations in $\Sigma^\approx$ is not necessarily preserved in passing from $\mathbf{A}$ to $\mathbf{A}^\delta$.

We also know from Example 2.1.48 that $\Diamond$ and $\Box$ have right and left adjoints, respectively, denoted by $\Box^{-1}$ and $\Diamond^{-1}$, respectively. Furthermore, $[\neg\varphi] > \bot$ in $\mathbf{A}^\delta$. So since $\mathbf{A}^\delta$ is atomic, there is some atom d in $\mathbf{A}^\delta$ such that $d \leq [\neg\varphi]$. Denote d by d_0. Suppose d_n is already defined, then let $d_{n+1} = \Diamond^{-1} d_n$ and

$$D = \bigvee_{n\in\mathbb{N}} d_n.$$

Define

$$\mathbf{A}_D \;\;=\;\; (A_D, \wedge^D, \vee^D, \neg^D, \bot^D, \top^D, \Diamond^D, \{s^D_{\mathbf{j}}\}_{\mathbf{j}\in\mathsf{NOM}}),$$

where $A_D = \{a \wedge D \mid a \in A\}$, $\wedge^D$ and $\vee^D$ are the restriction of $\wedge$ and $\vee$ to A_D, and

$$\begin{aligned} \neg^D a &= \neg a \wedge D & \qquad \Diamond^D a &= \Diamond a \wedge D \\ s^D_{\mathbf{j}} &= s_{\mathbf{j}} \wedge D & \top^D &= D \\ \bot^D &= \bot \end{aligned}$$

Now, by Lemma 3.1.3, $\mathfrak{A}_D$ is an algebra. We also know from Lemma 3.1.4 that $\mathbf{A}_D \models \mathbf{H} \oplus \Sigma^{\approx}$. To see that $\mathbf{A}_D \not\models \varphi \approx \top$, consider the assignment ν_D: $\mathsf{PROP} \to A_D$ given by $\nu_D(p) = h(\nu(p))$. Using the fact that h is a homomorphism, we can show by structural formula induction that $\widetilde{\nu_D}(\psi) = h(\widetilde{\nu}(\psi))$ for all $\mathcal{H}$-formulas ψ that use variables from PROP. Now, since $d \leq D$ and $d \leq [\neg\varphi]$,

$$\widetilde{\nu_D}(\neg\varphi) = h(\widetilde{\nu}(\neg\varphi)) = \widetilde{\nu}(\neg\varphi) \wedge D = [\neg\varphi] \wedge D \geq d > \bot$$

Hence, $\widetilde{\nu_D}(\neg\varphi) \neq \bot^d$, and so $\widetilde{\nu_D}(\varphi) \neq D$, i.e., $\widetilde{\nu_D}(\varphi) \neq \widetilde{\nu_D}(\top)$.

Now, if we can show that the constant interpretations of the nominals are atoms, we can just drop the constant interpretations and put all $s^D_{\mathbf{i}}$, $\mathbf{i} \in \mathsf{NOM}$, in our designated set of atoms. However, this is not necessarily the case. By Lemma 3.1.6, $s^D_{\mathbf{i}}$ is either an atom or $\bot$. So let $\mathfrak{A}_D = (\mathbf{A}^-_D, X_{A_D})$, where $\mathbf{A}^-_D$ is the modal algebra reduct of $\mathbf{A}_D$ obtained by omitting the constant interpretations of nominals, and $X_{A_D} = \{s^D_{\mathbf{i}} \mid s^D_{\mathbf{i}} > \bot\}$. But this is still not necessarily a hybrid algebra since it is possible that X can be empty. In fact, we have three possibilities, so our reasoning now splits into three cases depending on whether the constant interpretations of nominals in $\mathbf{A}_D$ are atoms or not:

Case 1: $s^D_{\mathbf{i}} > \bot$ for all $\mathbf{i} \in \mathsf{NOM}$. This is the simplest case. Since $X_{A_D} \neq \varnothing$, it follows from the foregoing that $\mathfrak{A}_D$ is a hybrid algebra. Furthermore, since Σ is closed under sorted substitution, $\mathfrak{A}_D \models \Sigma^{\approx}$. To see that $\mathfrak{A}_D \not\models \varphi \approx \top$, consider the assignment $\widetilde{\nu'_D}$ which extends ν_D from PROP to $\mathsf{PROP} \cup \mathsf{NOM}$, obtained by simply setting $\nu'_D(\mathbf{i}) = s^D_{\mathbf{i}}$ for each $\mathbf{i} \in \mathsf{NOM}$. It is clear that $\widetilde{\nu'_D}(\psi) = \widetilde{\nu_D}(\psi)$ for all $\mathcal{H}$-formulas ψ, and hence, $\widetilde{\nu'_D}(\varphi) \neq \widetilde{\nu'_D}(\top)$.

Case 2: $s^D_{\mathbf{i}} = \bot$ for some $\mathbf{i} \in \mathsf{NOM}$ but not all. From the foregoing, we know that $(\mathfrak{A}_D)_0 \models \mathbf{H} \oplus \Sigma^{\approx}$, and hence, by Proposition, 2.1.16, $\mathfrak{A}_D \times \mathfrak{A}_D \models \mathbf{H} \oplus \Sigma^{\approx}$. Now, let $\mathbf{j} \in \mathsf{NOM}$ such that $s^D_{\mathbf{j}} \neq \bot$. Then $X_{A_D \times A_D} \neq \varnothing$ since $(\bot, s^D_{\mathbf{j}}) \in X_{A_D \times A_D}$, so $\mathfrak{A}_D \times \mathfrak{A}_D$ is a hybrid algebra. Finally, consider the assignment ν''_D obtained by setting $\nu''_D(p) = (\nu_D(p), \nu_D(p))$ for all propositional variables $p \in \mathsf{PROP}$, and

$$\nu''_D(\mathbf{i}) = \begin{cases} (s^D_{\mathbf{i}}, \bot) & \text{if } s^D_{\mathbf{i}} > \bot \\ (\bot, s^D_{\mathbf{j}}) & \text{if } s^D_{\mathbf{i}} = \bot \end{cases}$$

for all nominals $\mathbf{i} \in \mathsf{NOM}$. It is straightforward to show (using structural induction) that for any $\mathcal{H}$-formula ψ, we have $\widetilde{\nu''_D}(\psi) = (\widetilde{\nu_D}(\psi), a_\psi)$, where a_ψ is some element of $\mathfrak{A}_D$. But then $\widetilde{\nu''_D}(\varphi) = (\widetilde{\nu_D}(\varphi), a_\varphi) \neq (\widetilde{\nu_D}(\top), D) = \widetilde{\nu''_D}(\top)$ since $\widetilde{\nu_D}(\varphi) \neq \widetilde{\nu_D}(\top)$.

Case 3: $s^D_{\mathbf{i}} = \bot$ for all $\mathbf{i} \in \mathsf{NOM}$. In this case, $X_{A_D} = \varnothing$, so $\mathfrak{A}_D$ is not a hybrid algebra. So lets get to work finding a hybrid algebra that will work here. First, we claim that $[\mathbf{i}] > \bot$ in $\mathbf{A}$ for all $\mathbf{i} \in \mathsf{NOM}$. To see this, suppose $[\mathbf{i}] = \bot$. Then $\vdash \mathbf{i} \leftrightarrow \bot$, and so $\vdash \neg\mathbf{i} \leftrightarrow \top$. Hence, $\vdash \neg\mathbf{i}$, which means that $\vdash \bot$ by the (*NameLite*) rule. However, this is a contradiction. We thus also have that $[\mathbf{i}] > \bot$ in $\mathbf{A}^\delta$ for all $\mathbf{i} \in \mathsf{NOM}$. Now, choose some nominal $\mathbf{j} \in \mathsf{NOM}$. Since $\mathbf{A}^\delta$ is atomic, there is an atom d' in $\mathbf{A}^\delta$ such that $d' \leq [\mathbf{j}]$. So define D' and $\mathbf{A}_{D'}$ in the same way as D and $\mathbf{A}_D$. In the same way as for $\mathbf{A}_D$, we can prove that $\mathbf{A}_{D'} \models \mathbf{H} \oplus \Sigma^{\approx}$, and that $s^{D'}_{\mathbf{i}}$ is either $\bot$ or an atom of $\mathbf{A}_{D'}$. Define $\nu_{D'}$ in the same way as ν_D. Note that we do not know if $\mathbf{A}_{D'}, \nu_{D'} \not\models \varphi \approx \top$. But this is not a problem as we will soon see. Now, let $\mathfrak{A}_{D'} = (\mathbf{A}^-_{D'}, X_{A_{D'}})$ where $\mathbf{A}^-_{D'}$ is the reduct of $\mathbf{A}_{D'}$ obtained by omitting the constant interpretations of nominals, and $X_{A_{D'}} = \{s^{D'}_{\mathbf{i}} \mid s^{D'}_{\mathbf{i}} > \bot\}$. We know that $X_{A_{D'}} \neq \varnothing$ since at least $s^{D'}_{\mathbf{j}} \neq \bot$. Furthermore, $(\mathfrak{A}_{D'})_0 \models \mathbf{H} \oplus \Sigma^{\approx}$, and so, since $(\mathfrak{A}_D)_0 \models \mathbf{H} \oplus \Sigma^{\approx}$, $\mathfrak{A}_D \times \mathfrak{A}_{D'} \models \mathbf{H} \oplus \Sigma^{\approx}$ by Proposition 2.1.16. To show that $\mathfrak{A}_D \times \mathfrak{A}_{D'} \not\models \varphi \approx \top$, let ν''' be defined by $\nu'''_D(p) = (\nu_D(p), \nu_{D'}(p))$ for all propositional variables $p \in \mathsf{PROP}$, and

$$\nu'''_D(\mathbf{i}) = \begin{cases} (\bot, s^{D'}_{\mathbf{i}}) & \text{if } s^{D'}_{\mathbf{i}} > \bot \\ (\bot, s^{D'}_{\mathbf{j}}) & \text{if } s^{D'}_{\mathbf{i}} = \bot \end{cases}$$

for all nominals $\mathbf{i} \in \mathsf{NOM}$. Using formula induction, we can show that $\widetilde{\nu'''_D}(\psi) = (\widetilde{\nu_D}(\psi), a'_\psi)$, where a'_ψ is some element in $\mathfrak{A}_{D'}$, for all $\mathcal{H}$-formulas ψ. But then $\widetilde{\nu'''_D}(\varphi) = (\widetilde{\nu_D}(\varphi), a'_\varphi) \neq (\widetilde{\nu_D}(\top), D') = \widetilde{\nu'''_D}(\top)$ since $\widetilde{\nu_D}(\varphi) \neq \widetilde{\nu_D}(\top)$. □

We will now prove the lemmas used in the proof of the above theorem. Unless stated otherwise, in what follows $\mathbf{A}$, $\mathbf{A}^\delta$, $\mathbf{A_D}$, ν and ν_D will be as in the proof of Theorem 3.1.1. The first lemma we need is that $\mathbf{A}_D$ is an algebra. To prove this, we have to show that A_D is closed under the operations defined in the proof of Theorem 3.1.1. But first we need to prove the following lemma:

Lemma 3.1.2. $D \leq \Box D$

Proof.

$$\begin{aligned}
\Box D &= \Box \bigvee_{n\in\mathbb{N}} d_n && \text{(by the definition of } D\text{)}\\
&\geq \bigvee_{n\in\mathbb{N}} \Box d_n && \text{(by the monotonicity of } \Box\text{)}\\
&= \left(\bigvee_{n\in\mathbb{N}-\{0\}} \Box\Diamond^{-1} d_{n-1}\right) \vee \Box d_0 && \text{(by the definition of } d_n\text{)}\\
&\geq \left(\bigvee_{n\in\mathbb{N}-\{0\}} \Box\Diamond^{-1} d_{n-1}\right) &&\\
&\geq \bigvee_{n\in\mathbb{N}-\{0\}} d_{n-1} && \text{(since } \Box \text{ and } \Diamond^{-1} \text{ are adjoint)}\\
&= D && \text{(by the definition of } D\text{)}
\end{aligned}$$

□

Lemma 3.1.3. *A_D is closed under the operations $\wedge^D$, $\vee^D$, $\neg^D$, and $\Diamond^D$.*

Proof. The cases for $\wedge^D$ and $\vee^D$ are straightforward, so we consider the cases for $\neg^D$ and $\Diamond^D$. Let $a \in A_D$. Then $a = a' \wedge D$ for some $a' \in A$. Now, for $\neg^D$,

$$\neg^D a = \neg(a' \wedge D) \wedge D = (\neg a' \vee \neg D) \wedge D = (\neg a' \wedge D) \vee \bot = \neg a' \wedge D$$

But A is closed under $\neg$, so $\neg a' \in A$, which means that $\neg^D a \in A_D$. Finally, for $\Diamond^D$,

$$\Diamond^D a = \Diamond(a' \wedge D) \wedge D \leq \Diamond a' \wedge D,$$

and, conversely,

$$\begin{aligned}
\Diamond^D a &= \Diamond(a' \wedge D) \wedge D\\
&\geq \Diamond a' \wedge \Box D \wedge D\\
&= \Diamond a' \wedge D,
\end{aligned}$$

where the last step follows from Lemma 3.1.2. Hence, $\Diamond^D a = \Diamond a' \wedge D$, and so, since A is closed under $\Diamond$, $\Diamond a' \in A$. We thus have that $\Diamond^D a \in A_d$. □

To show that the algebra $\mathbf{A}_D$ validates the equations in $\mathbf{H} \oplus \Sigma^{\approx}$, we prove that $\mathbf{A}_D$ is a homomorphic image of $\mathbf{A}$.

Lemma 3.1.4. *The map $h\colon A \to A_D$ defined by $h(a) = a \wedge D$ is a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{A}_D$.*

Proof. First, h is clearly surjective. In verifying that h is a homomorphism, all cases except those for $\neg$, and $\Diamond$ are straightforward. The case for $\neg$ is proved as follows:

$$\begin{aligned} h(\neg a) &= \neg a \wedge D \\ &= (\neg a \wedge D) \vee (\neg D \wedge D) \\ &= (\neg a \vee \neg D) \wedge D \\ &= \neg(a \wedge D) \wedge D \\ &= \neg^D (a \wedge D) \\ &= \neg^D h(a) \end{aligned}$$

The right-to-left inequality for $\Diamond$ is proved as follows:

$$\begin{aligned} h(\Diamond a) &= \Diamond a \wedge D \\ &\geq \Diamond(a \wedge D) \wedge D \\ &= \Diamond h(a) \wedge D \\ &= \Diamond^D h(a), \end{aligned}$$

where the inequality follows from the fact that $a \wedge D \leq a$, and the monotonicity of $\Diamond$. Conversely,

$$\begin{aligned} \Diamond^D h(a) &= \Diamond(a \wedge D) \wedge D \\ &\geq \Diamond a \wedge \Box D \wedge D \\ &= \Diamond a \wedge D \\ &= h(\Diamond a) \end{aligned}$$

Here the third step follows from Lemma 3.1.2. □

Our next goal is to show that the constant interpretation of a nominal is either an atom of $\mathbf{A}_D$ or $\perp$. But first, we need the following result:

Lemma 3.1.5. *Let $\mathbf{A}$ be a BAO, and let a and b be atoms of the canonical extension $\mathbf{A}^\delta$ of $\mathbf{A}$. Then $a \leq (\Diamond^{-1})^m b$ iff $b \leq \Diamond^m a$.*

Proof. For the left-to-right direction, let a and b be two atoms in $\mathbf{A}^\delta$, and assume $a \leq (\Diamond^{-1})^m b$. Suppose for the sake of a contradiction that $b \nleq \Diamond^m a$. Then $b \leq \neg\Diamond^m a$, and so $\Diamond^m a \leq \neg b$. By the monotonicity of $\Box^{-1}$, $\Box^{-1}\Diamond^m a \leq \Box^{-1}\neg b$. But $\Diamond^{m-1} a \leq \Box^{-1}\Diamond\Diamond^{m-1} a$, so $\Diamond^{m-1} a \leq \Box^{-1}\neg b$. Continuing this, we get that $a \leq (\Box^{-1})^m \neg b$, which means that $(\Diamond^{-1})^m b \leq \neg a$. Hence, by our assumption, $a \leq \neg a$, and so $\perp = a \wedge \neg a = a$, which is a contradiction.

For the converse, assume $b \leq \Diamond^m a$, and suppose $a \nleq (\Diamond^{-1})^m b$. Then $a \leq \neg(\Diamond^{-1})^m b$, and so $(\Diamond^{-1})^m b \leq \neg a$. Now, $(\Diamond^{-1})^{m-1} b \leq \Box(\Diamond^{-1})^m b \leq \Box\neg a$. If we continue this, we get that $b \leq \Box^m \neg a$. Hence, $\Diamond^m a \leq \neg b$, and so $b \leq \neg b$, which is a contradiction. □

Lemma 3.1.6. *For each $\mathbf{i} \in \mathsf{NOM}$, $s_{\mathbf{i}}^D$ is either $\perp$ or an atom of $\mathbf{A}^\delta$, and, hence, of $\mathbf{A}_D$.*

Proof. Suppose that $s_{\mathbf{i}}^D \neq \bot$. We show that if $a, b \in At\mathbf{A}^\delta$ such that $a, b \leq s_{\mathbf{i}}^D$, then $a = b$. So let $a, b \in At\mathbf{A}^\delta$ such that $a, b \leq s_{\mathbf{i}}^D$. Then $a, b \leq D$, and so $a \leq (\Diamond^{-1})^{n_1} d$ and $b \leq (\Diamond^{-1})^{n_2} d$ for some natural numbers n_1 and n_2. We thus have that $d \leq \Diamond^{n_1} a$ and $d \leq \Diamond^{n_2} b$ by Lemma 3.1.5, and so, since $a, b \leq s_{\mathbf{i}}^d$, $d \leq \Diamond^{n_1}(a \wedge s_{\mathbf{i}}^d)$ and $d \leq \Diamond^{n_2}(b \wedge s_{\mathbf{i}}^d)$. Hence,

$$d \leq \Diamond^{n_1}(a \wedge s_{\mathbf{i}} \wedge D) \leq \Diamond^{n_1}(a \wedge s_{\mathbf{i}})$$

and

$$d \leq \Diamond^{n_2}(b \wedge s_{\mathbf{i}} \wedge D) \leq \Diamond^{n_2}(b \wedge s_{\mathbf{i}}),$$

so, $d \leq \Box^m(\neg s_{\mathbf{i}} \vee a)$ and $d \leq \Box^m(\neg s_{\mathbf{i}} \vee b)$, where m is any natural number. So, since $\Diamond^{-1}$ and $\Box$ are adjoint,

$$(\Diamond^{-1})^m d \leq \neg s_{\mathbf{i}} \vee a$$

and

$$(\Diamond^{-1})^m d \leq \neg s_{\mathbf{i}} \vee b,$$

which means that, for all $m \in \mathbb{N}$, it holds that $(\Diamond^{-1})^m d \leq (\neg s_{\mathbf{i}} \vee a) \wedge (\neg s_{\mathbf{i}} \vee b)$. It thus follows that $D \leq (\neg s_{\mathbf{i}} \vee a) \wedge (\neg s_{\mathbf{i}} \vee b) = \neg s_{\mathbf{i}} \vee (a \wedge b)$. Now, if $a \neq b$, $D \leq \neg s_{\mathbf{i}}$, so $s_{\mathbf{i}} \leq \neg D$. Hence, $s_{\mathbf{i}} \wedge D \leq \bot$, i.e., $s_{\mathbf{i}}^D = \bot$, contradicting our assumption that $s_{\mathbf{i}}^D \neq \bot$. $\Box$

3.2 Algebraic completeness of $\mathbf{H}(@) \oplus \Sigma$

Recall that the validity of $\mathcal{H}(@)$-formulas is preserved under taking generated subframes. Normally the preservation of validity on frames follows from the invariance of satisfaction on models. However, there is a strangeness here: the truth of $\mathcal{H}(@)$-formulas is not generally transferred from the supermodel to the submodel when taking generated submodels. To see this, consider the models $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$, where $W = \{u, v\}, R = \{(u, u)\}, V(p) = \{v\}, V(q) = \{u\}, V(\mathbf{i}) = \{v\}, V(\mathbf{j}) = \{u\}, W' = \{u\}, R' = \{(u, u)\}, V'(q) = \{u\}$ and $V'(\mathbf{j}) = \{u\}$ (see Figure 3.1). It is not difficult to see that $\mathfrak{M}'$ is a generated submodel of $\mathfrak{M}$. However, $\mathfrak{M} \Vdash \mathbf{j} \wedge \Diamond q \wedge @_{\mathbf{i}} p$, while $\mathfrak{M}' \nVdash \mathbf{j} \wedge \Diamond q \wedge @_{\mathbf{i}} p$. The @-operators allow us to jump freely to a state named by a nominal, bypassing the accessibility relation, and when we generated with only u, we lost the part of the model where $@_{\mathbf{i}} p$ is true. This means that we cannot use the same approach as in the proof of Theorem 3.1.1 to prove the completeness of $\mathbf{H}(@) \oplus \Sigma$. Not to worry, this is certainly something we can fix: simply generate from all the interpretations of the nominals in the formula. This is also the route we will follow in our simulation process, so it is time to roll up our sleeves. We first give the statement of the completeness result and its proof, and subsequently prove the lemmas needed in the proof.

Theorem 3.2.1. *For any set Σ of $\mathcal{H}(@)$-formulas, the logic $\mathbf{H}(@) \oplus \Sigma$ is sound and complete with respect to the class of all hybrid @-algebras which validate Σ. That is to say, $\vdash_{\mathbf{H}(@) \oplus \Sigma} \varphi$ iff $\models_{\mathsf{H@A}(\Sigma)} \varphi \approx \top$.*

Proof. Suppose $\nvdash_{\mathbf{H}(@) \oplus \Sigma} \varphi$. We need to find a hybrid @-algebra $\mathfrak{A}$ and an assignment v such that $\mathfrak{A}, v \not\models \varphi \approx \top$. As in the proof of Theorem 3.1.1, we will work with the orthodox interpretation of $\mathcal{H}(@)$ for the time being.

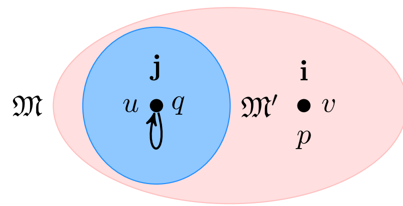

Figure 3.1: A model with a generated submodel

As before, we begin with the orthodox Lindenbaum-Tarski algebra of $\mathbf{H}(@) \oplus \Sigma$ over PROP. For simplicity, denote it by $\mathbf{A}$. In the usual way, we can show that $\mathbf{A}$ validates precisely the theorems of $\mathbf{H}(@) \oplus \Sigma$. Also, $\mathbf{A}, \nu \not\models \varphi \approx \top$, where ν is the natural map taking p to $[p]$. In the same way as in the proof of Theorem 3.1.1, we can also show that $[\neg\varphi] > \bot$ in $\mathbf{A}$.

Next, consider the orthodox canonical extension $\mathbf{A}^\delta$ of $\mathbf{A}$. We know that $[\neg\varphi] > \bot$ in $\mathbf{A}^\delta$, so, since $\mathbf{A}^\delta$ is atomic, there is an atom $d \in \mathbf{A}^\delta$ such that $d \leq [\neg\varphi]$. We will get back to this point shortly, but first denote the constant interpretations of the nominals occurring in φ by $s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_m}$. Note that, for each $1 \leq i \leq m$, $s_{\mathbf{i}_i} \neq \bot$ in $\mathbf{A}^\delta$, for otherwise, since $\mathbf{A}^\delta$ validates the axioms of $\mathbf{H}(@)$, $\top = @_{s_{\mathbf{i}_i}} s_{\mathbf{i}_i} = @_{s_{\mathbf{i}_i}} \bot = \bot$, a contradiction. Hence, we also have atoms $d_0^1, d_0^2, \ldots, d_0^m$ in $\mathbf{A}^\delta$ such that $d_0^1 \leq s_{\mathbf{i}_1}, d_0^2 \leq s_{\mathbf{i}_2}, \ldots, d_0^m \leq s_{\mathbf{i}_m}$. Now, let $d_0^0 = d$, and suppose that for each $0 \leq i \leq m$, d_n^i is already defined. For each $0 \leq i \leq m$, let $d_{n+1}^i = \Diamond^{-1} d_n^i$ and

$$D_i = \bigvee_{n \in \mathbb{N}} d_n^i.$$

Furthermore, let

$$D = \bigvee_{0 \leq i \leq m} D_i$$

and

$$\mathbf{A}_D = (A_D, \wedge^D, \vee^D, \neg^D, \bot^D, \top^D, \Diamond^D, @^D, \{s_{\mathbf{j}}^D\}_{\mathbf{j} \in \mathsf{NOM}}),$$

where $A_D = \{a \wedge D \mid a \in A\}$, $\wedge^D$ and $\vee^D$ are the restriction of $\wedge$ and $\vee$ to A_D, for all $\mathbf{j} \in \mathsf{NOM}$,

$$@^D_{s_{\mathbf{j}}^D} a = \begin{cases} D & \text{if } s_{\mathbf{j}}^D \leq a \\ \bot & \text{if } s_{\mathbf{j}}^D \leq \neg^D a \end{cases}$$

and

$$\neg^D a = \neg a \wedge D \qquad \Diamond^D a = \Diamond a \wedge D$$
$$\top^D = D \qquad \bot^D = \bot$$
$$s_{\mathbf{j}}^D = s_{\mathbf{j}} \wedge D$$

Now, by Lemma 3.2.3 below, A_D is closed under the above operations, so $\mathbf{A}_D$ is an algebra. Furthermore, from Lemma 3.2.4 it follows that $\mathbf{A}_D$ is an orthodox interpretation of $\mathcal{H}(@)$, and that $\mathbf{A}_D \models \mathbf{H}(@) \oplus \Sigma^{\approx}$. To show that $\mathbf{A}_D \not\models \varphi \approx \top$, let ν_D: $\mathsf{PROP} \to A_D$ be the assignment given by $\nu_D(p) = h(\nu(p))$. Using the fact that h is a homomorphism, we can show by structural formula induction that $\widetilde{\nu_D}(\psi) = h(\widetilde{\nu}(\psi))$ for all $\mathcal{H}(@)$-formulas ψ that use variables from PROP. Now, since $d \leq D$ and $d \leq [\neg\varphi]$,

$$\widetilde{\nu_D}(\neg\varphi) = h(\widetilde{\nu}(\neg\varphi)) = \widetilde{\nu}(\neg\varphi) \wedge D = [\neg\varphi] \wedge D \geq d > \bot.$$

Hence, $\widetilde{\nu_D}(\varphi) \neq D$.

Now, if we can find a suitable designated set of atoms in $\mathbf{A}_D$, we can drop the constant interpretations and we would be done. Luckily, unlike for the language $\mathcal{H}$, the @ operator makes things easier for us as it ensures that all $s^D_{\mathbf{j}}$ are atoms of $\mathbf{A}_D$ (see Lemma 3.2.5). So let $\mathfrak{A}_D = (\mathbf{A}^-_D, X_{A_D})$, where $\mathbf{A}^-_D$ is the reduct of $\mathbf{A}_D$ obtained by omitting the constant interpretations of nominals, and $X_{A_D} = \{s^D_{\mathbf{j}} \mid \mathbf{j} \in \mathsf{NOM}\}$. Then it follows from the foregoing that $\mathfrak{A}_D$ is a hybrid @-algebra, and, since Σ is closed under sorted substitution, $\mathfrak{A}_D \models \Sigma^{\approx}$. To show that $\mathfrak{A}_D \not\models \varphi \approx \top$, consider the assignment ν'_D which extends ν_D from PROP to $\mathsf{PROP} \cup \mathsf{NOM}$ by simply setting $\nu'_D(\mathbf{j}) = s^D_{\mathbf{j}}$ for each $\mathbf{j} \in \mathsf{NOM}$. Clearly, $\widetilde{\nu'_D}(\psi) = \widetilde{\nu_D}(\psi)$ for all $\mathcal{H}(@)$-formulas ψ. Hence, $\widetilde{\nu'_D}(\varphi) = \widetilde{\nu_D}(\varphi) \neq \widetilde{\nu_D}(\top) = \widetilde{\nu'_D}(\top)$. $\square$

In what follows, $\mathbf{A}$, its canonical extension $\mathbf{A}^\delta$, and $\mathbf{A}_D$ will be the algebras in the proof of Theorem 3.2.1. Now, as in the previous section, we have the following:

Lemma 3.2.2. $D \leq \Box D$

Proof.

$$\begin{aligned}
\Box D &= \Box \bigvee_{1 \leq i \leq m} D_i && \text{(by the definition of } D\text{)}\\
&= \Box \bigvee_{1 \leq i \leq m} \left(\bigvee_{n \in \mathbf{N}} d^i_n \right) && \text{(by the definition of } D_i\text{)}\\
&\geq \bigvee_{1 \leq i \leq m} \left(\bigvee_{n \in \mathbb{N}} \Box d^i_n \right) && \text{(by the monotonicity of } \Box\text{)}\\
&= \bigvee_{1 \leq i \leq m} \left(\bigvee_{n \in \mathbb{N} - \{0\}} \Box \Diamond^{-1} d^i_{n-1} \right) \vee \Box d^i_0 && \text{(by the definition of } d^i_n\text{)}\\
&\geq \bigvee_{1 \leq i \leq m} \left(\bigvee_{n \in \mathbb{N} - \{0\}} \Box \Diamond^{-1} d^i_{n-1} \right) &&\\
&\geq \bigvee_{1 \leq i \leq m} \left(\bigvee_{n \in \mathbb{N} - \{0\}} d^i_{n-1} \right) && \text{(since } \Diamond^{-1} \text{ and } \Box \text{ are adjoint)}\\
&= \bigvee_{1 \leq i \leq m} D_i && \text{(by the definition of } D_i\text{)}\\
&= D && \text{(by the definition } D\text{)}
\end{aligned}$$

$\square$

Using this result, we can prove the following two lemmas:

Lemma 3.2.3. *A_D is closed under the operations $\wedge^D$, $\vee^D$, $\neg^D$, $\Diamond^D$, and $@^D$.*

Proof. We only consider the case for $@^D$. The other cases are proved in the same way as in Lemma 3.1.3 using Lemma 3.2.2. Let $a \in A_D$. If $s_{\mathbf{j}}^D \leq a$, $@^D_{s_{\mathbf{j}}^D} a = D$. But $D \in A_D$, so $@^D_{s_{\mathbf{j}}^D} a \in A_D$. On the other hand, if $s_{\mathbf{j}}^D \leq \neg^D a$, then $@^D_{s_{\mathbf{j}}^D} a = \perp$, which means that $@^D_{s_{\mathbf{j}}^D} a \in A_D$ since $\perp$ is in A_D. $\square$

Lemma 3.2.4. *The map $h\colon A \to A_D$ defined by $h(a) = a \wedge D$ is a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{A}_D$.*

Proof. That h is surjective is obvious. In verifying that h is a homomorphism, all cases except those for $\neg$, $\Diamond$, and $@_{\mathbf{j}}$ are straightforward. $\neg$ and $\Diamond$ are proved in exactly the same way as in Lemma 3.1.4. So we need only check $@$. We consider the following two cases:

Case 1: $s_{\mathbf{j}} \leq a$**.** In this case, by Proposition 2.2.3, $h(@_{s_{\mathbf{j}}} a) = h(\top) = D$. From $s_{\mathbf{j}} \leq a$, we have $s_{\mathbf{j}} \wedge D \leq a \wedge D$. Hence, $h(s_{\mathbf{j}}) \leq h(a)$, which means that $@^D_{h(s_{\mathbf{j}})} h(a) = D$.

Case 2: $s_{\mathbf{j}} \leq \neg a$**.** Here $h(@_{s_{\mathbf{j}}} a) = h(\perp) = \perp$. From $s_{\mathbf{j}} \leq \neg a$, we have $s_{\mathbf{j}} \wedge D \leq \neg a \wedge D$. Hence, $h(s_{\mathbf{j}}) \leq h(\neg a) = \neg^D h(a)$, giving $@^D_{h(s_{\mathbf{j}})} h(a) = \perp$. $\square$

The final lemma tells us that all the $s_{\mathbf{i}}^D$ are actually atoms of $\mathbf{A}_D$.

Lemma 3.2.5. *For each $\mathbf{i} \in \mathsf{NOM}$, $s_{\mathbf{i}}^D$ an atom of $\mathbf{A}^\delta$, and hence, of $\mathbf{A}_D$.*

Proof. First, $s_{\mathbf{i}}^D \neq \perp$, for otherwise, $D = @^D_{s_{\mathbf{i}}^D} s_{\mathbf{i}}^D = @^D_{s_{\mathbf{i}}^D} \perp = \perp$, which is not possible. Now, let $a, b \in \mathbf{A}^\delta$ such that $a \leq s_{\mathbf{i}}^D$ and $b \leq s_{\mathbf{i}}^D$. We want to show that $a = b$, so suppose that they are not equal for the sake of a contradiction. From $a \leq s_{\mathbf{i}}^D$ and $b \leq s_{\mathbf{i}}^D$, we have $a, b \leq s_{\mathbf{i}}$ and $a, b \leq D$. This means there are $n_1, n_2 \in \mathbb{N}$ and $0 \leq j_1, j_2 \leq m$ such that $a \leq (\Diamond^{-1})^{n_1} d_0^{j_1}$ and $b \leq (\Diamond^{-1})^{n_2} d_0^{j_2}$. Hence, by Lemma 3.1.5, $d_0^{j_1} \leq \Diamond^{n_1} a$ and $d_0^{j_2} \leq \Diamond^{n_2} b$. But since $a, b \leq s_{\mathbf{i}}$, $d_0^{j_1} \leq \Diamond^{n_1}(s_{\mathbf{i}} \wedge a)$ and $d_0^{j_2} \leq \Diamond^{n_2}(s_{\mathbf{i}} \wedge b)$. We thus have

$$d_0^{j_1} \leq \Diamond^{n_1}(s_{\mathbf{i}} \wedge a) \leq \Diamond^{n_1} @_{s_{\mathbf{i}}} a \leq @_{s_{\mathbf{i}}} a$$

and

$$d_0^{j_2} \leq \Diamond^{n_2}(s_{\mathbf{i}} \wedge b) \leq \Diamond^{n_2} @_{s_{\mathbf{i}}} b \leq @_{s_{\mathbf{i}}} b.$$

This gives $@_{s_{\mathbf{i}}} a = \top$ and $@_{s_{\mathbf{i}}} b = \top$, for otherwise, $d_0^{j_1} = d_0^{j_2} = \perp$, a contradiction. Hence, $@_{s_{\mathbf{i}}} a \wedge @_{s_{\mathbf{i}}} b = \top$, which means that $@_{s_{\mathbf{i}}}(a \wedge b) = \top$. But $a \neq b$, so $\perp = @_{s_{\mathbf{i}}}(a \wedge b) = \top$, which is a contradiction. $\square$

3.3 Algebraic completeness of $\mathbf{H}(\mathsf{E}) \oplus \Sigma$

Recall that the validity of any modal formula containing the global modality is in general not preserved under taking generated subframes. We can also show that the truth of a $\mathcal{H}(\mathsf{E})$-formula is not transferred when we take generated submodels. To see this, consider the models $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$, where $W = \{u, v, w\}, R = \{(u, v), (v, u)\}, V(p) = W, W' = \{u, v\}, \{(u, v), (v, u)\}$ and $V'(p) = W'$ (see Figure 3.2). Clearly, $\mathfrak{M}'$ is a generated submodel of $\mathfrak{M}$. But $\mathfrak{M} \Vdash \mathsf{E}\neg\Diamond p$, while $\mathfrak{M}' \nVdash \mathsf{E}\neg\Diamond p$. The global modality allow us to jump freely to any state in a model, bypassing the accessibility relation, so when we generate, we may throw these states away and therefore validity is not transferred. To fix this, we have to use the accessibility relation of the global modality to generate, however, this simply gives us the original model back. So in order to build an algebra refuting a non-theorem like we did in the proofs of Theorems 3.1.1 and 3.2.1, we will have to take a different approach. As before, we will start with the orthodox Lindenbaum-Tarski algebra, and then consider an atom in the canonical extension. Since the canonical extension is isomorphic to the complex algebra of the ultrafilter frame of the Lindenbaum-Tarski algebra, we can think of this atom as some singleton of the domain of the ultrafilter frame. But instead of generating a submodel like we did before, we will 'cut' out a piece of the ultrafilter frame containing this ultrafilter falsifying the non-theorem, and where the accessibility relation of the global modality is the universal relation.

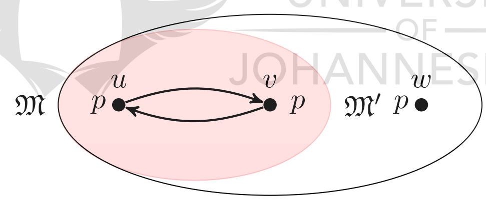

Figure 3.2: A model with a generated submodel demonstrating that the truth of $\mathcal{H}(\mathsf{E})$-formulas is not transferred when taking submodels

So let us get to work. As before, we first prove completeness, and subsequently prove the lemmas needed.

Theorem 3.3.1. *For any set Σ of $\mathcal{H}(\mathsf{E})$-formulas, the logic $\mathbf{H}(\mathsf{E}) \oplus \Sigma$ is sound and complete with respect to the class of all hybrid E-algebras which validate Σ. That is to say, $\vdash_{\mathbf{H}(\mathsf{E})\oplus\Sigma} \varphi$ iff $\models_{\mathsf{HEA}(\Sigma)} \varphi \approx \top$.*

Proof. Suppose $\nvdash_{\mathbf{H}(\mathsf{E})\oplus\Sigma} \varphi$. We need to find a hybrid E-algebra $\mathfrak{A}$ and an assignment v such that $\mathfrak{A}, v \not\models \varphi \approx \top$. Again, we will mostly work with the orthodox interpretation of $\mathcal{H}(\mathsf{E})$.

As before, we begin with the orthodox Lindenbaum-Tarski algebra of $\mathbf{H}(\mathsf{E}) \oplus \Sigma$ over PROP. For simplicity, we will denote it by $\mathbf{A}$. The fact that $\mathbf{A}$ validates precisely the theorems of $\mathbf{H}(\mathsf{E}) \oplus \Sigma$ is proved as usual. Also, $\mathbf{A}, \nu \not\models \varphi \approx \top$, where ν is the natural map taking p to $[p]$. We also have $[\neg\varphi] > \bot$ in $\mathbf{A}$ as before.

Next, consider the orthodox canonical extension $\mathbf{A}^\delta$ of $\mathbf{A}$. We know that $[\neg\varphi] > \bot$ in $\mathbf{A}^\delta$.

So let $c \in At\mathbf{A}^\delta$ such that $c \leq [\neg\varphi]$, and define

$$C = \bigwedge\{a \in \mathsf{A}^\delta \mid c \leq a \text{ and } \mathsf{E}a = a\}.$$

Let

$$\mathbf{A}_C \quad = \quad (A_C, \wedge^C, \vee^C, \neg^C, \bot^C, \top^C, \Diamond^C, \mathsf{E}^C, \{s^C_{\mathbf{j}}\}_{\mathbf{j}\in\mathsf{NOM}}),$$

where $A_C = \{a \wedge C \mid a \in A\}$, $\wedge^C$ and $\vee^C$ are the restriction of $\wedge$ and $\vee$ to A_C, and

$$\begin{aligned} \neg^C a &= \neg a \wedge C & \qquad \Diamond^C a &= \Diamond a \wedge C \\ s^C_{\mathbf{j}} &= s_{\mathbf{j}} \wedge C & \mathsf{E}^C a &= \mathsf{E}a \wedge C \\ \top^C &= C & \bot^C &= \bot \end{aligned}$$

First, by Lemma 3.3.4, $\mathbf{A}_C$ is an algebra. Second, by Lemma 3.3.5, we have that $\mathbf{A}_C \models \mathbf{H}(\mathsf{E}) \oplus \Sigma^{\approx}$. To show that $\mathbf{A}_C \not\models \varphi \approx \top$, let ν_C: $\mathsf{PROP} \to A_C$ be the assignment given by $\nu_C(p) = h(\nu(p))$. Using the fact that h is a homomorphism, we can show by structural induction that $\nu_C(\psi) = h(\nu(\psi))$ for all formulas ψ that use variables from PROP. Now, since $c \leq C$ and $c \leq [\neg\varphi]$,

$$\nu_C(\neg\varphi) = h(\nu(\neg\varphi)) = \nu(\neg\varphi) \wedge C = [\neg\varphi] \wedge C \geq c > \bot.$$

Hence, $\nu_C(\neg\varphi) \neq \bot$, which means that $\nu_C(\varphi) \neq C$.

Now, by Lemma 3.3.7, *all* $s^C_{\mathbf{i}}$ are atoms of the algebra $\mathbf{A}_C$. So let $\mathfrak{A}_C = (\mathbf{A}^-_C, X_{A_C})$, where $\mathbf{A}^-_C$ is the reduct of $\mathbf{A}_C$ obtained by omitting the constant interpretations of nominals, and $X_{A_C} = \{s^C_{\mathbf{i}} \mid \mathbf{i} \in \mathsf{NOM}\}$. Hence, it follows from the fact that $X_{A_C} \neq \varnothing$ and Lemma 3.3.6 that $\mathfrak{A}_C$ is a hybrid E-algebra. Furthermore, from the fact that Σ is closed under substitution, it follows that $\mathfrak{A}_C \models \Sigma^{\approx}$. To see that $\mathfrak{A}_C \not\models \varphi \approx \top$, let ν'_C be the assignment that extends ν_C from PROP to $\mathsf{PROP} \cup \mathsf{NOM}$ by simply setting $\nu'_C(\mathbf{i}) = s^C_{\mathbf{i}}$ for each $\mathbf{i} \in \mathsf{NOM}$. Clearly, $\widetilde{\nu'_C}(\psi) = \widetilde{\nu_C}(\psi)$ for all $\mathcal{H}(\mathsf{E})$-formulas ψ. Hence, $\widetilde{\nu'_C}(\varphi) = \widetilde{\nu_C}(\varphi) \neq \widetilde{\nu_C}(\top) = \widetilde{\nu'_C}(\top)$. $\square$

In the lemmas that follow, unless stated otherwise, $\mathbf{A}$, $\mathbf{A}^\delta$, $\mathbf{A}_C$, ν and ν_C will be as in the proof of Theorem 3.3.1.

Lemma 3.3.2. *Let $\mathbf{A}$ be an orthodox interpretation of $\mathcal{H}(\mathsf{E})$, and let $a \in A$. Then $\mathsf{E}a = a$ iff $\mathsf{A}a = a$.*

Proof. Let $a \in A$, and assume $\mathsf{E}a = a$. Then $\mathsf{AE}a = \mathsf{A}a$, and so, since $a \leq \mathsf{AE}a$ by (sym_{E}), $a \leq \mathsf{A}a$. But we also have $\mathsf{A}a \leq a$ by ($refl_{\mathsf{E}}$), so $\mathsf{A}a = a$. Conversely, assume $\mathsf{A}a = a$. Then $\mathsf{EA}a = \mathsf{E}a$. But by ($sym_{\mathsf{E}}$), $\mathsf{EA}a \leq a$, which gives $a \geq \mathsf{E}a$. Hence, since $a \leq \mathsf{E}a$ by ($refl_{\mathsf{E}}$), $\mathsf{E}a = a$. $\square$

Lemma 3.3.3. $\mathsf{A}C = C$*, and therefore,* $\mathsf{E}C = C$.

Proof. We first show that $\mathsf{A}C = C$:

$$\begin{aligned}
\mathsf{A}C &= \mathsf{A}\bigwedge\{a \in A^{\delta} \mid c \leq a \text{ and } \mathsf{E}a = a\} && \text{(by the definition of } C\text{)}\\
&= \bigwedge\{\mathsf{A}a \mid a \in A^{\delta}, c \leq \text{ and } \mathsf{E}a = a\} && \text{(since } \mathsf{A} \text{ distributes over arbitrary meets)}\\
&= \bigwedge\{a \in A^{\delta} \mid c \leq a \text{ and } \mathsf{E}a = a\} && \text{(by Lemma 3.3.2)}\\
&= C && \text{(by the definition of } C
\end{aligned}$$

It thus follows from Lemma 3.3.2 that $\mathsf{E}C = C$. □

Lemma 3.3.4. *A_C is closed under the operations $\wedge^C$, $\vee^C$, $\neg^C$, $\Diamond^C$, and E^C.*

Proof. The cases for $\wedge^C$ and $\vee^C$ are straightforward. For $\neg^C$, see Lemma 3.1.3. We consider the cases for $\Diamond^C$ and E^C. Let $a \in A_C$. Then there is some $a' \in A$ such that $a = a' \wedge C$. Now, by the monotonicity of $\Diamond$,

$$\Diamond^C a = \Diamond(a' \wedge C) \wedge C \leq \Diamond a' \wedge C.$$

Conversely,

$$\begin{aligned}
\Diamond^C a &= \Diamond(a' \wedge C) \wedge C\\
&\geq \Diamond a' \wedge \Box C \wedge C\\
&\geq \Diamond a' \wedge \mathsf{A}C \wedge C\\
&= \Diamond a' \wedge C,
\end{aligned}$$

where the third step follows from $(incl_{\Diamond})$ and the last step from Lemma 3.3.3. We thus have that $\Diamond^C a = \Diamond a' \wedge C$. But since A is closed under $\Diamond$, $\Diamond a' \in A$, which means that $\Diamond^C a \in A_c$.

For E^C,' let $a \in A_C$. Then $a = a' \wedge C$ for some $a' \in A$. Now, by the monotonicity of E,

$$\mathsf{E}^C a = \mathsf{E}(a' \wedge C) \wedge C \leq \mathsf{E}a' \wedge C,$$

and, conversely,

$$\begin{aligned}
\mathsf{E}^C a &= \mathsf{E}(a' \wedge C) \wedge C\\
&\geq \mathsf{E}a' \wedge \mathsf{A}C \wedge C\\
&= \mathsf{E}a' \wedge C.
\end{aligned}$$

Here the final step follows from Lemma 3.3.3. Hence, $\mathsf{E}^C a = \mathsf{E}a' \wedge C$. But A is closed under E, so $\mathsf{E}a' \in A$. This means that $\mathsf{E}^C a \in A_C$. □

Lemma 3.3.5. *Let $h\colon A \to A_C$ be the map defined by $h(a) = a \wedge C$. Then h is a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{A}_C$.*

Proof. It is clear from the definition of h that it is surjective. In verifying that h is a homomorphism, all cases except those for $\Diamond$ and E are proved in exactly the same way as in Lemma 3.1.4. So all that is left to check is $\Diamond$ and E. We first prove the case for $\Diamond$:

$$\begin{aligned} h(\Diamond a) &= \Diamond a \wedge C \\ &\geq \Diamond(a \wedge C) \wedge C \\ &= \Diamond h(a) \wedge C \\ &= \Diamond^C h(a), \end{aligned}$$

where the inequality follows from the fact that $a \wedge c \leq a$, and the monotonicity of $\Diamond$. Conversely,

$$\begin{aligned} \Diamond^C h(a) &= \Diamond h(a) \wedge C \\ &= \Diamond(a \wedge C) \wedge C \\ &\geq \Diamond a \wedge \Box C \wedge C \\ &\geq \Diamond a \wedge \mathsf{A}C \wedge C \\ &= \Diamond a \wedge C \\ &= h(\Diamond a) \end{aligned}$$

Here the fourth step follows from $(incl_\Diamond)$ and the fifth step from Lemma 3.3.3.

For E, the left-to-right inequality is proved as follows:

$$\begin{aligned} h(\mathsf{E}a) &= \mathsf{E}a \wedge C \\ &\geq \mathsf{E}(a \wedge C) \wedge C \\ &= \mathsf{E}h(a) \wedge C \\ &= \mathsf{E}^C h(a). \end{aligned}$$

Conversely,

$$\begin{aligned} \mathsf{E}^C h(a) &= \mathsf{E}(a \wedge C) \wedge C \\ &\geq \mathsf{E}a \wedge \mathsf{A}C \wedge C \\ &= \mathsf{E}a \wedge C \\ &= h(\mathsf{E}a), \end{aligned}$$

where the third step follows from Lemma 3.3.3. □

The lemma below shows that the global operator behaves like it should. Frame-theoretically, this lemma tells us that the accessibility relation of the global modality is the universal relation on the frame that we 'cut' out of the ultrafilter frame of the Lindenbaum-Tarski algebra.

Lemma 3.3.6. *For all $a \in A_C$,*

$$\mathsf{E}^C a = \begin{cases} C & \text{if } a > \bot \\ \bot & \text{otherwise.} \end{cases}$$

Proof. Let $a \in A_C$, and assume $a > \bot$. Then $a = a' \wedge C > \bot$ for some $a' \in A$. But this means that both $a' > \bot$ and $C > \bot$. Hence,

$$\begin{aligned} \mathsf{E}^C a &= \mathsf{E}(a' \wedge C) \wedge C \\ &\geq \mathsf{E}a' \wedge \mathsf{A}C \wedge C \\ &= \mathsf{E}a' \wedge C \\ &= \top \wedge C \\ &= C. \end{aligned}$$

On the other hand, if $a = \bot$, $\mathsf{E}^C \bot = \mathsf{E}\bot \wedge C = \bot \wedge C = \bot$. □

Finally, we show that all the constant interpretations of the nominals are actually atoms of $\mathbf{A}_C$.

Lemma 3.3.7. *For each* $\mathbf{i} \in \mathsf{NOM}$, $s_{\mathbf{i}}^C$ *is an atom of* $\mathbf{A}_C$.

Proof. First, $s_{\mathbf{i}}^C \neq \bot$ for all $\mathbf{i} \in \mathsf{NOM}$, for suppose not. Then $\mathsf{E}^C s_{\mathbf{i}}^C = \mathsf{E}^C \bot$, and so $C = \bot$, a contradiction.

We now show that if $a, b \in At\mathbf{A}_C$ such that $a, b \leq s_{\mathbf{i}}^C$, then $a = b$. So let $a, b \in At\mathbf{A}_C$ such that $a, b \leq s_{\mathbf{i}}^C$. Then $a \wedge s_{\mathbf{i}}^C = a > \bot$ and $b \wedge s_{\mathbf{i}}^C = b > \bot$, and so $\mathsf{E}^C(a \wedge s_{\mathbf{i}}^C) = C$ and $\mathsf{E}^C(b \wedge s_{\mathbf{i}}^C) = C$. Hence, by ($refl_{\mathsf{E}}$) and ($nom_{\mathsf{E}}$),

$$\neg^C s_{\mathbf{i}}^C \vee a \geq \mathsf{A}^C(\neg^C s_{\mathbf{i}}^C \vee a) \geq \mathsf{E}^C(a \wedge s_{\mathbf{i}}^C) = C,$$

and

$$\neg^C s_{\mathbf{i}}^C \vee b \geq \mathsf{A}^C(\neg^C s_{\mathbf{i}}^C \vee b) \geq \mathsf{E}^C(b \wedge s_{\mathbf{i}}^C) = C,$$

which means that $(\neg^C s_{\mathbf{i}}^C \vee a) \wedge (\neg^C s_{\mathbf{i}}^C \vee b) = C$. We thus have $\neg^C s_{\mathbf{i}}^C \vee (a \wedge b) = C$, so if $a \neq b$, $\neg^C s_{\mathbf{i}}^C = C$, giving $s_{\mathbf{i}}^C = \bot$, which is a contradiction. □

We could have taken a different approach. Note that in the orthodox Lindenbaum-Tarski algebra in the proof of Theorem 3.3.1, the global modality is simply an **S5**-modality, so when we take the orthodox canonical extension of the orthodox Lindenbaum-Tarski algebra, the accessibility relation of the global modality in the ultrafilter frame of the Lindenbaum-Tarski algebra is not the universal relation. So can we generate with the accessibility relation of the global modality? It is easy to see that if we generate with the accessibility relation of the global modality, the truth of an $\mathcal{H}(\mathsf{E})$-formula is transferred from the supermodel to the submodel. What is more, since we know that the accessibility relation of $\Diamond$ must be contained in the accessibility relation of the global modality, it is enough to generate with the accessibility relation of the global modality only. But how do we simulate this idea algebraically? Well, since A is completely meet-preserving in $\mathbf{A}^\delta$, it has a left adjoint, let us call it E^{-1}. Now, we can simply do our usual construction, but instead of using $\Diamond^{-1}$, we use E^{-1}.

3.4 Algebraic completeness of $\mathbf{H}^+ \oplus \Sigma$

As we will now show, the logic $\mathbf{H}^+ \oplus \Sigma$ is complete with respect to permeated hybrid algebras which validate Σ. We first state and prove the main theorem, and consequently prove the results needed.

Theorem 3.4.1. *For any set Σ of $\mathcal{H}$-formulas, the logic $\mathbf{H}^+ \oplus \Sigma$ is sound and complete with respect to the class of all permeated hybrid algebras which validate Σ. That is to say, $\vdash_{\mathbf{H}^+ \oplus \Sigma} \varphi$ iff $\models_{\mathsf{PHA}(\Sigma)} \varphi \approx \top$.*

Proof. Suppose $\nvdash_{\mathbf{H}^+ \oplus \Sigma} \varphi$. We have to find a permeated hybrid algebra $\mathfrak{A}$ and an assignment v such that $\mathfrak{A}, v \not\models \varphi \approx \top$. However, as before, we will work with the orthodox interpretation of $\mathcal{H}$ for the purpose of the proof. Let NOM' be a denumerably infinite set of nominals disjoint from NOM. We know from [72] that $\neg\varphi$ is contained in a $\mathbf{H}^+ \oplus \Sigma$-maximal consistent set of formulas Γ in the extended language such that

(i) Γ contains at least one nominal, say $\mathbf{i}_0$, and

(ii) for each formula of the form $\Diamond^n(\mathbf{i} \wedge \Diamond\varphi)$ in Γ, there is a nominal $\mathbf{j}$ for which the formula $\Diamond^n(\mathbf{i} \wedge \Diamond(\mathbf{j} \wedge \varphi))$ is in Γ.

Consider the orthodox Lindenbaum-Tarski algebra of $\mathbf{H}^+ \oplus \Sigma$ over PROP. For simplicity, denote this algebra by $\mathbf{A}$. In the usual way, we can show that $\mathbf{A} \models \mathbf{H}^+ \oplus \Sigma^{\approx}$ and $\mathbf{A}, \nu \not\models \varphi \approx \top$, where ν is the natural map taking p to $[p]$. Furthermore, by Lemma 3.4.2, the set $[\Gamma] = \{[\gamma] \mid \gamma \in \Gamma\}$ is an ultrafilter of $\mathbf{A}$. We thus know from the finite meet property that for every finite subset $\Gamma' \subseteq \Gamma$, it holds that $\bigwedge[\Gamma'] > \bot$ in $\mathbf{A}$.

Now, consider the orthodox canonical extension $\mathbf{A}^\delta$ of $\mathbf{A}$. Note that in $\mathbf{A}^\delta$ we have $\bigwedge[\Gamma] > \bot$. For suppose $\bigwedge[\Gamma] \leq \bot$. But we know that the elements of $\mathbf{A}$ are closed and that $\bot$ is open, so, by the compactness of the embedding of $\mathbf{A}$ into $\mathbf{A}^\delta$, there is a finite subset $\Gamma' \subseteq \Gamma$ such that $\bigwedge[\Gamma'] \leq \bot$ in $\mathbf{A}$, contradicting the claim above.

Since $\mathbf{A}^\delta$ is atomic, there is some atom d in $\mathbf{A}^\delta$ such that $d \leq \bigwedge[\Gamma]$. Denote d by d_0, and suppose d_n is already defined. Now, define d_{n+1}, D and $\mathbf{A}_D$ as in the proof of Theorem 3.1.1. We know that $\mathbf{A}_D$ is an algebra. Furthermore, using the map $h\colon A \to A_D$ defined as in Lemma 3.1.4, we can show that $\mathbf{A}_D$ is a homomorphic image of $\mathbf{A}$. Hence, $\mathbf{A}_D \models \mathbf{H}^+ \oplus \Sigma^{\approx}$. To see that $\mathbf{A}_D \not\models \varphi \approx \top$, consider the assignment $\nu_D(p)\colon \mathsf{PROP} \to A_D$ defined as in the proof of Theorem 3.1.1. Using the fact that h is a homomorphism, we can show by structural induction that $\widetilde{\nu_D}(\psi) = h(\widetilde{\nu}(\psi))$ for all $\mathcal{H}$-formulas ψ. Now, since $\bigwedge[\Gamma] \leq \bigwedge[\Gamma']$ for every finite subset $\Gamma' \subseteq \Gamma$, we have $\nu(\bigwedge \Gamma') \geq d$. It thus holds that

$$\widetilde{\nu_D}(\bigwedge \Gamma') = h(\widetilde{\nu}(\bigwedge \Gamma')) = \widetilde{\nu}(\bigwedge \Gamma') \wedge D \geq d > \bot$$

for every finite subset Γ' of Γ. But $\{\neg\varphi\}$ is a finite subset of Γ, so $\widetilde{\nu_D}(\varphi) \neq \top$.

In the same way as in the proof of Lemma 3.1.6, we can show that for each $\mathbf{i} \in \mathsf{NOM} \cup \mathsf{NOM}'$, $s^D_{\mathbf{i}}$ is either $\bot$ or an atom of $\mathbf{A}_D$. So let $\mathfrak{A}_D = (\mathbf{A}^-_D, X_{A_D})$, where $\mathbf{A}^-_D$ is the algebra reduct of $\mathbf{A}_D$ obtained by omitting the constant interpretations of nominals and $X_{A_D} = \{s^D_{\mathbf{i}} \mid s^D_{\mathbf{i}} > \bot\}$. We claim that $X_{A_D} \neq \varnothing$. To see this, recall that $\mathbf{i}_0 \in \Gamma$, so $d \leq \bigwedge[\Gamma] \leq [\mathbf{i}_0]$. It follows that

$d \leq s^D_{\mathbf{i}_0}$, and so, at least $s^D_{\mathbf{i}_0} > \bot$. In particular, since $s^D_{\mathbf{i}_0}$ and d are both atoms, $s^D_{\mathbf{i}_0} = d$. Now, from the fact that $\mathbf{H}^+ \oplus \Sigma$ is closed under (*Sorted substitution*), we have $\mathfrak{A}_D \models \mathbf{H}^+ \oplus \Sigma^{\approx}$. Furthermore, by Lemma 3.4.7, $\mathfrak{A}_D$ is also permeated. From here we split our reasoning into two cases depending on whether the constant interpretations of all nominals in $\mathbf{A}_D$ are atoms or not:

Case 1: $s^D_{\mathbf{i}} > \bot$ for all $\mathbf{i} \in \mathsf{NOM} \cup \mathsf{NOM}'$. In this case, consider the assignment ν'_D which extends ν_D from PROP to $\mathsf{PROP} \cup \mathsf{NOM} \cup \mathsf{NOM}'$, obtained by simply setting $\nu'_D(\mathbf{i}) = s^D_{\mathbf{i}}$ for each $\mathbf{i} \in \mathsf{NOM} \cup \mathsf{NOM}'$. It is clear that $\widetilde{\nu'_D}(\psi) = \widetilde{\nu_D}(\psi)$ for all $\mathcal{H}$-formulas ψ, and hence, $\widetilde{\nu'_D}(\varphi) \neq \widetilde{\nu'_D}(\top)$. This means that $\mathfrak{A}_D, \nu'_D \not\models \varphi \approx \top$, as required.

Case 2: $s^D_{\mathbf{i}} = \bot$ for some $\mathbf{i} \in \mathsf{NOM} \cup \mathsf{NOM}'$. Here we have that $(\mathfrak{A}_D)_0 \models \mathbf{H}^+ \oplus \Sigma^{\approx}$, and hence, by Proposition 2.1.16, $\mathfrak{A}_D \times \mathfrak{A}_D \models \mathbf{H}^+ \oplus \Sigma^{\approx}$. By Proposition 2.1.24, we know that $\mathfrak{A}_D \times \mathfrak{A}_D$ is also permeated. Now, consider the assignment ν''_D obtained by setting $\nu''_D(p) = (\nu_D(p), \nu_D(p))$ for all propositional variables $p \in \mathsf{PROP}$, and

$$\nu''_D(\mathbf{j}) = \begin{cases} (s^D_{\mathbf{j}}, \bot) & \text{if } s^D_{\mathbf{j}} > \bot \\ (\bot, s^D_{\mathbf{i}_0}) & \text{if } s^D_{\mathbf{j}} = \bot \end{cases}$$

for all nominals $\mathbf{j} \in \mathsf{NOM} \cup \mathsf{NOM}'$. For any $\mathcal{H}$-formula ψ, we have $\widetilde{\nu''_D}(\psi) = (\widetilde{\nu_D}(\psi), a_\psi)$, where a_ψ is some element of $\mathfrak{A}_D$. But then $\widetilde{\nu''_D}(\varphi) = (\widetilde{\nu_D}(\varphi), a_\varphi) \neq (\widetilde{\nu_D}(\top), D) = \widetilde{\nu''_D}(\top)$. □

Let us now prove the lemmas used in the proof of Theorem 3.4.1. In what follows, unless stated otherwise, Γ, $\mathbf{A}$, $\mathbf{A}^\delta$, $\mathbf{A}_D$, ν, ν_D and $\mathfrak{A}_D$ will be as in the proof of Theorem 3.4.1.

Lemma 3.4.2. *The set* $[\Gamma] = \{[\gamma] \mid \gamma \in \Gamma\}$ *is an ultrafilter of* $\mathbf{A}$.

Proof. First, we know that $\vdash \top$, so, since Γ is maximal consistent, $\mathbf{H}^+ \oplus \Sigma \subseteq \Gamma$, which means $\top \in \Gamma$. Hence, $[\top] \in [\Gamma]$.

To show that $[\Gamma]$ is closed under meets, let $[\gamma_1], [\gamma_2] \in [\Gamma]$. Then $\gamma_1 \in \Gamma$ and $\gamma_2 \in \Gamma$. But we know that $p \to (q \to (p \wedge q))$ is a classical tautology, and so, by *Sorted substitution*, $\vdash \gamma_1 \to (\gamma_2 \to (\gamma_1 \wedge \gamma_2))$. Now, since Γ is maximal consistent, $\mathbf{H}^+ \oplus \Sigma \subseteq \Gamma$, which means $\gamma_1 \to (\gamma_2 \to (\gamma_1 \wedge \gamma_2)) \in \Gamma$. We also know that Γ is closed under *Modus ponens*, so $\gamma_1 \wedge \gamma_2 \in \Gamma$, and hence, $[\gamma_1 \wedge \gamma_2] = [\gamma_1] \wedge [\gamma_2] \in [\Gamma]$.

Next, we show that $[\Gamma]$ is upward closed, so let $\gamma_1 \in [\Gamma]$, and assume $[\gamma_1] \leq [\gamma_2]$. Then $[\gamma_1] \wedge [\gamma_2] = [\gamma_1 \wedge \gamma_2] = [\gamma_1]$, and so $[\gamma_1 \wedge \gamma_2] \in [\Gamma]$. Hence, $\gamma_1 \wedge \gamma_2 \in \Gamma$. But $p \wedge q \to q$ is a classical tautology, so, by *Sorted substitution*, $\vdash \gamma_1 \wedge \gamma_2 \to \gamma_2$. Now, since $\mathbf{H}^+ \oplus \Sigma \subseteq \Gamma$, $\gamma_1 \wedge \gamma_2 \to \gamma_2 \in \Gamma$, and hence, by *Modus Ponens*, $\gamma_2 \in \Gamma$. Therefore, $[\gamma_2] \in [\Gamma]$, as required.

To show that $[\Gamma]$ is proper, assume $[\bot] \in [\Gamma]$. Then $\bot \in \Gamma$, and so, since $\vdash \bot \to \bot$, $\Gamma \vdash \bot$, contradicting the consistency of Γ.

Finally, assume $[\gamma_1] \in [\Gamma]$. Then $\gamma_1 \in \Gamma$, and so, since Γ is maximal consistent, $\neg\gamma_1 \notin \Gamma$. Hence, $[\neg\gamma_1] \notin [\Gamma]$. Likewise, if $[\neg\gamma_1] \in [\Gamma]$, $[\gamma_1] \notin [\Gamma]$. □

We now turn to the biggest task of this section: proving that the algebra $\mathfrak{A}_D$ is permeated. But first we need to prove a few lemmas. It is here where we crucially use items (i) and (ii) in the proof of Theorem 3.4.1.

Lemma 3.4.3. *Let* $\mathbf{A}$ *be a BAO,* $\mathbf{A}^\delta$ *its canonical extension, and* $a, b \in \mathbf{A}^\delta$*. Then we have* $a \wedge (\Diamond^{-1})^n b > \bot$ *iff* $b \leq \Diamond^n a$*.*

Proof. For the left-to-right direction, assume $a \wedge (\Diamond^{-1})^n b > \bot$. Then there is a $c \in At\mathbf{A}^\delta$ such that $c \leq a \wedge (\Diamond^{-1})^n b$, and so $c \leq a$ and $c \leq (\Diamond^{-1})^n b$. Hence, by Lemma 3.1.5, $b \leq \Diamond^n c$, which means that $b \leq \Diamond^n a$.

For the converse, suppose $a \wedge (\Diamond^{-1})^n b = \bot$. Then $(\Diamond^{-1})^n b \leq \neg a$, and so, since $\Diamond^{-1}$ and $\Box$ are adjoint, $b \leq \Box^n \neg a = \neg \Diamond^n a$. Hence, $b \nleq \Diamond^n a$. $\Box$

Lemma 3.4.4. *For any element* a *in* $\mathbf{A}_D$*, if* $d \leq \Diamond^n a$*, then* $d \leq (\Diamond^D)^n a$*.*

Proof. The proof is by induction on n. For $n = 1$, assume $d \leq \Diamond a$. Then $d \wedge D \leq \Diamond a \wedge D$, and so, since $d \leq D$, $d \leq \Diamond^D a$. Now, suppose that for all a in $\mathbf{A}_D$, the claim holds for $n = k$. For $n = k+1$, assume $d \leq \Diamond^{k+1} a$. Then $d \leq \Diamond^k \Diamond a$. But we know that $d \leq D = \Box^k D$, so

$$d \leq \Diamond^k \Diamond a \wedge \Box^k D \leq \Diamond^k (\Diamond a \wedge D) = \Diamond^k \Diamond^D a.$$

Now, since $\Diamond^D a \in A_D$, we can use the inductive hypothesis to get $d \leq (\Diamond^D)^k \Diamond a$. Hence,

$$d \leq (\Diamond^D)^k \Diamond a \wedge D \leq (\Diamond^D)^k \Diamond a \wedge (\Box^D)^k D \leq (\Diamond^D)^k (\Diamond a \wedge D) = (\Diamond^D)^k \Diamond^D a = (\Diamond^D)^{k+1} a,$$

as required. $\Box$

Lemma 3.4.5. *For all* $\mathcal{H}$*-formulas* γ*, it holds that* $\gamma \in \Gamma$ *iff* $d \leq \widetilde{\nu}(\gamma)$ *iff* $d \leq \widetilde{\nu_D}(\gamma)$*.*

Proof. Suppose $\gamma \in \Gamma$. Then $[\gamma] \in [\Gamma]$, and so $\bigwedge[\Gamma] \leq [\gamma]$. But $d \leq \bigwedge[\Gamma]$, so $d \leq [\gamma] = \widetilde{\nu}(\gamma)$. Conversely, assume $\gamma \notin \Gamma$. But Γ is a maximal consistent set of formulas, so $\neg\gamma \in \Gamma$. Hence, using the left-to-right direction, $d \leq \widetilde{\nu}(\neg\gamma) = \neg\widetilde{\nu}(\gamma)$, and so $d \nleq \widetilde{\nu}(\gamma)$. Now, assume $d \leq \widetilde{\nu}(\gamma)$. Then $d \wedge D \leq \widetilde{\nu}(\gamma) \wedge D$, and so $d \leq \widetilde{\nu_D}(\gamma)$. Conversely, assume $d \leq \widetilde{\nu_D}(\gamma)$. Then $d \leq \widetilde{\nu}(\gamma) \wedge D$, which gives $d \leq \widetilde{\nu}(\gamma)$. $\Box$

Lemma 3.4.6. *Let* a *be any element in* A_D*, and let* $n(a)$ *be the least* $n \in \mathbb{N}$ *such that* $a \wedge (\Diamond^{-1})^{n(a)} d > \bot$*, then there is an* $s^D_{\mathbf{j}}$ *such that* $a \wedge s^D_{\mathbf{j}} \wedge (\Diamond^{-1})^{n(a)} d > \bot$*.*

Proof. Let a be any element of A_D. The proof is by induction on $n(a)$. For $n(a) = 0$, we have $a \wedge d > \bot$. But $d = s^D_{\mathbf{i}_0}$, so we are done. Now, suppose that for every $a \in A_D$, the claim holds for $n(a) = k$. For $n(a) = k+1$, we have $a \wedge (\Diamond^{-1})^{k+1} d > \bot$, and so $d \leq \Diamond^{k+1} a = \Diamond^k \Diamond a$ by Lemma 3.4.3. Hence, $\Diamond a \wedge (\Diamond^{-1})^k d > \bot$ by Lemma 3.4.3, so

$$\Diamond a \wedge ((\Diamond^{-1})^k d \wedge D) = (\Diamond a \wedge D) \wedge (\Diamond^{-1})^k d = \Diamond^d a \wedge (\Diamond^{-1})^k d > \bot.$$

Using the inductive hypothesis, we know that there is a nominal $\mathbf{j} \in \mathsf{NOM} \cup \mathsf{NOM}'$ such that $\Diamond^D a \wedge s^D_{\mathbf{j}} \wedge (\Diamond^{-1})^k d > \bot$. But then $d \leq \Diamond^k (\Diamond^D a \wedge s^D_{\mathbf{j}})$. So, by Lemma 3.4.4, $d \leq (\Diamond^D)^k (\Diamond^D a \wedge s^D_{\mathbf{j}})$, and so, since ν and h are surjective, there is some ψ such that

$$\begin{array}{rcl} d & \leq & (\Diamond^D)^k (\Diamond^D \widetilde{\nu_D}(\psi) \wedge s^D_{\mathbf{j}}) \\ & = & \widetilde{\nu_D}(\Diamond^k (\Diamond \psi \wedge \mathbf{j})). \end{array}$$

We thus have that $\Diamond^k(\mathbf{j} \wedge \Diamond\psi) \in \Gamma$ by Lemma 3.4.5, which means there is a nominal $\mathbf{k}$ in $\mathsf{NOM} \cup \mathsf{NOM}'$ such that $\Diamond^k(\mathbf{j} \wedge \Diamond(\mathbf{k} \wedge \psi)) \in \Gamma$. Hence, by Lemma 3.4.5,

$$\begin{array}{rcl} d & \leq & \widetilde{\nu_D}(\Diamond^k(\mathbf{j} \wedge \Diamond(\mathbf{k} \wedge \psi))) \\ & = & (\Diamond^D)^k(s_{\mathbf{j}}^D \wedge \Diamond^d(s_{\mathbf{k}}^D \wedge a)) \\ & \leq & \Diamond^k(s_{\mathbf{j}}^D \wedge \Diamond(s_{\mathbf{k}}^D \wedge a)), \end{array}$$

so $\perp < (\Diamond^{-1})^k d \wedge s_{\mathbf{j}}^D \wedge \Diamond(s_{\mathbf{k}}^D \wedge a) \leq (\Diamond^{-1})^k d \wedge \Diamond(s_{\mathbf{k}}^D \wedge a)$. But then $d \leq \Diamond^k\Diamond(s_{\mathbf{k}}^D \wedge a) = \Diamond^{k+1}(s_{\mathbf{k}}^D \wedge a)$, and therefore, $(\Diamond^{-1})^{k+1} d \wedge s_{\mathbf{k}}^D \wedge a > \perp$. $\square$

Finally, we are ready to show that $\mathfrak{A}_D$ is permeated.

Lemma 3.4.7. *$\mathfrak{A}_D$ is permeated.*

Proof. For the first condition, let $b \in A_D$ such that $b > \perp$. Then $b \wedge (\Diamond^{-1})^m d > \perp$ for some $m \in \mathbb{N}$. Let $m(b)$ be the least m such that $b \wedge (\Diamond^{-1})^{m(b)} d > \perp$. By Lemma 3.4.6, there is an $s_{\mathbf{j}}^D$ such that $b \wedge s_{\mathbf{j}}^D \wedge (\Diamond^{-1})^{m(b)} d > \perp$. Then $d \leq \Diamond^{m(b)}(b \wedge s_{\mathbf{j}}^D)$ by Lemma 3.4.3. Now, $s_{\mathbf{j}}^D \neq \perp$, for otherwise, $d \leq \Diamond^{m(b)}\perp = \perp$, a contradiction. Hence, $s_{\mathbf{j}}^D$ is an atom by Lemma 3.1.6, and furthermore, $s_{\mathbf{j}}^D \in X_{A_D}$. We also claim that $s_{\mathbf{j}}^D \leq b$, for if not, $b \wedge s_{\mathbf{j}}^D = \perp$, giving $d \leq \perp$, a contradiction.

To prove the second condition, let $a \in A_D$ and $s_{\mathbf{i}}^D \in X_{A_D}$, and assume that $s_{\mathbf{i}}^D \leq \Diamond^D b$. Then $s_{\mathbf{i}}^D \leq \Diamond b \wedge D$, which means that $s_{\mathbf{i}}^D \leq \Diamond b \wedge (\Diamond^{-1})^m d$ for some $m \in \mathbb{N}$. Hence, $s_{\mathbf{i}}^D \wedge \Diamond b \wedge (\Diamond^{-1})^m d = s_{\mathbf{i}}^D > \perp$, and so $d \leq \Diamond^m(s_{\mathbf{i}}^D \wedge \Diamond b)$. We thus have $d \leq (\Diamond^D)^m(s_{\mathbf{i}}^D \wedge \Diamond b)$ by Lemma 3.4.4, so $d \wedge D \leq (\Diamond^D)^m(s_{\mathbf{i}}^D \wedge \Diamond b) \wedge D$. Hence,

$$\begin{array}{rcl} d & \leq & (\Diamond)^m(s_{\mathbf{i}}^D \wedge \Diamond b) \wedge (\Box^D)^m D \\ & \leq & (\Diamond^D)^m(s_{\mathbf{i}}^D \wedge \Diamond b \wedge D) \\ & = & (\Diamond^D)^m(s_{\mathbf{i}}^D \wedge \Diamond^D b) \\ & = & (\Diamond^D)^m(s_{\mathbf{i}}^D \wedge \Diamond^D \widetilde{\nu_D}(\psi)) \\ & = & \widetilde{\nu_D}(\Diamond^m(\mathbf{i} \wedge \Diamond\psi)) \end{array}$$

This means that $\Diamond^m(\mathbf{i} \wedge \Diamond\psi) \in \Gamma$, and so, there is a nominal $\mathbf{j} \in \mathsf{NOM} \cup \mathsf{NOM}'$ such that $\Diamond^m(\mathbf{i} \wedge \Diamond(\mathbf{j} \wedge \psi)) \in \Gamma$. This gives

$$\begin{array}{rcl} d & \leq & \widetilde{\nu_D}(\Diamond^m(\mathbf{i} \wedge \Diamond(\mathbf{j} \wedge \psi))) \\ & = & (\Diamond^D)^m(s_{\mathbf{i}}^D \wedge \Diamond^D(s_{\mathbf{j}}^D \wedge b)) \\ & \leq & \Diamond^m(s_{\mathbf{i}}^D \wedge \Diamond(s_{\mathbf{j}}^D \wedge b)). \end{array}$$

Now, first note that $s_{\mathbf{j}}^D \neq \perp$, for then $d = \perp$, which is a contradiction. This means that $s_{\mathbf{j}}^D$ is an atom and in X_{A_D}. Second, $s_{\mathbf{j}}^D \leq b$, for otherwise, $s_{\mathbf{j}}^D \wedge b = \perp$, giving $d = \perp$. Finally, to see that $s_{\mathbf{i}}^D \leq \Diamond^D s_{\mathbf{j}}^D$, suppose for the sake of a contradiction that it is not. Then $s_{\mathbf{i}}^D \wedge \Diamond^D s_{\mathbf{j}}^D = \perp$, and so

$$\begin{aligned} d &\leq \Diamond^m(s_{\mathbf{i}}^D \wedge \Diamond(s_{\mathbf{j}}^D \wedge b)) \\ &\leq \Diamond^m(s_{\mathbf{i}}^D \wedge \Diamond s_{\mathbf{j}}^D) \\ &\leq (\Diamond^D)^m(s_{\mathbf{i}}^D \wedge \Diamond s_{\mathbf{j}}^D) \wedge D \\ &\leq (\Diamond^D)^m(s_{\mathbf{i}}^D \wedge \Diamond s_{\mathbf{j}}^D) \wedge (\Box^D)^m D \\ &\leq (\Diamond^D)^m(s_{\mathbf{i}}^D \wedge \Diamond s_{\mathbf{j}}^D \wedge D) \\ &= (\Diamond^D)^m(s_{\mathbf{i}}^D \wedge \Diamond^D s_{\mathbf{j}}^D) \\ &= \bot, \end{aligned}$$

a contradiction. □

3.5 Algebraic completeness of $\mathbf{H}^+(@) \oplus \Sigma$

The lemmas needed in the proof of the following theorem follow afterwards.

Theorem 3.5.1. *For any set Σ of $\mathcal{H}(@)$-formulas, the logic $\mathbf{H}^+(@) \oplus \Sigma$ is sound and complete with respect to the class of all permeated hybrid @-algebras which validate Σ. That is to say, $\vdash_{\mathbf{H}^+(@)\oplus\Sigma} \varphi$ iff $\models_{\mathsf{PH@A}(\Sigma)} \varphi \approx \top$.*

Proof. Suppose $\nvdash_{\mathbf{H}^+(@)\oplus\Sigma} \varphi$. We need to find a permeated hybrid @-algebra $\mathfrak{A}$ and an assignment v such that $\mathfrak{A}, v \not\models \varphi \approx \top$.

Let NOM' be a denumerably infinite set of nominals disjoint from NOM. We know from Lemma 7.25 in [10] that $\neg\varphi$ is contained in a $\mathbf{H}^+(@) \oplus \Sigma$-maximal consistent set of formulas Γ in the extended language such that

(i) Γ contains at least one nominal, say $\mathbf{i}_0$, and

(ii) for each formula of the form $@_{\mathbf{i}}\Diamond\varphi \in \Gamma$, there exists a nominal $\mathbf{j}$ such that $@_{\mathbf{i}}\Diamond\mathbf{j} \in \Gamma$ and $@_{\mathbf{j}}\varphi \in \Gamma$.

Now, consider the orthodox Lindenbaum-Tarski algebra of $\mathbf{H}^+(@) \oplus \Sigma$ over PROP. Denote it by $\mathbf{A}$. In the usual way, $\mathbf{A} \models \mathbf{H}^+(@) \oplus \Sigma^{\approx}$, while $\mathbf{A}, \nu \not\models \varphi \approx \top$, where ν is the natural map taking p to $[p]$. Furthermore, in the same way as in Lemma 3.4.2, we can show that $[\Gamma] = \{[\gamma] \mid \gamma \in \Gamma\}$ is an ultrafilter of $\mathbf{A}$, and thus, by the finite meet property, $\bigwedge[\Gamma'] > \bot$ in $\mathbf{A}$ for every finite $\Gamma' \subseteq \Gamma$.

Next, consider the orthodox canonical extension $\mathbf{A}^\delta$ of $\mathbf{A}$. Then $\bigwedge[\Gamma] > \bot$ in $\mathbf{A}^\delta$, for if $\bigwedge[\Gamma] \leq \bot$, by the compactness of the embedding of $\mathbf{A}$ into $\mathbf{A}^\delta$, there is a finite subset $\Gamma' \subseteq \Gamma$ such that $\bigwedge[\Gamma'] \leq \bot$ in $\mathbf{A}$, which is not true according to the previous paragraph.

Let d in $At\mathbf{A}^\delta$ such that $d \leq \bigwedge[\Gamma]$, and denote it by d_0. Furthermore, let $s_{\mathbf{i}_1}, s_{\mathbf{i}_2}, \ldots, s_{\mathbf{i}_m}$ be the constant interpretations of the nominals occurring in φ. Since $s_{\mathbf{i}_i} \neq \bot$ in $\mathbf{A}^\delta$ for each $1 \leq i \leq m$, there are atoms $d_0^1, d_0^2, \ldots, d_0^m$ in $\mathbf{A}^\delta$ such that $d_0^1 \leq s_{\mathbf{i}_1}, d_0^2 \leq s_{\mathbf{i}_2}, \ldots, d_0^m \leq s_{\mathbf{i}_m}$. Now, suppose d_n^i is already defined for each $1 \leq i \leq m$, and then define d_{n+1}^i, D and $\mathbf{A}_D$ as in the proof of Theorem 3.2.1. In the same way as in Lemma 3.2.3, we can then show that

$\mathbf{A}_D$ is an algebra, and furthermore, by Lemma 3.2.4, we have that $\mathbf{A}_D \models \mathbf{H}^+(@)\Sigma^{\approx}$. To see that $\mathbf{A}_D \not\models \varphi \approx \top$, consider the assignment ν_D defined as in the proof of Theorem 3.2.1. We then get that $\widetilde{\nu_D}(\bigwedge \Gamma') > \bot$ for every finite subset Γ' of Γ. But $\{\neg\varphi\}$ is a finite subset of Γ, so $\widetilde{\nu_D}(\varphi) \neq \top$.

Now, in the same way as in the proof of Theorem 3.2.1, we can show that for each $\mathbf{i} \in \mathsf{NOM} \cup \mathsf{NOM}'$, $s^D_{\mathbf{i}}$ is an atom of $\mathbf{A}_D$. So let $\mathfrak{A}_D = (\mathbf{A}^-_D, X_{A_D})$, where $\mathbf{A}^-_D$ is the reduct of $\mathbf{A}_D$ obtained by omitting the constant interpretations of nominals, and $X_{A_D} = \{s^D_{\mathbf{i}} \mid \mathbf{i} \in \mathsf{NOM} \cup \mathsf{NOM}'\}$. X_{A_D} is thus clearly non-empty. Furthermore, since $\mathbf{H}^+(@) \oplus \Sigma$ is closed under sorted substitution, it follows from the foregoing that $\mathfrak{A}_D \models \mathbf{H}(@)^+ \oplus \Sigma^{\approx}$. To see that $\mathfrak{A}_D \not\models \varphi \approx \top$, consider the assignment ν'_D which extends ν_D from PROP to $\mathsf{PROP} \cup \mathsf{NOM} \cup \mathsf{NOM}'$, obtained by simply setting $\nu'_D(\mathbf{i}) = s^D_{\mathbf{i}}$ for each $\mathbf{i} \in \mathsf{NOM} \cup \mathsf{NOM}'$. It is clear that $\widetilde{\nu'_D}(\psi) = \widetilde{\nu_D}(\psi)$ for all $\mathcal{H}(@)$-formulas ψ, and hence, $\widetilde{\nu'_D}(\varphi) \neq \widetilde{\nu'_D}(\top)$. Finally, by Lemma 3.5.3, $\mathfrak{A}_D$ is permeated. $\square$

As in the previous section, the biggest task is proving that $\mathfrak{A}_D$ is permeated. But first we need the lemma below. In what follows, unless stated otherwise, Γ, $\mathbf{A}$, $\mathbf{A}^\delta$, $\mathbf{A}_D$, ν, ν_D and $\mathfrak{A}_D$ will be as in the proof of the above theorem.

Lemma 3.5.2. *Let a be an element of $\mathbf{A}_D$. Then $d \leq @^D_{s^D_{\mathbf{i}}}(\Diamond^D)^n a$ implies there is a nominal $\mathbf{j} \in \mathsf{NOM} \cup \mathsf{NOM}'$ such that $s^D_{\mathbf{j}} \leq a$.*

Proof. The proof is by induction on n. For $n = 0$, $d \leq @^D_{s^D_{\mathbf{i}}} a$. This implies that $s^D_{\mathbf{i}} \leq a$, for if not, $d \leq @^D_{s^D_{\mathbf{i}}} a = \bot$, which is a contradiction. Hence, $s^D_{\mathbf{i}}$ works. Now, suppose that for every $a \in A_D$, the claim holds for $n = k$. For $n = k+1$, assume $d \leq @^D_{s^D_{\mathbf{i}}}(\Diamond^D)^{k+1} a = @^D_{s^D_{\mathbf{i}}}(\Diamond^D)^k \Diamond^D a$. Hence, since A_D is closed under $\Diamond^D$, $\Diamond^D a \in A_D$. Using the inductive hypothesis, we have a nominal $\mathbf{k} \in \mathsf{NOM} \cup \mathsf{NOM}'$ such that $s^D_{\mathbf{k}} \leq \Diamond^D a$. We then have that $D = @^D_{s^D_{\mathbf{k}}} s^D_{\mathbf{k}} \leq @^D_{s^D_{\mathbf{k}}} \Diamond^D a$, and so $d \leq @^D_{s^D_{\mathbf{k}}} \Diamond^D a$. Now, since both ν and h are surjective, there is a ψ such that $d \leq @^D_{s^D_{\mathbf{k}}} \Diamond^D \widetilde{\nu_D}(\psi) = \widetilde{\nu_D}(@_{\mathbf{k}} \Diamond \psi)$. In the same way as in Lemma 3.4.5, we can show that for every $\mathcal{H}(@)$-formula γ, $\gamma \in \Gamma$ iff $d \leq \widetilde{\nu_D}(\gamma)$. But then $@_{\mathbf{k}} \Diamond \psi \in \Gamma$, which means there is a nominal $\mathbf{j} \in \mathsf{NOM} \cup \mathsf{NOM}'$ such that $@_{\mathbf{k}} \Diamond \mathbf{j} \in \Gamma$ and $@_{\mathbf{j}} \psi \in \Gamma$. Hence, $d \leq \widetilde{\nu_D}(@_{\mathbf{j}} \psi) = @^D_{s^D_{\mathbf{j}}} \widetilde{\nu_D}(\psi) = @^D_{s^D_{\mathbf{j}}} a$, and so $D = @^D_{s^D_{\mathbf{i}_0}} d \leq @^D_{s^D_{\mathbf{i}_0}} @^D_{s^D_{\mathbf{j}}} a \leq @^D_{s^D_{\mathbf{j}}} a$. Therefore, $@^D_{s^D_{\mathbf{j}}} a = D$, so $s^D_{\mathbf{j}} \leq a$. $\square$

Lemma 3.5.3. *$\mathfrak{A}_D$ is permeated.*

Proof. To prove the first condition, let $b \in \mathbf{A}_D$ such that $b > \bot$. Then

$$b \wedge \bigvee_{0 \leq i \leq m} D_i > \bot,$$

which means there is a $0 \leq j \leq m$ such that $b \wedge D_j > \bot$. Hence,

$$b \wedge \bigvee_{n \in \mathbb{N}} (\Diamond^{-1})^n d^j_0 > \bot,$$

and so there is some $n_j \in \mathbb{N}$ such that $b \wedge (\Diamond^{-1})^{n_j} d_0^j > \bot$. We thus know from Lemma 3.4.3 that $d_0^j \leq \Diamond^{n_j} b$. Now, since $d_0^j \leq s_{\mathbf{i}_j}$, $d_0^j \leq s^D_{\mathbf{i}_j}$. But both are atoms, so $d_0^j = s^D_{\mathbf{i}_j}$. Hence, $D = @^D_{s^D_{\mathbf{i}_j}} s^D_{\mathbf{i}_j} \leq @^D_{s^D_{\mathbf{i}_j}} \Diamond^{n_j} b$, which means that $d \leq @^D_{s^D_{\mathbf{i}_j}} \Diamond^{n_j} b$. It thus follows from Lemma 3.5.2 that there is an atom $s^D_{\mathbf{j}} \in X_{A_D}$ such that $s^D_{\mathbf{j}} \leq b$.

For the second condition, let $s^D_{\mathbf{i}} \in X_{A_D}$ and $b \in A_D$ such that $s^D_{\mathbf{i}} \leq \Diamond^D b$. Then $D = @^D_{s^D_{\mathbf{i}}} s^D_{\mathbf{i}} \leq @^D_{s_{\mathbf{i}}} \Diamond^D b$. Hence,

$$d \leq D = @^D_{s^D_{\mathbf{i}}} \Diamond^D b = @^D_{s^D_{\mathbf{i}}} \Diamond^D \widetilde{\nu_D}(\psi) = \widetilde{\nu_D}(@_{\mathbf{i}} \Diamond \psi),$$

and so $@_{\mathbf{i}} \Diamond \psi \in \Gamma$. We thus know that there is a nominal $\mathbf{j} \in \mathsf{NOM} \cup \mathsf{NOM}'$ such that $@_{\mathbf{i}} \Diamond \mathbf{j} \in \Gamma$ and $@_{\mathbf{j}} \psi \in \Gamma$. Hence, $d \leq @^D_{s^D_{\mathbf{i}}} \Diamond^D s^D_{\mathbf{j}}$ and $d \leq @^D_{s^D_{\mathbf{j}}} b$. We now make the following claims:

Claim 1. $s^D_{\mathbf{j}} \leq b$.

Proof of claim. To see this, suppose for the sake of a contradiction that $s^D_{\mathbf{j}} \not\leq b$. Then $s^D_{\mathbf{j}} \leq \neg^D b$, and so $@^D_{s^D_{\mathbf{j}}} b = \bot$. However, this is a contradiction, since $@^D_{s^D_{\mathbf{j}}} b \geq d > \bot$.

Claim 2. $s^D_{\mathbf{i}} \leq \Diamond^D s^D_{\mathbf{j}}$.

Proof of claim. Suppose $s^D_{\mathbf{i}} \not\leq \Diamond^D s^D_{\mathbf{j}}$. Then $s^D_{\mathbf{i}} \leq \neg^D \Diamond^D s^D_{\mathbf{j}}$, and so $@^D_{s^D_{\mathbf{i}}} \Diamond^D s^D_{\mathbf{j}} = \bot$. However, this is a contradiction, since $@^D_{s^D_{\mathbf{i}}} \Diamond^D s^D_{\mathbf{j}} \geq d > \bot$. $\square$

3.6 Algebraic completeness of $\mathbf{H}^+(\mathsf{E}) \oplus \Sigma$

Theorem 3.6.1. *For any set Σ of $\mathcal{H}(\mathsf{E})$-formulas, the logic $\mathbf{H}^+(\mathsf{E}) \oplus \Sigma$ is sound and complete with respect to the class of all permeated hybrid E-algebras which validate Σ. That is, $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \varphi$ iff $\models_{\mathsf{PHEA}(\Sigma)} \varphi \approx \top$.*

Proof. Suppose $\nvdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \varphi$. We need to find a permeated hybrid E-algebra $\mathfrak{A}$ and an assignment v such that $\mathfrak{A} \models \Sigma^{\approx}$ and $\mathfrak{A}, v \not\models \varphi \approx \top$. We will again work with the orthodox interpretation of $\mathcal{H}(\mathsf{E})$ for the time being.

Let NOM' be a denumerably infinite set of nominals such that $\mathsf{NOM} \cap \mathsf{NOM}' = \varnothing$. We know from Lemma 3.6.2 that $\neg\varphi$ is contained in a $\mathbf{H}^+(\mathsf{E}) \oplus \Sigma$-maximal consistent set of formulas Γ in the extended language such that

(i) Γ contains at least one nominal, say $\mathbf{i}_0$,

(ii) for all $\mathsf{E}(\mathbf{i} \wedge \Diamond\varphi) \in \Gamma$, there is a nominal $\mathbf{j}$ such that $\mathsf{E}(\mathbf{i} \wedge \Diamond\mathbf{j}) \in \Gamma$ and $\mathsf{E}(\mathbf{j} \wedge \varphi) \in \Gamma$, and

(iii) for all $\mathsf{E}(\mathbf{i} \wedge \mathsf{E}\varphi) \in \Gamma$, there is a nominal $\mathbf{j}$ such that $\mathsf{E}(\mathbf{i} \wedge \mathsf{E}\mathbf{j}) \in \Gamma$ and $\mathsf{E}(\mathbf{j} \wedge \varphi) \in \Gamma$.

Consider the orthodox Lindenbaum-Tarski algebra of $\mathbf{H}^+(\mathsf{E}) \oplus \Sigma$ over PROP and denote it by $\mathbf{A}$ for simplicity. As usual, $\mathbf{A} \models \mathbf{H}^+(\mathsf{E}) \oplus \Sigma^{\approx}$ and $\mathbf{A}, \nu \not\models \varphi \approx \top$, where ν is the natural map taking p to $[p]$. Furthermore, Γ is an ultrafilter of $\mathbf{A}$, so, by the finite meet property, $\bigwedge[\Gamma'] > \bot$ in $\mathbf{A}$ for every finite subset Γ' of Γ.

Next, consider the orthodox canonical extension $\mathbf{A}^\delta$ of $\mathbf{A}$. By the compactness of the embedding of $\mathbf{A}$ into $\mathbf{A}^\delta$, $\bigwedge[\Gamma] > \bot$ in $\mathbf{A}^\delta$. So since $\mathbf{A}^\delta$ is atomic, there is some $c \in At\mathbf{A}^\delta$ such that $c \leq \bigwedge[\Gamma]$. Now, define C and $\mathbf{A}_C$ as in the proof of Theorem 3.3.1. We then know that $\mathbf{A}_C$ is an algebra, and furthermore, $\mathbf{A}_C \models \mathbf{H}^+(\mathsf{E}) \oplus \Sigma^{\approx}$ and $\mathbf{A}_C, \nu_C \not\models \varphi \approx \top$, where ν_C is also defined as in the proof of Theorem 3.3.1. We can also show that $s_{\mathbf{i}}^C$ is an atom for all $\mathbf{i} \in \mathsf{NOM} \cup \mathsf{NOM}'$. So let $\mathfrak{A}_C = (\mathbf{A}_C^-, X_{A_C})$, where $\mathbf{A}_C^-$ is the reduct of $\mathbf{A}_C$ obtained by omitting the constant interpretations of the nominals, and $X_{A_C} = \{s_{\mathbf{i}}^C \mid \mathbf{i} \in \mathsf{NOM} \cup \mathsf{NOM}'\}$. It thus follows that $\mathfrak{A}_C$ is a hybrid E-algebra, and, since Σ is closed under sorted substitution, we have $\mathfrak{A}_C \models \Sigma^{\approx}$. To see that $\mathfrak{A}_C \not\models \varphi \approx \top$, consider the assignment ν'_C which extends ν_C from PROP to $\mathsf{PROP} \cup \mathsf{NOM} \cup \mathsf{NOM}'$ obtained by setting $\nu'_C(\mathbf{i}) = s_{\mathbf{i}}^C$ for each $\mathbf{i} \in \mathsf{NOM} \cup \mathsf{NOM}'$. Clearly, $\widetilde{\nu'_C}(\psi) = \widetilde{\nu_C}(\psi)$ for all $\mathcal{H}(\mathsf{E})$-formulas ψ. Hence, $\widetilde{\nu'_C}(\varphi) = \widetilde{\nu_C}(\varphi) \neq \widetilde{\nu_C}(\top) = \widetilde{\nu'_C}(\top)$. Finally, by Lemma 3.6.3 below, $\mathfrak{A}_C$ is permeated. $\square$

The following lemma is not readily available in the literature:

Lemma 3.6.2 (**Extended Lindenbaum Lemma**). *Let NOM' be a countable infinite collection of nominals disjoint from NOM. Then every $\mathbf{H}^+(\mathsf{E}) \oplus \Sigma$-consistent set Γ of formulas in the original language can be extended to a maximal $\mathbf{H}^+(\mathsf{E}) \oplus \Sigma$-consistent set Γ^+ of formulas in the extended language satisfying the following:*

(i) *Γ^+ contains at least one nominal.*

(ii) *For all $\mathsf{E}(\mathbf{i} \wedge \Diamond\varphi) \in \Gamma^+$, there is a nominal $\mathbf{j}$ such that $\mathsf{E}(\mathbf{i} \wedge \Diamond\mathbf{j}) \in \Gamma$ and $\mathsf{E}(\mathbf{j} \wedge \varphi) \in \Gamma^+$.*

(iii) *For all $\mathsf{E}(\mathbf{i} \wedge \mathsf{E}\varphi) \in \Gamma^+$, there is a nominal $\mathbf{j}$ such that $\mathsf{E}(\mathbf{i} \wedge \mathsf{E}\mathbf{j}) \in \Gamma^+$ and $\mathsf{E}(\mathbf{j} \wedge \varphi) \in \Gamma^+$.*

Proof. Let $(\mathbf{i}_n)_{n\in\mathbb{N}}$ be an enumeration of NOM'. Define $\Gamma_{\mathbf{i}_0}$ to be $\Gamma \cup \{\mathbf{i}_0\}$. We now show that $\Gamma_{\mathbf{i}_0}$ is consistent. So suppose it is not. Then there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \mathbf{i}_0 \wedge \psi_1 \wedge \cdots \wedge \psi_n \to \bot$. Hence, $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \mathbf{i}_0 \to \neg(\psi_1 \wedge \cdots \wedge \psi_n)$. But $\mathbf{i}_0$ is a new nominal, and therefore does not occur in $\psi_1 \wedge \cdots \wedge \psi_n$, so, by $(Name_{\mathsf{E}})$, $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \neg(\psi_1 \wedge \cdots \wedge \psi_n)$. This means $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \psi_1 \wedge \cdots \wedge \psi_n \to \bot$, contradicting the consistency of Γ.

Now, let $(\gamma_n)_{n\in\mathbb{N}}$ be an enumeration of all $\mathcal{H}(\mathsf{E})$-formulas in the extended language. Define Γ^0 to be $\Gamma_{\mathbf{i}_0}$. Suppose we already defined Γ^m, where $m \geq 0$. Let γ_{m+1} be the $(m+1)$-th formula in our enumeration. We then define Γ^{m+1} as follows: if $\Gamma^m \cup \{\gamma_{m+1}\}$ is inconsistent, let $\Gamma^{m+1} = \Gamma^m$; otherwise,

(i) $\Gamma^{m+1} = \Gamma^m \cup \{\gamma_{m+1}\}$, if γ_{m+1} is not of the form $\mathsf{E}(\mathbf{i} \wedge \Diamond\varphi)$ or $\mathsf{E}(\mathbf{i} \wedge \mathsf{E}\varphi)$.

(ii) $\Gamma^{m+1} = \Gamma^m \cup \{\gamma_{m+1}\} \cup \{\mathsf{E}(\mathbf{i} \wedge \Diamond\mathbf{j}) \wedge \mathsf{E}(\mathbf{j} \wedge \varphi)\}$, if γ_{m+1} is of the form $\mathsf{E}(\mathbf{i} \wedge \Diamond\varphi)$ (here $\mathbf{j}$ is the first new nominal in the nominal enumeration that does not occur in Γ^m or γ_{m+1}).

(iii) $\Gamma^{m+1} = \Gamma^m \cup \{\gamma_{m+1}\} \cup \{\mathsf{E}(\mathbf{i} \wedge \mathsf{E}\mathbf{j}) \wedge \mathsf{E}(\mathbf{j} \wedge \varphi)\}$, if γ_{m+1} is of the form $\mathsf{E}(\mathbf{i} \wedge \mathsf{E}\varphi)$ (here $\mathbf{j}$ is the first new nominal in the nominal enumeration that does not occur in Γ^m or γ_{m+1}).

Each step preserves consistency. The only non-trivial cases are items (ii) and (iii). Let us first check the case for item (ii). Suppose $\Gamma^m \cup \{\gamma_{m+1}\} \cup \{\mathsf{E}(\mathbf{i} \wedge \Diamond\mathbf{j}) \wedge \mathsf{E}(\mathbf{j} \wedge \varphi)\}$ is not consistent. Then there are formulas $\psi_1, \ldots, \psi_n \in \Gamma^m \cup \{\gamma_{m+1}\}$ such that $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma}$

$\psi_1 \wedge \cdots \wedge \psi_n \wedge \mathsf{E}(\mathbf{i} \wedge \Diamond \mathbf{j}) \wedge \mathsf{E}(\mathbf{j} \wedge \varphi) \to \bot$. Hence, $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \mathsf{E}(\mathbf{i} \wedge \Diamond \mathbf{j}) \wedge \mathsf{E}(\mathbf{j} \wedge \varphi) \to \neg(\psi_1 \wedge \cdots \wedge \psi_n)$. But $\mathbf{j}$ does not occur in φ or Γ^m, so, by $(BG_{\mathsf{E}\Diamond})$, $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \mathsf{E}(\mathbf{i} \wedge \Diamond \varphi) \to \neg(\psi_1 \wedge \cdots \wedge \psi)$. Hence, $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \mathsf{E}(\mathbf{i} \wedge \Diamond \varphi) \wedge \psi_1 \wedge \cdots \wedge \psi \to \bot$, contradicting the consistency of $\Gamma^m \cup \{\gamma_{m+1}\}$. For (iii), suppose $\Gamma^m \cup \{\gamma_{m+1}\} \cup \{\mathsf{E}(\mathbf{i} \wedge \mathsf{E}\mathbf{j}) \wedge \mathsf{E}(\mathbf{j} \wedge \varphi)\}$ is not consistent. Then there are $\psi_1, \ldots, \psi_n \in \Gamma^m \cup \{\gamma_{m+1}\}$ such that $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \psi_1 \wedge \cdots \wedge \psi_n \wedge \mathsf{E}(\mathbf{i} \wedge \mathsf{E}\mathbf{j}) \wedge \mathsf{E}(\mathbf{j} \wedge \varphi) \to \bot$. Hence, $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \mathsf{E}(\mathbf{i} \wedge \mathsf{E}\mathbf{j}) \wedge \mathsf{E}(\mathbf{j} \wedge \varphi) \to \neg(\psi_1 \wedge \cdots \wedge \psi_n)$. But $\mathbf{j}$ does not occur in φ or Γ^m, so by $(BG_{\mathsf{E}\mathsf{E}})$, $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \mathsf{E}(\mathbf{i} \wedge \mathsf{E}\varphi) \to \neg(\psi_1 \wedge \cdots \wedge \psi_n)$, which means $\vdash_{\mathbf{H}^+(\mathsf{E})\oplus\Sigma} \mathsf{E}(\mathbf{i} \wedge \mathsf{E}\varphi) \wedge \psi_1 \wedge \cdots \wedge \psi_n \to \bot$, contradicting the consistency of $\Gamma^m \cup \{\gamma_{m+1}\}$.

Now, let $\Gamma^+ = \bigcup_{m \geq 0} \Gamma^m$. First, since each step preserves consistency, Γ^+ is also consistent. Furthermore, since every formula or its negation is in our enumeration, Γ^+ must also be maximal. Finally, it is also clear from the way we constructed Γ^+ that is must satisfy conditions (i) – (iii). □

Before we show that $\mathfrak{A}_C$ is permeated, note that since $\mathbf{i}_0 \in \Gamma$, $c \leq \bigwedge[\Gamma] \leq [\mathbf{i}_0]$, and so $c \leq [\mathbf{i}_0] \wedge C = s^C_{\mathbf{i}_0}$. But both c and $s^C_{\mathbf{i}_0}$ are atoms, so $c = s^C_{\mathbf{i}_0}$. Furthermore, since Γ is a *maximal* consistent set of formulas, we can show in the same way as in Lemma 3.4.5 that for each $\mathcal{H}(\mathsf{E})$-formula ψ, $\psi \in \Gamma$ iff $c \leq \widetilde{\nu}(\psi)$ iff $c \leq \widetilde{\nu_C}(\psi)$.

Lemma 3.6.3. *$\mathfrak{A}_C$ is permeated.*

Proof. For the first condition, let $b \in A_C$ such that $b \neq \bot$. Then we have $\mathsf{E}^C b = C$, and so $c = s^C_{\mathbf{i}_0} \leq C = \mathsf{E}^C b$. Hence, $c \leq s^C_{\mathbf{i}_0} \wedge \mathsf{E}^C b$, which gives $c \leq \mathsf{E}^C c \leq \mathsf{E}^C(s^C_{\mathbf{i}_0} \wedge \mathsf{E}^C b)$. But since both ν and h are surjective, there is a formula ψ such that

$$c \leq \mathsf{E}^C(s^C_{\mathbf{i}_0} \wedge \mathsf{E}^C \widetilde{\nu_C}(\psi)) = \widetilde{\nu_C}(\mathsf{E}(\mathbf{i}_0 \wedge \mathsf{E}\psi)).$$

So $\mathsf{E}(\mathbf{i}_0 \wedge \mathsf{E}\psi) \in \Gamma$, and hence, there is a nominal $\mathbf{j} \neq \mathbf{i}_0$ not occurring in the formula ψ such that $\mathsf{E}(\mathbf{i}_0 \wedge \mathsf{E}\mathbf{j})$ and $\mathsf{E}(\mathbf{j} \wedge \psi) \in \Gamma$. This means that

$$c \leq \widetilde{\nu_C}(\mathsf{E}(\mathbf{j} \wedge \psi)) = \mathsf{E}^C(s^C_{\mathbf{j}} \wedge b).$$

We now claim that $s^C_{\mathbf{j}} \leq b$. For suppose not, then $s^C_{\mathbf{j}} \leq \neg^C b$, which means that $s^C_{\mathbf{j}} \wedge b = \bot$. This gives $c \leq \bot$, a contradiction.

To prove the second condition, let $s^C_{\mathbf{i}} \in X_{A_C}$ and $b \in A_C$ such that $s^C_{\mathbf{i}} \leq \Diamond^C b$. Then $s^C_{\mathbf{i}} = s^C_{\mathbf{i}} \wedge \Diamond^C b$, and so $\mathsf{E}^C(s^C_{\mathbf{i}} \wedge \Diamond^C b) = C$. Hence, since both h and ν are surjective, there is a ψ such that

$$c \leq C = \mathsf{E}^C(s^C_{\mathbf{i}} \wedge \Diamond^C b) = \mathsf{E}^C(s^C_{\mathbf{i}} \wedge \Diamond^C \widetilde{\nu_C}(\psi)) = \widetilde{\nu_C}(\mathsf{E}(\mathbf{i} \wedge \Diamond\psi)),$$

which means that $\mathsf{E}(\mathbf{i} \wedge \Diamond\psi) \in \Gamma$. We thus know that there is some nominal $\mathbf{j} \neq \mathbf{i}$ such that $\mathsf{E}(\mathbf{i} \wedge \Diamond\mathbf{j}) \in \Gamma$ and $\mathsf{E}(\mathbf{j} \wedge \psi) \in \Gamma$, so $c \leq \mathsf{E}^C(s^C_{\mathbf{i}} \wedge \Diamond^C s^C_{\mathbf{j}})$ and $c \leq \mathsf{E}^C(s^C_{\mathbf{j}} \wedge b)$. We now claim the following:

Claim 1. $s^C_{\mathbf{j}} \leq b$.

Proof of claim. Suppose $s^C_{\mathbf{j}} \nleq b$. Then $s^C_{\mathbf{j}} \leq \neg^C b$, and so $s^C_{\mathbf{j}} \wedge b = \bot$. But this means that $c \leq \mathsf{E}^C(s^C_{\mathbf{j}} \wedge b) = \bot$, which is a contradiction.

Claim 2. $s_{\mathbf{i}}^C \leq \Diamond^C s_{\mathbf{j}}^C$.

Proof of claim. For the sake of a contradiction, assume that $s_{\mathbf{i}}^C \nleq \Diamond^C s_{\mathbf{j}}^C$. Then $s_{\mathbf{i}}^C \leq \neg^C \Diamond^C s_{\mathbf{j}}^C$, which means that $s_{\mathbf{i}}^C \wedge \Diamond^C s_{\mathbf{j}}^C = \bot$. Hence, $c \leq \mathsf{E}^C(s_{\mathbf{i}}^C \wedge \Diamond^C s_{\mathbf{j}}^C) = \bot$, a contradiction. $\square$

Chapter 4

Sahlqvist theory for hybrid logics

A natural research direction is to investigate the transfer of Sahlqvist theory from modal logic to hybrid logics. The reader will recall that every modal Sahlqvist formula enjoys two properties: firstly, it has a local-first order frame correspondent and, secondly, it is canonical. The second property implies that any normal modal logic axiomatized with Sahlqvist formulas (in addition to the axioms of the base logic **K**) is strongly complete with respect to its Kripke frames.

As regards Sahlqvist theory for hybrid logic, it is fairly straightforward to see that nominals may be freely introduced into modal Sahlqvist formulas without destroying the first property. The second property is more tricky. In this regard, Ten Cate, Marx and Viana [73] show that any hybrid logic obtained by adding *modal* Sahlqvist formulas to the basic hybrid logic **H** is strongly complete. Also, one of the very first results in the study of hybrid logic was the fact that any extension of **H** with pure axioms is strongly complete [40]. In [73], it is shown that these two results cannot be combined in general, since there is a modal Sahlqvist formula and a pure formula which together give a Kripke-incomplete logic when added to **H**.

The intention of this chapter is to see to what extent these two results *can* be combined and, in so doing, to develop a genuinely *hybrid* Sahlqvist theory. Some initial results in this direction appear in [28]. We define a hybrid version of the inductive formulas [55], encompassing both the modal Sahlqvist formulas and the pure formulas. In addition we define two subclasses, called the skeletal and nominally skeletal hybrid inductive formulas. We show that members of these subclasses are respectively preserved under canonical extensions and Dedekind-MacNeille completions of certain hybrid algebras, which is enough to ensure that these formulas axiomatize relationally complete logics.

The key methodological tool in proving the above results is a hybrid version of the ALBA algorithm [35], which we formulate and call hybrid-ALBA. This algorithm manipulates formulas by applying a calculus of rewrite rules. In line with the philosophy of unified correspondence theory [32], these rules are entirely predicated upon the order-theoretic behaviour of the connectives. However, applying this philosophy in the setting of hybrid algebras requires certain innovations. In particular, the linchpin of the canonicity strategy employed in works like [35] and [32], namely the equivalence of validity and admissible validity of pure formulas, fails in this setting. This necessitates an investigation of the preservation of pure inequalities under completions of hybrid algebras.

4.1 Preliminaries

In this section, we collect some essential details we will be using.

4.1.1 Expanded language of $\mathcal{H}(@)$

The expansion of $\mathcal{H}(@)$ discussed here will play a critical role in the Sahlqvist theory for hybrid logics, and includes the connectives corresponding to all the adjoint operations. In particular, the computations used to calculate first-order correspondents takes place in the expanded language.

Let us now introduce the expanded language formally. The formulas of the expanded language $\mathcal{H}^+(@)$[1] are defined as follows:

$$\varphi \quad ::= \quad \bot \mid p \mid \mathbf{j} \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Diamond^{-1}\varphi \mid @_{\mathbf{j}}\varphi \mid \mathsf{E}\varphi,$$

where $p \in \mathsf{PROP}$ and $\mathbf{j} \in \mathsf{NOM}$. As usual, $\Box^{-1}\varphi := \neg\Diamond^{-1}\neg\varphi$.

A *quasi-inequality* of the language $\mathcal{H}^+(@)$ is an expression of the form

$$\varphi_1 \leq \psi_1 \ \& \ \cdots \& \ \varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi,$$

where $\varphi_i, \psi_i, \varphi$ and ψ are formulas of $\mathcal{H}^+(@)$.

4.1.2 Semantics

We interpret $\mathcal{H}^+(@)$-formulas in Kripke frames, general frames and models. Given a model $\mathfrak{M}$ and a state w in $\mathfrak{M}$, then the truth definition is extended with the following clause:

$$\mathfrak{M}, w \Vdash \Diamond^{-1}\varphi \text{ iff there exists } v \text{ such that } vRw \text{ and } \mathfrak{M}, v \Vdash \varphi.$$

Fix a model $\mathfrak{M}$ and a state w in $\mathfrak{M}$. For any φ and ψ in $\mathcal{H}^+(@)$, $\mathfrak{M}, w \Vdash \varphi \leq \psi$ iff $\mathfrak{M}, w \Vdash \varphi \to \psi$. For $\varphi_i, \psi_i, \varphi$ and ψ in $\mathcal{H}^+(@)$, $\mathfrak{M}, w \Vdash \varphi_1 \leq \psi_1 \ \& \ \cdots \& \ \varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$ iff $\mathfrak{M}, w \nVdash \varphi_i \leq \psi_i$ for some $1 \leq i \leq n$, or $\mathfrak{M}, w \Vdash \varphi \leq \psi$.

Algebraically, $\mathcal{H}^+(@)$ is interpreted in complete hybrid @-algebras. However, in this section, we will drop the @ and talk about hybrid algebras instead. In particular, $\Diamond^{-1}$ is interpreted as the left adjoint of $\Box$ in complete hybrid algebras.

A quasi-inequality $\varphi_1 \leq \psi_1 \ \& \ \cdots \& \ \varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$ is *true* in a hybrid algebra $\mathfrak{A}$ under assignment v, if $\varphi_i \leq \psi_i$ is not true in $\mathfrak{A}$ under v for some $1 \leq i \leq n$, or $\varphi \leq \psi$ is true in $\mathfrak{A}$ under v.

[1] Note that the logic $\mathbf{H}^+(@)$ does not refer to the logic corresponding to the extended language $\mathcal{H}^+(@)$, as the common superscript + might suggest. These notations are well established in the hybrid logic and algorithmic correspondence literatures, respectively. We opt to keep with these traditions, and risk the potential confusion.

4.1.3 Standard translation and frame correspondents

The $\mathcal{H}^+(@)$-formulas are translated into $\mathcal{L}_1$ by means of the usual *standard translation* function ST_x, defined by induction on $\mathcal{H}^+(@)$-formulas. In particular, we have $ST_x(\Diamond^{-1}\varphi) := \exists y(yRx \wedge ST_y(\varphi))$, where y is a variable that has not been used in the translation.

In the usual way, one can prove by induction on formulas that $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M} \models ST_x(\varphi)[x := w]$, that $\mathfrak{M}, w \Vdash \varphi \leq \psi$ iff $\mathfrak{M} \models ST_x(\varphi) \to ST_x(\psi)[x := w]$ and that $\mathfrak{M} \Vdash \&_{i=1}^{n} \varphi_i \leq \psi_i \Rightarrow \varphi \leq \psi$ iff $\mathfrak{M} \models \bigwedge_{i=1}^{n} \forall x(ST_x(\varphi_i) \to ST_x(\psi_i)) \to \forall x(ST_x(\varphi) \to ST_x(\psi))$. Here the suffix '$[x := w]$' means that the variable x is interpreted as w. Similarly, on frames we have that $\mathfrak{F}, w \Vdash \varphi$ iff $\mathfrak{F} \models \forall\overline{P}\forall\overline{y}ST_x(\varphi)[x := w]$, where $\overline{P}$ is the vector of all predicate symbols P_i corresponding to propositional variables p_i occurring in φ and $\overline{y}$ is the vector of all variables $y_{\mathbf{i}}$ corresponding to nominals $\mathbf{i}$ occurring in φ. Similar equivalences hold for inequalities and quasi-inequalities interpreted on frames.

A $\mathcal{H}^+(@)$ formula φ and a $\mathcal{L}_0$ formula $\alpha(x)$ with one free variable x are *local frame correspondents* if $\mathfrak{F}, w \Vdash \varphi$ iff $\mathfrak{F} \models \alpha(x)[x := \alpha]$.

4.1.4 Residuals of the satisfaction operators

Now, we know from Proposition 2.2.5 that @ is completely join-preserving (meet-preserving) in the second coordinate, and therefore it has a right (left) residual (see e.g. [36]), which will be denoted by $@^{+2}$ and $@^{-2}$, respectively. The following proposition derives definitions of $@^{+2}$ and $@^{-2}$ from the general theory of residuals:

Proposition 4.1.1. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a complete hybrid algebra. Then*

$$@_x^{+2}a = \begin{cases} \top & \textit{if } a = \top \\ \neg x & \textit{otherwise} \end{cases} \qquad \textit{and} \qquad @_x^{-2}a = \begin{cases} \bot & \textit{if } a = \bot \\ x & \textit{otherwise} \end{cases}.$$

Proof. (i) Since @ is completely join-preserving in the second coordinate,

$$@_x^{+2}a = \bigvee\{b \in A \mid @_x b \leq a\}.$$

Now, if $a = \top$, then we have that $\{b \in A \mid @_x b \leq a\} = A$, so $@_x^{+2}a = \bigvee A = \top$. If $a < \top$, then $\{b \in A \mid @_x b \leq a\} = \{b \in A \mid b \leq \neg x\}$, which means $@_x^{+2}a = \bigvee\{b \in A \mid b \leq \neg x\} = \neg x$.

(ii) We know that @ is also completely meet preserving in the second coordinate, so

$$@_x^{-2}a = \bigwedge\{b \in A \mid a \leq @_x b\}.$$

Now, if $a = \bot$. Then $\{b \in A \mid a \leq @_x a\} = A$, and so $@_x^{-2}a = \bigwedge A = \bot$. On the other hand, if $a > \bot$, then $\{b \in A \mid a \leq @_x b\} = \{b \in A \mid x \leq b\}$. But this means that $@_x^{-2}a = \bigwedge\{b \in A \mid x \leq b\} = x$. $\square$

4.1.5 Dedekind MacNeille completions

Let $\mathbf{A}$ be a Boolean algebra with operators. Recall that the *Dedekind MacNeille completion* $\mathbf{B}$ of $\mathbf{A}$ is a completion of $\mathbf{A}$ such that $\mathbf{A}$ is both meet-dense and join-dense in $\mathbf{B}$. It can be shown that the Dedekind MacNeille completion of a BAO always exists and that it is unique up to isomorphism (see for instance [68]). This justifies our speaking of 'the' Dedekind MacNeille completion of BAO $\mathbf{A}$, and from now on this algebra will be denoted by $\mathbf{A}^{dm}$.

The complete extension of $\Diamond^{\mathbf{A}}$ on A is the operation $\Diamond^{\mathbf{A}^{dm}}$ on A^{dm} defined by

$$\Diamond^{\mathbf{A}^{dm}} a = \bigvee\{\Diamond b \mid b \in A \text{ and } b \leq a\}.$$

If we know that $\Diamond^{\mathbf{A}}$ preserves all existing joins, then $\Diamond^{\mathbf{A}^{dm}}$ is completely join preserving on $\mathbf{A}^{dm}$. Indeed, the case for non-empty joins follows from Lemma 13 in [45], which tells us that the complete extension of a completely additive map between Boolean algebras is completely additive. The case for the empty join follows since we have $\Diamond^{\mathbf{A}^{dm}} \bigvee \varnothing = \Diamond^{\mathbf{A}^{dm}} \bot = \bigvee\{\Diamond^{\mathbf{A}} b \mid b \in A \text{ and } b \leq a\} = \bigvee\{\Diamond^{\mathbf{A}} \bot\} = \bot = \bigvee \varnothing = \bigvee\{\Diamond^{\mathbf{A}^{dm}} b \mid b \in \varnothing\}$.

We will also make use of the following lemma later:

Lemma 4.1.2. *If a BAO is atomic, then its the Dedekind MacNeille completion is also atomic, and moreover, it has exactly the same atoms as the original BAO.*

Proof. For the first claim, let $\mathbf{A}$ be an atomic BAO, and consider its Dedekind MacNeille completion $\mathbf{A}^{dm}$. To show that $\mathbf{A}^{dm}$ is atomic, let $b \in A^{dm}$ such that $b \neq \bot$. Since $\mathbf{A}$ is join-dense in $\mathbf{A}^{dm}$, $b = \bigvee\{a \in A \mid a \leq b\}$. But since $b \neq \bot$, there is some $a_0 \in A$ such that $a_0 \leq b$, and we are done. To see that $\mathbf{A}$ and $\mathbf{A}^{dm}$ have the same atoms, first let $y \in At\mathbf{A}$, and suppose there is some $b \in \mathbf{A}^{dm}$ such that $\bot < b \leq y$. But since $\mathbf{A}$ is join-dense in $\mathbf{A}^{dm}$, $b = \bigvee\{a \in A \mid a \leq b\}$, and hence, since $b \neq \bot$, there is some $a_0 \in A$ such that $a \leq b \leq y$. Hence, since $y \in At\mathbf{A}$, $y = a_0$, which means that $y = b$. Conversely, let $y \in At\mathbf{A}^{dm}$. But since $\mathbf{A}$ is atomic, $y = \bigvee\{x \in At\mathbf{A} \mid x \leq y\}$. But we know that y is an atom, so this means that y must be an atom of $\mathbf{A}$. $\square$

Now, we define the Dedekind MacNeille completion of a hybrid algebra $\mathfrak{A} = (\mathbf{A}, X_A)$ as the pair $\mathfrak{A}^{dm} = (\mathbf{A}^{dm}, At\mathbf{A}^{dm})$. We say that a hybrid algebra $\mathfrak{A} = (\mathbf{A}, X_A)$ is *atomic* if $\mathbf{A}$ is atomic and $X_A = At\mathbf{A}$. Clearly, the Dedekind MacNeille completion of an atomic hybrid algebra is also atomic, and it has exactly the same atoms as the original hybrid algebra.

Finally, it turns out that preservation under Dedekind MacNeille completions implies di-persistence. This originates from the fact that the Dedekind MacNeille completion of the underlying hybrid algebra $\mathfrak{g}^* = (A, \cap, \cup, -, \varnothing, W, \langle R\rangle, X_B)$ of the *discrete* two-sorted general frame $\mathfrak{g} = (W, R, A, B)$ is the complex algebra $(\mathfrak{g}_\sharp)^+$ of the underlying Kripke frame $\mathfrak{g}_\sharp$ of $\mathfrak{g}$. The notion of di-persistence then becomes: $\varphi \approx \psi$ is di-persistent, if $\mathfrak{g}^* \models \varphi \approx \psi$ implies $(\mathfrak{g}_\sharp)^+ \models \varphi \approx \psi$. But this is just preservation under Dedekind MacNeille completions.

4.1.6 Admissible validity

Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a hybrid subalgebra of $\mathfrak{B} = (\mathbf{B}, X_B)$, i.e, $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $X_A \subseteq X_B$. An *admissible assignment in* $\mathfrak{B}$ *relative to* $\mathfrak{A}$ is any assignment sending propositional variables into A and nominals into X_A. We say that an equality $\varphi \approx \psi$ is *admissibly valid in* $\mathfrak{B}$ *relative to* $\mathfrak{A}$, denoted $\mathfrak{B} \models_{\mathfrak{A}} \varphi \approx \psi$, if $\mathfrak{B}, v \models \varphi \approx \psi$ for every assignment v admissible relative to $\mathfrak{A}$. Admissible validity of inequalities and quasi-inequalities is defined in the obvious, analogous way. Note that if $\varphi, \psi \in \mathcal{H}(@)$, then $\mathfrak{B} \models_{\mathfrak{A}} \varphi \approx \psi$ iff $\mathfrak{A} \models \varphi \approx \psi$. Therefore, the notion of admissible validity is only interesting for formulas from the extended language $\mathcal{H}^+(@)$. In particular, we will be interested in the cases when $\mathfrak{B}$ is the canonical extension or the Dedekind-MacNeille completion of $\mathfrak{A}$.

Notice that our definition of admissible validity differ from the usual definition used for instance in [30] and [35] in the sense that since we have nominals in the language $\mathcal{H}(@)$, we require nominals to range over X_A rather than over X_B.

4.1.7 Signed generation trees

To any formula/term in $\mathcal{H}^+(@)$ we assign two *signed generation trees*. That is, for $\varphi \in \mathcal{H}(@)$ we consider two trees $+\varphi$ and $-\varphi$, each beginning at the root with the main connective and then branching out into n-nodes at each n-ary connective. Each leaf is either a propositional variable, a constant, or a nominal. The nodes are signed as follows:

- the root node of $+\varphi$ is signed $+$ and the root node of $-\varphi$ is signed $-$;
- if a node is labelled with $\vee, \wedge, \Diamond, \Box$, $\Diamond^{-1}$, or $\Box^{-1}$, its children inherit its sign;
- if a node is labelled with $\neg$, its child is assigned the opposite sign;
- if a node is labelled with $\rightarrow$, the right child inherits its sign, while the left child is assigned the opposite sign;
- if a node is labelled with @ (corresponding to a subformula $@_{\mathbf{i}}\alpha$), the right child (corresponding to α) inherits its sign, while the left child (corresponding to $\mathbf{i}$) is assigned the sign $\pm$.

Note that $\mathsf{E}, \mathsf{A}, \Diamond^{-1}$ and $\Box^{-1}$ do not belong to the language $\mathcal{H}(@)$, so we do not assign signed generation trees to the formulas $\mathsf{E}\varphi, \mathsf{A}\varphi, \Diamond^{-1}\varphi$ and $\Box^{-1}\varphi$.

A node in a signed generation tree is said to be *positive* if it is signed "+", *negative* if it is signed "−", and *bi-polar* if its is signed "±". Examples of signed generation trees can be found in Figure 4.1 and Figure 4.2.

A formula φ is *positive* (*negative*) in a propositional variable p if every occurrence of p in a leaf of the generation tree $+\varphi$ is signed $+$ $(-)$. A formula φ is *positive* (*negative*) in a nominal $\mathbf{i}$ if every occurrence of $\mathbf{i}$ in a leaf of the generation tree $+\varphi$ is signed $+$ or $\pm$ ($-$ or $\pm$).

Note that the only nodes signed "±" are those corresponding to the nominal "subscript" argument of the @ operator. The intuition behind this is the fact that $@_{\mathbf{i}}\varphi$ is equivalent to both $\mathsf{A}(\mathbf{i} \rightarrow \varphi)$ and $\mathsf{E}(\mathbf{i} \wedge \varphi)$ and can therefore be seen as negative or positive depending on the needs of the context.

4.1.8 Order types

We will often use formulas in n variables and therefore write $\overline{p}$ to denote an n-tuple of variables. An *order-type* over $n \in \mathbb{N}$ is an n-tuple $\epsilon \in \{1, \partial\}^n$. Given an order-type $\epsilon = (\epsilon_{p_1}, \ldots, \epsilon_{p_n})$, its opposite order-type, denoted ϵ^∂, is defined by $\epsilon^\partial_{p_i} = 1$ iff $\epsilon_{p_i} = \partial$ for each $1 \leq i \leq n$.

Skeleton (P_3)		PIA (P_1)
Δ-adjoints		SRA
Primary	Secondary	
$+ \quad \vee$	$+ \quad \wedge$	$+ \quad \Box \quad \wedge \quad \neg$
$- \quad \wedge$	$- \quad \vee$	$- \quad \Diamond \quad \vee \quad \neg$
SLR		SRR
$+ \quad \Diamond \quad \neg \quad @$		$+ \quad \vee \quad @ \quad \rightarrow$
$- \quad \Box \quad \neg \quad @ \quad \rightarrow$		$- \quad \wedge \quad @$

Table 4.1: Skeleton and PIA nodes.

4.2 Syntactic classes

Here we introduce a new syntactically defined class of $\mathcal{H}(@)$-inequalities. This class expands the Sahlqvist and inductive formulas (introduced and studied by Goranko and Vakarelov in [53], [54] and [55]) to $\mathcal{H}(@)$. What is more, it generalizes the nominalized Sahlqvist–van Benthem formulas in [35]. In this chapter, we prove that the formulas in this class have first-order local frame correspondents. We also define subclasses, and prove that they are preserved under canonical extensions and Dedekind MacNeille completions, respectively.

For any $\mathcal{H}(@)$ formula $\varphi(p_1, \ldots, p_n)$, any order-type $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, and any $1 \leq i \leq n$, an *ϵ-critical node* in a signed generation tree of φ is a (leaf) node labelled with $+p_i$ if $\epsilon_i = 1$, or $-p_i$ if $\epsilon_i = \partial$. An *ϵ-critical branch* in the tree is a branch terminating in an ϵ-critical node. The intuition, which will be built upon later, is that variable occurrences corresponding to ϵ-critical nodes are *to be solved for, according to ϵ*. We say that $+\varphi$ (resp. $-\varphi$) *agrees with* ϵ, and write $\epsilon(+\varphi)$ (resp. $\epsilon(-\varphi)$), if every leaf node in the signed generation tree of $+\varphi$ (resp. $-\varphi$) which is labelled with a propositional variable is ϵ-critical. In other words, $\epsilon(+\varphi)$ (resp. $\epsilon(-\varphi)$) means that all propositional variable occurrences corresponding to leaves of $+\varphi$ (resp. $-\varphi$) are to be solved for according to ϵ.

We will also make use of the *sub-tree relation* $\gamma \prec \varphi$, which extends to signed generation trees, and we will write $\epsilon(\gamma) \prec *\varphi$ to indicate that γ, regarded as a sub-tree of $*\varphi$, agrees with ϵ.

We will refer to the nodes in signed generation trees as *skeleton nodes* and *PIA nodes*, and further classify them as Δ-adjoints, SLR, SRA and SRR, according to the specification given in Table 4.1[2]. The acronym PIA stands for "positive antecedent implies atom" and is due to van Benthem [75]. The abbreviations SLR, SRA and SRR stand for syntactically left residual, right adjoint and right residual, respectively. The nodes are therefore classified according to the order-theoretic properties of their interpretations.

Why P_3 and not P_2? Notice that in [30], nodes are classified as P_1, P_2 and P_3, and that our P_1 and P_3 correspond to their P_1 and P_3, respectively. So following [30], we use P_3 instead of P_2.

[2]The classification P_1 and P_3 agrees with that in [31] where nodes are classified as P_1, P_2 and P_3. The absence of fixed points in the current setting accounts for the missing P_2.

While reading the following definition, the reader might find it useful to refer to Example 4.2.6 for an illustration of the concepts being introduced.

Definition 4.2.1. Let $\varphi(p_1, \ldots, p_n)$ be a formula in the propositional variables $p_1, \ldots, p_n$, let ϵ be an order type on $\{1, \ldots, n\}$, and $<_\Omega$ a strict partial order on the variables $p_1, \ldots, p_n$. A branch in a signed generation tree $*\varphi$, $* \in \{+, -\}$, ending in a propositional variable is an (ϵ, Ω)-*conforming branch* if, apart from the leaf, it is the concatenation of two paths P_1 and P_3, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting only of PIA-nodes, P_3 consists only of skeleton-nodes, and moreover, it satisfies the conditions (CB1) and (CB2) below:

(CB1) For every SRR-node in P_1 of the form $\gamma \odot \beta$ or $\beta \odot \gamma$, where β is the side where the branch lies, $\epsilon^\partial(\gamma) \prec *\varphi$ (i.e., γ contains no variable occurrences to be solved for — see above). In particular:

(i) if $\gamma \odot \beta$ is $+(\gamma \vee \beta)$ or $+(\beta \to \gamma)$, then $\epsilon^\partial(+\gamma)$;

(ii) if $\gamma \odot \beta$ is $+(\gamma \to \beta)$ or $-(\gamma \wedge \beta)$, then $\epsilon^\partial(-\gamma)$ (equivalently, $\epsilon(+\gamma)$);

(iii) if $\gamma \odot \beta$ is $+@_\gamma\beta$ or $-@_\gamma\beta$, then, by the definition of the syntax, γ must be a nominal, so the condition is met.

(CB2) For every SRR-node in P_1 of the form $\gamma \odot \beta$ or $\beta \odot \gamma$, where β is the side where the branch lies, $p_j <_\Omega p_i$ for every p_j occurring in γ, where p_i is the propositional variable labelling the leaf of the branch.

A branch is called *skeletal* if, apart from the leaf node, it consists entirely of P_3 nodes.

Definition 4.2.2. A signed generation tree $*\psi$, $* \in \{+, -\}$, is said to be (ϵ, Ω)-*inductive* if every ϵ-critical branch in it is (ϵ, Ω)-conforming. A formula ψ is (ϵ, Ω)-*inductive* if $-\psi$ is (ϵ, Ω)-inductive. The formula ψ is *inductive* if it is (ϵ, Ω)-inductive for some ϵ and some Ω. We say that an inequality $\varphi \leq \psi$ is (ϵ, Ω)-inductive if both the generation trees $+\varphi$ and $-\psi$ are (ϵ, Ω)-inductive. The inequality $\varphi \leq \psi$ is *inductive* if it is (ϵ, Ω)-inductive for some ϵ and some Ω.

Remark 4.2.3. Why $+\varphi$ and $-\psi$, and not the other way around, as one would expect? This goes back to [42], and this tradition has been followed in for example [30] and [35]. So we will keep with tradition to make comparisons easier. However, after first approximation (defined in Section 4.3), φ is on the right of the inequality sign, while ψ is on the left, and from there, it is as one would expect.

Definition 4.2.4. We say that a signed generation tree $*\psi$, $* \in \{+, -\}$, is (ϵ, Ω)-*skeletal* if each ϵ-critical branch is skeletal. A formula ψ is said to be (ϵ, Ω)-*skeletal* if $-\psi$ is (ϵ, Ω)-skeletal. A formula ψ is *skeletal* if it is (ϵ, Ω)-skeletal for some order type ϵ and some Ω. An inequality $\varphi \leq \psi$ is (ϵ, Ω)-*skeletal* if both the generation trees $+\varphi$ and $-\psi$ are (ϵ, Ω)-skeletal. An inequality $\varphi \leq \psi$ is *skeletal* if it is (ϵ, Ω)-*skeletal* for some order type ϵ and Ω.

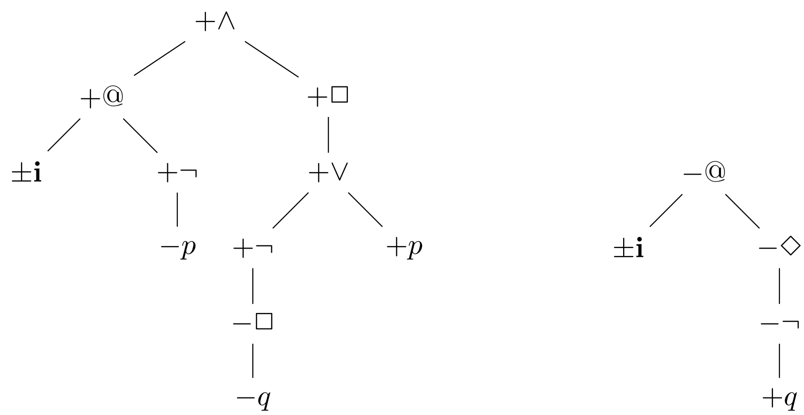

Figure 4.1: The signed generation trees of $+@_{\mathbf{i}}\neg p \wedge \Box(\Box q \to p)$ and $-@_{\mathbf{i}}\Diamond\neg q$.

Note that every (ϵ, Ω)-*skeletal* signed generation tree is (ϵ, Ω)-*inductive.* Indeed, every ϵ-critical branch is the concatenation of paths P_1 and P_3, where P_1 has length 0, and (CB1) and (CB2) trivially holds since the signed generation tree does not contain any SRR nodes.

Definition 4.2.5. A signed generation tree $*\psi$, $* \in \{+, -\}$, is said to be *nominally skeletal* if every branch with leaf $+\mathbf{j}$, for some nominal $\mathbf{j}$,

(NS1) is skeletal, and

(NS2) it shares no secondary Δ-adjoint node ($+\wedge$ or $-\vee$) with another branch also ending in the same positively signed nominal $\mathbf{j}$.

A formula ψ is said to be *nominally skeletal* if $-\psi$ is nominally skeletal. A nominally skeletal formula ψ is *singular* if, for each nominal $\mathbf{i}$, the generation tree $-\psi$ contains at most one leaf labelled $+\mathbf{i}$. An inequality $\varphi \leq \psi$ is *nominally skeletal* if both the generation trees $+\varphi$ and $-\psi$ are nominally skeletal and no positively signed nominal occurs as a leaf in both trees. A nominally skeletal inequality $\varphi \leq \psi$ is *singular* if, for each nominal $\mathbf{i}$, there is at most one leaf labelled $+\mathbf{i}$ between the generation trees $+\varphi$ and $-\psi$.

It is an easy exercise to check that every Sahlqvist formula, every inductive formula in [53], [54] and [55], and every nominalized Sahlqvist–van Benthem formula in [35] is inductive in the sense of Definition 4.2.2.

Example 4.2.6. The inequality $@_{\mathbf{i}}\neg p \wedge \Box(\Box q \to p) \leq @_{\mathbf{i}}\Diamond\neg q$ is inductive for (i) $\epsilon_p = 1 = \epsilon_q$ and $q <_\Omega p$, and (ii) $\epsilon_p = \partial$, $\epsilon_q = 1$ and Ω anything, as can be seen from the signed generation trees in Figure 4.1. Note that the option $\epsilon_p = \partial = \epsilon_q$ does not work since the branch labelled with $-q$ does not divide correctly into P_1 and P_3 parts. This inequality is nominally skeletal. However, it is *not* skeletal for the ϵ's given in (i) and (ii), as both signed generation trees have ϵ-critical branches with P_1 nodes.

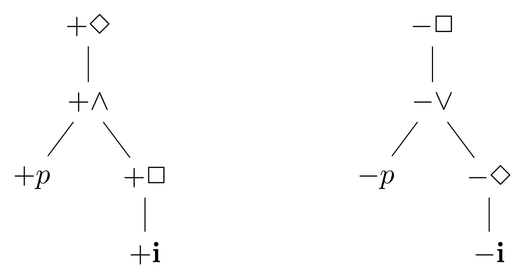

Figure 4.2: The signed generation trees of $+\Diamond(p \wedge \Box\mathbf{i})$ and $-\Box(p \vee \Diamond\mathbf{i})$.

Example 4.2.7. The formula $\Diamond(p \wedge \Box\mathbf{i}) \rightarrow \Box(p \vee \Diamond\mathbf{i})$ is inductive for $\epsilon_p = 1$, $\epsilon_p = \partial$ and $\Omega = \varnothing$, as can be seen from the signed generation trees in Figure 4.2. Moreover, it is skeletal. However, this formula is *not* nominally skeletal, as the branch labelled with $+\mathbf{i}$ contains a P_1 node and therefore is not skeletal.

Example 4.2.8. The formula $@_{\mathbf{i}}\Box(p \rightarrow \Box q) \wedge \Box(\Diamond\mathbf{i} \rightarrow p) \rightarrow \Diamond\Box q$ is inductive for $\epsilon_p = 1 = \epsilon_q$ and $p <_\Omega q$. Moreover, it is nominally skeletal. However, it is *not* skeletal, as the signed generation tree $+@_{\mathbf{i}}\Box(p \rightarrow \Box q) \wedge \Box(\Diamond\mathbf{i} \rightarrow p)$ has ϵ-critical branches that contains P_1 nodes.

Example 4.2.9. The formula $@_{\mathbf{i}}p \wedge \Diamond\Box(p \wedge \mathbf{i}) \rightarrow \Box\Diamond(p \vee \mathbf{i})$ is inductive for $\epsilon_p = 1$ and $\Omega = \varnothing$. However, it is neither skeletal nor nominally skeletal.

Figure 4.3 illustrates the relationships between the inductive, skeletal and nominally skeletal inductive $\mathcal{H}(@)$-formulas and some other relevant 'Sahlqvist-type' classes found in the literature. These other classes are the *Sahlqvist formulas*, the *inductive formulas* in the basic modal language [55, 33], here referred to as *inductive formulas* (ML) to distinguish them form the inductive formulas in $\mathcal{H}(@)$), and the *nominalized Sahlqvist–van Benthem formulas* [28]. We now discuss these relationships in a little more detail.

First we discuss the inclusions. The inclusion of the skeletal and nominally skeletal inductive $\mathcal{H}(@)$-formulas in the inductive $\mathcal{H}(@)$-formulas is immediate by the definitions. It is easy to see that all nominalized Sahlqvist–van Benthem formulas are nominally skeletal inductive $\mathcal{H}(@)$-formulas. In fact, they are are exactly those which contain no @ operators and where the P_1 parts of critical branches contain no SRR nodes. The relationship between inductive and Sahlqvist formulas is well known, see e.g. [55]. The nominalized Sahlqvist–van Benthem formulas without nominal occurrences are, modulo some propositional equivalences, exactly the ordinary modal Sahlqvist formulas, so the intersection between the inductive and nominalized Sahlqvist–van Benthem formulas is exactly the Sahlqvist formulas. Since a pure formula contains no critical branches it is inductive and, in particular, skeletal. A skeletal inductive formula in ML must be Sahlqvist as the only difference between Sahlqvist and inductive formulas is in the PIA (P_1) part of critical branches and these parts are vacuous for skeletal formulas. Therefore, the subset shaded in black in figure 4.3 is empty.

We next demonstrate the non-emptiness of each of the subsets labelled 1 to 12 in the diagram. The formula in Example 4.2.9 is neither skeletal nor nominally skeletal, and so belongs to subset 1. The formula (corresponding to the inequality) in Example 4.2.6 is

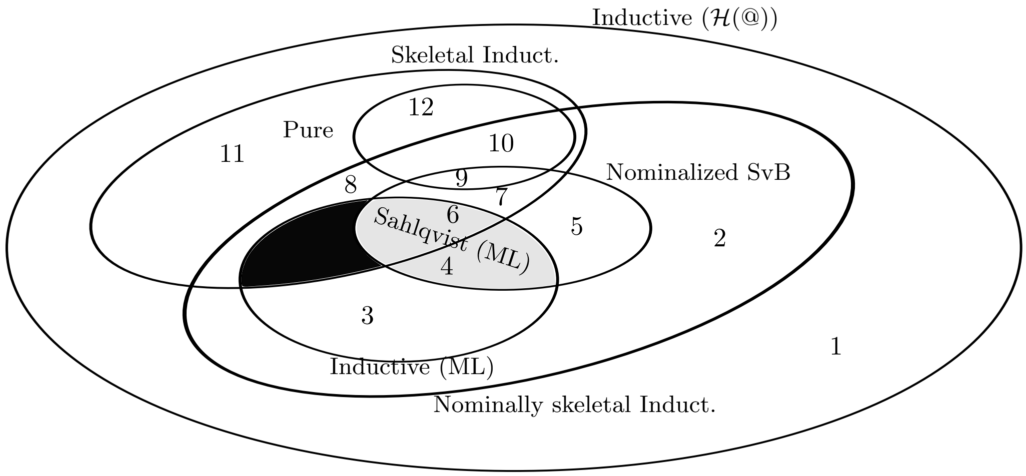

Figure 4.3: Relationships between the syntactic classes.

nominally skeletal but not skeletal and the presence of the @ disqualifies it as inductive (ML) and nominalized Sahlqvist–van Benthem, so it belongs to subset 2. The inductive (ML) formula $p\wedge\Box(\Diamond p \to \Box q) \to \Diamond\Box\Box q$ is not Sahlqvist (and in fact, its frame class is not definable by any Sahlqvist formula [55]), not skeletal and not nominalized Sahlqvist–van Benthem, so it belongs to subset 3. The Geach formula $\Diamond\Box p \to \Box\Diamond p$ is Sahlqvist but not skeletal and so belongs to subset 4. The nominalized Sahlqvist–van Benthem formula $\Box(\neg\mathbf{i}\vee\Diamond\neg p)\vee\Diamond(\mathbf{i}\vee\Box p)$ is neither skeletal nor inductive (ML), and hence witnesses the non-emptiness of subset 5. The formula $p \to \Box p$ belongs to subset 6, being Sahlqvist and skeletal. The nominalized Sahlqvist–van Benthem formula $\Box(\neg p\vee\Box\neg\mathbf{i})\vee\Diamond\Box p$ is Skeletal but neither pure nor Sahlqvist, and so is a member of subset 7. To see that subset 8 is non-empty, consider the non-pure skeletal and nominally skeletal inductive formula $@_{\mathbf{i}}\Diamond p \to \Box(\mathbf{i}\wedge\Box p)$ which, because of the presence of the @, is neither inductive (ML) nor nominalized Sahlqvist–van Benthem. The formula $\Diamond\mathbf{i} \to \mathbf{i}$ belongs to subset 9 since it is pure and a nominalized Sahlqvist–van Benthem formula. In subset 10 we find formulas like $@_{\mathbf{i}}\Diamond\neg i$ which is pure, nominally skeletal and (trivially) skeletal but, because of the presence of the @, not nominalized Sahlqvist–van Benthem. The formula in Example 4.2.7 is skeletal but neither pure nor nominally skeletal, and is therefore to be found in subset 11. Lastly, subset 12 contains formulas like $\Diamond\Box\mathbf{i} \to \mathbf{i}$ which is pure but not nominally skeletal.

Let us now give the main results we will prove in this chapter. The first result concerns correspondence and is a consequence of Theorems 4.5.1 and 4.9.1.

Theorem 4.2.10. *Every inductive formula has a local first order frame correspondent.*

The next two theorems deal with preservation. In particular, the first preservation result says that all skeletal formulas are preserved under Dedekind MacNeile completions. This result follows from Theorems 4.5.4 and 4.9.2.

Theorem 4.2.11. *Every skeletal formula is preserved under Dedekind MacNeille completions of atomic hybrid algebras in which $\Diamond$ preserves all existing joins.*

Since the canonical general frame of any hybrid logic $\mathbf{H}^+(@) \oplus \Sigma$ is discrete [10], and given the connection between di-persistence and preservation under MacNeille completions outlined in Subsection 4.1.5, we have the following:

Corollary 4.2.12. *For any set Σ of skeletal formulas, the logic $\mathbf{H}^+(@) \oplus \Sigma$ is sound and strongly complete with respect to its class of Kripke frames (i.e., the class of Kripke frames defined by the first-order correspondents of the axioms in Σ).*

The final result says that if an inductive formula is nominally skeletal, then it is preserved under canonical extensions of permeated hybrid algebras. This theorem follows from Theorems 4.5.3 and 4.9.4.

Theorem 4.2.13. *Every inductive formula that is nominally skeletal is preserved under canonical extensions of permeated hybrid algebras.*

Since every hybrid logic $\mathbf{H}^+(@) \oplus \Sigma$ is strongly sound and complete with respect its class of permeated hybrid algebras (or dually, with respect to its class of strongly descriptive two-sorted general frames, we have the following:

Corollary 4.2.14. *For any set Σ of inductive formulas that are nominally skeletal, the logic $\mathbf{H}^+(@) \oplus \Sigma$ is sound and strongly complete with respect to its class of Kripke frames (i.e., the class of Kripke frames defined by the first-order correspondents of the axioms in Σ).*

4.3 A calculus and algorithm for correspondence — hybrid-ALBA

The algorithm ALBA was introduced in [35]. Here we present an adaptation called hybrid-ALBA. The aim of this algorithm is to eliminate all propositional variables from a given $\mathcal{H}(@)$-formula or inequality through application of the rules. If this is successful, applying the standard translation to the set of pure quasi-inequalities produced, yields a first-order frame correspondent for the given formula or inequality. Moreover, under certain conditions, success also guarantees canonicity or preservation under Dedekind MacNeille completions.

4.3.1 Strategy and rules

The algorithm proceeds in four phases. The first phase takes an $\mathcal{H}(@)$ inequality $\varphi \leq \psi$, or formula ψ (in this case ψ is replaced with the inequality $\top \leq \psi$), and preprocesses it. To each inequality resulting from this preprocessing, the *first approximation rule* is applied in the second phase. The aim of the third phase is to eliminate all occurring propositional variables from the set of inequalities obtained in the second phase. If this is possible, we proceed to the fourth phase; if not, hybrid-ALBA reports failure and terminates. The fourth phase is the output phase, which returns a first-order frame correspondent for the inequality.

We will now specify these stages and the rules used in each. Although it would be possible to specify rules only for a minimal adequate set of primitive connectives, we will give rules for all connectives. This makes it easier to argue about the algorithm's performance on syntactic classes like the Sahlqvist and inductive formulas, which are typically given in terms of a full complement of connectives. The names approximation, adjunction and residuation derive from the order-theoretic properties that justify the soundness of the rules in these groupings, namely, approximation by atoms or co-atoms and distribution, adjunction, and residuation.

Phase 1: preprocessing

The purpose of preprocessing is to equivalently break up an inequality $\varphi \leq \psi$, given in input, into smaller inequalities through the application of the rules ($\vee$-Adj) and ($\wedge$-Adj) given in Subsection 4.3.1, in this context referred to as *splitting rules*. To facilitate this, consider the positive generation tree of φ and the negative generation tree of ψ, and surface positive occurrences of $\vee$ and negative occurrences of $\wedge$ by applying the following standard equivalences:

$$
\begin{array}{rclcrcl}
\alpha \wedge (\beta \vee \gamma) & \equiv & (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) & \qquad & \alpha \vee (\beta \wedge \gamma) & \equiv & (\alpha \vee \beta) \wedge (\alpha \vee \gamma) \\
\neg(\alpha \vee \beta) & \equiv & \neg\alpha \wedge \neg\beta & & \neg(\alpha \wedge \beta) & \equiv & \neg\alpha \vee \neg\beta \\
\Diamond(\alpha \vee \beta) & \equiv & \Diamond\alpha \vee \Diamond\beta & & \Box(\alpha \wedge \beta) & \equiv & \Box\alpha \wedge \Box\beta \\
@_{\mathbf{i}}(\alpha \vee \beta) & \equiv & @_{\mathbf{i}}\alpha \vee @_{\mathbf{i}}\beta & & @_{\mathbf{i}}(\alpha \wedge \beta) & \equiv & @_{\mathbf{i}}\alpha \wedge @_{\mathbf{i}}\beta
\end{array}
$$

Phase 2: first approximation

Each inequality produced by preprocessing is turned into a quasi-inequality by applying the *first approximation rule* (First-Approx) given below. The algorithm now proceeds separately on each of the quasi-inequalities obtained.

First approximation rule

Let $\{\varphi_i \leq \psi_i \mid i \in I\}$ be the set of inequalities obtained in Phase 1. Then the following *first-approximation rule* is applied to each $\varphi_i \leq \psi_i$ only once:

$$\frac{\varphi_i \leq \psi_i}{\mathbf{i}_0 \leq \varphi_i \;\&\; \psi_i \leq \neg\mathbf{j}_0 \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0}\ \text{(First-Approx)}$$

Here $\mathbf{i}_0$ and $\mathbf{j}_0$ are special reserved nominals which do not occur in any inequality received in input.

Phase 3: reduction and elimination

The aim of this phase is to eliminate all propositional variables from the quasi-inequalities resulting from Phase 2 through the application of the Ackermann rules (RH-Ack) and (LH-Ack), or their special cases (RH-Ack-0) and (LH-Ack-0). To bring the quasi-inequality into the shape to which one of these rules is applicable, the approximation, residuation and adjunction rules are used. All these rules transform one quasi-inequality into another by rewriting part of the antecedent. The only exceptions are the rules (@-R-Res) and (@-L-Res), which introduce a disjunction $\parr$ in the antecedent, and thus cause the quasi-inequality to be rewritten as two quasi-inequalities by distributing the $\Rightarrow$ over the $\parr$. We proceed separately on each of the latter. If all propositional variable occurrences have been eliminated, we denote the resulting set of pure quasi-inequalities by $\mathtt{pure}(\varphi \leq \psi)$. If some propositional variable occurrences could not be eliminated, the algorithm reports failure.

Adjunction rules

The first batch of rules is called *adjunction rules* and is given below:

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \ \& \ \alpha \leq \gamma}\,(\wedge\text{-Adj}) \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \ \& \ \beta \leq \gamma}\,(\vee\text{-Adj})$$

$$\frac{\alpha \leq \Box\beta}{\Diamond^{-1}\alpha \leq \beta}\,(\Box\text{-Adj}) \qquad \frac{\Diamond\alpha \leq \beta}{\alpha \leq \Box^{-1}\beta}\,(\Diamond\text{-Adj})$$

$$\frac{\alpha \leq \neg\beta}{\beta \leq \neg\alpha}\,(\neg\text{-R-Adj}) \qquad \frac{\neg\alpha \leq \beta}{\neg\beta \leq \alpha}\,(\neg\text{-L-Adj})$$

The rules ($\wedge$-Adj) and ($\vee$-Adj) follow from the fact that $\wedge$ is a right adjoint and $\vee$ a left adjoint of the diagonal map $\Delta\colon A \to A \times A$ given by $\Delta\colon a \mapsto (a, a)$. The rules ($\Box$-Adj) and ($\Diamond$-Adj) are justified by the fact that $\Box$ is the right adjoint of $\Diamond^{-1}$ and $\Diamond$ the left adjoint of $\Box^{-1}$. Finally, the last two rules follow from the fact that $\neg$ is its own adjoint.

Residuation rules

The next batch of rules is called *residuation rules* and is as follows:

$$\frac{\alpha \wedge \beta \leq \gamma}{\alpha \leq \beta \to \gamma}\,(\wedge\text{-Res}) \qquad \frac{\alpha \leq \beta \vee \gamma}{\alpha \wedge \neg\beta \leq \gamma}\,(\vee\text{-Res}) \qquad \frac{\alpha \leq \beta \to \gamma}{\alpha \wedge \beta \leq \gamma}\,(\to\text{-Res})$$

$$\frac{\alpha \leq @_{\mathbf{j}}\beta}{\alpha \leq \bot \ \parr \ \mathbf{j} \leq \beta}\,(@\text{-R-Res}) \qquad \frac{@_{\mathbf{j}}\alpha \leq \beta}{\top \leq \beta \ \parr \ \alpha \leq \neg\mathbf{j}}\,(@\text{-L-Res})$$

The residuation rules are based on the residuation properties of the interpretations of the connectives. But why are (@-R-Res) and (@-L-Res) morally residuation rules? Well, by Proposition 4.1.1, these are the right and left residuals of $\alpha \leq @_{\mathbf{j}}\beta$ and $@_{\mathbf{j}}\alpha \leq \beta$, respectively.

Approximation rules

The *approximation rules* are based on the fact that in a complete and atomic hybrid algebra each element is the join of atoms below it and the meet of co-atoms above it, and on the infinitary distribution properties of a complete and atomic hybrid algebra. These rules are given below:

$$\frac{\Box\alpha \leq \neg\mathbf{i}}{\exists\mathbf{j}(\Box\neg\mathbf{j} \leq \neg\mathbf{i} \ \&\ \alpha \leq \neg\mathbf{j})}\ (\Box\text{-Approx}) \qquad \frac{\mathbf{i} \leq \Diamond\alpha}{\exists\mathbf{j}(\mathbf{i} \leq \Diamond\mathbf{j} \ \&\ \mathbf{j} \leq \alpha)}\ (\Diamond\text{-Approx})$$

$$\frac{\mathbf{i} \leq @_{\mathbf{j}}\alpha}{\mathbf{j} \leq \alpha}\ (@\text{-R-Approx}) \qquad \frac{@_{\mathbf{j}}\alpha \leq \neg\mathbf{i}}{\alpha \leq \neg\mathbf{j}}\ (@\text{-L-Approx})$$

Note that the introduced nominal $\mathbf{j}$ in ($\Box$-Approx) and ($\Diamond$-Approx) must be fresh, i.e., $\mathbf{j}$ is not allowed to occur in the computation thus far.

Remark 4.3.1. It would be possible to give approximation for negation, but these would reduce to special cases of the adjunction rules.

Ackermann rules

The *Ackermann rules* are used to eliminate the propositional variables, and therefore form the core of hybrid-ALBA. In contrast to the adjunction, approximation and residuation rules, which are applied to individual inequalities within the antecedents of quasi-inequalities, the Ackermann rules are applied to whole antecedents of quasi-inequalities. These rules are as follows:

$$\frac{\&_{i=1}^{n} \alpha_i \leq p \ \&\ \&_{j=1}^{m} \beta_j(p) \leq \gamma_j(p)}{\&_{j=1}^{m} \beta_j(\bigvee_{i=1}^{n} \alpha_i) \leq \gamma_j(\bigvee_{i=1}^{n} \alpha_i)}\ (\text{RH-Ack})$$

$$\frac{\&_{i=1}^{n} p \leq \alpha_i \ \&\ \&_{j=1}^{m} \gamma_j(p) \leq \beta_j(p)}{\&_{j=1}^{m} \gamma_j(\bigwedge_{i=1}^{n} \alpha_i) \leq \beta_j(\bigwedge_{i=1}^{n} \alpha_i)}\ (\text{LH-Ack})$$

Here

(i) the α_i are p-free,

(ii) the β_j are positive in p, and

(iii) the γ_i are negative in p.

If $n = 0$, $\bigvee_{i=1}^{n} \alpha_i \equiv \bot$ and $\bigwedge_{i=1}^{n} \alpha_i \equiv \top$, so we have the following special cases of (RH-Ack) and (LH-Ack):

$$\frac{\&_{j=1}^{m} \beta_j(p) \leq \gamma_j(p)}{\&_{j=1}^{m} \beta_j(\bot) \leq \gamma_j(\bot)}\ (\text{RH-Ack-0}) \qquad \frac{\&_{j=1}^{m} \gamma_j(p) \leq \beta_j(p)}{\&_{j=1}^{m} \gamma_j(\top) \leq \beta_j(\top)}\ (\text{LH-Ack-0})$$

Phase 4: translation

The resulting set of quasi-inequalities $\mathtt{pure}(\varphi \leq \psi)$ is rewritten as a (conjunction of) $\mathcal{H}^+(@)$ formula(s) and translated into first-order logic by means of the standard translation. The result is a first order frame correspondent for the input formula or inequality.

4.3.2 Safe and topological reductions

An application of (RH-Ack) or (LH-Ack) in which the α_i are all formulas of the original language $\mathcal{H}(@)$ is said to be *safe*. A run of hybrid-ALBA is called *safe* if all applications of (RH-Ack) and (LH-Ack) in it are.

An application of an Ackermann rule is called *topological*, if each inequality in the antecedent of the quasi-inequality to which it is applied is either p-free, or has a syntactically pre-closed left-hand side and a syntactically pre-open right-hand side (for the definition of pre-closed and pre-open formulas, see Definition 4.7.3). More precisely, an application of (RH-Ack) is called *topological*, if the α_i are syntactically pre-closed, and for each $1 \leq j \leq m$, the inequality $\beta_j(p) \leq \gamma_j(p)$ is p-free, or $\beta_j(p)$ is syntactically pre-closed and γ_j is syntactically pre-open. Similarly, an application of (LH-Ack) is called *topological*, if the α_i are syntactically pre-open, and for each $1 \leq j \leq m$, the inequality $\gamma_j(p) \leq \beta_j(p)$ is p-free, or $\gamma_j(p)$ is syntactically pre-closed and β_j is syntactically pre-open. A run of hybrid-ALBA is called *topological* if all applications of (RH-Ack) and (LH-Ack) in it are.

4.4 Examples

In this section, we provide some examples of the algorithm at work.

Example 4.4.1. Consider the formula $@_{\mathbf{i}}\neg p \wedge \Box(\Box q \to p) \to @_{\mathbf{i}}\Diamond\neg q$ in Example 4.2.6. The corresponding inequality $@_{\mathbf{i}}\neg p \wedge \Box(\neg\Box q \vee p) \leq @_{\mathbf{i}}\Diamond\neg q$ remains unchanged under preprocessing, and first approximation turns it into

$$\&\left\{\ \mathbf{i}_0 \leq @_{\mathbf{i}}\neg p \wedge \Box(\neg\Box q \vee p) \qquad @_{\mathbf{i}}\Diamond\neg q \leq \neg\mathbf{j}_0\ \right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Now, applying ($\wedge$-Adj) to $\mathbf{i}_0 \leq @_{\mathbf{i}}\neg p \wedge \Box(\neg\Box q \vee p)$ gives

$$\&\left\{\begin{array}{ll} \mathbf{i}_0 \leq @_{\mathbf{i}}\neg p & @_{\mathbf{i}}\Diamond\neg q \leq \neg\mathbf{j}_0 \\ \mathbf{i}_0 \leq \Box(\neg\Box q \vee p) & \end{array}\right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Next, we apply ($\Box$-Adj) to $\mathbf{i}_0 \leq \Box(\neg\Box q \vee p)$ and (@-L-Approx) to $@_{\mathbf{i}}\Diamond\neg q \leq \neg\mathbf{j}_0$ to get

$$\&\left\{\begin{array}{ll} \mathbf{i}_0 \leq @_{\mathbf{i}}\neg p & \Diamond\neg q \leq \neg\mathbf{i} \\ \Diamond^{-1}\mathbf{i}_0 \leq \neg\Box q \vee p & \end{array}\right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Applying ($\vee$-Res) to $\Diamond^{-1}\mathbf{i}_0 \leq \neg\Box q \vee p$ and ($\Diamond$-Adj) to $\Diamond\neg q \leq \neg\mathbf{i}$ yields

$$\&\left\{\begin{array}{ll} \mathbf{i}_0 \leq @_{\mathbf{i}}\neg p & \neg q \leq \Box^{-1}\neg\mathbf{i} \\ \Diamond^{-1}\mathbf{i}_0 \wedge \neg\neg\Box q \leq p & \end{array}\right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

We next apply ($\neg$-L-Adj) to $\neg q \leq \Box^{-1}\neg\mathbf{i}$ to obtain

$$\&\left\{\begin{array}{ll} \mathbf{i}_0 \leq @_{\mathbf{i}}\neg p & \neg\Box^{-1}\neg\mathbf{i} \leq q \\ \Diamond^{-1}\mathbf{i}_0 \wedge \neg\neg\Box q \leq p & \end{array}\right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Applying (RH-Ack) gives

$$\& \left\{ \begin{array}{l} \mathbf{i}_0 \leq @_{\mathbf{i}}\neg p \\ \Diamond^{-1}\mathbf{i}_0 \wedge \neg\neg\Box\neg\Box^{-1}\neg\mathbf{i} \leq p \end{array} \right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Finally, another application of (RH-Ack) produces

$$\mathbf{i}_0 \leq @_{\mathbf{i}}\neg(\Diamond^{-1}\mathbf{i}_0 \wedge \neg\neg\Box\neg\Box^{-1}\neg\mathbf{i}) \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

At this point all propositional variables have been eliminated and the the standard translations of the above quasi-inequality will be a first-order frame correspondent of this formula. Before attempting this translation, however, we first simplify the above quasi-inequality to get

$$\mathbf{i}_0 \leq @_{\mathbf{i}}(\Box^{-1}\neg\mathbf{i}_0 \vee \Diamond\Box^{-1}\neg\mathbf{i}) \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Now,

$$\begin{array}{rl} & \forall\mathbf{i}\forall\mathbf{j}_0 \left(\mathbf{i}_0 \leq @_{\mathbf{i}}(\Box^{-1}\neg\mathbf{i}_0 \vee \Diamond\Box^{-1}\neg\mathbf{i}) \Rightarrow \mathbf{i}_0 \neq \mathbf{j}_0\right) \\ \equiv & \forall\mathbf{i}\forall\mathbf{j}_0 \left(\mathbf{i}_0 \leq @_{\mathbf{i}}\Box^{-1}\neg\mathbf{i}_0 \vee @_{\mathbf{i}}\Diamond\Box^{-1}\neg\mathbf{i} \Rightarrow \mathbf{i}_0 \neq \mathbf{j}_0\right) \\ \equiv & \forall\mathbf{i} \left(\mathbf{i}_0 \leq @_{\mathbf{i}}\Box^{-1}\neg\mathbf{i}_0 \vee @_{\mathbf{i}}\Diamond\Box^{-1}\neg\mathbf{i} \Rightarrow \forall\mathbf{j}_0(\mathbf{i}_0 \neq \mathbf{j}_0)\right) \\ \equiv & \forall\mathbf{i} \left(\mathbf{i}_0 \leq @_{\mathbf{i}}\Box^{-1}\neg\mathbf{i}_0 \vee @_{\mathbf{i}}\Diamond\Box^{-1}\neg\mathbf{i} \Rightarrow \bot\right) \\ \equiv & \forall\mathbf{i} \left(\mathbf{i}_0 \leq @_{\mathbf{i}}\Box^{-1}\neg\mathbf{i}_0 \Rightarrow \bot\right) \\ \equiv & \forall\mathbf{i}(\mathbf{i}_0 \leq @_{\mathbf{i}}\Diamond^{-1}\mathbf{i}_0). \end{array}$$

Translating this gives

$$ST_{y_{\mathbf{i}_0}} \left(\forall\mathbf{i}(\mathbf{i}_0 \leq @_{\mathbf{i}}\Diamond^{-1}\mathbf{i}_0)\right) = \forall y_{\mathbf{i}} \left(\exists y(yRy_{\mathbf{i}} \wedge y = y_{\mathbf{i}_0})\right) = \forall y_{\mathbf{i}}(y_{\mathbf{i}_0}Ry_{\mathbf{i}}).$$

Example 4.4.2. Consider the formula $\Diamond(p \wedge \Box\mathbf{i}) \to \Box(p \vee \Diamond\mathbf{i})$ in Example 4.2.7. The corresponding inequality $\Diamond(p \wedge \Box\mathbf{i}) \leq \Box(p \vee \Diamond\mathbf{i})$ remains unchanged under preprocessing, and first approximation turns it into

$$\& \left\{ \begin{array}{ll} \mathbf{i}_0 \leq \Diamond(p \wedge \Box\mathbf{i}) & \qquad \Box(p \vee \Diamond\mathbf{i}) \leq \neg\mathbf{j}_0 \end{array} \right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Applying ($\Diamond$-Approx) to $\mathbf{i}_0 \leq \Diamond(p \wedge \Box\mathbf{i})$ produces the quasi-inequality

$$\& \left\{ \begin{array}{ll} \mathbf{i}_0 \leq \Diamond\mathbf{j} & \qquad \Box(p \vee \Diamond\mathbf{i}) \leq \neg\mathbf{j}_0 \\ \mathbf{j} \leq p \wedge \Box\mathbf{i} & \end{array} \right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Now, apply ($\wedge$-Adj) to the inequality $\mathbf{j} \leq p \wedge \Box\mathbf{i}$ to obtain

$$\& \left\{ \begin{array}{ll} \mathbf{i}_0 \leq \Diamond\mathbf{j} & \qquad \Box(p \vee \Diamond\mathbf{i}) \leq \neg\mathbf{j}_0 \\ \mathbf{j} \leq p & \\ \mathbf{j} \leq \Box\mathbf{i} & \end{array} \right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Applying (RH-Ack) to the above quasi-inequality gives

$$\& \left\{ \begin{array}{ll} \mathbf{i}_0 \leq \Diamond\mathbf{j} & \qquad \Box(\mathbf{j} \vee \Diamond\mathbf{i}) \leq \neg\mathbf{j}_0 \\ \mathbf{j} \leq \Box\mathbf{i} & \end{array} \right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

At this point all propositional variables have been eliminated from all quasi-inequalities, and it can be further simplified and then translated. This is left for the reader.

Example 4.4.3. Consider the formula $@_{\mathbf{i}}p \wedge \Diamond\Box(p \wedge \mathbf{i}) \rightarrow \Box\Diamond(p \vee \mathbf{i})$ in Example 4.2.9. The corresponding inequality $@_{\mathbf{i}}p \wedge \Diamond\Box(p \wedge \mathbf{i}) \leq \Box\Diamond(p \vee \mathbf{i})$ remains unchanged under preprocessing, and first approximation turns it into

$$\&\left\{\begin{array}{ll} \mathbf{i}_0 \leq @_{\mathbf{i}}p \wedge \Diamond\Box(p \wedge \mathbf{i}) & \qquad \Box\Diamond(p \vee \mathbf{i}) \leq \neg\mathbf{j}_0 \end{array}\right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Now, applying ($\wedge$-Adj) to the inequality $\mathbf{i}_0 \leq @_{\mathbf{i}}p \wedge \Diamond\Box(p \wedge \mathbf{i})$ gives

$$\&\left\{\begin{array}{ll} \mathbf{i}_0 \leq @_{\mathbf{i}}p & \qquad \Box\Diamond(p \vee \mathbf{i}) \leq \neg\mathbf{j}_0 \\ \mathbf{i}_0 \leq \Diamond\Box(p \wedge \mathbf{i}) & \end{array}\right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Applying (@-R-Approx) to $\mathbf{i}_0 \leq @_{\mathbf{i}}p$ and ($\Diamond$-Approx) to $\mathbf{i}_0 \leq \Diamond\Box(p \wedge \mathbf{i})$ yields

$$\&\left\{\begin{array}{ll} \mathbf{i} \leq p & \qquad \Box\Diamond(p \vee \mathbf{i}) \leq \neg\mathbf{j}_0 \\ \mathbf{i}_0 \leq \Diamond\mathbf{j} & \\ \mathbf{j} \leq \Box(p \wedge \mathbf{i}) & \end{array}\right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

We now apply ($\Box$-Adj) to $\mathbf{j} \leq \Box(p \wedge \mathbf{i})$ to get

$$\&\left\{\begin{array}{ll} \mathbf{i} \leq p & \qquad \Box\Diamond(p \vee \mathbf{i}) \leq \neg\mathbf{j}_0 \\ \mathbf{i}_0 \leq \Diamond\mathbf{j} & \\ \Diamond^{-1}\mathbf{j} \leq p \wedge \mathbf{i} & \end{array}\right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Another application of ($\wedge$-Adj) yields

$$\&\left\{\begin{array}{ll} \mathbf{i} \leq p & \qquad \Box\Diamond(p \vee \mathbf{i}) \leq \neg\mathbf{j}_0 \\ \mathbf{i}_0 \leq \Diamond\mathbf{j} & \\ \Diamond^{-1}\mathbf{j} \leq p & \\ \Diamond^{-1}\mathbf{j} \leq \mathbf{i} & \end{array}\right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

Now, apply (RH-Ack) to the above quasi-inequality to get

$$\&\left\{\begin{array}{ll} \mathbf{i}_0 \leq \Diamond\mathbf{j} & \qquad \Box\Diamond(\mathbf{i} \vee \Diamond^{-1}\mathbf{j} \vee \mathbf{i}) \leq \neg\mathbf{j}_0 \\ \Diamond^{-1}\mathbf{j} \leq \mathbf{i} & \end{array}\right\} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0.$$

At this point all propositional variables have been eliminated, and we leave it to the reader to simplify and translate.

4.5 Correspondence and preservation

Sahlqvist theory consists of two parts: *correspondence* and *preservation*. In the first part of this section, we will focus on correspondence. In particular, we show that whenever hybrid-ALBA succeeds in eliminating all propositional variables from an inequality, the first-order formula returned is locally equivalent on frames to the inequality. We first give the main theorem and its proof, and subsequently give the lemma needed to prove this theorem.

Theorem 4.5.1. *If* hybrid-ALBA *succeeds on a* $\mathcal{H}(@)$*-formula* ψ*, then* $\forall\overline{y}\forall x ST_x(\mathtt{pure}(\psi))$ *is a local first-order frame correspondent for* ψ*. Here* $ST_x(\mathtt{pure}(\psi))$ *is the conjunction of the standard translations of the quasi-inequalities in* $\mathtt{pure}(\psi)$ *and* $\overline{y}$ *is the vector off all variables* $y_{\mathbf{i}}$ *corresponding to nominals* $\mathbf{i}$ *other than* $\mathbf{i}_0$ *occurring in* $\mathtt{pure}(\psi)$*.*

Proof. Let $\varphi_i \leq \psi_i$, $1 \leq i \leq n$ be the inequalities produced by preprocessing the inequality $\top \leq \psi$. Then we have the following equivalences:

$$
\begin{aligned}
\mathfrak{F}, w \Vdash \psi &\iff \mathfrak{F}, w \Vdash \bigwedge_{i=1}^{n} \varphi_i \leq \psi_i \\
&\iff \text{ for all } 1 \leq i \leq n,\ \mathfrak{F}, w \Vdash \varphi_i \leq \psi_i \\
&\iff \text{ for all } 1 \leq i \leq n,\ \mathfrak{F} \Vdash \mathbf{i}_0 \leq \varphi_i \Rightarrow \mathbf{i}_0 \leq \psi_i[\mathbf{i}_0 := w] \\
&\iff \text{ for all } 1 \leq i \leq n,\ \mathfrak{F} \Vdash \mathbf{i}_0 \leq \varphi_i \,\&\, \psi_i \leq \neg\mathbf{j}_0 \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0[\mathbf{i}_0 := w] \\
&\iff \text{ for all } 1 \leq i \leq n,\ \mathfrak{F}^+ \models \mathbf{i}_0 \leq \varphi_i \,\&\, \psi_i \leq \neg\mathbf{j}_0 \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0[\mathbf{i}_0 := w] \\
&\iff \mathfrak{F}^+ \models \mathtt{pure}(\psi)[\mathbf{i}_0 := w]. \\
&\iff \mathfrak{F} \Vdash \mathtt{pure}(\psi)[\mathbf{i}_0 := w] \\
&\iff \mathfrak{F} \models \forall\overline{y}\forall x ST_x(\mathtt{pure}(\psi))[\mathbf{i}_0 := w].
\end{aligned}
$$

In the last line above $\overline{y}$ is the vector off all variables $y_{\mathbf{i}}$ corresponding to nominals $\mathbf{i}$ *other than* $\mathbf{i}_0$ occurring in $\mathtt{pure}(\psi)$. Thus the only free variable in $\forall\overline{y}\forall x ST_x(\mathtt{pure}(\psi))$ is $y_{\mathbf{i}_0}$. The first equivalence is a consequence of Lemma 4.5.2 below, while the third last equivalence follows from Proposition 4.6.2. $\square$

The following lemma is immediate as the equivalences and rules applied during preprocessing are obviously valid on all hybrid algebras and also (locally) on frames.

Lemma 4.5.2. *Let* $\mathfrak{F}$ *be a frame. If* $\varphi \leq \psi$ *is an* $\mathcal{H}(@)$*-inequality and* $\{\varphi_i \leq \psi_i \mid 1 \leq i \leq n\}$ *is the set of inequalities obtained from it by the application of one or more preprocessing steps, then*

(i) *for any hybrid algebra* $\mathfrak{B} = (\mathbf{B}, X_B)$ *and hybrid subalgebra* $\mathfrak{A} = (\mathbf{A}, X_A)$*, it holds that*

$$\mathfrak{B} \models_{\mathfrak{A}} \varphi \leq \psi \ \textit{ iff } \ \mathfrak{B} \models_{\mathfrak{A}} \varphi_i \leq \psi_i \ \textit{ for all } 1 \leq i \leq n;$$

(ii) *for any* $\mathfrak{F} = (W, R)$ *and* $w \in W$*, it holds that*

$$\mathfrak{F}, w \Vdash \varphi \leq \psi \ \textit{ iff } \ \mathfrak{F}, w \Vdash \varphi_i \leq \psi_i \ \textit{ for all } 1 \leq i \leq n.$$

Let us now move on to preservation. In particular, we look at canonicity and preservation under Dedekind-MacNeille completions.

Our approach can be illustrated by the "U-shaped argument" in Figure 4.4. In this diagram, $\mathfrak{A}$ is an atomic hybrid algebra, and $\mathfrak{B}$ some completion of $\mathfrak{A}$. Let us now focus on the left-hand arm of the diagram. The first equivalence follows from the fact that the validity of $\mathcal{H}(@)$-formulas in $\mathfrak{A}$ coincides with admissible validity of $\mathcal{H}(@)$-formulas in $\mathfrak{B}$. The aim now is to transform the inequality into a set of pure inequalities, denoted $\mathtt{pure}(\varphi \leq \psi)$ in Figure 4.4. It is here where the algorithm hybrid-ALBA comes in. Moving on to the equivalence in the base of the "U", note that since our idea of admissible validity differ from the usual definition, this equivalence is not a given. If $\mathfrak{B}$ is the Dedekind MacNeille completion of $\mathfrak{A}$, the equivalence forming the base of the "U" follows from the fact that $\mathfrak{B}$ and $\mathfrak{A}$ have the same atoms, and hence that the validity and admissible validity of pure formulas coincide. However, if $\mathfrak{B}$ is the canonical extension of $\mathfrak{A}$, we need an additional requirement that the inequalities obtained have a specific shape. But what guarantees the second and third equivalences in the left-hand arm and the equivalences on the right-hand arm? Unlike the other equivalences, these require a bit more work. But before we give formal proofs of these equivalences, we give the preservation results.

$$
\begin{array}{ccc}
\mathfrak{A} \models \varphi \leq \psi & & \mathfrak{B} \models \varphi \leq \psi \\
\Updownarrow & & \\
\mathfrak{B} \models_{\mathfrak{A}} \varphi \leq \psi & & \Big\Updownarrow \\
\Updownarrow & & \\
\mathfrak{B} \models_{\mathfrak{A}} \mathbf{i}_0 \leq \varphi \ \&\ \psi \leq \mathbf{j}_0 \Rightarrow \mathbf{i}_0 \leq \neg \mathbf{i}_0 & & \mathfrak{B} \models \mathbf{i}_0 \leq \varphi \ \&\ \psi \leq \mathbf{j}_0 \Rightarrow \mathbf{i}_0 \leq \neg \mathbf{i}_0 \\
\Updownarrow & & \Updownarrow \\
\mathfrak{B} \models_{\mathfrak{A}} \mathtt{pure}(\varphi \leq \psi) & \Longleftrightarrow & \mathfrak{B} \models \mathtt{pure}(\varphi \leq \psi)
\end{array}
$$

Figure 4.4: The U-shaped argument for preservation of inequalities interpreted on a hybrid algebra $\mathfrak{A}$.

We begin with canonicity.

Theorem 4.5.3. *If a topological run of* hybrid-ALBA *succeeds on an $\mathcal{H}(@)$-inequality $\varphi \leq \psi$, and every quasi-inequality produced is of the form prescribed by Proposition 4.8.5, then $\varphi \leq \psi$ is canonical.*

Proof. Note that if we can show that each of the inequalities we obtain after preprocessing is canonical, then their conjunction is also canonical, so we may assume without loss of generality that the preprocessing process yields a single inequality $\varphi \leq \psi$. Now, let $\mathtt{pure}(\varphi \leq \psi)$ be the quasi-inequalities obtained after executing hybrid-ALBA. Then we have the following:

$$
\begin{array}{ccc}
\mathfrak{A} \models \varphi \leq \psi & & \mathfrak{A}^\delta \models \varphi \leq \psi \\
\Updownarrow & & \\
\mathfrak{A}^\delta \models_{\mathfrak{A}} \varphi \leq \psi & & \Big\Updownarrow \\
\Updownarrow & & \\
\mathfrak{A}^\delta \models_{\mathfrak{A}} \mathbf{i}_0 \leq \varphi \ \&\ \psi \leq \mathbf{j}_0 \Rightarrow \mathbf{i}_0 \leq \neg \mathbf{i}_0 & & \mathfrak{A}^\delta \models \mathbf{i}_0 \leq \varphi \ \&\ \psi \leq \mathbf{j}_0 \Rightarrow \mathbf{i}_0 \leq \neg \mathbf{i}_0 \\
\Updownarrow & & \Updownarrow \\
\mathfrak{A}^\delta \models_{\mathfrak{A}} \mathtt{pure}(\varphi \leq \psi) & \Longleftrightarrow & \mathfrak{A}^\delta \models \mathtt{pure}(\varphi \leq \psi)
\end{array}
$$

The first equivalence in the left-hand arm of the "U" follows from the fact that the formulas

φ and ψ are in $\mathcal{H}(@)$. For the second equivalence in the left-hand arm, we use Lemma 4.6.1. The third equivalence in the left-hand arm follows from Proposition 4.6.4. The equivalence in the base of the "U" follows from Proposition 4.8.5. For the second equivalence in the right-hand arm, we use Proposition 4.6.2. Finally, the first equivalence in the right-hand arm follows from Lemma 4.6.1. $\square$

In the case of Dedekind-MacNeille completions, we have the following preservation result:

Theorem 4.5.4. *If a safe run of* hybrid-ALBA *succeeds on an $\mathcal{H}(@)$-inequality $\varphi \leq \psi$, then $\varphi \leq \psi$ is preserved under Dedekind-MacNeille completions of atomic hybrid algebras in which $\Diamond$ preserves all existing joins.*

Proof. Note that if we can show that each of the inequalities we obtain after preprocessing is preserved under Dedekind MacNeille completions, then their conjunction is also preserved under Dedekind MacNeille completions, so we may assume without loss of generality that the preprocessing process yields a single inequality $\varphi \leq \psi$. Now, let $\mathtt{pure}(\varphi \leq \psi)$ be the quasi-inequalities obtained after a safe run of hybrid-ALBA. We then have the following:

$$
\begin{array}{ccc}
\mathfrak{A} \models \varphi \leq \psi & & \mathfrak{A}^{dm} \models \varphi \leq \psi \\
\Updownarrow & & \\
\mathfrak{A}^{dm} \models_{\mathfrak{A}} \varphi \leq \psi & & \Big\Updownarrow \\
\Updownarrow & & \\
\mathfrak{A}^{dm} \models_{\mathfrak{A}} \mathbf{i}_0 \leq \varphi \;\&\; \psi \leq \mathbf{j}_0 \Rightarrow \mathbf{i}_0 \leq \neg \mathbf{i}_0 & & \mathfrak{A}^{dm} \models \mathbf{i}_0 \leq \varphi \;\&\; \psi \leq \mathbf{j}_0 \Rightarrow \mathbf{i}_0 \leq \neg \mathbf{i}_0 \\
\Updownarrow & & \Updownarrow \\
\mathfrak{A}^{dm} \models_{\mathfrak{A}} \mathtt{pure}(\varphi \leq \psi) & \Longleftrightarrow & \mathfrak{A}^{dm} \models \mathtt{pure}(\varphi \leq \psi)
\end{array}
$$

The first equivalence in the left-hand arm follows from the fact that the formulas φ and ψ are in the language $\mathcal{H}(@)$. For the second equivalence in the left-hand arm, we apply Lemma 4.6.1. The third equivalence in the left-hand arm is a consequence of Proposition 4.6.5. The equivalence in the base of the "U" follows from the fact that $\mathtt{pure}(\varphi \leq \psi)$ contains no propositional variables and that $\mathfrak{A}$ and $\mathfrak{A}^{dm}$ have the same atoms, and hence that its admissible validity and validity coincide. The second equivalence in the right-hand arm follows from Proposition 4.6.2. Finally, the first equivalence in the right-hand arm is a consequence of Lemma 4.6.1. $\square$

4.6 Soundness

Here we prove the results that complete the "U"-shaped arguments in Theorems 4.5.3 and 4.5.4. We begin with the second equivalence in the left-hand arm and the first equivalence in the right-hand arm.

Lemma 4.6.1. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be an atomic hybrid algebra and $\varphi \leq \psi$ an $\mathcal{H}(@)$ inequality. Then $\mathfrak{A} \models \varphi \leq \psi$ iff $\mathfrak{A} \models \mathbf{i}_0 \leq \varphi \,\&\, \psi \leq \neg \mathbf{j}_0 \Rightarrow \mathbf{i}_0 \leq \neg \mathbf{j}_0$, where $\mathbf{i}_0$ and $\mathbf{j}_0$ are any two nominals not occurring in $\varphi \leq \psi$.*

Proof. The implication from left to right is immediate. For the sake of the other direction, assume that $\mathfrak{A} \models \mathbf{i}_0 \leq \varphi \,\&\, \psi \leq \neg\mathbf{j}_0 \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$, and let v be a arbitrary assignment on $\mathfrak{A}$. We need to show that $v(\varphi) \leq v(\psi)$. Since $\mathfrak{A}$ is atomic, $v(\varphi) = \bigvee\{x \in X_A \mid x \leq v(\varphi)\}$, so it is enough to show that $x \leq v(\varphi)$ implies $x \leq v(\psi)$ for all $x \in X_A$. Accordingly, suppose that $x \leq v(\varphi)$, while $x \not\leq v(\psi)$. Then $x \leq \neg v(\psi)$, and hence, $v(\psi) \leq \neg x$. Let v' the variant of v sending both $\mathbf{i}_0$ and $\mathbf{j}$ to x. Then $v'(\mathbf{i}_0) \leq v'(\varphi)$ and $v'(\psi) \leq v'(\neg\mathbf{j}_0)$ but $v'(\mathbf{i}_0) \not\leq v'(\neg\mathbf{j}_0)$, contradicting our assumption. $\square$

Let us now move on to the second equivalences in the right-hand arms of the "U"-shaped arguments in Theorems 4.5.3 and 4.5.4. Note that the implication from top to bottom tells us that hybrid-ALBA must be sound on complete and atomic hybrid algebras, while the implication from bottom to top says that the inverses of the rules of hybrid-ALBA must also be sound on complete and atomic hybrid algebras. This is exactly what the next proposition tells us. In order to prove this proposition, we need algebraic versions of the *Ackermann lemmas*, which will be proved in Section 4.7.

Proposition 4.6.2 (Parameterized soundness on complete and atomic hybrid algebras). *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a complete and atomic hybrid algebra. If $\varphi_1 \leq \psi_1 \,\&\cdots\&\, \varphi_n \leq \psi_n \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$ is an $\mathcal{H}^+(@)$-quasi-inequality, and $\varphi'_1 \leq \psi'_1 \,\&\cdots\&\, \varphi'_m \leq \psi'_m \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$ is obtained from it by any approximation, residuation, adjunction, or Ackermann rule, then for $x_0 \in X_A$, $\mathfrak{A} \models \varphi_1 \leq \psi_1 \,\&\cdots\&\, \varphi_n \leq \psi_n \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0[\mathbf{i}_0 := x_0]$ iff $\mathfrak{A} \models \varphi'_1 \leq \psi'_1 \,\&\cdots\&\, \varphi'_m \leq \psi'_m \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0[\mathbf{i}_0 := x_0]$[3].*

Proof. We need to verify the claim for each approximation, residuation, adjunction and Ackermann rule. We will treat the cases for ($\Box$-Approx), ($\Diamond$-Adj), (@-L-Res) and (RH-Ack) as a representative sample, the cases for the other rules each being similar to one of these.

($\Box$-Approx) Let C be a conjunction of inequalities, and assume that we have $\mathfrak{A} \models C \;\&\; \Box\alpha \leq \neg\mathbf{i} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0[\mathbf{i}_0 := x_0]$. Suppose that under assignment v with $v(\mathbf{i}_0) = x_0$, it is the case that $C \;\&\; \alpha \leq \neg\mathbf{j} \;\&\; \Box\neg\mathbf{j} \leq \neg\mathbf{i}$. We need to show that $\mathbf{i}_0 \leq \neg\mathbf{j}_0$ under v. But by the monotonicity of $\Box$, it follows that $\Box\alpha \leq \neg\mathbf{i}$, i.e., under v, $C \;\&\; \Box\alpha \leq \neg\mathbf{i}$, so by our assumption, $\mathbf{i}_0 \leq \neg\mathbf{j}_0$.

Conversely, assume $\mathfrak{A} \models C \,\&\, \alpha \leq \neg\mathbf{j} \,\&\, \Box\neg\mathbf{j} \leq \neg\mathbf{i} \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0[\mathbf{i}_0 := x_0]$. Suppose that under assignment v with $v(\mathbf{i}_0) = x_0$, $C \;\&\; \Box\alpha \leq \neg\mathbf{i}$. Now, $\Box\bigwedge\{\neg x \mid \alpha \leq \neg x \text{ and } x \in X_A\} \leq \neg\mathbf{i}$, so $\bigwedge\{\Box\neg x \mid \alpha \leq \neg x \text{ and } x \in X_A\} \leq \neg\mathbf{i}$. This means that $\mathbf{i} \leq \neg\bigwedge\{\Box\neg x \mid \alpha \leq \neg x \text{ and } x \in X_A\}$, and hence, $\mathbf{i} \leq \bigvee\{\neg\Box\neg x \mid \alpha \leq \neg x \text{ and } x \in X_A\}$. Since the value of $\mathbf{i}$ under v must be an atom, $\mathbf{i} \leq \neg\Box\neg y_0$ for some $y_0 \in X_A$ such that $\alpha \leq \neg y_0$. Let v' be the $\mathbf{j}$-variant of v which sends the fresh nominal $\mathbf{j}$ to y_0. Then, under v', we have $\alpha \leq \neg\mathbf{j}$ and $\mathbf{i} \leq \neg\Box\neg\mathbf{j}$, i.e., $\alpha \leq \neg\mathbf{j}$ and $\Box\neg\mathbf{j} \leq \neg\mathbf{i}$. Thus, by our assumption, it follows that $\mathbf{i}_0 \leq \neg\mathbf{j}_0$.

($\Diamond$-Adj) Let C be a conjunction of inequalities, and assume that we have $\mathfrak{A} \models C \;\&\; \Diamond\alpha \leq \beta \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0[\mathbf{i}_0 := x_0]$. Suppose that under assignment v with $v(\mathbf{i}_0) = x_0$, it is the case that $C \;\&\; \alpha \leq \Box^{-1}\beta$. We have to show that under v, $\mathbf{i}_0 \leq \neg\mathbf{j}_0$. Now, by the definition of an adjoint pair, $\Diamond\alpha \leq \beta$. Hence, by our assumption, $\mathbf{i}_0 \leq \neg\mathbf{j}_0$.

[3]Note that the rules (@-L-Res) and (@-R-Res) introduce disjunctions, so the resulting expressions are of course not strictly speaking quasi-inequalities.

For the converse, assume $\mathfrak{A} \models C \mathbin{\&} \alpha \leq \Box^{-1}\beta \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0[\mathbf{i}_0 := x_0]$. Furthermore, suppose that under assignment v with $v(\mathbf{i}_0) = x_0$, $C \mathbin{\&} \Diamond\alpha \leq \beta$. Again, by the definition of an adjoint pair, $\alpha \leq \Box^{-1}\beta$, so, by our assumption, $\mathbf{i}_0 \leq \neg\mathbf{j}_0$.

(@-L-Res) Let C be a conjunction of inequalities, and assume that we have $\mathfrak{A} \models C \mathbin{\&} @_{\mathbf{j}}\alpha \leq \beta \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0[\mathbf{i}_0 := x_0]$. Suppose that under assignment v with $v(\mathbf{i}_0) = x_0$, $C \mathbin{\&} (\top \leq \beta \parr \alpha \leq \neg\mathbf{j})$. We have to show that under v, $\mathbf{i}_0 \leq \neg\mathbf{j}_0$. If $\top \leq \beta$, $@_{\mathbf{j}}\alpha \leq \beta$, so by our assumption $\mathbf{i}_0 \leq \neg\mathbf{j}_0$. On the other hand, if $\alpha \leq \neg\mathbf{j}$, $@_{\mathbf{j}}\alpha = \bot$, which means that $@_{\mathbf{j}}\alpha \leq \beta$, so, again, by our assumption, $\mathbf{i}_0 \leq \neg\mathbf{j}_0$.

Conversely, assume $\mathfrak{A} \models C \mathbin{\&} (\top \leq \beta \parr \alpha \leq \neg\mathbf{j}) \Rightarrow \mathbf{i}_0 \leq \mathbf{j}_0[\mathbf{i}_0 := x_0]$. Suppose that under assignment v with $v(\mathbf{i}_0) = x_0$, $C \mathbin{\&} @_{\mathbf{j}}\alpha \leq \beta$. First, assume $\mathbf{j} \leq \alpha$. Then $@_{\mathbf{j}}\alpha = \top$, which means that $\top \leq \beta$. Next, assume $\mathbf{j} \nleq \alpha$. But the value of $\mathbf{j}$ under v must be an atom, so $\mathbf{j} \leq \neg\alpha$. Hence, $\alpha \leq \neg\mathbf{j}$, and so, we have $C \mathbin{\&} (\top \leq \beta \parr \alpha \leq \neg\mathbf{j})$ under v. Thus, by our assumption $\mathbf{i}_0 \leq \neg\mathbf{j}_0$.

(RH-Ack) Assume that $\mathfrak{A} \models \&_{i=1}^{n} \alpha_i \leq p \mathbin{\&} \&_{j=1}^{m} \beta_j(p) \leq \gamma_j(p) \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0[\mathbf{i}_0 := x_0]$. Suppose that under assignment v with $v(\mathbf{i}_0) = x_0$, $\&_{j=1}^{m} \beta_j(\bigvee_{i=1}^{n} \alpha_i) \leq \gamma_j(\bigvee_{i=1}^{n} \alpha_i)$. Now, by Lemma 4.7.1, there is some $a \in A$ such that $\bigvee_{i=1}^{n} \alpha_i \leq a$ and $\beta_j(a) \leq \gamma_j(a)$ for each $1 \leq j \leq m$. Hence, by our assumption, $\mathbf{i}_0 \leq \neg\mathbf{j}_0$.

For the converse, assume $\mathfrak{A} \models \&_{j=1}^{m} \beta_j(\bigvee_{i=1}^{n} \alpha_i) \leq \gamma_j(\bigvee_{i=1}^{n} \alpha_i) \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0[i_0 := x_0]$. Suppose that under assignment v with $v(\mathbf{i}_0) = x_0$, $\&_{i=1}^{n} \alpha_i \leq p \mathbin{\&} \&_{j=1}^{m} \beta_j(p) \leq \gamma_j(p)$. But since the value of p under v must be an element of A, we can apply Lemme 4.7.1 again to get that for each $1 \leq j \leq m$, $\beta_j(\bigvee_{i=1}^{n} \alpha_i) \leq \gamma_j(\bigvee_{i=1}^{n} \alpha_i)$. Therefore, by our assumption, $\mathbf{i}_0 \leq \neg\mathbf{j}_0$. ∎

Remark 4.6.3. Note that Proposition 4.6.2 implies the version of itself where the parameterization $[\mathbf{i}_0 := x_0]$ is omitted.

We now consider the third equivalences in the left-hand arms of the "U"-shaped arguments in Theorems 4.5.3 and 4.5.4. For both canonical extensions and Dedekind-MacNeille completions, this equivalence requires the rules of hybrid-ALBA, as well as their inverses, to be sound with respect to admissible validity. We first prove this for permeated hybrid algebras and their canonical extensions. This proposition makes use of the topological Ackermann lemmas, also proved in Section 4.7.

Proposition 4.6.4 (Soundness: permeated hybrid algebras and their canonical extensions). *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a permeated hybrid algebra, and let $\mathfrak{A}^\delta = (\mathbf{A}^\delta, At\mathbf{A}^\delta)$ be its canonical extension. If $\varphi_1 \leq \psi_1 \mathbin{\&} \cdots \mathbin{\&} \varphi_n \leq \psi_n \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$ is an $\mathcal{H}^+(@)$-quasi-inequality, and $\varphi'_1 \leq \psi'_1 \mathbin{\&} \cdots \mathbin{\&} \varphi'_m \leq \psi'_m \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$ is obtained from it by any approximation, residuation or adjunction rule, or through a topological application of an Ackermann rule, then $\mathfrak{A}^\delta \models_{\mathfrak{A}} \varphi_1 \leq \psi_1 \mathbin{\&} \cdots \mathbin{\&} \varphi_n \leq \psi_n \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$ iff $\mathfrak{A}^\delta \models_{\mathfrak{A}} \varphi'_1 \leq \psi'_1 \mathbin{\&} \cdots \mathbin{\&} \varphi'_m \leq \psi'_m \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$.*

Proof. We need to verify the claim for each approximation, residuation and adjunction rule, as well as for topological applications of the Ackermann rule. The cases for the residuation and adjunction rules follow, as in the proof of Proposition 4.6.2, from the adjunction and residuation properties of the interpretations of the connectives. The cases for the approximation rules also follow as in the proof of Proposition 4.6.2, making use of the complete distributivity of the interpretations in $\mathfrak{A}^\delta$ of the involved connectives and the fact that every element in $\mathfrak{A}$ is equal to a join of atoms in X_A. The cases for the Ackermann rules follow from Lemmas 4.7.10 and 4.7.11. □

Finally, we show that the rules of hybrid-ALBA, as well as their inverses, are sound on the Dedekind MacNeille completion of an atomic hybrid algebra with respect to admissible validity. In this case we make use of the safe Ackermann lemmas.

Proposition 4.6.5 (**Soundness: atomic hybrid algebras and their MacNeille completions**). *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be atomic with $\Diamond^{\mathbf{A}}$ preserving all existing joins. Let $\mathfrak{A}^{dm} = (\mathbf{A}^{dm}, X_A)$ be its Dedekind-MacNeille completion. If $\varphi_1 \leq \psi_1 \,\&\cdots\&\, \varphi_n \leq \psi_n \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$ is an $\mathcal{H}^+(@)$-quasi-inequality, and $\varphi'_1 \leq \psi'_1 \,\&\cdots\&\, \varphi'_m \leq \psi'_m \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$ is obtained from it by any approximation, residuation or adjunction rule, or through a safe application of an Ackermann rule, then $\mathfrak{A}^{dm} \models_{\mathfrak{A}} \varphi_1 \leq \psi_1 \,\&\cdots\&\, \varphi_n \leq \psi_n \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$ iff $\mathfrak{A}^{dm} \models_{\mathfrak{A}} \varphi'_1 \leq \psi'_1 \,\&\cdots\&\, \varphi'_m \leq \psi'_m \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$.*

Proof. It is sufficient to verify the claim for each approximation, residuation and adjunction rule, as well as for safe applications of the Ackermann rule. The cases for the Ackermann rules follow from Lemmas 4.7.12 and 4.7.13. □

4.7 Ackermann lemmas

In this section, we collect the tools used to prove the results in Section 4.6. To begin with, we prove the Ackermann lemmas needed to prove the second equivalences in the right-handed arms of the "U"-shaped arguments in Theorems 4.5.3 and 4.5.4.

Lemma 4.7.1 (**Right-handed Ackermann lemma**). *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a complete and atomic hybrid algebra. Let $\alpha(\overline{q}, \overline{\mathbf{i}})$ be an $\mathcal{H}^+(@)$-formula that does not contain any occurrences of p, and let $\beta(p, \overline{q}, \overline{\mathbf{i}})$ and $\gamma(p, \overline{q}, \overline{\mathbf{i}})$ be $\mathcal{H}^+(@)$-formulas such that*

(i) *$\beta(p, \overline{q}, \overline{\mathbf{i}})$ is positive in p, and*

(ii) *$\gamma(p, \overline{q}, \overline{\mathbf{i}})$ is negative in p.*

Then for any $\overline{b} \in A$ and $\overline{x} \in X_A$, the following are equivalent:

1. *there exists $a \in A$ such that $\alpha(\overline{b}, \overline{x}) \leq a$ and $\beta(a, \overline{b}, \overline{x}) \leq \gamma(a, \overline{b}, \overline{x})$, and*

2. *$\beta(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x}) \leq \gamma(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x})$.*

Proof. The implication from top to bottom follows from the monotonicity of β in p and the anti-monotonicity of γ in p. For the converse, if $\beta(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x}) \leq \gamma(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x})$ we simply take $a = \alpha(\overline{b}, \overline{x})$. □

Lemma 4.7.2 (**Left-handed Ackermann lemma**). *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a complete and atomic hybrid algebra. Let $\alpha(\overline{q}, \overline{\mathbf{i}})$ be an $\mathcal{H}^+(@)$-formula that does not contain any occurrences of p, and let $\beta(p, \overline{q}, \overline{\mathbf{i}})$ and $\gamma(p, \overline{q}, \overline{\mathbf{i}})$ be $\mathcal{H}^+(@)$-formulas such that*

(i) *$\beta(p, \overline{q}, \overline{\mathbf{i}})$ is negative in p, and*

(ii) *$\gamma(p, \overline{q}, \overline{\mathbf{i}})$ is positive in p.*

Then for any $\overline{b} \in A$ and $\overline{x} \in X_A$, the following are equivalent:

1. *there exists $a \in A$ such that $a \leq \alpha(\overline{b}, \overline{x})$ and $\beta(a, \overline{b}, \overline{x}) \leq \gamma(a, \overline{b}, \overline{x})$, and*

2. *$\beta(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x}) \leq \gamma(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x})$.*

So can we use the Ackermann lemmas above to prove the second equivalence in the right-hand arm of the "U-shaped argument" in Figure 4.4? Well, in this case, we cannot take $a = \alpha(\overline{b}, \overline{x})$ since $\alpha(\overline{q}, \overline{\mathbf{i}})$ is in the expanded language $\mathcal{H}^+(@)$, and therefore might contain operators under which A is not closed. If $\mathfrak{B}$ is the canonical extension of $\mathfrak{A}$, we can make use of the compactness of $\mathfrak{A}$ in its canonical extension to find a suitable a in A. But there is a price to pay: we require additional syntactic restrictions on formulas in terms of openness and closedness. So we need the following definitions:

Definition 4.7.3 (**Syntactically closed and open formulas**). An $\mathcal{H}^+(@)$ formula is *syntactically open* if, in it, all occurrences of nominals and $\Diamond^{-1}$ are negative, while all occurrences of $\Box^{-1}$ are positive. Dually, an $\mathcal{H}^+(@)$ formula is *syntactically closed* if, in it, all occurrences of nominals and $\Diamond^{-1}$ are positive, while all occurrences of $\Box^{-1}$ are negative. We obtain the notions of *syntactically pre-open* and *syntactically pre-closed* formulas by dropping the constraints on nominal occurrences in the definitions of syntactically open and syntactically closed formulas, respectively.

In order to prove the topological Ackermann lemmas (Lemmas 4.7.10 and 4.7.11), we will need some technical results, the proofs of which will make extensive use of Lemmas 4.7.4 and 4.7.5 below.

Lemma 4.7.4. *For all $x \in X_{A^\delta}$, $a \in A^\delta$, $c \in \mathbb{K}(\mathfrak{A}^\delta)$ and $o \in \mathbb{O}(\mathfrak{A}^\delta)$,*

1. *$\Box c \in \mathbb{K}(\mathfrak{A}^\delta)$;*
2. *$\Box o \in \mathbb{O}(\mathfrak{A}^\delta)$;*
3. *$\Diamond o \in \mathbb{O}(\mathfrak{A}^\delta)$;*
4. *$\Diamond c \in \mathbb{K}(\mathfrak{A}^\delta)$;*
5. *$\Box^{-1} o \in \mathbb{O}(\mathfrak{A}^\delta)$;*
6. *$\Diamond^{-1} c \in \mathbb{K}(\mathfrak{A}^\delta)$;*
7. *$\mathsf{A}a \in \mathbb{K}(\mathfrak{A}^\delta) \cap \mathbb{O}(\mathfrak{A}^\delta)$;*
8. *$\mathsf{E}a \in \mathbb{K}(\mathfrak{A}^\delta) \cap \mathbb{O}(\mathfrak{A}^\delta)$;*
9. *$@_x a \in \mathbb{K}(\mathfrak{A}^\delta) \cap \mathbb{O}(\mathfrak{A}^\delta)$.*

Proof. The proofs of items 1 – 6 can be found in [33] and [35]. We will prove items 7, 8 and 9.

7. If $a = \top$, $\mathsf{A}a = \top$, and so, since $\top$ is both open and closed, $\mathsf{A}a \in \mathbb{K}(\mathfrak{A}^\delta) \cap \mathbb{O}(\mathfrak{A}^\delta)$. On the other hand, if $a \neq \top$, $\mathsf{A}a = \bot$. But $\bot$ is also both open and closed, so $\mathsf{A}a \in \mathbb{K}(\mathfrak{A}^\delta) \cap \mathbb{O}(\mathfrak{A}^\delta)$.

8. If $a = \bot$, $\mathsf{E}a = \bot$, which means that $\mathsf{E}a \in \mathbb{K}(\mathfrak{A}^\delta) \cap \mathbb{O}(\mathfrak{A}^\delta)$. On the other hand, if $a \neq \bot$, then $\mathsf{E}a = \top$, so $\mathsf{E}a \in \mathbb{K}(\mathfrak{A}^\delta) \cap \mathbb{O}(\mathfrak{A}^\delta)$.

9. If $x \leq a$, $@_x a = \top$, and so $@_x a \in \mathbb{K}(\mathfrak{A}^\delta) \cap \mathbb{O}(\mathfrak{A}^\delta)$. On the other hand, if $x \not\leq a$, $@_x a = \bot$, which means $@_x a \in \mathbb{K}(\mathfrak{A}^\delta) \cap \mathbb{O}(\mathfrak{A}^\delta)$. $\square$

Lemma 4.7.5. *Let $\varnothing \neq D \subseteq \mathbb{K}(\mathfrak{A}^\delta)$ be down-directed, $\varnothing \neq U \subseteq \mathbb{O}(\mathfrak{A}^\delta)$ up-directed, $S \subseteq A$, and $x \in AtA^\delta$. Then, in $\mathfrak{A}^\delta$,*

1. $\Diamond \bigvee S = \bigvee_{s\in S} \Diamond s$;
2. $\Diamond \bigwedge D = \bigwedge_{d\in D} \Diamond d$;
3. $\Box \bigwedge S = \bigwedge_{s\in S} \Box s$;
4. $\Box \bigvee U = \bigvee_{u\in U} \Box u$;
5. $\Diamond^{-1} \bigvee S = \bigvee_{s\in S} \Diamond^{-1} s$;
6. $\Diamond^{-1} \bigwedge D = \bigwedge_{d\in D} \Diamond^{-1} d$;
7. $\Box^{-1} \bigwedge S = \bigwedge_{s\in S} \Box^{-1} s$;
8. $\Box^{-1} \bigvee U = \bigvee_{u\in U} \Box^{-1} u$;
9. $\mathsf{E} \bigvee S = \bigvee_{s\in S} \mathsf{E} s$;
10. $\mathsf{E} \bigwedge D = \bigwedge_{d\in D} \mathsf{E} d$;
11. $\mathsf{A} \bigwedge S = \bigwedge_{s\in S} \mathsf{A} s$;
12. $\mathsf{A} \bigvee U = \bigvee_{u\in U} \mathsf{A} u$;
13. $@_x \bigvee S = \bigvee_{s\in S} @_x s$;
14. $@_x \bigwedge S = \bigwedge_{s\in S} @_x s$;

Proof. Item 2 is Esakia's lemma (see [37]) and item 4 is its dual. Proofs of items 1 to 8 can be found in [33] and in [35]. We prove items 9, 10, 11, 12, 13 and 14.

9. The right-to-left inequality follows from the monotonicity of E. For the converse inequality, first suppose $\bigvee S = \bot$. Then $\mathsf{E}\bigvee S = \bot$, and so $\mathsf{E}\bigvee S \leq \bigvee_{s\in S} \mathsf{E}s$. Next, suppose $\bigvee S \neq \bot$. Then $\mathsf{E}\bigvee S = \top$. But this also means that there is some $s_0 \in S$ such that $s_0 \neq \bot$. Hence, $\mathsf{E}s_0 = \top$, which gives $\bigvee_{s\in S} \mathsf{E}s = \top$. We therefore have $\mathsf{E}\bigvee S = \bigvee_{s\in S} \mathsf{E}\bigwedge s$.

10. The left-to-right inequality follows from the monotonicity of E. For the converse inequality, first suppose $\bigwedge D \neq \bot$. Then we have $\mathsf{E}\bigwedge D = \top$, and so $\bigwedge_{d\in D} \mathsf{E}d \leq \mathsf{E}\bigwedge D$. Now, suppose $\bigwedge D = \bot$. Then $\mathsf{E}\bigwedge D = \bot$. But we also know that $D \subseteq \mathbb{K}(\mathfrak{A}^\delta)$ and $\bot \in \mathbb{O}(\mathfrak{A}^\delta)$, so, by compactness, there is a finite subset $D_0 \subseteq D$ such that $\bigwedge D_0 \leq \bot$. Since D is down-directed, there is some $d_0 \in D_0$ such that $d_0 \leq \bigwedge D_0 \leq \bot$. Hence, $\mathsf{E}d_0 = \bot$, which means that $\bigwedge_{d\in D} \mathsf{E}d = \bot$. So we have $\bigwedge_{d\in D} \mathsf{E}d = \mathsf{E}\bigwedge D$.

11. We use duality and 9:

$$\mathsf{A}\bigwedge S = \neg\mathsf{E}\neg\bigwedge S = \neg\mathsf{E}\bigvee_{s\in S}\neg s = \neg\bigvee_{s\in S}\mathsf{E}\neg s = \bigwedge_{s\in S}\neg\mathsf{E}\neg s = \bigwedge_{s\in S}\mathsf{A}s$$

12. Here we make use of the fact that $\{\neg u \mid u \in U\}$ is a set of closed elements and down-directed if U is up-directed, duality and 10:

$$\mathsf{A}\bigvee U = \neg\mathsf{E}\neg\bigvee U = \neg\mathsf{E}\bigwedge_{u\in U}\neg u = \neg\bigwedge_{u\in U}\mathsf{E}\neg u = \bigvee_{u\in U}\neg\mathsf{E}\neg u = \bigvee_{u\in U}\mathsf{A}u$$

13. The right-to-left inequality follows from the monotonicity of @ in the second coordinate. For the other inequality, first assume $x \not\leq \bigvee S$. Then $@_x\bigvee S = \bot$, which gives $@_x\bigvee S \leq \bigvee_{s\in S} @_x s$. Next, suppose $x \leq \bigvee S$. Then $@_x\bigvee S = \top$. But since x is an atom, this also means that $x \leq s_0$ for some $s_0 \in S$. Hence, $@_x s_0 = \top$, and so $\bigvee_{s\in S} @_x s = \top$. Therefore, $@_x\bigvee S = \bigvee_{s\in S} @_x s$.

14. We use duality and 13:

$$@_x\bigwedge S = \neg @_x\neg\bigwedge S = \neg @_x\bigvee_{s\in S}\neg s = \neg\bigvee_{s\in S} @_x\neg s = \bigwedge_{s\in S}\neg @_x\neg s = \bigwedge_{s\in S} @_x s \qquad \Box$$

The intention behind the definition of syntactically open and closed $\mathcal{H}^+(@)$ formulas is that admissible assignments will always interpret them as open and closed elements of $\mathfrak{A}^\delta$, respectively.

Lemma 4.7.6. *Let $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ be syntactically closed and $\psi(p, \overline{q}, \overline{\mathbf{i}})$ syntactically open. Let $\overline{b} \in A$, $\overline{x} \in At\mathfrak{A}^\delta$, $k \in \mathbb{K}(\mathfrak{A}^\delta)$ and $u \in \mathbb{O}(\mathfrak{A}^\delta)$.*

1. (i) *If $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ is positive in p, then $\varphi(k, \overline{b}, \overline{x}) \in \mathbb{K}(\mathfrak{A}^\delta)$.*
   (ii) *If $\psi(p, \overline{q}, \overline{\mathbf{i}})$ is negative in p, then $\psi(k, \overline{b}, \overline{x}) \in \mathbb{O}(\mathfrak{A}^\delta)$.*

2. (i) *If $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ is negative in p, then $\varphi(u, \overline{b}, \overline{x}) \in \mathbb{K}(\mathfrak{A}^\delta)$.*
   (ii) *If $\psi(p, \overline{q}, \overline{\mathbf{i}})$ is positive in p, then $\psi(u, \overline{b}, \overline{x}) \in \mathbb{O}(\mathfrak{A}^\delta)$.*

Proof. The proof is by simultaneous mutual induction on φ and ψ. For the language without the @ operator this has been proved in [33] and in [35] in the setting of descriptive general frames. We therefore only need to consider the cases for φ of the form $@_{\mathbf{i}}\varphi_1$, $\mathsf{E}\varphi_1$ or $\mathsf{A}\varphi_1$ and ψ of the form $@_{\mathbf{i}}\psi_1$, $\mathsf{E}\psi_1$ or $\mathsf{A}\psi_1$. Since each of these always evaluate to either the top or the bottom element of the algebra this is immediate. $\square$

The next corollary says that if, in Lemma 4.7.6, we take the $\overline{x}$ in X_A instead of in $At(\mathfrak{A}^\delta)$, then we may loosen the conditions on φ and ψ to syntactic pre-closedness and pre-openness and still obtain the same result. The proof is identical except for the extra base case for ψ of the form $\mathbf{i}$ which now also follows since the $\overline{x}$ are clopen.

Corollary 4.7.7. *Let $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ be syntactically pre-closed and $\psi(p, \overline{q}, \overline{\mathbf{i}})$ syntactically pre-open. Let $\overline{b} \in A$, $\overline{x} \in At(\mathfrak{A})$, $k \in \mathbb{K}(\mathfrak{A}^\delta)$ and $u \in \mathbb{O}(\mathfrak{A}^\delta)$.*

1. (i) *If $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ is positive in p, then $\varphi(k, \overline{b}, \overline{x}) \in \mathbb{K}(\mathfrak{A}^\delta)$.*
   (ii) *If $\psi(p, \overline{q}, \overline{\mathbf{i}})$ is negative in p, then $\psi(k, \overline{b}, \overline{x}) \in \mathbb{O}(\mathfrak{A}^\delta)$.*

2. (i) *If $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ is negative in p, then $\varphi(u, \overline{b}, \overline{x}) \in \mathbb{K}(\mathfrak{A}^\delta)$.*
   (ii) *If $\psi(p, \overline{q}, \overline{\mathbf{i}})$ is positive in p, then $\psi(u, \overline{b}, \overline{x}) \in \mathbb{O}(\mathfrak{A}^\delta)$.*

Lemma 4.7.8. *Let $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ be syntactically closed and $\psi(p, \overline{q}, \overline{\mathbf{i}})$ be syntactically open. Let $D \subseteq \mathbb{K}(\mathfrak{A}^\delta)$ be down-directed, $U \subseteq \mathbb{O}(\mathfrak{A}^\delta)$ up-directed, $\overline{b} \in A$ and $\overline{x} \in At\mathfrak{A}^\delta$.*

1. (i) *If $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ is positive in p, then $\varphi(\bigwedge D, \overline{b}, \overline{x}) = \bigwedge\{\varphi(d, \overline{b}, \overline{x}) \mid d \in D\}$.*
   (ii) *If $\psi(p, \overline{q}, \overline{\mathbf{i}})$ is negative in p, then $\psi(\bigwedge D, \overline{b}, \overline{x}) = \bigvee\{\psi(d, \overline{b}, \overline{x}) \mid d \in D\}$.*

2. (i) *If $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ is negative in p, then $\varphi(\bigvee U, \overline{b}, \overline{x}) = \bigwedge\{\varphi(u, \overline{b}, \overline{x}) \mid u \in U\}$.*
   (ii) *If $\psi(p, \overline{q}, \overline{\mathbf{i}})$ is positive in p, then $\psi(\bigvee U, \overline{b}, \overline{x}) = \bigvee\{\psi(u, \overline{b}, \overline{x}) \mid u \in U\}$.*

Proof. The proof is again by simultaneous mutual induction on φ and ψ. Again, for the language without the @ operator this has been proved in [33] and in [35] in the setting of descriptive general frames. We therefore only need to consider the cases when the main connective of φ or ψ is @, E or A.

Suppose that $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ is of the form $@_{\mathbf{i}_0}\varphi_1(p, \overline{q}, \overline{\mathbf{i}})$, where $\mathbf{i}_0$ is among the $\overline{\mathbf{i}}$, and φ_1 is syntactically closed and positive in p. Then $@_{x_0}\varphi_1(\bigwedge D, \overline{b}, \overline{x}) = \mathsf{E}(x_0 \wedge \varphi_1(\bigwedge D, \overline{b}, \overline{x}))$. By the inductive hypothesis, the latter is equal to $\mathsf{E}(x_0 \wedge \bigwedge_{d \in D} \varphi_1(d, \overline{b}, \overline{x}))$, which, by the associativity of $\wedge$, is equal to $\mathsf{E} \bigwedge_{d \in D}(x_0 \wedge \varphi_1(d, \overline{b}, \overline{x}))$. By the monotonicity of φ_i in p, the family $\{x_0 \wedge \varphi_1(d, \overline{b}, \overline{x}) \mid d \in D\}$ is down-directed and its members are closed by Lemma 4.7.6. We may thus apply Lemma 4.7.5 to conclude that $\mathsf{E} \bigwedge_{d \in D}(x_0 \wedge \varphi_1(d, \overline{b}, \overline{x})) = \bigwedge_{d \in D} \mathsf{E}(x_0 \wedge \varphi_1(d, \overline{b}, \overline{x}))$.

The other cases are similar. $\square$

The next corollary follows from Lemma 4.7.8, but using Corollary 4.7.7, whereas Lemma 4.7.8 uses Lemma 4.7.6.

Corollary 4.7.9. *Let $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ be syntactically pre-closed and $\psi(p, \overline{q}, \overline{\mathbf{i}})$ be syntactically pre-open. Let $D \subseteq \mathbb{K}(\mathfrak{A}^\delta)$ be down-directed, $U \subseteq \mathbb{O}(\mathfrak{A}^\delta)$ up-directed, $\overline{b} \in A$ and $\overline{x} \in At(\mathfrak{A})$.*

1. (i) *If $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ is positive in p, then $\varphi(\bigwedge D, \overline{b}, \overline{x}) = \bigwedge\{\varphi(d, \overline{b}, \overline{x}) \mid d \in D\}$.*
   (ii) *If $\psi(p, \overline{q}, \overline{\mathbf{i}})$ is negative in p, then $\psi(\bigwedge D, \overline{b}, \overline{x}, \overline{o}) = \bigvee\{\psi(d, \overline{b}, \overline{x}, \overline{o}) \mid d \in D\}$.*

2. (i) *If $\varphi(p, \overline{q}, \overline{\mathbf{i}})$ is negative in p, then $\varphi(\bigvee U, \overline{b}, \overline{x}) = \bigwedge\{\varphi(u, \overline{b}, \overline{x}) \mid u \in U\}$.*
   (ii) *If $\psi(p, \overline{q}, \overline{\mathbf{i}})$ is positive in p, then $\psi(\bigvee U, \overline{b}, \overline{x}) = \bigvee\{\psi(u, \overline{b}, \overline{x}) \mid u \in U\}$.*

We are now ready to prove the topological Ackermann lemmas.

Lemma 4.7.10 (Right-handed topological Ackermann lemma). *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a permeated hybrid algebra, and let $\alpha(\overline{q}, \overline{\mathbf{i}})$, $\beta(p, \overline{q}, \overline{\mathbf{i}})$ and $\gamma(p, \overline{q}, \overline{\mathbf{i}})$ be $\mathcal{H}^+(@)$-formulas such that*

(i) *$\alpha(\overline{q}, \overline{\mathbf{i}})$ is syntactically pre-closed and does not contain any occurrences of p,*

(ii) *$\beta(p, \overline{q}, \overline{\mathbf{i}})$ is syntactically pre-closed and positive in p, and*

(iii) *$\gamma(p, \overline{q}, \overline{\mathbf{i}})$ is syntactically pre-open and negative in p.*

Then for any $\overline{b} \in A$ and $\overline{x} \in X_A$, the following are equivalent:

1. *there exists $a \in A$ such that $\alpha(\overline{b}, \overline{x}) \leq a$ and $\beta(a, \overline{b}, \overline{x}) \leq \gamma(a, \overline{b}, \overline{x})$ as interpreted in $\mathfrak{A}^\delta$, and*

2. *$\beta(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x}) \leq \gamma(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x})$ as interpreted in $\mathfrak{A}^\delta$.*

Proof. For the implication from top to bottom, we appeal to the monotonicity of β in p and the anti-monotonicity of γ in p to get $\beta(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x}) \leq \beta(a, \overline{b}, \overline{x}) \leq \gamma(a, \overline{b}, \overline{x}) \leq \gamma(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x})$.

For the converse direction, assume $\beta(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x}) \leq \gamma(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x})$. By Corollary 4.7.7, $\alpha(\overline{b}, \overline{x})$ is closed, and so $\alpha(\overline{b}, \overline{x}) = \bigwedge\{s \in A \mid \alpha(\overline{b}, \overline{x}) \leq s\}$. Now, let $S = \{s \in A \mid \alpha(\overline{b}, \overline{x}) \leq s\}$. Then we have $\beta(\bigwedge S, \overline{b}, \overline{x}) \leq \gamma(\bigwedge S, \overline{b}, \overline{x})$. But S is a down-directed set of clopen elements, and furthermore, β is syntactically pre-closed and positive in p, while γ is syntactically pre-open and negative in p, so, by Corollary 4.7.9, $\bigwedge\{\beta(s, \overline{b}, \overline{x}) \mid s \in S\} \leq \bigvee\{\gamma(s, \overline{b}, \overline{x}) \mid s \in S\}$. By Corollary 4.7.7, $\beta(s, \overline{b}, \overline{x})$ is closed and $\gamma(s, \overline{b}, \overline{x})$ open for all $s \in A$, so, by compactness,

$$\bigwedge_{i=1}^{n} \beta(u_i, \overline{b}, \overline{x}) \leq \bigvee_{j=1}^{m} \gamma(u'_j, \overline{b}, \overline{x})$$

for some $s_1, \ldots, s_n \in A$ such that $\alpha(\overline{b}, \overline{x}) \leq s_i$ for $1 \leq i \leq n$, and $s'_1, \ldots, s'_m \in A$ such that $\alpha(\overline{b}, \overline{x}) \leq s'_j$ for $1 \leq j \leq m$. Now, let $s_1 \wedge \cdots \wedge s_n \wedge s'_1 \wedge \cdots \wedge s'_m = a$. Clearly then $\alpha(\overline{b}, \overline{x}) \leq a$. Furthermore, by the monotonicity of β is p and the anti-monotonicity of γ in p,

$$\beta(a, \overline{b}, \overline{x}) \leq \bigwedge_{i=1}^{n} \beta(u_i, \overline{b}, \overline{x}) \leq \bigvee_{j=1}^{m} \gamma(u'_j, \overline{b}, \overline{x}) \leq \gamma(a, \overline{b}, \overline{x}).$$

$\square$

The proof of the next lemma is similar.

Lemma 4.7.11 (Left-handed topological Ackermann lemma). *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a permeated hybrid algebra, and $\alpha(\overline{q}, \overline{\mathbf{i}})$, $\beta(p, \overline{q}, \overline{\mathbf{i}})$ and $\gamma(p, \overline{q}, \overline{\mathbf{i}})$ be $\mathcal{H}^+(@)$-formulas such that*

(i) *$\alpha(\overline{q}, \overline{\mathbf{i}})$ is syntactically pre-open and does not contain any occurrences of p,*

(ii) *$\beta(p, \overline{q}, \overline{\mathbf{i}})$ is syntactically pre-open and positive in p, and*

(iii) *$\gamma(p, \overline{q}, \overline{\mathbf{i}})$ is syntactically pre-closed and negative in p.*

Then for any $\overline{b} \in A$ and $\overline{x} \in X_A$, the following are equivalent:

1. *there exists $a \in A$ such that $a \leq \alpha(\overline{b}, \overline{x})$ and $\gamma(a, \overline{b}, \overline{x}) \leq \beta(a, \overline{b}, \overline{x})$ as interpreted in $\mathfrak{A}^\delta$, and*
2. *$\gamma(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x}) \leq \beta(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x})$ as interpreted in $\mathfrak{A}^\delta$.*

Unfortunately, an algebra need not be compact in its Dedekind-MacNeille completion. This creates an obvious impediment for the proof of the direction from bottom to top of the equivalence in the topological Ackermann lemma. We need versions of the above lemmas which apply to atomic hybrid algebras seen as embedded in their Dedekind MacNeille completions. However, this is not enough. The only way to push the direction from bottom to top through is to require that $\alpha(\overline{q}, \overline{\mathbf{i}})$ be an $\mathcal{H}(@)$-formula like in Lemmas 4.7.1 and 4.7.2. We will refer to these Ackermann lemmas as the *safe Ackerman lemmas.*

Lemma 4.7.12 (Right-handed safe Ackermann lemma). *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be an atomic hybrid algebra with $\Diamond^{\mathfrak{A}}$ preserving all existing joins. Let $\alpha(\overline{q}, \overline{\mathbf{i}})$ be an $\mathcal{H}(@)$-formula such that $\alpha(\overline{q}, \overline{\mathbf{i}})$ does not contain any occurrences of p, and let $\beta(p, \overline{q}, \overline{\mathbf{i}})$ and $\gamma(p, \overline{q}, \overline{\mathbf{i}})$ be $\mathcal{H}^+(@)$-formulas such that*

(i) *$\beta(p, \overline{q}, \overline{\mathbf{i}})$ is positive in p, and*

(ii) *$\gamma(p, \overline{q}, \overline{\mathbf{i}})$ is negative in p.*

Then for any $\overline{b} \in A$ and $\overline{x} \in X_A$, the following are equivalent:

1. *there exists $a \in A$ such that $\alpha(\overline{b}, \overline{x}) \leq a$ and $\beta(a, \overline{b}, \overline{x}) \leq \gamma(a, \overline{b}, \overline{x})$ as interpreted in $\mathfrak{A}^{dm}$, and*
2. *$\beta(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x}) \leq \gamma(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x})$ as interpreted in $\mathfrak{A}^{dm}$.*

Proof. The implication from top to bottom follows from the monotonicity of β in p and the anti-monotonicity of γ in p. For the converse inequality, assume $\beta(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x}) \leq \gamma(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x})$. Now, since $\alpha(\overline{q}, \overline{\mathbf{i}})$ is a $\mathcal{H}(@)$-formula, $\alpha(\overline{b}, \overline{x}) \in A$, so let $a = \alpha(\overline{b}, \overline{x})$. Furthermore, by our assumption, $\beta(a, \overline{b}, \overline{x}) \leq \gamma(a, \overline{b}, \overline{x})$. $\square$

Lemma 4.7.13 (Left-handed safe Ackermann lemma). *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be an atomic hybrid algebra with $\Diamond^{\mathfrak{A}}$ preserving all existing joins. Let $\alpha(\overline{q}, \overline{\mathbf{i}})$ be an $\mathcal{H}(@)$-formula such that $\alpha(\overline{q}, \overline{\mathbf{i}})$ does not contain any occurrences of p, and let $\beta(p, \overline{q}, \overline{\mathbf{i}})$ and $\gamma(p, \overline{q}, \overline{\mathbf{i}})$ be $\mathcal{H}^+(@)$-formulas such that*

(i) *$\beta(p, \overline{q}, \overline{\mathbf{i}})$ is negative in p, and*

(ii) *$\gamma(p, \overline{q}, \overline{\mathbf{i}})$ is positive in p.*

Then for any $\overline{b} \in A$ and $\overline{x} \in X_A$, the following are equivalent:

1. *there exists $a \in A$ such that $a \leq \alpha(\overline{b}, \overline{x})$ and $\beta(a, \overline{b}, \overline{x}) \leq \gamma(a, \overline{b}, \overline{x})$ as interpreted in $\mathfrak{A}^{dm}$, and*

2. *$\beta(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x}) \leq \gamma(\alpha(\overline{b}, \overline{x}), \overline{b}, \overline{x})$ as interpreted in $\mathfrak{A}^{dm}$.*

4.8 Canonicity of pure quasi-inequalities

In this section, we give some definitions and lemmas needed to complete the base of the "U"-shaped argument in Theorem 4.5.3.

Definition 4.8.1. A *diamond-link formula* is a formula of the form $\mathbf{i} \to \Diamond\mathbf{j}$ or $\Box\neg\mathbf{j} \to \neg\mathbf{i}$. Note that these two implications are equivalent. Analogously, a *diamond-link inequality* is an inequality of the form $\mathbf{i} \leq \Diamond\mathbf{j}$ or $\Box\neg\mathbf{j} \leq \neg\mathbf{i}$. A *flat-link formula* is a formula of the from $\mathbf{i} \to \mathbf{j}$, and a *flat-link inequality* is an inequality of the form $\mathbf{i} \leq \mathbf{j}$. A *nominal-link formula* is any diamond-link formula or flat-link formula. A *compound nominal-link formula* is a conjunction of one or more nominal link formulas.

Let φ be a compound nominal-link formula. Define an equivalence relation $\sim$ on the set of nominals occurring in φ such that $\mathbf{i} \sim \mathbf{j}$ iff $\mathbf{i} \to \mathbf{j}$ or $\mathbf{j} \to \mathbf{i}$ is a conjunct of φ. The *dependency digraph* $D(\varphi)$ of φ has the set of equivalence classes of all nominals appearing in φ as vertex set, while its arc set consists of all ordered pairs $([\mathbf{i}], [\mathbf{j}])$ such that $\mathbf{i}' \to \Diamond\mathbf{j}'$ or $\Box\neg\mathbf{j}' \to \neg\mathbf{i}'$ is a conjunct of φ for some $\mathbf{i}' \in [\mathbf{i}]$ and $\mathbf{j}' \in [\mathbf{j}]$. A compound nominal-link formula φ is called *forest-like* if every vertex in $D(S)$ has at most one predecessor. The *dependency digraph* $D(S)$ of a set S of nominal-link inequalities is defined analogously, and S is called *forest-like* if every vertex in $D(S)$ has at most one predecessor.

Example 4.8.2. The following formula is an example of a forest-like compound nominal-link formula:

$$(\mathbf{j}_1 \to \Diamond\mathbf{j}_2) \wedge (\mathbf{j}_1 \to \Diamond\mathbf{j}_3) \wedge (\mathbf{j}_2 \to \Diamond\mathbf{j}_4) \wedge (\mathbf{j}_3 \to \Diamond\mathbf{j}_5) \wedge (\mathbf{j}_3 \to \Diamond\mathbf{j}_6) \wedge (\mathbf{j}_3 \to \mathbf{j}_7) \wedge$$
$$(\mathbf{j}_8 \to \Diamond\mathbf{j}_9) \wedge (\mathbf{j}_8 \to \Diamond\mathbf{j}_{10}) \wedge (\mathbf{j}_{10} \to \Diamond\mathbf{j}_{11}).$$

Lemma 4.8.3. *Let $\gamma(\mathbf{j}_1, \ldots, \mathbf{j}_n)$ be a forest-like compound nominal-link formula, and let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a permeated hybrid algebra. If $x_1, \ldots, x_n \in At\mathfrak{A}^\delta$ and $a_1, \ldots, a_n \in A$ such that $x_i \leq a_i$ for all $1 \leq i \leq n$, and $\mathfrak{A}^\delta \models \gamma(x_1, \ldots, x_n) = \top$, then there exists $y_1, \ldots, y_n \in X_A$ such that $\mathfrak{A}^\delta \models \gamma(y_1, \ldots, y_n) = \top$.*

Proof. Keeping track of the tree structures makes the proof of this lemma extremely tedious, so we will illustrate the idea of the proof with an example instead. This should make the general proof idea clear. Let γ be the forest-like compound nominal-link formula in Example 4.8.2. The dependency digraph of this formula is given in Figure 4.5. Let $x_1, \ldots, x_{11} \in At\mathfrak{A}^\delta$

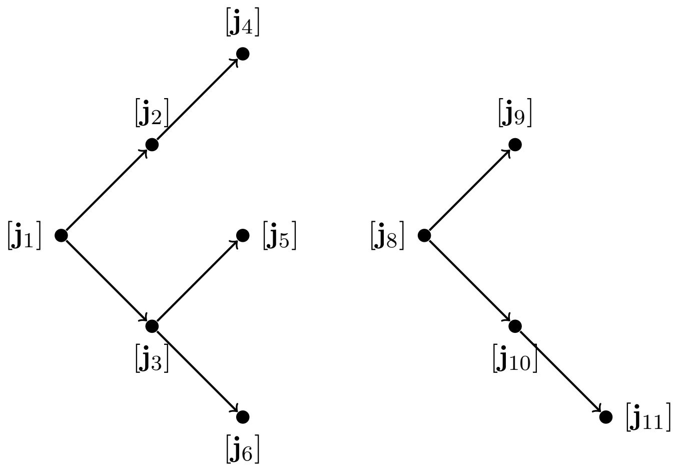

Figure 4.5: The dependency digraph of the forest-like compound nominal-link formula in Example 4.8.2.

and $a_1, \dots, a_{11} \in A$ such that $x_i \leq a_i$ for all $1 \leq i \leq 11$, and $\mathfrak{A}^\delta \models \gamma(x_1, \dots, x_{11}) = \top$. Then $x_1 \leq \Diamond x_2, x_1 \leq \Diamond x_3, x_2 \leq \Diamond x_4, x_3 \leq \Diamond x_5, x_3 \leq \Diamond x_6, x_3 \leq x_7, x_8 \leq \Diamond x_9, x_8 \leq \Diamond x_{10}$, and $x_{10} \leq \Diamond x_{11}$, so we have that $x_1 \leq a_1 \wedge \Diamond(a_2 \wedge \Diamond a_4) \wedge \Diamond(a_3 \wedge \Diamond a_5 \wedge \Diamond a_6 \wedge a_7)$ and $x_8 \leq a_8 \wedge \Diamond a_9 \wedge \Diamond(a_{10} \wedge \Diamond a_{11})$. But we know that $\mathfrak{A}^\delta$ is permeated, so there are y_1 and y_8 in X_A such that $y_1 \leq a_1 \wedge \Diamond(a_2 \wedge \Diamond a_4) \wedge \Diamond(a_3 \wedge \Diamond a_5 \wedge \Diamond a_6 \wedge a_7)$ and $y_8 \leq a_8 \wedge \Diamond a_9 \wedge \Diamond(a_{10} \wedge \Diamond a_{11})$. From the first inequality we have $y_1 \leq \Diamond(a_2 \wedge \Diamond a_4)$ and $y_1 \leq \Diamond(a_3 \wedge \Diamond a_5 \wedge \Diamond a_6 \wedge a_7)$, and hence, since $\mathfrak{A}^\delta$ is permeated, there is a $y_2 \in X_A$ such that $y_2 \leq a_2 \wedge \Diamond a_4$ and $y_1 \leq \Diamond y_2$. Likewise, there is a $y_3 \in X_A$ such that $y_3 \leq a_3 \wedge \Diamond a_5 \wedge \Diamond a_6 \wedge a_7$ and $y_1 \leq \Diamond y_3$. From $y_2 \leq \Diamond a_4$, we know there is a $y_4 \in X_A$ such that $y_4 \leq a_4$ and $y_2 \leq \Diamond y_4$. Similarly, from $y_3 \leq \Diamond a_5 \wedge \Diamond a_6$, we have $y_5, y_6 \in X_A$ such that $y_5 \leq a_5, y_3 \leq \Diamond y_5, y_6 \leq a_6$ and $y_3 \leq \Diamond y_6$. We therefore have that

$$(y_1 \to \Diamond y_2) \wedge (y_1 \to \Diamond y_3) \wedge (y_2 \to \Diamond y_4) \wedge (y_3 \to \Diamond y_5) \wedge (y_3 \to \Diamond y_6) \wedge (y_3 \to y_3) = \top.$$

In a similar way, from $y_8 \leq a_8 \wedge \Diamond a_9 \wedge \Diamond(a_{10} \wedge \Diamond a_{11})$, we can find $y_8, y_9, y_{10}, y_{11} \in X_A$ such that $y_8 \leq \Diamond y_9, y_8 \leq \Diamond y_{10}, y_{10} \leq \Diamond y_{11}$. Hence, $(y_8 \to \Diamond y_9) \wedge (y_8 \to \Diamond y_{10}) \wedge (y_{10} \to \Diamond y_{11}) = \top$, and so $\mathfrak{A}^\delta \models \gamma(y_1, \dots, y_{11}) = \top$. □

Lemma 4.8.4. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a permeated hybrid algebra. Let $\beta(\mathbf{j}_1, \dots, \mathbf{j}_n) \in \mathcal{H}^+(@)$ be pure and syntactically open*[4]*, and let $\gamma(\mathbf{j}_\ell, \dots, \mathbf{j}_n)$, $\ell \leq n$, be a forest-like nominal-link formula. If $x_1, \dots, x_n \in At\mathbf{A}^\delta$ such that $\beta(x_1, \dots, x_n) \wedge \gamma(x_\ell, \dots, x_n) = \top$, then there exist $y_1, \dots, y_n \in X_A$ such that $\beta(y_1, \dots, y_n) \wedge \gamma(y_\ell, \dots, y_n) = \top$.*

Proof. Let $x_1, \dots, x_n \in At\mathbf{A}^\delta$ such that we have $\beta(x_1, \dots, x_n) \wedge \gamma(x_\ell, \dots x_n) = \top$. Then $\neg\beta(x_1, \dots, x_n) = \bot$. But we know that the atoms of $At\mathbf{A}^\delta$ are closed, so $x_i = \bigwedge A_i$ with

[4] We only assume that the nominals occurring in β are among $\mathbf{j}_1, \dots, \mathbf{j}_n$, not that all these nominals necessarily occur in β.

$A_i = \{a \in A \mid x_i \leq a\}$ for each $1 \leq i \leq n$, and $\neg\beta(\bigwedge A_1, \ldots, \bigwedge A_n) = \bot$. Now, since $\neg\beta(p_1, \ldots, p_n)$ is syntactically closed and positive in the p_i and the A_i are down directed sets of closed elements, Lemma 4.7.8 gives

$$\bigwedge_{a_1 \in A_1} \cdots \bigwedge_{a_n \in A_n} \neg\beta(a_1, \ldots, a_n) = \bot.$$

But $\neg\beta$ is syntactically closed and positive in the p_i, so, by Lemma 4.7.6, each $\neg\beta(a_1, \ldots, a_n)$ is closed. Hence, by compactness, for each $1 \leq i \leq n$, there is a $a'_i \in A_i$ such that $\neg\beta(a'_1, \ldots, a'_n) = \bot$. Now, we know that $x_\ell \leq a'_\ell, \ldots, x_n \leq a'_n$ and that $\gamma(x_\ell, \ldots, x_n) = \top$, so, by Lemma 4.8.3, there are $y_\ell, \ldots, y_n \in X_A$ such that $y_i \leq a'_i$ for each $\ell \leq i \leq n$, and $\gamma(y_\ell, \ldots, y_n) = \top$. Since $\mathfrak{A}$ is permeated, we can choose arbitrary $y_1, \ldots, y_{\ell-1} \in X_A$ such that $y_i \leq a'_i$, $1 \leq i \leq \ell - 1$. But $\neg\beta$ is positive in all nominals, so, by monotonicity, $\neg\beta(y_1, \ldots, y_n) = \bot$, and therefore, $\beta(y_1, \ldots, y_n) \wedge \gamma(y_\ell, \ldots, y_n) = \top$. $\square$

Proposition 4.8.5. *Let S be a finite forest-like set of nominal-link inequalities such that $[\mathbf{i}_0]$ and $[\mathbf{j}_0]$ have no predecessors in $D(S)$. Let φ_i be syntactically closed and pure, while ψ_i is syntactically open and pure, $1 \leq i \leq k$. Then $\mathfrak{A}^\delta \models_{\mathfrak{A}} \,\&\, S \,\&\, \&_{i=1}^k \varphi_i \leq \psi_i \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$ iff $\mathfrak{A}^\delta \models \,\&\, S \,\&\, \&_{i=1}^k \varphi_i \leq \psi_i \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$.*

Proof. We only prove the implication from left to right, as the other direction is immediate from Lemma 4.8.6 below. We proceed by contraposition, so assume that $\mathfrak{A}^\delta \not\models \,\&\, S \,\&\, \&_{i=1}^k \varphi_i \leq \psi_i \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$. We rewrite $\&_{i=1}^k \varphi_i \leq \psi_i$ and $\&\, S$ as formulas β and γ, respectively, and set $\alpha(\mathbf{j}_1, \ldots, \mathbf{j}_n, \mathbf{i}_0, \mathbf{j}_0) := \beta \wedge \gamma$. Note that the nominals $\mathbf{i}_0$ and $\mathbf{i}_0$ may or may not actually occur in α. By assumption, there are $x_1, \ldots, x_n, d_1, d_2 \in At\mathfrak{A}^\delta$ such that $\alpha(x_1, \ldots, x_n, d_1, d_2) = \top$ and $d_1 \not\leq \neg d_2$, i.e., $d_1 = d_2$. Let $\alpha'(\mathbf{j}_1, \ldots, \mathbf{j}_n, \mathbf{i}_0)$ be obtained from α by identifying $\mathbf{i}_0$ and $\mathbf{j}_0$, i.e, it is the formula $\beta' \wedge \gamma'$, where β' and γ' are obtained by substituting $\mathbf{i}_0$ for all occurrences of $\mathbf{j}_0$ in β an γ, respectively. Since $\mathbf{i}_0$ and $\mathbf{j}_0$ have no predecessors in $D(\gamma)$, it follows that $D(\gamma')$ is still forest-like — we are identifying the roots of disjoint trees. Now, by Lemma 4.8.4, there are $y_1, \ldots, y_n, c_1 \in X_A$ such that $\alpha'(y_1, \ldots, y_n, c_1) = \top$. If we let v be an admissible assignment sending $\mathbf{j}_i$ to y_i, $1 \leq i \leq n$, and sending the nominals $\mathbf{i}_0$ and $\mathbf{j}_0$ both to c_1, then $v(\alpha') = \top$, while $v(\mathbf{i}_1) \not\leq \neg v(\mathbf{i}_2)$. Hence,

$$\mathfrak{A}^\delta, v \not\models \gamma(y_1, \ldots, y_n, c_1, c_1) = \top \,\&\, \beta(y_1, \ldots, y_n, c_1, c_1) = \top \Rightarrow c_1 \leq \neg c_1,$$

and so $\mathfrak{A}^\delta \not\models_{\mathfrak{A}} \,\&\, S \,\&\, \&_{i=1}^k \varphi_i \leq \psi_i \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$. $\square$

Lemma 4.8.6. *If $\mathbf{A}$ is an atomic BAO, then every atom of $\mathbf{A}$ is also an atom of $\mathbf{A}^\delta$. Consequently, if $(\mathbf{A}, X_A)$ is permeated, then every atom of X_A is also an atom of $(\mathbf{A}^\delta, X_{A^\delta})$.*

Proof. Assume x is an atom of $\mathbf{A}$, and let $b \in A^\delta$ such that $\bot < b \leq x$. Now, we know that $b = \bigvee\{c \in \mathbb{K}(\mathbf{A}^\delta) \mid c \leq b\}$. Since $b \neq \bot$, there is some $c_0 \in \mathbb{K}(\mathbf{A}^\delta)$ such that $\bot < c_0 \leq b \leq x$. By definition, $c_0 = \bigwedge\{a \in A \mid c_0 \leq a\}$. But since $\mathbf{A}$ is an atomic BAO, we have $c_0 = \bigwedge\{\bigvee\{y \in At\mathbf{A} \mid y \leq a\} \mid c_0 \leq a\}$. Distributing this, we see that c_0 can be written as a join of meets of atoms of $\mathbf{A}$. Now, since different atoms meet at $\bot$, it follows that c_0 can be written as a join of atoms of $\mathbf{A}$. Hence, there is some $y_0 \in At\mathbf{A}$ such that $\bot < y_0 \leq c_0 \leq b \leq x$. Since $y_0, x_0 \in At\mathbf{A}$, we have $y_0 = x$, and hence $b = x$, as required. $\square$

4.9 The algorithm **hybrid-ALBA** is complete for all inductive $\mathcal{H}(@)$-formulas.

In [35], it is proven that ALBA succeeds on all inductive inequalities in the language of distributive modal logic (DML) [42]. It is also shown how this implies that ALBA succeeds on all inductive inequalities in the language of basic modal logic, where these inequalities can be conveniently defined by simply restricting Definition 4.2.2 to the basic modal language. In outline, the proof consists in showing that, given an (ϵ, Ω)-inductive inequality, the propositional variable occurrences corresponding to the leaves of all ϵ-critical branches can be 'surfaced' or 'solved for', and hence, the quasi-inequalities can be brought into the shape required by the Ackermann rule and the propositional variables can be eliminated. This surfacing process proceeds from the root to the leaves by removing the skeleton nodes through preprocessing and the application of ($\wedge$-Adj), ($\vee$-Adj) and approximation rules, and then removing the PIA nodes by applying adjunction and residuation rules. An Ackermann rule may be applied to eliminate a particular propositional variable from the quasi-inequality as soon as that propositional variable has been solved for, or one can solve for all propositional variables first, and then eliminate them by consecutive applications of the Ackermann rules, thus postponing these applications to the very end of the run. The latter strategy is followed in the proof in [35]. We also followed this strategy in Example 4.4.1.

In fact, the proof in [35] can also be used to show that hybrid-ALBA succeeds on all inductive $\mathcal{H}(@)$- inequalities. What we need to account for is how the novel language elements — negation, nominals and the @ operator — are accommodated by the strategy outlined above. Negation plays exactly the same role as the unary order-reversing connectives $\lhd$ and $\rhd$ of the DML language in [35]. Unlike propositional variables, nominals do not need to be solved for or eliminated: they do not occur in ϵ-critical branches and can therefore be completely ignored as far as it concerns the success of a run of hybrid-ALBA. Occurrences of $+@$ and $-@$ in the skeleton are handled like any binary residuated connective (like $\rightarrow$ for example) by applying the appropriate approximation rule — indeed, as we showed in Proposition 4.1.1, the rules (@-R-Approx) and (@-L-Approx) are refinements of the generic approximation rule for residuated connectives. Similarly, occurrences of $+@$ and $-@$ on the PIA parts of critical branches are handled like any binary residuated connective (like $\rightarrow$ for example) by applying the appropriate residuation rules. Again, the rules (@-R-Res) and (@-L-Res) are refinements of the generic residuation rule.

Theorem 4.9.1. *The algorithm* hybrid-ALBA *succeeds on all inductive $\mathcal{H}(@)$ inequalities and formulas.*

We can also show that hybrid-ALBA succeeds on skeletal inductive formulas.

Theorem 4.9.2. *The algorithm* hybrid-ALBA *succeeds on all skeletal inductive $\mathcal{H}(@)$ inequalities and formulas by means of safe runs.*

Proof. Since skeletal inductive inequalities and formulas are inductive, Theorem 4.9.1 guarantees that hybrid-ALBA succeeds on them. We therefore just need to argue that it does so by means of safe runs. For this it is sufficient to show that the α_i in the applications of the Ackermann rules do not contain occurrences of $\Diamond^{-1}$ or $\Box^{-1}$. Indeed, something stronger holds:

neither $\Diamond^{-1}$ nor $\Box^{-1}$ ever appears in any expression during the entire run. To see this, note that $\Diamond^{-1}$ and $\Box^{-1}$ do not appear in the input, nor can they be introduced by preprocessing or the approximation rules ($\wedge$-Adj) and ($\vee$-Adj). They can only be introduced by applying ($\Box$-Adj) and ($\Diamond$-Adj), which only happens when processing occurrences of $\Box$ and $\Diamond$ on the PIA parts of critical branches. By definition the PIA parts of the ϵ-critical branches of a skeletal (ϵ, Ω)-inductive inequality are vacuous. The claim follows. $\Box$

Finally, we show that for every nominally skeletal inductive $\mathcal{H}(@)$-inequality, there is a topological run of hybrid-ALBA that succeeds on it, but first we need the following lemma:

Lemma 4.9.3. *The* hybrid-ALBA *preprocessing reduces any nominally skeletal inductive $\mathcal{H}(@)$-inequality to a set of singular nominally skeletal inductive $\mathcal{H}(@)$-inequalities.*

Proof. That each inequality produced by preprocessing is again inductive follows directly from Lemma 10.4 in [35]. An easy inspection reveals that '*all branches with positively signed nominals as leaves satisfy conditions* (NS1) *and* (NS2)' is a condition which is invariant under the application of preprocessing rules. We thus have that each inequality produced by preprocessing is skeletal inductive. We need to argue that these inequalities will be singular. To this end, suppose that $+\varphi$ or $-\psi$ contains at least two leaves labelled $+\mathbf{i}$. These two leaves must share a binary ancestor which, given the conditions (NS1) and (NS2), can only be labelled $+\vee$ or $-\wedge$. Since this $+\vee$ or $-\wedge$ only has SLR and primary Δ-adjoints as ancestors, it may be made the root of the tree by applying preprocessing rules. Moreover, it will 'surface' as $+\vee$ in the transformed tree $+\varphi'$, or as $-\wedge$ in the transformed tree $-\psi'$. Preprocessing will therefore separate the two occurrences of $+\mathbf{i}$ into two different inequalities by applying either ($\vee$-Adj) or ($\wedge$-Adj) to the inequality $\varphi' \leq \psi'$. Continuing in this way, all leaves labelled with positively signed nominals are separated, leaving only singular inequalities. $\Box$

Theorem 4.9.4. *For every nominally skeletal inductive $\mathcal{H}(@)$-inequality, there is a topological run of* hybrid-ALBA *which succeeds on it. Moreover, each member of the set of pure quasi-inequalities* $\texttt{pure}(\varphi \leq \psi)$ *resulting from this run has the form* $\&\, S \,\&\, \&_{i=1}^{k} \varphi_i \leq \psi_i \Rightarrow \mathbf{i}_0 \leq \neg\mathbf{j}_0$*, where*

(i) *S is a finite forest-like set of nominal-link inequalities such that $[\mathbf{i}_0]$ and $[\mathbf{j}_0]$ have no predecessors in $D(S)$, and*

(ii) *φ_i (ψ_i) is pure and syntactically closed (open) for each $1 \leq i \leq k$.*

Proof. By Lemma 4.9.3, preprocessing turns any nominally skeletal inequality into a set of singular ones. It is therefore sufficient to prove the claim for singular nominally skeletal inductive inequalities. Let $\varphi(\overline{p}, \mathbf{i}_1, \ldots, \mathbf{i}_n) \leq \psi(\overline{p}, \mathbf{i}_1, \ldots, \mathbf{i}_n)$ be a nominally skeletal (ϵ, Ω)-inductive inequality with its propositional variables among $\overline{p}$ and nominals among $\mathbf{i}_1, \ldots, \mathbf{i}_n$. Let $\varphi'(\overline{p}, q_1, \ldots, q_n, \mathbf{i}_1, \ldots, \mathbf{i}_n)$ be the formula obtained from φ by substituting q_i for each occurrence of $\mathbf{i}_i$ that is *not* the first (subscript) argument of an @, for each $1 \leq i \leq n$. Let the formula $\psi'(\overline{p}, q_1, \ldots, q_n, \mathbf{i}_1, \ldots, \mathbf{i}_n)$ be defined similarly. Then $\varphi(\overline{p}, \mathbf{i}_1, \ldots, \mathbf{i}_n)$ is equal to $\varphi'(\overline{p}, \mathbf{i}_1/q_1, \ldots, \mathbf{i}_n/q_n, \mathbf{i}_1, \ldots, \mathbf{i}_n)$, and similarly, the formula $\psi(\overline{p}, \mathbf{i}_1, \ldots, \mathbf{i}_n)$ is equal to $\psi'(\overline{p}, \mathbf{i}_1/q_1, \ldots, \mathbf{i}_n/q_n, \mathbf{i}_1, \ldots, \mathbf{i}_n)$. By the assumption that $\varphi \leq \psi$ is singular, it follows that, for each $1 \leq i \leq n$, there is at most one leaf labelled $+q_i$ between $+\varphi'$ and $-\psi'$.

Now, we claim that $+\varphi'$ and $-\psi'$ are (ϵ', Ω')-inductive, where $\epsilon_{q_i} = 1$ for all $1 \leq i \leq n$ and $\Omega' = \Omega \cup \{(q_i, p) \mid 1 \leq i \leq n, p \in \overline{p}\}$. Indeed, since every branch in $+\varphi$ and $-\psi$ ending in $+\mathbf{i}_i$ is skeletal, it consists only of skeleton-nodes, and hence is (ϵ', Ω')-conforming; in particular, since these branches contain no SRR-nodes, conditions (CB1) and (CB2) are vacuously satisfied. It follows from Theorem 4.9.1 that hybrid-ALBA succeeds on $\varphi' \leq \psi'$. By a simple induction on the number of rule application after first approximation, one can prove that during any hybrid-ALBA run on $\varphi' \leq \psi'$, the following hold:

(i) Every inequality in the antecedent of each quasi-inequality is either a diamond-link inequality, or has a syntactically closed left-hand side and a syntactically open right-hand side (the fact that nominals occur only as the first arguments of @ is used).

(ii) The set S of all diamond-link inequalities occurring in any given quasi-inequality is forest-like and $[\mathbf{i}_0]$ and $[\mathbf{j}_0]$ have no predecessors in $D(S)$. We use the facts that every occurring diamond-link inequality is the result of applying the of the rules ($\Box$-Approx) or ($\Diamond$-Approx). Indeed, item (i) helps to guarantee that diamond-link inequalities cannot be introduced in another way.

As described above, all applications of Ackermann rules can be postponed to the end of the run. Consider the point in the run where all propositional variables among $\overline{p}$ have been eliminated and all propositional variables $q_1, \ldots, q_n$ have been solved for but have not yet been eliminated. It follows from the forgoing argument that, at this point, the antecedent of every quasi-inequality will consist of a forest-like set of diamond-link inequalities, inequalities of the form $\beta \leq \gamma$ where β (γ) is syntactically closed (open) and positive (negative) in $q_1, \ldots, q_n$, and inequalities of the form $\mathbf{j} \leq q_i$. Moreover, since there was at most one leaf labelled $+q_i$ between $+\varphi'$ and $-\psi'$, there is at most one inequality of the form $\mathbf{j} \leq q_i$, for each $1 \leq i \leq n$ (approximation, residuation and adjunction rules do not multiply variable occurrences, and since the branches leading to $+q$'s are skeletal, they will never feature in formulas being substituted with in Ackermann rule applications w.r.t *other* variables).

Taking the entire run, up to this point, and substituting $\mathbf{i}_i$ for q_i, $1 \leq i \leq n$, we obtain a successful hybrid-ALBA run on $\varphi \leq \psi$. Moreover, in this run every inequality in the antecedent of each quasi-inequality is either a nominal-link inequality, or has a syntactically *pre*-closed left-hand side and a syntactically *pre*-open right-hand side. Therefore, every application of an Ackermann rule is topological. It follows that the antecedent of every final pure quasi-inequality obtained consists of:

(i) a forest-like set S of diamond-link inequalities such that $[\mathbf{i}_0]$ and $[\mathbf{j}_0]$ have no predecessors in $D(S)$,

(ii) inequalities of the form $\beta(\mathbf{i}_1/q_1, \ldots, \mathbf{i}_n/q_n) \leq \gamma(\mathbf{i}_1/q_1, \ldots, \mathbf{i}_n/q_n)$, where β (γ) is syntactically closed (open), and

(iii) at most one inequality of the form $\mathbf{j} \leq \mathbf{i}_i$, for each $1 \leq i \leq n$.

But then the union of S and the set of these latter inequalities is forest-like and $[\mathbf{i}_0]$ and $[\mathbf{j}_0]$ still have no predecessors in the resulting graph. $\Box$

Chapter 5

Hybrid logics with the finite model property

In 1965, R.A. Bull showed that the systems obtained by extending **S4** with any of the members of a certain inductively defined class of formulas have the finite model property [18]. Shortly thereafter, he famously proved that each normal extension of **S4.3** has the finite model property [20]. In this chapter, we give hybrid analogues of these results. Like the original proofs of Bull's results, ours are algebraic, and thus our secondary aim with this work is to illustrate the usefulness of algebraic methods within hybrid logic research, a field where such methods have been largely ignored.

Recall that in relational semantics, a normal modal logic has the *finite model property* (FMP) if every non-theorem of the logic is refuted in some finite model for the logic. If there is a function f from natural numbers to natural numbers such that every non-theorem φ of a normal modal logic is refuted in some $f(n)$-size model, then the logic has the *strong finite model property*. We say that it has the finite frame property, if every non-theorem of the logic is refuted in some finite frame for the logic. Algebraically, we say that a normal modal logic has the finite algebra property, if every non-theorem of the logic is refuted in some finite algebra for the logic. It is a well-known fact that these notions coincide for normal modal logics. See for instance [10] and [23]. We can therefore prove that a normal modal logic has the finite model property using an algebraic toolkit.

Now, for hybrid logics, we say that a hybrid logic has the *finite hybrid algebra property* if every non-theorem φ of the hybrid logic is refuted in some finite hybrid algebra for the hybrid logic. If there is a function f from natural numbers to natural numbers such that this hybrid algebra has size $f(n)$, we say that the hybrid logic has the *strong finite hybrid algebra property*. In the first section, we prove hybrid analogues for Bull's theorem. Well, we will prove something stronger: we will prove the *strong* finite hybrid algebra property. In the second section, we show that hybrid extensions of **S4** with certain inductively defined hybrid formulas have the strong finite hybrid algebra property. Note that we will sometimes use the phrase 'finite model property' as an abbreviation for 'finite hybrid algebra property'.

5.1 Analogues of Bull's Theorem for $\mathcal{H}$, $\mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$

Why **S4.3**? The normal logics extending **S4.3** are particularly well-behaved. Not only do such logics have the finite model property, but they are also finitely axiomatizable and have a negative characterization in terms of finite sets of finite frames (see [39]).

The logics extending **S4.3** are logics of the frames that are rooted, transitive and connected ($\forall x \forall y(xRy \vee yRx)$). To see this, recall that **S4.3** has as axioms (T), (4) and (.3). These formulas are canonical for reflexivity, transitivity, and no branching to the right ($\forall x \forall y \forall z(xRy \wedge xRz \rightarrow yRz \vee y = z \vee zRy)$), respectively. Note that branching to the left is allowed. By taking a point generated subframe, however, we obtain a frame that inherits all three of these properties, but which is also rooted and connected. Now, any connected frame is reflexive, so *rootedness*, *transitivity* and *connectedness* are fundamental. Note that we can view any **S4.3** frame as a chain of clusters[1] (see Figure 5.1), a perspective which will be useful in what follows.

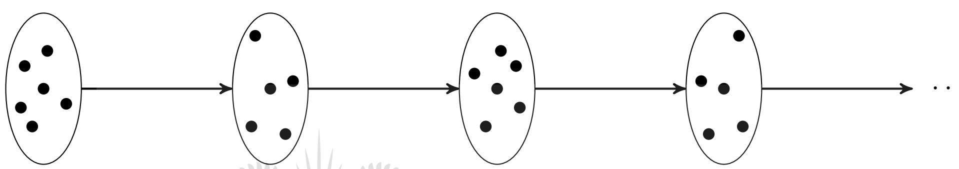

Figure 5.1: A chain of clusters

As we already mentioned, Bull worked with the algebraic semantics, more precisely, closure algebras. A *closure algebra* is a BAO $(A, \wedge, \vee, \neg, \bot, \top, \Diamond)$ such that for all $a \in A$,

(i) $a \leq \Diamond a$, and

(ii) $\Diamond\Diamond a \leq \Diamond a$.

Why 'closure' algebra? The answer is simple: $\Diamond$ is a closure operator. A *closure operator* is a map $C\colon A \rightarrow A$ satisfying

(extensiveness) $a \leq C(a)$,

(idempotency) $C(C(a)) = C(a)$, and

(isotoness) $a \leq b$ implies $C(a) \leq C(b)$.

The first and second conditions follow from the reflexivity and transitivity of $\Diamond$, respectively, while the third condition follows from the fact that $\Diamond$ is monotone.

In the first part of Bull's proof, he falls back on a result of McKinsey and Tarski in [62], as well as that of Birkhoff in [8]. McKinsey and Tarski proved that each normal extension of **S4** is sound and complete with respect to the corresponding class of closure algebras. Birkhoff

[1] A *cluster* on a transitive frame (W, R) is a maximal, nonempty equivalence class under R. That is, $C \subseteq W$ is a cluster if the restriction of R to C is an equivalence relation, and this is *not* the case for any subset D properly extending C.

showed that any closure algebra is sub-directly reducible to well-connected closure algebras (a well-connected BAO is one in which $\Diamond a \wedge \Diamond b = \bot$ iff $a = \bot$ or $b = \bot$). Bull put these facts together and concluded that every normal extension of **S4** is sound and complete with respect to the corresponding class of well-connected closure algebras.

Next, Bull generated a finite Boolean subalgebra of a well-connected closure algebra from a finite subset X of elements of the well-connected closure algebra. He then proceeded by defining a new operation on this finite Boolean subalgebra that is an operator and, furthermore, preserves $\Diamond$ in X. What Bull essentially did, was a filtration. So let us give a short overview on algebraic filtrations. For more details on algebraic filtrations, see [34].

Let $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top, f)$ be a BAO, and let S be a finite subset of A. The subalgebra of $\mathbf{A}$ generated by S is in general infinite, due to the modal operator f, and we cannot expect to generate a finite BAO in this way. However, the *Boolean* subalgebra of $\mathbf{A}$ generated by S (denoted by $\mathbf{A}_S$) is finite and, clearly, preserves all existing Boolean operations in S. The goal is to define a new operation f' on this finite Boolean subalgebra such that f' is an operator and f' preserves f in S (i.e., if $a \in S$ and $f(a) \in S$, then $f'(a) = f(a)$).

Next, let $\underline{S}$ be a subset of S such that $f(a) \in S$ whenever $a \in \underline{S}$. If f' is such that $f'(a) = f(a)$ whenever $a \in \underline{S}$, we say that f' extends $f \restriction_{\underline{S}}$.

Before we define this new operation on A_S, recall that for any function g: $At\mathbf{A}_S \to A_S$ (and therefore any operator on $\mathbf{A}_S$), there is an associated binary relation R^g on $At\mathbf{A}_S$ defined by

$$xR^g y \text{ iff } x \leq g(y).$$

Conversely, any binary relation R on $At\mathbf{A}_S$ has an associated function g^R: $At\mathbf{A}_S \to A_S$ defined by

$$g^R(y) = \bigvee\{x \in At\mathbf{A}_S \mid xRy\}$$

It is not difficult to show that the functions g: $At\mathbf{A}_S \to A_S$ and binary relations $R \subseteq A_S \times A_S$ are in a one-to-one correspondence.

Given a binary relation R on $At\mathbf{A}_S$, the function g^R: $At\mathbf{A}_S \to A_S$ defined above extends uniquely to an operator f^R on A_S as follows:

$$\begin{aligned} f^R(a) &= f\left(\bigvee\{y \in At\mathbf{A}_S \mid y \leq b\}\right) \\ &= \bigvee\{g^R(y) \mid y \in At\mathbf{A}_S \text{ and } y \leq a\} \\ &= \bigvee\left\{\bigvee\{x \in At\mathbf{A}A_S \mid xRy\} \mid y \in At\mathbf{A}_S \text{ and } y \leq a\right\} \\ &= \bigvee\{x \in At\mathbf{A}_S \mid \exists y \in At\mathbf{A}_S(y \leq a \text{ and } xRy)\} \end{aligned}$$

The operation f^R is an operator on $\mathbf{A}_S$. First, $f^R(\bot) = \bigvee \varnothing = \bot$, so f^R is normal. For the additivity, let $a, b \in A_S$, and denote $\{x \in At\mathbf{A}_S \mid \exists y \in At\mathbf{A}_S(y \leq a \text{ and } xRy)\}$ and

$\{x \in At\mathbf{A}_S \mid \exists y \in At\mathbf{A}_S(y \leq b \text{ and } xRy)\}$ by X and Y for simplicity. Now,

$$\begin{aligned}
& x_0 \in \{x \in At\mathbf{A}_S \mid \exists y \in At\mathbf{A}_S(y \leq a \vee b \text{ and } xRy)\} \\
\iff\ & \exists y_0 \in At\mathbf{A}_S(y_0 \leq a \vee b \text{ and } x_0Ry_0) \\
\iff\ & \exists y_0 \in At\mathbf{A}_S(y_0 \leq a \text{ or } y_0 \leq b \text{ and } x_0Ry_0) \\
\iff\ & \exists y_0 \in At\mathbf{A}_S((y_0 \leq a \text{ and } x_0Ry_0) \text{ or } (y_0 \leq b \text{ and } x_0Ry_0)) \\
\iff\ & \exists y_0 \in At\mathbf{A}_S(y_0 \leq a \text{ and } x_0Ry_0) \text{ or } \exists y_0 \in At\mathbf{A}_S(y_0 \leq b \text{ and } x_0Ry_0) \\
\iff\ & x_0 \in X \text{ or } x_0 \in Y \\
\iff\ & x_0 \in X \cup Y,
\end{aligned}$$

so

$$\begin{aligned}
& f^R(a \vee b) \\
=\ & \bigvee\{x \in At\mathbf{A}_S \mid \exists y \in At\mathbf{A}_S(y \leq a \vee b \text{ and } xRy)\} \\
=\ & \bigvee (X \cup Y) \\
=\ & \bigvee X \vee \bigvee Y \\
=\ & f^R(a) \vee f^R(b)
\end{aligned}$$

To ensure that f^R extends $f \restriction_{\underline{S}}$ it suffices that R satisfy the following condition:

$$(\forall a \in \underline{S})(\forall x \in At\mathbf{A}_S)(x \leq f(a) \iff (\exists y \in At\mathbf{A}_S)(y \leq a \text{ and } xRy)) \tag{R}$$

To see this, let R be a binary relation on $At\mathbf{A}_S$ such that (R) holds, and let $a \in \underline{S}$. Then

$$\begin{aligned}
f^R(a) &= \bigvee\{x \in At\mathbf{A}_S \mid \exists y \in At\mathbf{A}_S(y \leq a \text{ and } xRy)\} \\
&= \bigvee\{x \in At\mathbf{A}_S \mid x \leq f(a)\} \\
&= f(a),
\end{aligned}$$

where the second equality follows from (R) and the last equality from the fact that $f(a) \in S \subseteq A_S$.

Now, the BAO $(\mathbf{A}_S, f^R)$ is called the *algebraic filtration of* $\mathbf{A}$ *through* $(S, \underline{S})$ *with* R.

We now return to our discussion of Bull's proof that every normal extension of **S4.3** has the finite model property. Bull had his finite algebra. All that he needed to show is that this algebra validates the axioms of the logic. He proved this by showing that the finite algebra can be imbedded into the original well-connected algebra. This forms the crux of his work in [20]. It is here where he crucially used the fact that the original algebra is well-connected.

5.1.1 An analogue of Bull's Theorem for $\mathcal{H}$

In this section, we give an analogue of Bull's Theorem for the language $\mathcal{H}$. Our approach is not entirely the same as that of Bull. Recall that the validity of $\mathcal{H}$-formulas is generally not preserved under taking products of hybrid algebras. We therefore don't have a result similar to Birkhoff's result in [8] that says that any closure algebra is sub-directly reducible to well-connected closure algebras for our hybrid algebras. However, all is not lost — well-connectedness is closely related, in the dual relational semantics, to the ability to take point-generated submodels, and we have a way of simulating the process of taking generated submodels algebraically, so we do not need a result similar to Birkhoff's result for our hybrid algebras to obtain a completeness result with respect to well-connected hybrid algebras. The rest of our approach is pretty much the same as that of Bull: we obtain a finite hybrid algebra by a filtration and embed this finite hybrid algebra into the original hybrid algebra.

To begin with, recall that every normal extension of **S4** is sound and complete with respect to the corresponding class of closure algebras. We can obtain a similar result for extensions of the hybrid logic obtained by extending **H** with (T) and (4). Formally:

Definition 5.1.1. The logic **HS4** is the smallest set of $\mathcal{H}$-formulas containing all propositional tautologies, the axioms in Table 5.1, except for (.3), and which is closed under the inference rules in Table 5.1.

Axioms:	
$(Taut)$	$\vdash \varphi$ for all propositional tautologies φ.
(K)	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$
$(Dual)$	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$
(Nom)	$\vdash \Diamond^n(\mathbf{i} \wedge p) \to \Box^m(\mathbf{i} \to p)$ for all $n, m \in \mathbb{N}$.
(4)	$\vdash \Diamond\Diamond p \to \Diamond p$
(T)	$\vdash p \to \Diamond p$
(.3)	$\vdash \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(q \wedge \Diamond p)$
Rules of inference:	
$(Modus\ ponens)$	If $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$.
$(Sorted\ substitution)$	$\vdash \varphi'$ whenever $\vdash \varphi$, where φ' is obtained from φ by sorted substitution.
(Nec)	If $\vdash \varphi$, then $\vdash \Box\varphi$.
$(NameLite)$	If $\vdash \neg\mathbf{i}$, then $\vdash \bot$.

Table 5.1: Axioms and inference rules of **HS4** and **HS4.3**

Algebraically, hybrid extensions of **HS4** are characterized by classes of hybrid closure algebras (defined below).

Definition 5.1.2 (Hybrid closure algebra). A *hybrid closure algebra* is a hybrid algebra $\mathfrak{A} = (\mathbf{A}, X_A)$ such that for all $a, b \in A$ the following holds:

(*refl*) $a \leq \Diamond a$ and

(*trans*) $\Diamond\Diamond a \leq \Diamond a$.

Theorem 5.1.3. *Let Σ be a set of $\mathcal{H}$-formulas. Then every normal hybrid logic* $\mathbf{HS4} \oplus \Sigma$ *is sound and complete with respect to the class of all hybrid closure algebras validating* Σ.

Proof. This result follows directly from Theorem 3.1.1. $\square$

Bull next showed that every normal extension of the logic **S4** is sound and complete with respect to the corresponding class of well-connected closure algebras by falling back on Birkhoff's result that any closure algebra is sub-directly reducible to well-connected closure algebras in [8]. As we mentioned earlier, here we will fall back on the fact that well-connectedness is equivalent, in the dual relational semantics, to the ability to take generated submodels, and make use of our construction on page 63 to obtain a similar result for **HS4**. However, we work with piecewise well-connected hybrid closure algebras instead of well-connected hybrid algebras.

Definition 5.1.4. A hybrid algebra $\mathfrak{A}$ is *piecewise well-connected*, if there are $a_1, a_2, \ldots, a_m$ in A such that

(i) $a_i \wedge a_j = \bot$ for $i \neq j$,

(ii) $a_1 \vee a_2 \vee \cdots \vee a_m = \top$,

(iii) $\Diamond a_i = a_i$ for each $1 \leq i \leq m$, and

(iv) $\Diamond a \wedge \Diamond b = \bot$ iff $a = \bot$ or $b = \bot$ for all $a, b \leq a_i$ and $1 \leq i \leq m$.

We will often refer to $a_1, \ldots, a_m$ as 'pieces' of the algebra.

So let us prove our claim before Definition 5.1.4. The lemmas needed to prove this will follow afterwards.

Theorem 5.1.5. *Every normal hybrid logic* $\mathbf{HS4} \oplus \Sigma$ *is sound and complete with respect to the class of all piecewise well-connected hybrid closure algebras validating* Σ. *Moreover, at most two pieces will always suffice.*

Proof. We only prove the completeness direction as the soundness direction is just a special case of the soundness direction of Theorem 5.1.3. So suppose $\varphi \notin \mathbf{HS4} \oplus \Sigma$. By Theorem 5.1.3, there is a hybrid closure algebra $\mathfrak{A} = (\mathbf{A}, X_A)$ and an assignment ν such that $\mathfrak{A} \models \Sigma^{\approx}$ but $\mathfrak{A}, \nu \not\models \varphi \approx \top$. Next, consider the canonical extension $\mathfrak{A}^\delta$ of $\mathfrak{A}$. We know that $\nu(\neg\varphi) > \bot$ in $\mathfrak{A}^\delta$, so, since $\mathfrak{A}^\delta$ is atomic, there is some atom d in $\mathfrak{A}^\delta$ such that $d \leq \nu(\neg\varphi)$. Let $d_0 = d$, and suppose d_n is already defined. Then define $d_{n+1} = \Diamond^{-1} d_n$. Using Lemma 5.1.6 below, we see that $d_{n+1} = \Diamond^{-1} d$ for all $n \in \mathbb{N}$, so let

$$D = \bigvee \Diamond^{-1} d.$$

Next, let $\mathbf{A}_D = (A_D, \wedge^D, \vee^D, \neg^D, \bot^D, \top^D, \Diamond^D)$, where $A_D = \{a \wedge D \mid a \in A\}$, $\wedge^D$ and $\vee^D$ are the restriction of $\wedge$ and $\vee$ to A_D,

$$\neg^D a = \neg a \wedge D \qquad \Diamond^D a = \Diamond a \wedge D$$
$$\bot^D = \bot \qquad \top^D = D.$$

Finally, let $\mathfrak{A}_D = (\mathbf{A}_D, X_D)$, where $X_D = \{x \in X_A \mid x \leq D\}$. Now, in a similar way as in Lemma 3.1.3, we can show that A_D is closed under the operations $\wedge^D$, $\vee^D$, $\neg^D$, and $\Diamond^D$. Furthermore, by Lemma 5.1.7 below, $\mathfrak{A}_D$ is well-connected. From here we break the proof up into three cases:

Case 1: $x \leq D$ for all $x \in X_A$. This is the easiest case. First, note that in this case $X_A = X_D$. Now, let h: $A \to A_D$ be the map defined by $h(a) = a \wedge D$. Then clearly h is surjective from $\mathbf{A}$ onto $\mathbf{A}_D$, and h maps elements of X_A to elements of X_D. To see that h is surjective from X_A onto X_D, let $x \in X_D$. We know that $x \leq D$, so $x = x \wedge D = h(x)$. Hence, since $x \in X_A$, x is its own pre-image. Showing that h is a homomorphism is done in a similar way as in Lemma 3.1.4. This means that $\mathfrak{A}_D \models \mathbf{HS4}\Sigma^{\approx}$. Furthermore, $\mathfrak{A}_D \not\models \varphi \approx \top$. To see this, consider the assignment ν_D: $\mathsf{PROP} \cup \mathsf{NOM} \to A_D$ defined by $\nu_D(p) = h(\nu(p))$ and $\nu_D(\mathbf{j}) = h(\nu(\mathbf{j}))$. It is easy to show using structural induction that $\nu_D(\psi) = h(\nu(\psi))$ for all formulas ψ that use propositional variables from PROP and nominals from NOM. Since $d \leq D$ and $d \leq \nu(\neg\varphi)$,

$$\nu_D(\neg\varphi) = h(\nu(\neg\varphi)) = \nu(\neg\varphi) \wedge D \geq d > \bot,$$

which gives $\nu_D(\varphi) \neq D = \nu_D(\top)$. Finally, we also know that $\Diamond^D D = \Diamond D \wedge D \geq D \wedge D = D$, so, since $\mathfrak{A}_D$ is well-connected, $\mathfrak{A}_D$ is piecewise well-connected.

Case 2: $x \not\leq D$ for some $x \in X_A$ but not all. In this case, we also work with products like we did in the completeness theorem of $\mathbf{H} \oplus \Sigma$. We want to show that $(\mathfrak{A}_D)_0 \models \mathbf{HS4}\Sigma^{\approx}$. So let h: $A \to A_D$ be the map defined by $h(a) = a \wedge D$. As before we can show that h is a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{A}_D$ in a similar way as in Lemma 3.1.4. To show that h is surjective from X_A onto $X_D \cup \{\bot\}$, let $x \in X_D \cup \{\bot\}$. The case where $x \in X_D$ is the same as for Case 1, so assume $x = \bot$. We know there is an $x_0 \in X_A$ such that $x_0 \not\leq D$, which means that $h(x) = x \wedge D = \bot$. Hence, x_0 is the pre-image of $\bot$. We therefore now have that $(\mathfrak{A}_D)_0 \models \mathbf{HS4}\Sigma^{\approx}$, and hence that $\mathfrak{A}_D \times \mathfrak{A}_D \models \mathbf{HS4}\Sigma^{\approx}$ by Proposition 2.1.16. To see that $(\mathfrak{A}_D)_0 \not\models \varphi \approx \top$, let ν_D: $\mathsf{PROP} \cup \mathsf{NOM} \to A_D$ be defined as before. Then $\nu_D(\varphi) \neq \nu_D(\top)$. Now, let ν': $\mathsf{PROP} \cup \mathsf{NOM} \to A_D \times A_D$ be defined by $\nu'(p) = (\nu_D(p), \nu_D(p))$ and

$$\nu'(\mathbf{j}) = \begin{cases} (\nu_D(\mathbf{j}), \bot) \text{ if } \nu_D(\mathbf{j}) \neq \bot \\ (\bot, x_1) \text{ if } \nu_D(\mathbf{j}) = \bot \end{cases}$$

for some $x_1 \in X_D$. As before, using structural induction on ψ, we can show that $\nu'(\psi) = (\nu_D(\psi), a_\psi)$ for some $a_\psi \in A_D$. Thus, $\nu'(\varphi) = (\nu_D(\varphi), a_\varphi) \neq (\nu_D(\top), D) = \nu'(\top)$.

Finally, we must just show that $\mathfrak{A}_D \times \mathfrak{A}_D$ is piecewise well-connected. First, we have $(D, \bot) \wedge (\bot, D) = (\bot, \bot)$ and $(\bot, D) \vee (D, \bot) = (D, D)$. Furthermore,

$$\Diamond(D, \bot) = (\Diamond^D D, \Diamond^D \bot) = (D, \bot)$$

and

$$\Diamond(\bot, D) = (\Diamond^D \bot, \Diamond^D D) = (\bot, D).$$

Now, let $(a,b),(a',b') \in \mathfrak{A}_D \times \mathfrak{A}_D$ such that $(a,b) \leq (D,\bot)$ and $(a',b') \leq (D,\bot)$. Then $b = \bot$ and $b' = \bot$. Assume $\Diamond(a,b) \wedge \Diamond(a',b') = (\bot,\bot)$. Then $(\Diamond a \wedge \Diamond a', \bot) = (\bot,\bot)$, and so, since $\mathfrak{A}_D$ is well-connected, $a = \bot$ or $a' = \bot$. Hence, $(a,b) = (\bot,\bot)$ or $(a',b') = (\bot,\bot)$. Similarly for $(\bot, D)$.

Case 3: $x \not\leq D$ for all $x \in X_A$. As in Case 2, $(\mathfrak{A}_D)_0 \models \mathbf{HS4}\Sigma^{\approx}$. Furthermore, $(\mathfrak{A}_D)_0, \nu_D \not\models \varphi \approx \top$, where ν_D is defined as before. However, since $X_D = \varnothing$, we are not done. Now, we know that $X_A \neq \varnothing$, so choose some $x_0 \in X_A$ and denote it by d'. Then define

$$D' = \Diamond^{-1} d',$$

and let $\mathbf{A}_{D'} = (A_{D'}, \wedge^{D'}, \vee^{D'}, \neg^{D'}, \bot^{D'}, \top^{D'}, \Diamond^{D'})$, where $A_{D'}, \wedge^{D'}, \vee^{D'}, \neg^{D'}, \top^{D'}, \bot^{D'}$, and $\Diamond^{D'}$ are defined as before. Set

$$X_{D'} = \{x \in X_A \mid x \leq D'\},$$

and then let $\mathfrak{A}_{D'} = (\mathbf{A}_{D'}, X_{D'})$. In the same way as in Lemma 5.1.7, we can show that $\mathfrak{A}_{D'}$ is well-connected. Now, if $x \leq D'$ for all $x \in X_A$, we let $h\colon A \to A_{D'}$ be defined by $h(a) = a \wedge D'$. As before, we can show that h is a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{A}_{D'}$ and that h is surjective from X_A onto $X_{D'}$. Therefore, $\mathfrak{A}_{D'} \models \mathbf{HS4}\Sigma^{\approx}$, and so $\mathfrak{A}_D \times \mathfrak{A}_{D'} \models \mathbf{HS4}\Sigma^{\approx}$ by Proposition 2.1.19. On the other hand, if $x \not\leq D'$ for some $x \in X_A$, let $h\colon A \to A_{D'}$ be defined by $h(a) = a \wedge D'$. Then h is a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{A}_{D'}$, and furthermore, h is surjective from X_A onto $X_{D'} \cup \{\bot\}$. Hence, $(\mathfrak{A}_{D'})_0 \models \mathbf{HS4}\Sigma^{\approx}$, which means that $\mathfrak{A}_D \times \mathfrak{A}_{D'} \models \mathbf{HS4}\Sigma^{\approx}$ by Proposition 2.1.16. Now, in both cases, define $\nu_{D'}$ in the same way as we defined ν_D. Note that we do not know if $\mathfrak{A}_{D'}, \nu_{D'} \not\models \varphi \approx \top$. This is not a problem, as we will now show. Consider the assignment $\nu''\colon \mathsf{PROP} \cup \mathsf{NOM} \to A_D \times A_{D'}$ defined by $\nu''(p) = (\nu_D(p), \nu_{D'}(p))$ and

$$\nu''(\mathbf{j}) = \begin{cases} (\bot, \nu_{D'}(\mathbf{j})) \text{ if } \nu_{D'}(\mathbf{j}) \neq \bot \\ (\bot, x_0) \text{ if } \nu_{D'}(\mathbf{j}) = \bot. \end{cases}$$

Using structural induction on ψ, we can show that $\nu''(\psi) = (\nu_D(\psi), a_\psi)$ for some $a_\psi \in A_{D'}$. Thus, $\nu''(\varphi) = (\nu_D(\varphi), a_\varphi) \neq (\nu_D(\top), D') = \nu''(\top)$.

All that is left to check is that $\mathfrak{A}_D \times \mathfrak{A}_{D'}$ is piecewise well-connected. First,

$$(D,\bot) \wedge (\bot, D') = (\bot,\bot)$$

and

$$(\bot, D') \vee (D,\bot) = (D, D').$$

Furthermore,

$$\Diamond(D,\bot) = (\Diamond^D D, \Diamond^{D'} \bot) = (D,\bot)$$

and

$$\Diamond(\bot, D') = (\Diamond^D \bot, \Diamond^{D'} D') = (\bot, D').$$

Finally, since both $\mathfrak{A}_D$ and $\mathfrak{A}_{D'}$ are both well-connected, we can show in the same way as in Case 2 that $\mathfrak{A}_D \times \mathfrak{A}_{D'}$ is piecewise well-connected. $\square$

Lemma 5.1.6. *Let $\mathfrak{A}$ be a hybrid closure algebra, and let a be an element of the canonical extension of $\mathfrak{A}$. Then*

(i) $a \leq \Diamond^{-1} a$, *and*

(ii) $\Diamond^{-1}\Diamond^{-1} a \leq \Diamond^{-1} a$.

Proof. (i) Note that since all axioms of **S4** are Sahlqvist, it follows from the canonicity of Sahlqvist equations that the validity of the axioms of **S4** is preserved in passing from $\mathfrak{A}$ to its canonical extension. Hence, for all b in the canonical extension of $\mathfrak{A}$, $\neg b \leq \Diamond\neg b$ by (*refl*). This means that $\neg\Diamond\neg b \leq \neg\neg b$ for all b in the canonical extension of $\mathfrak{A}$, and so $\Box b \leq b$ for all b in the canonical extension of $\mathfrak{A}$. We therefore have that $\Box\Diamond^{-1} a \leq \Diamond^{-1} a$. But $\Box$ and $\Diamond^{-1}$ are adjoint, so $a \leq \Box\Diamond^{-1} a$, which gives $a \leq \Diamond^{-1} a$.

(ii) First, for all b in the canonical extension of $\mathfrak{A}$, $\Diamond\Diamond\neg b \leq \Diamond\neg b$ by (*trans*), so we have $\neg\Diamond\neg b \leq \neg\Diamond\Diamond\neg b$. Hence, $\Box b \leq \Box\Box b$ for all b in the canonical extension of $\mathfrak{A}$, and so $\Box\Diamond^{-1} a \leq \Box\Box\Diamond^{-1} a$. This means that $a \leq \Box\Box\Diamond^{-1} a$, and therefore, $\Diamond^{-1}\Diamond^{-1} a \leq \Diamond^{-1} a$. □

For the lemma that follows, $\mathfrak{A}_D$ will be the hybrid algebra constructed in the proof of Theorem 5.1.5. The proof of this lemma establishes our earlier claim that well-connectedness corresponds, in the dual relational semantics, to the ability to take generated submodels.

Lemma 5.1.7. *$\mathfrak{A}_D$ is well-connected.*

Proof. It is easy to see that $a = \bot$ or $b = \bot$ implies $\Diamond a \wedge \Diamond b = \bot$. For the converse direction, let $a, b \in A_D$ such that $a, b \neq \bot$. Then $a = a' \wedge D$ and $b = b' \wedge D$ for some $a', b' \in A$, and so $a' \wedge D \neq \bot$ and $b' \wedge D \neq \bot$. Hence, $a' \wedge \Diamond^{-1} d \neq \bot$ and $b' \wedge \Diamond^{-1} d \neq \bot$. This means that $d \leq \Diamond a'$ and $d \leq \Diamond b'$ by Lemma 3.4.3. Now, since $d \leq D$, $d \leq \Diamond a' \wedge D$ and $d \leq \Diamond b' \wedge D$. We also know that $D \leq \Box D$, so $d \leq \Diamond a' \wedge \Box D$ and $d \leq \Diamond b' \wedge \Box D$, and therefore, $d \leq \Diamond(a' \wedge D)$ and $d \leq \Diamond(b' \wedge D)$. We thus have that $d \leq \Diamond(a' \wedge D) \wedge \Diamond(b' \wedge D) \wedge D = \Diamond^D a \wedge \Diamond^D b$, which means that $\Diamond^D a \wedge \Diamond^D b \neq \bot$. □

Of course Bull proved that normal extensions of **S4.3** have the finite model property, so we are interested in the hybrid logics extending **HS4.3** (defined below).

Definition 5.1.8. The logic **HS4.3** is the smallest set of formulas containing all propositional tautologies, the axioms in Table 5.1, and which is closed under the inference rules in Table 5.1.

Note that by Theorem 5.1.5, **HS4.3** $\oplus\, \Sigma$ is sound and complete with respect to the class of piecewise well-connected hybrid algebras. This means that frame-theoretically, any **HS4.3** frame will consist of at most two chains of clusters.

Definition 5.1.9. An **HS4.3**-*algebra* is a hybrid closure algebra satisfying in addition

(.3) $\Diamond a \wedge \Diamond b \leq \Diamond(a \wedge \Diamond b) \vee \Diamond(b \wedge \Diamond a) \vee \Diamond(a \wedge b)$.

We also need the following lemmas about **HS4.3**-algebras:

Lemma 5.1.10. *Let $\mathfrak{A}$ be an* **HS4.3**-*algebra, and let a and b be elements of A. Then*

(i) $\Diamond(\Box a \wedge \neg\Box b) \wedge \Diamond(\Box b \wedge \neg\Box a) = \bot$, *and*

(ii) $\Diamond(\Box a \wedge \neg\Box b) \wedge \Diamond(\Box(\neg\Box a \vee \Box b) \wedge \neg\Box b) = \bot$.

Proof. (i) Let a and b be elements of A. By (.3),

$$\Diamond(\Box a \wedge \neg\Box b) \wedge \Diamond(\Box b \wedge \neg\Box a) \leq \Diamond(\Box a \wedge \neg\Box b \wedge \Diamond(\Box b \wedge \neg\Box a)) \vee \Diamond(\Box b \wedge \neg\Box a \wedge \Diamond(\Box a \wedge \neg\Box b)),$$

and so, by the monotonicity of $\Diamond$ and (*refl*),

$$\Diamond(\Box a \wedge \neg\Box b) \wedge \Diamond(\Box b \wedge \neg\Box a) \leq \Diamond(\Box a \wedge \Diamond\neg\Box a) \vee \Diamond(\Box b \wedge \Diamond\neg\Box b).$$

Hence, by the definition of $\Box$,

$$\Diamond(\Box a \wedge \neg\Box b) \wedge \Diamond(\Box b \wedge \neg\Box a) \leq \Diamond(\Box a \wedge \Diamond\Diamond\neg a) \vee \Diamond(\Box b \wedge \Diamond\Diamond\neg b).$$

Therefore, using (*trans*), we get

$$\Diamond(\Box a \wedge \neg\Box b) \wedge \Diamond(\Box b \wedge \neg\Box a) \leq \Diamond(\Box a \wedge \Diamond\neg a) \vee \Diamond(\Box b \wedge \Diamond\neg b),$$

and so

$$\begin{aligned}
&\Diamond(\Box a \wedge \neg\Box b) \wedge \Diamond(\Box b \wedge \neg\Box a) \\
&\leq \Diamond(\Box a \wedge \neg\Box a) \vee \Diamond(\Box b \wedge \neg\Box b) \\
&= \Diamond\bot \vee \Diamond\bot \\
&= \bot \vee \bot \\
&= \bot.
\end{aligned}$$

We thus have $\Diamond(\Box a \wedge \neg\Box b) \wedge \Diamond(\Box b \vee \neg\Box a) = \bot$.

(ii) Let a and b be elements of A, and furthermore, for simplicity, let $c = \Box a \wedge \neg\Box b$ and $d = \Box(\neg\Box a \vee \Box b) \wedge \neg\Box b$. Now, by (.3), we have

$$\Diamond c \wedge \Diamond d \leq \Diamond(c \wedge \Diamond d) \vee \Diamond(c \wedge d) \vee \Diamond(\Diamond a \wedge d).$$

Using (*refl*) and (*trans*), and simplifying, we get

$$\Diamond c \wedge \Diamond d \leq \Diamond(c \wedge \Diamond(\Box(\neg\Box a \vee \Box b) \wedge \Diamond\Diamond\neg b)) \vee \Diamond(c \wedge (\neg\Box\Box a \vee \Box\Box b)) \vee \Diamond\Diamond(c \wedge (\neg\Box a \vee \Box b)),$$

which means

$$\Diamond c \wedge \Diamond d \leq \Diamond(c \wedge \Diamond\Diamond((\neg\Box a \vee \Box b) \wedge \Diamond\neg b)) \vee \Diamond(c \wedge (\neg\Box a \vee \Box b)) \vee \Diamond(c \wedge (\neg\Box a \vee \Box b)).$$

But then

$$\Diamond c \wedge \Diamond d \leq \Diamond(\Box a \wedge \neg\Box b \wedge \neg\Box\Box a \wedge \neg\Box\Box b),$$

and so, by (*refl*) and (*trans*),

$$\Diamond c \wedge \Diamond d \leq \Diamond(\Box a \wedge \neg\Box b \wedge \neg\Box a \wedge \neg\Box b) = \Diamond\bot = \bot.$$

Hence, $\Diamond(\Box a \wedge \neg\Box b) \wedge \Diamond(\Box(\neg\Box a \vee \Box b) \wedge \neg\Box b) = \bot$, as required. $\Box$

We can prove a similar result as Lemma 2 in [20] for our piecewise well-connected hybrid algebras. This lemma will enable us to label the atoms in each 'piece' of the piecewise well-connected hybrid algebra in the same way Bull labeled the atoms in his finite algebra. We will also fall back later to this lemma to prove that our map used to show that the finite hybrid algebra is embeddable into the original hybrid algebra preserves the diamond.

Lemma 5.1.11. *Let $a_1, a_2, \ldots, a_m$ be the pieces of a piecewise well-connected* **HS4.3***-algebra $\mathfrak{A}$, and let a and b be elements of $\mathfrak{A}$ such that $a \leq a_i$ and $b \leq a_i$, $i = 1, \ldots, m$. Then*

(i) $\Diamond a \leq \Diamond b$ *or* $\Diamond b \leq \Diamond a$,

(ii) $\Diamond(a \wedge \neg\Diamond b) = \Diamond(\Diamond a \wedge \neg\Diamond b)$, *and*

(iii) $\Diamond b < \Diamond a$ *implies* $\Diamond(a \wedge \neg\Diamond b) = \Diamond a$.

Proof. (i) We have $\Diamond(\neg\Diamond a \wedge \Diamond b) \wedge \Diamond(\neg\Diamond b \wedge \Diamond a) = \bot$ by Lemma 5.1.10, and so, by (*refl*), $\Diamond(\neg\Diamond a \wedge b) \wedge \Diamond(\neg\Diamond b \wedge a) = \bot$. But since $a \leq a_i$ and $b \leq a_i$, we have $a \wedge \neg\Diamond b \leq a \leq a_i$ and $b \wedge \neg\Diamond a \leq b \leq a_i$. Hence, by the definition of a piecewise connected hybrid algebra, $b \wedge \neg\Diamond a = \bot$ or $a \wedge \neg\Diamond b = \bot$. Thus, $b \leq \Diamond a$ or $a \leq \Diamond b$, and so $\Diamond b \leq \Diamond\Diamond a = \Diamond a$ or $\Diamond a \leq \Diamond\Diamond b = \Diamond b$.

(ii) By (*refl*), $a \leq \Diamond a$, so $a \wedge \neg\Diamond b \leq \Diamond a \wedge \neg\Diamond b$. Hence, by the monotonicity of $\Diamond$, $\Diamond(a \wedge \neg\Diamond b) \leq \Diamond(\Diamond a \wedge \neg\Diamond b)$. Conversely, by (*trans*), $\Diamond\Diamond b = \Diamond b$, so

$$\begin{aligned} \Diamond a \wedge \neg\Diamond b &= \Diamond a \wedge \neg\Diamond\Diamond b \\ &= \Diamond a \wedge \Box\neg\Diamond b \\ &\leq \Diamond(a \wedge \neg\Diamond b). \end{aligned}$$

Therefore, by the monotonicity of $\Diamond$ and (*trans*), $\Diamond(\Diamond a \wedge \neg\Diamond b) \leq \Diamond\Diamond(a \wedge \neg\Diamond b) = \Diamond(a \wedge \neg\Diamond b)$.

(iii) First, by (ii), $\Diamond(a \wedge \neg\Diamond b) = \Diamond(\Diamond a \wedge \neg\Diamond b)$. But by the monotonicity of $\Diamond$ and (*trans*), $\Diamond(\Diamond a \wedge \neg\Diamond b) \leq \Diamond\Diamond a = \Diamond a$, so if we can show that $\Diamond a \leq \Diamond(\Diamond a \wedge \neg\Diamond b)$, we are done. Now,

$$\Diamond(\Diamond a \wedge \neg\Diamond b) \wedge \Diamond(\Diamond a \wedge \neg\Diamond(\Diamond a \wedge \neg\Diamond b)) = \bot.$$

by Lemma 5.1.10, and hence, $\Diamond(a \wedge \neg\Diamond b) \wedge \Diamond(a \wedge \neg\Diamond(\Diamond a \wedge \neg\Diamond b)) = \bot$ by (*refl*). Since $a \leq a_i$ and $b \leq a_i$, $a \wedge \neg\Diamond b \leq a \leq a_i$ and $a \wedge \neg\Diamond(\Diamond a \wedge \neg\Diamond b) \leq a \leq a_i$, so, by the definition of a piecewise well-connected hybrid algebra, $a \wedge \neg\Diamond b = \bot$ or $a \wedge \neg\Diamond(\Diamond a \wedge \neg\Diamond b) = \bot$. We thus have $a \leq \Diamond b$ or $a \leq \Diamond(\Diamond a \wedge \neg\Diamond b)$, and so $\Diamond a \leq \Diamond\Diamond b = \Diamond b$ or $\Diamond a \leq \Diamond\Diamond(\Diamond a \wedge \neg\Diamond b) = \Diamond(\Diamond a \wedge \neg\Diamond b)$. But $\Diamond b < \Diamond a$, which means that $\Diamond a \leq \Diamond(\Diamond a \wedge \neg\Diamond b)$, as required. □

We now give the main result of this section.

Theorem 5.1.12. *Every normal hybrid logic* **HS4.3** $\oplus \Sigma$ *has the strong finite hybrid algebra property.*

Proof. Suppose $\varphi \notin$ **HS4.3**$\oplus\Sigma$. By Theorem 5.1.5, there is a piecewise well-connected closure hybrid algebra $\mathfrak{A} = (\mathbf{A}, X_A)$ and an assignment ι such that $\mathfrak{A} \models \Sigma^{\approx}$ but $\mathfrak{A}, \iota \not\models \varphi \approx \top$. We also know that there are a_1 and a_2 in A that satisfy the conditions of Definition 5.1.4 (possibly

$a_1 = a_2 = \top$). Now, let S_0 be the set of elements of $\mathfrak{A}$ used in the evaluation of φ and $\top$ under ι. Then let $S_1 = S_0 \cup \{x_0\} \cup \{a_1, a_2\}$, where x_0 is an arbitrary atom in X_A. Furthermore, let $X_S = X_A \cap S_1$, and finally, let $S = S_1 \cup \{\Diamond x \mid x \in X_S\}$. Define $\mathbf{B}_S$ as the Boolean subalgebra of $\mathfrak{A}$ generated by S. Since S is a finite subset of A, $\mathbf{B}_S$ is finite. Also, $\mathbf{B}_S$ clearly preserves all Boolean operations. Further, define a relation R on $At\mathbf{B}_S$ by

$$\forall x \forall y \in At\mathbf{B}_S (xRy \iff \Diamond x \leq \Diamond y),$$

and let

$$\Diamond^S b = \bigvee \{x \in At\mathbf{B}_S \mid \exists y \in At\mathbf{B}_S (y \leq b \text{ and } xRy)\}. \tag{5.1}$$

Then consider the structure $\mathfrak{B}_S = (\mathbf{B}_S, \Diamond^S, X_S)$. Note that $X_S \neq \varnothing$ since $x_0 \in X_S$. If we can now show that R satisfies condition (R), we know that $\Diamond^S$ is a normal operator extending $\Diamond$, and hence that $\mathfrak{B}_S \not\models \varphi \approx \top$. So let $\underline{S}$ be the subset of S satisfying $\Diamond b \in S$ whenever $b \in \underline{S}$, and let $b \in \underline{S}$ and $x \in At\mathbf{B}_S$. For the left-to-right direction, assume $x \leq \Diamond b$. But since $b \in \mathbf{B}_S$, we have

$$x \leq \Diamond b = \Diamond \bigvee \{y \in At\mathbf{B}_S \mid y \leq b\} = \bigvee \{\Diamond y \mid y \in At\mathbf{B}_S \text{ and } y \leq b\}.$$

Hence, $x \leq \Diamond y_0$ for some $y_0 \leq b$. But $\Diamond x \leq \Diamond\Diamond y_0 = \Diamond y_0$, so xRy_0. For the converse, assume $y_0 \leq b$ and $\Diamond x \leq \Diamond y_0$. From the first inequality we get $\Diamond y_0 \leq \Diamond b$, while from the second inequality we have $x \leq \Diamond y_0$ by (*refl*). Therefore, $x \leq \Diamond b$, as required, and so $\mathfrak{B}_S \not\models \varphi \approx \top$. Finally, by Lemma 5.1.13 below, $\mathfrak{B}_S$ can be embedded into $\mathfrak{A}$, so $\mathfrak{B}_S \models \mathbf{HS4.3}\Sigma^{\approx}$.

We conclude this proof with a calculation of an upper bound for the number of elements in the algebra $\mathfrak{B}_S$. First, let us calculate an upper bound for the number of elements in S. Let $l(\varphi)$ be the sum of the number of different propositional variables and different nominals in φ. From here on we will refer to this as the *length* of φ. Then S_0 contains at most $l(\varphi)+1$ elements. This means that S_1 has at most $l(\varphi) + 1 + 2 + 1 = l(\varphi) + 4$ elements. Now, we also know that $X \cap S_1$ contains at most $l(\varphi) + 1$ atoms, so $\{\Diamond x \mid x \in X_B\}$ contains at most $l(\varphi) + 1$ elements. Therefore,

$$|S| \leq l(\varphi) + 4 + l(\varphi) + 1 = 2l(\varphi) + 5.$$

We can thus conclude that $\mathfrak{B}_S$ contains at most $2^{2l(\varphi)+5}$ atoms, and hence, at most $2^{2^{2l(\varphi)+5}}$ elements. $\square$

For the lemma below, let $\mathfrak{A}$ and $\mathfrak{B}_S$ be the hybrid algebras in the proof of Theorem 5.1.12. In this lemma, we borrow Bull's map in his Lemma 4 in [20]. However, we have to modify this map to accommodate the designated atoms and 'pieces' of our hybrid algebra, which in turn means that we have to do a lot more work to show that this map is indeed an embedding, as we will soon see.

Lemma 5.1.13. *$\mathfrak{B}_S$ can be embedded into $\mathfrak{A}$.*

Proof. Let $b_1, b_2, \ldots, b_n$ be the atoms of $\mathfrak{B}_S$. We have to make sure that the pieces a_1 and a_2 form a partition of the atoms of $\mathfrak{B}_S$, so let $b_j \in At\mathfrak{B}_S$ such that $b_j \leq a_1$ and $b_j \leq a_2$. Then $b_j \leq a_1 \wedge a_2 = \perp$, contradicting the fact that b_j is an atom. Now, let $b^1_1, b^1_2, \ldots, b^1_{n_1}, b^2_1, \ldots, b^2_{n_2}$ be the atoms of $\mathfrak{B}_S$ such that $b^1_1, b^1_2, \ldots, b^1_{n_1} \leq a_1$ and $b^2_1, \ldots, b^2_{n_2} \leq a_2$. Appealing to Lemma 5.1.11, we choose to index the atoms in such a way that in their indexed order

$$\Diamond b^i_{k(1)} = \cdots = \Diamond b^i_{k(2)-1} < \Diamond b^i_{k(2)} = \cdots = \Diamond b^i_{k(3)-1} < \cdots < \Diamond b^i_{k(m_i)} = \cdots = \Diamond b^i_{k(m_i+1)-1}$$

for $i = 1, 2$ and $1 = k(1) < k(2) < \cdots < k(m_i + 1) = n_i + 1$. For convenience, define $b^i_{k(0)} = \perp$ for $i = 1, 2$. Furthermore, let $X^i_j = \left\{b_{k(j)}, b_{k(j)+1}, \ldots, b_{k(j+1)-1}\right\} \cap X_S$ and $Y^i_j = X_S \backslash X^i_j$ for $i = 1, 2$. Consider the map θ: $B_S \to A$ defined as follows:

(i) $\theta(\perp) = \perp$

(ii) For each $i = 1, 2$ and $1 \leq j \leq m_i$,

$$\theta\left(b^i_{k(j)}\right) = \begin{cases} b^i_{k(j)} \text{ if } b^i_{k(j)} \in X_S \\ \Diamond b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee b^i_{k(j)+2} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)} & \text{otherwise.} \end{cases}$$

(iii) For $i = 1, 2$ and $k(j) + 1 \leq k(j) + l \leq k(j + 1) - 1$,

$$\theta\left(b^i_{k(j)+l}\right) = \begin{cases} b^i_{k(j)+l} \text{ if } b^i_{k(j)+l} \in X_S \\ b^i_{k(j)+l} \wedge \neg\Diamond b^i_{k(j-1)} \text{ otherwise.} \end{cases}$$

(iv) For any $b \in B_S$,

$$\theta(b) = \bigvee_{b_j \leq b} \theta(b_j).$$

First, θ obviously respects $\perp$, and furthermore, it maps designated atoms of X_S to designated atoms of X_A. We break the rest of the proof up into claims.

Claim 1. For all $i = 1, 2$ and $1 \leq j \leq m_i$,

$$\Diamond^S b^i_{k(j)} = \bigvee\left\{b^i_{k(1)}, b^i_{k(1)+1}, \ldots, b^i_{k(2)-1}, b^i_{k(2)}, \ldots, b^i_{k(j)}, b^i_{k(j)+1}, \ldots, b^i_{k(j+1)-1}\right\}.$$

Proof of claim. Since $\Diamond^S b^i_{k(j)} = \bigvee\left\{b_l \in At\mathfrak{B}_S \mid \Diamond b_l \leq \Diamond b^i_{k(j)}\right\}$, we have to prove that

$$\left\{b_l \in At\mathfrak{B}_S \mid \Diamond b_l \leq \Diamond b^i_{k(j)}\right\} = \left\{b^i_{k(1)}, b^i_{k(1)+1}, \ldots, b^i_{k(2)-1}, b^i_{k(2)}, \ldots, b^i_{k(j)}, b^i_{k(j)+1}, \ldots, b^i_{k(j+1)-1}\right\}$$

for each $i = 1, 2$ and $1 \leq j \leq m_i$. To prove the right-to-left inclusion, note that for each $k(1) \leq k(j) + l \leq k(j + 1) - 1$, $\Diamond b^i_{k(j)+l} \leq \Diamond b^i_{k(j)}$ by our ordering on the atoms, so we have $b^i_{k(j)+l} \in \left\{b_l \in At\mathfrak{B}_S \mid \Diamond b_l \leq \Diamond b^i_{k(j)}\right\}$. For the converse, let $b_{l_0} \in \left\{b_l \in At\mathfrak{B}_S \mid \Diamond b_l \leq \Diamond b^i_{k(j)}\right\}$. First, we show that $b_{l_0} \leq a_i$. Now, we claim that $\Diamond b^i_{k(j)} \leq a_i$. To see this, note that

$b^i_{k(j)} \leq a_i$, so $\Diamond b^i_{k(j)} \leq \Diamond a_i$ by the monotonicity of $\Diamond$. But by the definition of piecewise well-connectedness, $\Diamond a_i = a_i$, so $\Diamond b^i_{k(j)} \leq a_i$. Since $b_{l_0} \in \left\{ b_l \in At\mathfrak{B}_S \mid \Diamond b_l \leq \Diamond b^i_{k(j)} \right\}$, we have $\Diamond b_{l_0} \leq \Diamond b^i_{k(j)}$. Hence, by (*refl*), $b_{l_0} \leq a_i$. To show that $l_0 = k(j) + l$ for some $k(1) \leq k(j) + l \leq k(j+1) - 1$, suppose $l_0 \geq k(j+1)$. Then $\Diamond b_{l_0} > \Diamond b^i_{k(j)}$ by our ordering. But we also have that $\Diamond b_{l_0} \leq \Diamond b^i_{k(j)}$, so we have a contradiction.

Claim 2. For each $i = 1, 2$, $1 \leq j \leq m_i$ and $k(j)+1 \leq k(j)+l \leq k(j+1)-1$, $b^i_{k(j)+l} \nleq \Diamond b^i_{k(j-1)}$.

Proof of claim. For the sake of a contradiction, suppose that $b^i_{k(j)+l} \leq \Diamond b^i_{k(j-1)}$. Then $\Diamond b^i_{k(j)+l} \leq \Diamond\Diamond b^i_{k(j-1)} = \Diamond b^i_{k(j-1)}$ by the monotonicity of $\Diamond$ and (*trans*). However, this contradicts our ordering.

Claim 3. For each $i = 1, 2$ and $1 \leq j \leq m_i$,

$$\theta\left(b^i_{k(j)}\right) \vee \theta\left(b^i_{k(j)+1}\right) \vee \cdots \vee \theta\left(b^i_{k(j+1)-1}\right) = \Diamond b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)}.$$

Proof of claim. We consider the following cases:

Case 1: $b^i_{k(j)}, b^i_{k(j)+1}, \ldots, b^i_{k(j+1)-1}$ are all non-designated. Then

$$\begin{aligned}
& \theta\left(b^i_{k(j)}\right) \vee \theta\left(b^i_{k(j)+1}\right) \vee \cdots \vee \theta\left(b^i_{k(j+1)-1}\right) \\
= \; & \left(\Diamond b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)}\right) \vee \left(b^i_{k(j)+1} \wedge \neg\Diamond b^i_{k(j-1)}\right) \vee \cdots \vee \\
& \left(b^i_{k(j+1)-1} \wedge \neg\Diamond b^i_{k(j-1)}\right) \\
= \; & \left(\left(\Diamond b^i_{k(j)} \wedge \neg(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1})\right) \vee (b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1})\right) \wedge \neg\Diamond b^i_{k(j-1)} \\
= \; & \left(\Diamond b^i_{k(j)} \vee b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b_{k(j-1)} \\
= \; & \Diamond b^i_{k(j)} \wedge \neg\Diamond b_{k(j-1)},
\end{aligned}$$

where the last equality follows from the fact that for all $k(j) + 1 \leq k(j) + l \leq k(j+1) - 1$, $b^i_{k(j)+l} \leq \Diamond b^i_{k(j)}$.

Case 2: $b^i_{k(j)}, b^i_{k(j)+1}, \ldots, b^i_{k(j+1)-1}$ is a mixture of designated and non-designated atoms. We may assume without loss of generality that $b^i_{k(j)}$ is non-designated (just modify the indexes). We now show that we get $\theta\left(b^i_{k(j)+l}\right) = b^i_{k(j)+l} \wedge \neg\Diamond b^i_{k(j-1)}$ for all $i = 1, 2$, $1 \leq j \leq m_i$ and $k(j) + 1 \leq k(j) + l \leq k(j+1) - 1$. We know from Claim 2 that $b^i_{k(j)+l} \nleq \Diamond b^i_{k(j-1)}$. So if $b^i_{k(j)+l} \in X_S$, $b^i_{k(j)+l} \in At\mathfrak{A}$, which means that $b^i_{k(j)+l} \leq \neg\Diamond b^i_{k(j-1)}$. Hence, for all $i = 1, 2$, $1 \leq j \leq m_i$ and $k(j) + 1 \leq k(j) + l \leq k(j+1) - 1$ such that $b^i_{k(j)+l} \in X_S$,

$$\theta\left(b^i_{k(j)+l}\right) = b^i_{k(j)+l} = b_{k(j)+l} \wedge \neg\Diamond b_{k(j-1)}.$$

On the other hand, if $b^i_{k(j)+l} \notin X_S$, we know from the definition of θ that

$$\theta\left(b^i_{k(j)+l}\right) = b^i_{k(j)+l} \wedge \neg\Diamond b^i_{k(j-1)}$$

Now, as in Case 1,

$$\theta\left(b^i_{k(j)}\right) \vee \theta\left(b^i_{k(j)+1}\right) \vee \cdots \vee \theta\left(b^i_{k(j+1)-1}\right) = \Diamond b^i_{k(j)} \wedge \neg\Diamond b_{k(j-1)},$$

Case 3: $b^i_{k(j)}, b^i_{k(j)+1}, \ldots, b^i_{k(j+1)-1}$ are all designated. By the definition of S, $\Diamond b^i_{k(j)} \in \mathfrak{B}$, so $\Diamond^S b^i_{k(j)} = \Diamond b^i_{k(j)}$, which means that

$$\Diamond b^i_{k(j)} = \bigvee\left\{b_l \in At\mathfrak{B}_S \mid \Diamond b_l \leq \Diamond b^i_{k(j)}\right\}.$$

Hence, by Claim 1,

$$\Diamond b^i_{k(j)} = \bigvee\left\{b^i_{k(1)}, b^i_{k(1)+1}, \ldots, b^i_{k(2)-1}, b^i_{k(2)}, \ldots, b^i_{k(i-1)}, \ldots, b^i_{k(j)}, b^i_{k(j)+1}, \ldots, b^i_{k(j+1)-1}\right\},$$

and so, since

$$\bigvee\left\{b^i_{k(1)}, b^i_{k(1)+1}, \ldots, b^i_{k(2)}, \ldots, b^i_{k(j)}, b^i_{k(j)+1}, \ldots, b^i_{k(j+1)-1}\right\} \leq \Diamond b^i_{k(j-1)} \vee b^i_{k(j)} \vee \cdots \vee b^i_{k(j+1)-1},$$

$\Diamond b^i_{k(j)} \leq \Diamond b^i_{k(j-1)} \vee b^i_{k(j)} \vee \cdots \vee b^i_{k(j+1)-1}$. But $\Diamond b^i_{k(j-1)} \vee b^i_{k(j)} \vee \cdots \vee b^i_{k(j+1)-1} \leq \Diamond b^i_{k(j)}$, which gives $\Diamond b^i_{k(j)} = \Diamond b^i_{k(j-1)} \vee b^i_{k(j)} \vee \cdots \vee b^i_{k(j+1)-1}$. Hence,

$$\begin{aligned}\Diamond b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)} &= \left(\Diamond b^i_{k(j-1)} \vee b^i_{k(j)} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)} \\ &= \left(b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)}\right) \vee \cdots \vee \left(b^i_{k(j+1)-1} \wedge \neg\Diamond b^i_{k(j-1)}\right).\end{aligned}$$

But by Claim 2, $b^i_{k(j)+l} \not\leq \Diamond b^i_{k(j-1)}$, so since $b^i_{k(j)+l} \in At\mathfrak{A}$, $b^i_{k(j)+l} \leq \neg\Diamond b^i_{k(j-1)}$. We therefore get

$$b^i_{k(j)} \vee \cdots \vee b^i_{k(j+1)-1} = \Diamond b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)}.$$

Thus,

$$\begin{aligned} & \theta\left(b^i_{k(j)}\right) \vee \theta\left(b^i_{k(j)+1}\right) \vee \cdots \vee \theta\left(b^i_{k(j+1)-1}\right) \\ = \; & b^i_{k(j)} \vee b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1} \\ = \; & \Diamond b_{k(i)} \wedge \neg\Diamond b_{k(i-1)}.\end{aligned}$$

Claim 4. For each $i = 1, 2$,

$$\bigvee_{1 \leq j \leq m_i}\left(\Diamond b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)}\right) = \Diamond b^i_{k(m_i)}.$$

Proof of claim. Let $i = 1, 2$. Then we have

$$\begin{aligned} & \bigvee_{1\leq j\leq m_i} \left(\Diamond b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)}\right) \\ = & \bigvee_{1\leq j\leq m_i} \Diamond b^i_{k(j)} \\ = & \Diamond b^i_{k(m_i)}, \end{aligned}$$

where the last step follows from our ordering.

Claim 5. $\theta(b_1), \theta(b_2), \ldots, \theta(b_n)$ cover $\mathfrak{A}$.

Proof of claim. Since the elements $b^i_1, b^i_2, \ldots, b^i_{n_i}$ cover a_i (i.e. $\bigvee_{1\leq j\leq n_i} b^i_j = a_i$), and, for every $1 \leq j \leq n_i$, $\Diamond b^i_j \leq \Diamond b^i_{k(m_i)}$ by the choice of indexing,

$$\begin{aligned} \Diamond b^i_{k(m_i)} & \geq \bigvee_{1\leq j\leq n_i} \Diamond b^i_j \\ & = \Diamond \bigvee_{1\leq j\leq n_i} b^i_j \\ & = \Diamond a_i \\ & \geq a_i \end{aligned}$$

But we also have $b^i_{k(m_i)} \leq a_i$, so $\Diamond b^i_{k(m_i)} \leq \Diamond a_i = a_i$, which means that $\Diamond b^i_{k(m_i)} = a_i$. Hence, by Claim 4,

$$\bigvee_{1\leq j\leq m_i} \left(\Diamond b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)}\right) = \Diamond b^i_{k(m_i)} = a_i,$$

and so

$$\theta(b_1) \vee \theta(b_2) \vee \cdots \vee \theta(b_n) = \bigvee_{i=1,2} a_i = \top.$$

Claim 6. For each $i = 1, 2$, if $b^i_{k(j)} \notin X_S$, then

$$\theta(b^i_{k(j)}) = \Diamond b^i_{k(j)} \wedge \neg \left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)} \wedge \left(\bigwedge_{x\in Y^i_j} \neg x\right).$$

Proof of claim. First, assume $x \in X^i_1 \cup \cdots \cup X^i_{j-1}$. Then $\Diamond x \leq \Diamond b^i_{k(j-1)}$ by the ordering, so $\neg\Diamond b^i_{k(j-1)} \leq \neg x$. On the other hand, if $x \in X^i_{j+1} \cup \cdots \cup X^i_{m_i}$, then $\Diamond b^i_{k(j)} \wedge x = \bot$. To see this, suppose $\Diamond b^i_{k(j)} \wedge x > \bot$. But since $x \in At\mathfrak{A}$, $x \leq \Diamond b^i_{k(j)}$, which means that $\Diamond x \leq \Diamond\Diamond b^i_{k(j)} = \Diamond b^i_{k(j)}$, contradicting our ordering. Hence, $\Diamond b^i_{k(j)} \leq \neg x$. Finally, if $x \in X^{i'}_{j'}, i \neq i'$,

$\Diamond b^i_{k(j)} \wedge x \leq a_i \wedge a_{i'} = \bot$. This means we also get $\Diamond b^i_{k(j)} \leq \neg x$. We therefore have

$$\begin{aligned} & \theta\left(b^i_{k(j)}\right) \\ = \ & \Diamond b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)} \\ = \ & \Diamond b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)} \wedge \left(\bigwedge_{x \in Y^i_j} \neg x\right). \end{aligned}$$

Claim 7. $\theta(b_1), \theta(b_2), \ldots, \theta(b_n)$ are pairwise disjoint.

Proof of claim. Let $b^i_l, b^{i'}_{l'} \in At\mathfrak{B}_S$. We break the proof of this claim into cases. But first note that for each $k(j) \leq k(j) + l \leq k(j+1) - 1$, $b^i_{k(j)+l} \leq \Diamond b^i_{k(j)+l} \leq a_i$, so since $a_i \wedge a_{i'} = \bot$ for $i \neq i'$, we need only to consider the cases where $i = i'$.

Case 1: $b^i_l, b^i_{l'} \in X_S$, $b^i_l = b^i_{k(j)}$ and $b^i_{l'} = b^i_{k(j')}$. Then

$$\begin{aligned} & \theta(b^i_l) \wedge \theta(b^i_{l'}) \\ = \ & \theta\left(b^i_{k(j)}\right) \wedge \theta\left(b^i_{k(j')}\right) \\ = \ & b^i_{k(j)} \wedge b^i_{k(j')} \\ = \ & \bot. \end{aligned}$$

Case 2: $b^i_{l'} \in X_S, b^i_l \notin X_S, b^i_l = b^i_{k(j)}$ and $b^i_{l'} = b^i_{k(j')}$. We then have that $b^i_{k(j')} \in Y^i_j$, so

$$\begin{aligned} & \theta(b^i_{l'}) \wedge \theta(b^i_l) \\ = \ & \theta\left(b^{i'}_{k(j')}\right) \wedge \theta\left(b^i_{k(j)}\right) \\ = \ & b^i_{k(j')} \wedge \Diamond b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)} \wedge \left(\bigwedge_{x \in Y^i_j} \neg x\right) \\ \leq \ & b^i_{k(j')} \wedge \neg b^i_{k(j')} \\ = \ & \bot, \end{aligned}$$

where the first equality follows from Claim 6.

Case 3: $b^i_l, b^i_{l'} \in X_S$, $b^i_l = b^i_{k(j)}$ and $b^i_{l'} = b^i_{k(j')+r'}$. Then

$$\begin{aligned} & \theta(b^i_l) \wedge \theta(b^i_{l'}) \\ = \ & \theta\left(b^i_{k(j)}\right) \wedge \theta\left(b^i_{k(j')+r'}\right) \\ = \ & b^i_{k(j)} \wedge b^i_{k(j')+r'} \\ = \ & \bot. \end{aligned}$$

Case 4: $b^i_{l'} \in X_S, b^i_l \notin X_S, b^i_l = b^i_{k(j)+r}$ and $b^i_{l'} = b^i_{k(j')}$. We then have

$$\begin{aligned} & \theta(b^i_{l'}) \wedge \theta(b^i_l) \\ = \ & \theta\left(b^i_{k(j')}\right) \wedge \theta\left(b^i_{k(j)+r}\right) \\ = \ & b^i_{k(j')} \wedge b^i_{k(j)+r} \wedge \neg\Diamond b^i_{k(j-1)} \\ = \ & \bot \end{aligned}$$

Case 5: $b^i_l, b^i_{l'} \notin X_S, b^i_l = b^i_{k(j)}$ and $b^i_{l'} = b^i_{k(j')}$. First, if we have $\Diamond b^i_{k(j')} < \Diamond b^i_{k(j)}$, then $\Diamond b^i_{k(j')} \leq \Diamond b^i_{k(j-1)}$. Hence,

$$\begin{aligned} & \theta(b^i_l) \wedge \theta(b^i_{l'}) \\ = \ & \theta\left((b^i_{k(j)}\right) \wedge \theta\left(b^i_{k(j')}\right) \\ = \ & \left(\Diamond b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)}\right) \wedge \\ & \left(\Diamond b^i_{k(j')} \wedge \neg\left(b^i_{k(j')+1} \vee \cdots \vee b^i_{k(j'+1)-1}\right) \wedge \neg\Diamond b^i_{k(j'-1)}\right) \\ \leq \ & \neg\Diamond b^i_{k(j-1)} \wedge \Diamond b^i_{k(j-1)} \\ = \ & \bot. \end{aligned}$$

On the other hand, if $\Diamond b^i_{k(j)} < \Diamond b^i_{k(j')}$, then $\Diamond b^i_{k(j)} \leq \Diamond b^i_{k(j'-1)}$. So

$$\begin{aligned} & \theta(b^i_l) \wedge \theta(b^i_{l'}) \\ = \ & \theta\left(b^i_{k(j)}\right) \wedge \theta\left(b^i_{k(j')}\right) \\ = \ & \left(\Diamond b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)}\right) \wedge \\ & \left(\Diamond b^i_{k(j')} \wedge \neg\left(b^i_{k(j')+1} \vee \cdots \vee b^i_{k(j'+1)-1}\right) \wedge \neg\Diamond b^i_{k(j'-1)}\right) \\ \leq \ & \Diamond b^i_{k(j'-1)} \wedge \neg\Diamond b^i_{k(j'-1)} \\ = \ & \bot. \end{aligned}$$

Case 6: $b^i_l \notin X_S, b^i_{l'} \in X_S, b^i_l = b^i_{k(j)}$ and $b^i_{l'} = b^i_{k(j')+r'}$. We know that $b^i_{k(j')+r'} \in Y^i_j$, so

$$\begin{aligned} & \theta(b^i_l) \wedge \theta(b^i_{l'}) \\ = \ & \theta\left(b^i_{k(j)}\right) \wedge \theta\left(b^i_{k(j')}\right) \\ = \ & \Diamond b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)} \wedge \left(\bigwedge_{x \in Y^i_j} \neg x\right) \wedge b^i_{k(j')+r'} \\ \leq \ & \neg b^i_{k(j')+r'} \wedge b^i_{k(j')+r'} \\ = \ & \bot \end{aligned}$$

Case 7: $b^i_l, b^i_{l'} \notin X_S, b^i_l = b^i_{k(j)}$ and $b^i_{l'} = b^i_{k(j')+r'}$. If we have $\Diamond b^i_{k(j')} < \Diamond b^i_{k(j)}$, then $\Diamond b^i_{k(j')} \leq \Diamond b^i_{k(j-1)}$. Hence,

$$\begin{aligned}
& \theta(b^i_l) \wedge \theta(b^i_{l'}) \\
= \;& \theta\left(b^i_{k(j)}\right) \wedge \theta\left(b^i_{k(j')+r'}\right) \\
= \;& \left(\Diamond b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)}\right) \wedge \left(b^i_{k(j')+r'} \wedge \neg\Diamond b^i_{k(j'-1)}\right) \\
\leq \;& \neg\Diamond b^i_{k(j-1)} \wedge \Diamond b^i_{k(j')+r'} \\
= \;& \neg\Diamond b^i_{k(j-1)} \wedge \Diamond b^i_{k(j')} \\
\leq \;& \neg\Diamond b^i_{k(j-1)} \wedge \Diamond b^i_{k(j-1)} \\
= \;& \bot.
\end{aligned}$$

On the other hand, if $\Diamond b^i_{k(j)} < \Diamond b^i_{k(j')}$, then $\Diamond b^i_{k(j)} \leq \Diamond b^i_{k(j'-1)}$. So

$$\begin{aligned}
& \theta(b^i_l) \wedge \theta(b^i_{l'}) \\
= \;& \theta\left(b^i_{k(j)}\right) \wedge \theta\left(b^i_{k(j')+r'}\right) \\
= \;& \left(\Diamond b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)}\right) \wedge \left(b^i_{k(j')+r'} \wedge \neg\Diamond b^i_{k(j'-1)}\right) \\
\leq \;& \Diamond b^i_{k(j'-1)} \wedge \neg\Diamond b^i_{k(j'-1)} \\
= \;& \bot.
\end{aligned}$$

Case 8: $b^i_l, b^i_{l'} \in X_S, b^i_l = b^i_{k(j)+r}$ and $b^i_{l'} = b^i_{k(j')+r'}$. Then we have

$$\begin{aligned}
& \theta(b^i_l) \wedge \theta(b^i_{l'}) \\
= \;& \theta\left(b^i_{k(j)+r}\right) \wedge \theta\left(b^i_{k(j')+r'}\right) \\
= \;& b^i_{k(j)+r} \wedge b^i_{k(j')+r'} \\
= \;& \bot.
\end{aligned}$$

Case 9: $b^i_l \in X_S, b^i_{l'} \notin X_S, b^i_l = b^i_{k(j)+r}$ and $b^i_{l'} = b^i_{k(j')+r'}$. Then

$$\begin{aligned}
& \theta(b^i_l) \wedge \theta(b^i_{l'}) \\
= \;& \theta\left(b^i_{k(j)+r}\right) \wedge \theta\left(b^i_{k(j')+r'}\right) \\
= \;& b^i_{k(j)+r} \wedge \left(b^i_{k(j')+r'} \wedge \neg\Diamond b^i_{k(j')+r'}\right) \\
= \;& \bot.
\end{aligned}$$

Case 10: $b^i_l, b^i_{l'} \notin X_S, b^i_l = b^i_{k(j)+r}$ and $b^i_{l'} = b^i_{k(j')+r'}$. Here we have

$$\begin{aligned}
& \theta(b^i_l) \wedge \theta(b^i_{l'}) \\
= \; & \theta\left(b^i_{k(j)+r}\right) \wedge \theta\left(b^i_{k(j')+r'}\right) \\
= \; & \left(b^i_{k(j)+r} \wedge \neg\Diamond b^i_{k(j)+r}\right) \wedge \left(b^i_{k(j')+r'} \wedge \neg\Diamond b^i_{k(j')+r'}\right) \\
= \; & \bot.
\end{aligned}$$

Claim 8. For any $b \in B_S$, $\theta(\neg b) = \neg\theta(b)$.

Proof of claim.

$$\begin{aligned}
\neg\theta(b) &= \neg\left(\bigvee_{b_j \leq b} \theta(b_j)\right) \\
&= \top \wedge \neg\left(\bigvee_{b_j \leq b} \theta(b_j)\right) \\
&= \bigvee_{1 \leq i \leq n} \theta(b_i) \wedge \neg\left(\bigvee_{b_j \leq b} \theta(b_j)\right) \\
&= \bigvee_{1 \leq i \leq n} \theta(b_i) \wedge \bigwedge_{b_j \leq b} \neg\theta(b_j) \\
&= \bigvee_{b_j \leq b} (\theta(b_j) \wedge \neg\theta(b_j)) \vee \bigvee_{b_i \not\leq b} \left(\theta(b_i) \wedge \bigwedge_{b_j \leq b} \neg\theta(b_j)\right) \\
&= \bot \vee \bigvee_{b_i \not\leq b} \theta(b_i) \\
&= \bigvee_{b_i \not\leq b} \theta(b_i) \\
&= \bigvee_{b_i \leq \neg b} \theta(b_i) \\
&= \theta(\neg b)
\end{aligned}$$

Here the first part of the fifth equality and the second part of the sixth equality follow from the fact that for $i \neq j$, $\theta(b_i) \wedge \theta(b_j) = \bot$, and hence that $\theta(b_i) \leq \neg\theta(b_j)$. The second-to-last equality follows from the fact that b_i is an atom of $\mathfrak{B}_S$.

Claim 9. For any $a, b \in B_S$, $\theta(a \vee b) = \theta(a) \vee \theta(b)$.

Proof of claim. We first show that

$$\{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq a \vee b\} = \{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq a\} \cup \{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq b\} :$$

$$\begin{aligned}
& x \in \{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq a \vee b\} \\
\Longleftrightarrow \;& \exists b_{i_0} \in At\mathfrak{B}_S(x = \theta(b_{i_0}) \text{ and } b_{i_0} \leq a \vee b) \\
\Longleftrightarrow \;& \exists b_{i_0} \in At\mathfrak{B}_S(x = \theta(b_{i_0}) \text{ and } (b_{i_0} \leq a \text{ or } b_{i_0} \leq b)) \\
\Longleftrightarrow \;& \exists b_{i_0} \in At\mathfrak{B}_S((x = \theta(b_{i_0}) \text{ and } b_{i_0} \leq a) \text{ or } (x = \theta(b_{i_0}) \text{ and } b_{i_0} \leq b)) \\
\Longleftrightarrow \;& \exists b_{i_0} \in At\mathfrak{B}_S(x = \theta(b_{i_0}) \text{ and } b_{i_0} \leq a) \text{ or } \exists b_{i_0} \in At\mathfrak{B}_S(x = \theta(b_{i_0}) \text{ and } b_{i_0} \leq b) \\
\Longleftrightarrow \;& x \in \{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq a\} \text{ or } x \in \{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq b\} \\
\Longleftrightarrow \;& x \in \{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq a\} \cup \{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq b\}.
\end{aligned}$$

But then

$$\begin{aligned}
& \theta(a \vee b) \\
= \;& \bigvee \{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq a \vee b\} \\
= \;& \bigvee \left(\{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq a\} \cup \{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq b\}\right) \\
= \;& \bigvee \{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq a\} \vee \bigvee \{\theta(b_i) \mid b_i \in At\mathfrak{B}_S \;\&\; b_i \leq b\} \\
= \;& \theta(a) \vee \theta(b).
\end{aligned}$$

Claim 10. For each $i = 1, 2$, $1 \leq j \leq m_i$ and $k(j) \leq k(j) + l \leq k(j+1) - 1$,

$$\Diamond\theta\left(b^i_{k(j)+l}\right) = \Diamond\left(b^i_{k(j)+l} \wedge \neg\Diamond b^i_{k(j-1)}\right).$$

Proof of claim. Consider the following two cases:

Case 1: $b^i_{k(j)}$ is not designated. We first prove the left-to-right inequality:

$$\begin{aligned}
\theta\left(b^i_{k(j)}\right) &= \Diamond b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)} \\
&\leq \Diamond b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)}.
\end{aligned}$$

For the right-to-left inequality, note that we have $b^i_{k(j)} \wedge b^i_{k(j)+l} = \bot$ for each $k(j) + 1 \leq k(j) + l \leq k(j+1) - 1$), so $b^i_{k(j)} \leq \neg b^i_{k(j)+l}$. Hence,

$$\begin{aligned}
\theta\left(b^i_{k(j)}\right) &= \Diamond b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)} \\
&\geq b^i_{k(j)} \wedge \neg\left(b^i_{k(j)+1} \vee \cdots \vee b^i_{k(j+1)-1}\right) \wedge \neg\Diamond b^i_{k(j-1)} \\
&= b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)}.
\end{aligned}$$

We thus have

$$\Diamond\left(b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)}\right) \leq \Diamond\theta\left(b^i_{k(j)}\right) \leq \Diamond\left(\Diamond b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)}\right) = \Diamond\left(b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)}\right),$$

where the equality follows from Lemma 5.1.11. Therefore,

$$\Diamond\theta\left(b_{k(i)}\right) = \Diamond\left(b_{k(i)} \wedge \neg\Diamond b_{k(i-1)}\right).$$

Case 2: $b^i_{k(j)}$ is designated. Since $b^i_{k(j)}, b^i_{k(j-1)} \leq a_i$ and $\Diamond b^i_{k(j-1)} < \Diamond b^i_{k(j)}$, we have

$$\Diamond\theta\left(b^i_{k(j)}\right) = \Diamond b^j_{k(j)} = \Diamond\left(b^i_{k(j)} \wedge \neg\Diamond b^i_{k(j-1)}\right)$$

by Lemma 5.1.11.

Case 3: $b^i_{k(j)+l}$ is not designated. Here we have $\Diamond\theta\left(b^i_{k(j)+l}\right) = \Diamond\left(b^i_{k(j)+l} \wedge \neg\Diamond b^i_{k(j-1)}\right)$ by the definition of θ.

Case 4: $b^i_{k(j)+l}$ is designated. Since $\Diamond b^i_{k(j)+l} = \Diamond b^i_{k(j)}$ and $\Diamond b^i_{k(j-1)} < \Diamond b^i_{k(j)}$, we have $\Diamond b^i_{k(j-1)} < \Diamond b^i_{k(j)+l}$. Hence,

$$\Diamond\theta\left(b^i_{k(j)+l}\right) = \Diamond b^i_{k(j)+l} = \Diamond\left(b^i_{k(j)+l} \wedge \neg\Diamond b^i_{k(j-1)}\right)$$

by Lemma 5.1.11.

Claim 11. For each $i = 1, 2$, $1 \leq j \leq m_i$ and $k(j) \leq k(j) + l \leq k(j+1) - 1$,

$$\Diamond\theta\left(b^i_{k(j)+l}\right) = \theta\left(\Diamond^S b^i_{k(j)+l}\right).$$

Proof of claim.

$$\begin{aligned}
\Diamond\theta(b^i_{k(j)+l}) &= \Diamond\left(b^i_{k(j)+l} \wedge \neg\Diamond b^i_{k(j-1)}\right) && \text{(by Claim 10)}\\
&= \Diamond b^i_{k(j)+l} && \text{(by Lemma 5.1.11)}\\
&= \Diamond b^i_{k(j)} && \text{(by our choice of indexing)}\\
&= \bigvee_{1\leq r\leq j} \left(\Diamond b_{k(r)} \wedge \neg\Diamond b_{k(r-1)}\right) && \text{(by Claim 4)}\\
&= \bigvee_{1\leq r\leq j} \left(\theta\left(b^i_{k(r)}\right) \vee \theta\left(b^i_{k(r)+1}\right) \vee \cdots \vee \theta\left(b^i_{k(r+1)-1}\right)\right) && \text{(by Claim 3)}\\
&= \theta\left(\bigvee_{1\leq r\leq j} \left(b^i_{k(r)} \vee b^i_{k(r)+1} \vee \cdots \vee b^i_{k(r+1)-1}\right)\right) && \text{(by Claim 9)}\\
&= \theta\left(\Diamond^S b^i_{k(j)+l}\right) && \text{(by Claim 1)}
\end{aligned}$$

Claim 12. For any $b \in B_S$, $\Diamond\theta(b) = \theta(\Diamond^S b)$.

Proof of claim. First, for $b = \bot$,

$$\theta(\Diamond^S(\bot)) = \theta(\bot) = \bot = \Diamond\bot = \Diamond\theta(\bot).$$

Now, assume $b > \bot$. Then

$$\begin{aligned}
\Diamond\theta(b) &= \Diamond \bigvee_{b_j \leq b} \theta(b_j) && \text{(by the definition of } \theta\text{)} \\
&= \bigvee_{b_j \leq b} \Diamond\theta(b_j) && \text{(since } \Diamond \text{ is a normal modal operator)} \\
&= \bigvee_{b_j \leq b} \theta(\Diamond^S b_j) && \text{(by Claim 11)} \\
&= \theta(\bigvee_{b_j \leq b} \Diamond^S b_j) && \text{(by Claim 9)} \\
&= \theta(\Diamond^S \bigvee_{b_j \leq b} b_j) && \text{(since } \Diamond^S \text{ is a normal modal operator)} \\
&= \theta(\Diamond^S b) && \text{(since } \mathfrak{B}_S \text{ is atomic).}
\end{aligned}$$

Claim 13. θ is injective.

Proof of claim. First, let $b_i, b_j \in At\mathfrak{B}_S$ such that $b_i \neq b_j$. But we know that $\theta(b_i) \wedge \theta(b_j) = \bot$, so $\theta(b_i) \leq \neg\theta(b_j)$. Hence, $\theta(b_i) \not\leq \theta(b_j)$, which means that $\theta(b_i) \neq \theta(b_j)$. Now, let a and b be any elements of $\mathfrak{B}_S$ such that $a \neq b$. Then

$$\bigvee_{b_i \leq a} b_i \not\leq \bigvee_{b_j \leq b} b_j \text{ or } \bigvee_{b_j \leq b} b_j \not\leq \bigvee_{b_i \leq a} b_a.$$

In the first case, there is a $b_k \leq a$ such that

$$b_k \not\leq \bigvee_{b_j \leq b} b_j,$$

and so $b_k \not\leq b_j$ for all $b_j \leq b$. Hence, $\theta(b_k) \neq \theta(b_j)$ for all $b_j \leq b$, and so $\theta(b_k) \in \{\theta(b_i) \mid b_i \leq a\}$ but $\theta(b_k) \notin \{\theta(b_j) \mid b_j \leq b\}$. This means that $\{\theta(b_i) \mid b_i \leq a\} \neq \{\theta(b_j) \mid b_j \leq b\}$, which gives

$$\bigvee \Big\{\theta(b_i) \mid b_i \leq a\} \neq \bigvee\{\theta(b_j) \mid b_j \leq b\Big\}.$$

Thus, $\theta(a) \neq \theta(b)$, as required. The proof of the other case is the same. $\square$

Theorem 5.1.12 yields decision procedures for a large number of extensions of **HS4.3** through the following corollary.

Corollary 5.1.14. *If Σ is finite, then* **HS4.3** $\oplus \Sigma$ *is decidable.*

To conclude this subsection, recall that for modal logics, a finite algebra is always complete and atomic, which means it is dual to a Kripke frame. However, for hybrid logics, although a finite hybrid algebra is also complete and atomic, not all atoms are designated, which means it is not dual to a Kripke frame. So unlike for model logics, the fact that a hybrid logic has the finite hybrid algebra property does not imply that it also have the finite model property with respect to relational models.

5.1.2 An analogue of Bull's Theorem for $\mathcal{H}(@)$

To what extent does Bull's theorem hold for extensions of the logic obtained by adding the axioms (T), (4) and (.3) to $\mathbf{H}(@)$? We will denote this logic by $\mathbf{H}(@)\mathbf{S4.3}$. It turns out that this generalization is not straightforward.

Recall that well-connectedness plays a crucial role in Bull's proof that all normal extensions of **S4.3** have the finite model property. Well-connectedness is closely related, in the dual relational semantics, to the ability to take point-generated submodels. Luckily, we have a way of simulating the process of taking generated submodels algebraically. However, as we showed on page 85, the truth of $\mathcal{H}(@)$-formulas is in general not transferred from the supermodel to the submodel when taking point-generated submodels. To ensure that the truth of a formula is transferred, we have to generate not only from the state where this formula is true, but also all states named by nominals in this formula. Unfortunately, the algebra obtained by simulating this idea algebraically need not be well-connected.

Frame-theoretically, what does this mean? Recall that any **S4.3**-frame can be turned into a rooted, transitive and connected frame by taking a point-generated subframe. However, since we cannot take point-generated subframes in this case, we do not have connectedness. So the question now is: can we enforce connectedness axiomatically? Unfortunately, it is not clear at this stage if this is indeed possible. However, the formula $@_{\mathbf{i}}\Diamond\mathbf{j} \vee @_{\mathbf{j}}\Diamond\mathbf{i}$ defines the class of two-sorted general frames $\mathfrak{g} = (W, R, A, B)$ in which R is connected on B. First, assume $\mathfrak{g} \not\Vdash @_{\mathbf{i}}\Diamond\mathbf{j} \vee @_{\mathbf{j}}\Diamond\mathbf{i}$. Then there is some admissible valuation V and a state u such that $(\mathfrak{g}, V), u \not\Vdash @_{\mathbf{i}}\Diamond\mathbf{j}$ and $(\mathfrak{g}, V), u \not\Vdash @_{\mathbf{j}}\Diamond\mathbf{i}$. But this means that $(\mathfrak{g}, V), v \not\Vdash \Diamond\mathbf{j}$ and $(\mathfrak{g}, V), w \not\Vdash \Diamond\mathbf{i}$, where $V(\mathbf{i}) = \{v\}$ and $V(\mathbf{j}) = \{w\}$. Now, since V is admissible, both v and w are in B. Furthermore, both vRw and wRv do not hold, for otherwise we have a contradiction. For the converse, suppose R is not connected on B. then there are two states v and w in B such that both vRw and wRv do not hold. Define $V(\mathbf{i}) = \{v\}$ and $V(\mathbf{j}) = \{w\}$. Clearly, V is admissible, and furthermore, $(\mathfrak{g}, V), v \not\Vdash @_{\mathbf{i}}\Diamond\mathbf{j}$ and $(\mathfrak{g}, V), v \not\Vdash @_{\mathbf{j}}\Diamond\mathbf{i}$.

The above result tells us that the elements in B can be seen as a chain of clusters or a linear order of pre-orders. For this reason, we will denote the axiom $@_{\mathbf{i}}\Diamond\mathbf{j} \vee @_{\mathbf{j}}\Diamond\mathbf{i}$ by (*lpa*), which is short for linear pre-order axiom. Furthermore, we will denote the logic obtained by adding this formula as an axiom to $\mathbf{H}(@)\mathbf{S4.3}$ by $\mathbf{LP}(@)$. More precisely:

Definition 5.1.15. The minimal hybrid logic $\mathbf{LP}(@)$ is the smallest set of formulas containing all propositional tautologies, the axioms in Table 5.2, and which is closed under the inference rules in Table 5.2.

Sadly, we have more bad news. Recall that to transfer the truth of a $\mathcal{H}(@)$-formula, we have to generate from the state where the formula is true, as well as all the states named by a nominal in this formula. So what happens if the state where the formula is true does not belong to B? Of course our axiom then cannot get a grip on this state, which means we are back to square one. We therefore have to make sure that this state is in B. One way to ensure this is to add the rules $(Name_{@})$ and $(BG_{@})$ and work with strongly descriptive general frames, or algebraically, permeated hybrid @-algebras. However, as we will show at the end of this section, the natural way of constructing a finite hybrid @-algebra from a permeated hybrid @-algebra that is also permeated fails. As a last possibility, note that if the formula

Axioms:	
$(Taut)$	$\vdash \varphi$ for all propositional tautologies φ.
(K)	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$
$(Dual)$	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$
$(K_{@})$	$\vdash @_{\mathbf{j}}(p \to q) \to (@_{\mathbf{j}}p \to @_{\mathbf{j}}q)$
$(Selfdual)$	$\vdash \neg @_{\mathbf{j}}p \leftrightarrow @_{\mathbf{j}}\neg p$
(Ref)	$\vdash @_{\mathbf{j}}\mathbf{j}$
$(Intro)$	$\vdash \mathbf{j} \wedge p \to @_{\mathbf{j}}p$
$(Back)$	$\vdash \Diamond @_{\mathbf{j}}p \to @_{\mathbf{j}}p$
$(Agree)$	$\vdash @_{\mathbf{i}}@_{\mathbf{j}}p \to @_{\mathbf{j}}p$
(4)	$\vdash \Diamond\Diamond p \to \Diamond p$
(T)	$\vdash p \to \Diamond p$
$(.3)$	$\vdash \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(q \wedge \Diamond p)$
(lpa)	$\vdash @_{\mathbf{i}}\Diamond \mathbf{j} \vee @_{\mathbf{j}}\Diamond \mathbf{i}$
Rules of inference:	
$(Modus\ ponens)$	If $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$
$(Sorted\ substitution)$	$\vdash \varphi'$ whenever $\vdash \varphi$, where φ' is obtained from φ by sorted substitution.
(Nec)	If $\vdash \varphi$, then $\vdash \Box\varphi$.
$(Nec_{@})$	If $\vdash \varphi$, then $\vdash @_{\mathbf{j}}\varphi$.

Table 5.2: Axioms and inference rules of $\mathbf{H}(@)\mathbf{S4.3}$ and $\mathbf{LP}(@)$

of which we want to transfer the truth has the form $@_{\mathbf{i}}\varphi$, we are where we want to be. So for now, will show that the fragment of $\mathbf{LP}(@)$ that contains only formulas of this form has the finite hybrid algebra property.

Definition 5.1.16. The *named fragment* of $\mathbf{LP}(@)$, denoted $Name(\mathbf{LP}(@))$, is the subset of formulas from $\mathbf{LP}(@)\Sigma$ of the form $@_{\mathbf{j}}\psi$.

The named fragment of the logic $\mathbf{LP}(@)\Sigma$ is characterized by the class of well-connected $\mathbf{LP}(@)$-algebras (defined below).

Definition 5.1.17. A *hybrid closure @-algebra* is a hybrid @-algebra satisfying the following conditions:

(*refl*) $a \leq \Diamond a$, and

(*trans*) $\Diamond\Diamond a \leq \Diamond a$.

Definition 5.1.18. An $\mathbf{LP}(@)$-algebra is a hybrid closure @-algebra satisfying in addition

(.3) $\Diamond a \wedge \Diamond b \leq \Diamond(a \wedge \Diamond b) \vee \Diamond(b \wedge \Diamond a) \vee \Diamond(a \wedge b)$ and

(*lin*) $@_x \Diamond y \vee @_y \Diamond x = \top$.

We prove the lemmas needed for this theorem after the proof.

Theorem 5.1.19. *The named fragment of every normal hybrid logic* $\mathbf{LP}(@)\Sigma$ *is sound and complete with respect to the class of all well-connected hybrid closure @-algebras validating* Σ.

Proof. Suppose $@_\mathbf{i}\varphi \notin Name(\mathbf{LP}(@))$. Then $@_\mathbf{j}\varphi \notin \mathbf{LP}(@)\Sigma$. By Theorem 3.2.1, there is an $\mathbf{LP}(@)$-algebra $\mathfrak{A} = (\mathbf{A}, X_A)$ and an assignment ν such that $\mathfrak{A} \models \Sigma$ but $\mathfrak{A}, \nu \not\models @_\mathbf{i}\varphi \approx \top$. Now, consider the canonical extension $\mathfrak{A}^\delta$ of $\mathfrak{A}$. Let

$$D = \bigvee_{x \in X_A} \Diamond^{-1} x,$$

and then let $A_D = \{a \wedge D \mid a \in A\}$ and $X_D = X_A$. Define

$$\mathbf{A}_D = (A_D, \wedge^D, \vee^D, \neg^D, \bot^D, \top^D, \Diamond^D, @^D),$$

where $\wedge^D$ and $\vee^D$ are the restriction of $\wedge$ and $\vee$ to A_D,

$$\neg^D a = \neg a \wedge D \qquad \Diamond^D a = \Diamond a \wedge D$$
$$\bot^D = \bot \qquad \top^D = D$$

and, for $x \in X_D$,

$$@_x^D a = \begin{cases} D & x \leq a \\ \bot & \text{otherwise.} \end{cases}$$

Finally, let $\mathfrak{A}_D = (\mathbf{A}_D, X_D)$. Note that since $X_A \neq \varnothing$, $X_D \neq \varnothing$. Using Lemma 5.1.20, we can prove in the same way as in Lemma 3.2.3 that A_D is closed under the operations $\wedge^D$, $\vee^D$, $\neg^D$, $\Diamond^D$, and $@^D$. We also have that $\mathfrak{A}_D$ is well-connected by Lemma 5.1.23, and furthermore, by Lemma 5.1.24, $\mathfrak{A}_D \models \mathbf{LP}(@)\Sigma^{\approx}$. Finally, $\mathfrak{A}_D \not\models @_\mathbf{i}\varphi \approx \top$. To see this, consider the assignment $\nu_D\colon \mathsf{PROP} \cup \mathsf{NOM} \to A_D$ defined by $\nu_D(p) = h(\nu(p))$ and $\nu_D(\mathbf{j}) = h(\nu(\mathbf{j}))$. Now, we know that $\nu(\neg @_\mathbf{i}\varphi) \neq \bot$ in $\mathfrak{A}$, so $\nu(@_\mathbf{i}\neg\varphi) \neq \bot$. But then $@_{\nu(\mathbf{i})}\nu(\neg\varphi) \neq \bot$, which means that $\nu(\mathbf{i}) \leq \nu(\neg\varphi)$ by Proposition 2.2.5. Hence,

$$\nu_D(\neg\varphi) = h(\nu(\neg\varphi)) = \nu(\neg\varphi) \wedge D \geq \nu(\mathbf{i}) \wedge D = \nu(\mathbf{i}) > \bot$$

which gives $\nu_D(\varphi) \neq D = \nu_D(\top)$. $\square$

We have to show that the algebra $\mathfrak{A}_D$ constructed in the above proof is well-connected. But first we need the following lemmas. For the lemma below, let D be defined as in the proof of Theorem 5.1.19.

Lemma 5.1.20. $D \leq \Box D$

Proof.

$$
\begin{aligned}
\Box D &= \Box \left(\bigvee_{x \in X_A} \Diamond^{-1} x \right) && \text{(by the definition of } D\text{)} \\
&\geq \bigvee_{x \in X_A} (\Box \Diamond^{-1} x) && \text{(by the monotonicity of } \Box\text{)} \\
&\geq \bigvee_{x \in X_A} (\Box \Diamond^{-1} \Diamond^{-1} x) && \text{(by Lemma 5.1.6)} \\
&\geq \bigvee_{x \in X_A} \Diamond^{-1} x && \text{(since } \Diamond^{-1} \text{ and } \Box \text{ are adjoint)} \\
&= D && \text{(by the definition of } D\text{)}
\end{aligned}
$$

□

Lemma 5.1.21. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be an $\mathbf{LP}(@)$-algebra, and let $x \in X_A$ and $a \in A$. Then $a \wedge \Diamond^{-1} x \neq \bot$ iff $x \leq \Diamond a$.*

Proof. For the left-to-right direction, assume $x \nleq \Diamond a$. But then $x \leq \neg \Diamond a = \Box \neg a$ since x is an atom. Now, by the monotonicity of $\Diamond^{-1}$, $\Diamond^{-1} x \leq \Diamond^{-1} \Box \neg a$, and so, since $\Diamond^{-1}$ and $\Box$ are ajoint, $\Diamond^{-1} x \leq \neg a$. Hence, $a \wedge \Diamond^{-1} = \bot$.

For the converse, assume $a \wedge \Diamond^{-1} x = \bot$. Then $\Diamond^{-1} x \leq \neg a$, and so, by the monotonicity of $\Box$, $\Box \Diamond^{-1} x \leq \Box \neg a = \neg \Diamond a$. But $\Diamond^{-1}$ and $\Box$ are adjoint, so $x \leq \neg \Diamond a$. Hence, $x \nleq \Diamond a$. □

Lemma 5.1.22. *Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be an $\mathbf{LP}(@)$-algebra, and let $x, y \in X_A$. Then $x \leq \Diamond y$ or $y \leq \Diamond x$.*

Proof. Assume $x \nleq \Diamond y$ and $y \nleq \Diamond x$. Then $x \leq \neg \Diamond y$ and $y \leq \neg \Diamond x$, and so $@_x x \leq @_x \neg \Diamond y = \neg @_x \Diamond y$ and $@_y y \leq @_y \neg \Diamond x = \neg @_y \Diamond x$. But $@_x x = \top$ and $@_y y = \top$, so $\neg @_x \Diamond y = \top$ and $\neg @_y \Diamond x = \top$. Hence,

$$\neg @_x \Diamond y \wedge \neg @_y \Diamond x = \neg (@_x \Diamond y \vee @_y \Diamond x) = \top,$$

and so, since $(@_x \Diamond y \vee @_y \Diamond x) = \top$ by (*lin*), $\top = \neg \top = \bot$, which is a contradiction. □

For the following two lemmas, $\mathfrak{A}$ and $\mathfrak{A}_D$ will be the algebras used in the proof of Theorem 5.1.19.

Lemma 5.1.23. *$\mathfrak{A}_D$ is well-connected.*

Proof. Let $a, b \in A_D$ and assume $a \neq \bot$ and $b \neq \bot$. Then $a' \wedge D = a$ and $b' \wedge D = b$ for some $a', b' \in A$, and so $a' \wedge D \neq \bot$ and $b' \wedge D \neq \bot$. Hence,

$$a' \wedge \bigvee_{x \in X_A} \Diamond^{-1} x = \bigvee_{x \in X_A} (a' \wedge \Diamond^{-1} x) \neq \bot$$

and

$$b' \wedge \bigvee_{x \in X_A} \Diamond^{-1} x = \bigvee_{x \in X_A} (b' \wedge \Diamond^{-1} x) \neq \bot,$$

which means that $a' \wedge \Diamond^{-1}x' \neq \bot$ and $b' \wedge \Diamond^{-1}x'' \neq \bot$ for some $x', x'' \in X_A$. We therefore have that $x' \leq \Diamond a'$ and $x'' \leq \Diamond b'$ by Lemma 5.1.21. But then $\Diamond x' \leq \Diamond\Diamond a' = \Diamond a'$ and $\Diamond x'' \leq \Diamond\Diamond b' = \Diamond b'$ by the monotonicity of $\Diamond$, (*refl*) and (*trans*). Now, using Lemma 5.1.22, we get $x' \leq \Diamond x''$ or $x'' \leq \Diamond x'$. In the first case, we thus have $x' \leq \Diamond a'$ and $x' \leq \Diamond b'$. But $x' \leq D$ by definition, so, since $D \leq \Box D$ by Lemma 5.1.20,

$$x' \leq \Diamond a' \wedge D \leq \Diamond a' \wedge \Box D \leq \Diamond(a' \wedge D)$$

and

$$x' \leq \Diamond b' \wedge D \leq \Diamond b' \wedge \Box D \leq \Diamond(b' \wedge D),$$

which gives

$$x' \leq \Diamond(a' \wedge D) \wedge \Diamond(b' \wedge D) \wedge D = \Diamond^D a \wedge \Diamond^D b.$$

The proof of the other case is similar. We therefore get $\Diamond^D a \wedge \Diamond^D b > \bot$ in both cases. □

Finally, to show that $\mathfrak{A}_D \models \mathbf{LP}(@)\Sigma^{\approx}$, we show that $\mathfrak{A}_D$ is a homomorphic image of $\mathfrak{A}$.

Lemma 5.1.24. *The map $h\colon A \to A_D$ defined by $h(a) = a \wedge D$ is a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{A}_D$, and, furthermore, h is surjective from X_A onto X_D.*

Proof. It is clear that h is surjective from $\mathbf{A}$ onto $\mathbf{A}_D$. To see that h is surjective from X_A onto X_D, let $x \in X_D$. Then $x \in X_A$ by definition. But we know that $\Diamond^{-1}x \leq D$ by the definition of D, so $x \leq \Diamond^{-1}x \leq D$ by Lemma 5.1.6. Hence, $x = x \wedge D = h(x)$, which means that x is its own pre-image. To show that h maps elements of X_A to elements of X_D, let $x \in X_A$. Then $x \in X_D$. But we know that $x \leq D$, so $h(x) = x$, which means that $h(x) \in X_D$. We also have to show that h is a homomorphism. The cases for $\wedge$, $\vee$, $\neg$ and $\Diamond$ are proven in the same way as in Lemma 3.1.4, so we only need to check @. Let $x \in X_A$ and $a \in A$. Assume $x \leq a$. Then $h(@_x a) = h(\top) = D$, and, since $x \leq a$ and $x \leq D$ implies that $x \leq a \wedge D$,

$$@^D_{h(x)} h(a) = @^D_{x \wedge D}(a \wedge D) = @^D_x(a \wedge D) = D.$$

Now, assume $x \not\leq a$. Here $h(@_x a) = h(\bot) = \bot$. But $x \not\leq a \wedge D$, for otherwise, $x \leq a$, a contradiction. Hence,

$$@^D_{h(x)} h(a) = @^D_{x \wedge D}(a \wedge D) = @^D_x(a \wedge D) = \bot.$$ □

Before we give the main result, we give a lemma similar to Lemma 5.1.11.

Lemma 5.1.25. *Let a and b be elements of a well-connected $\mathbf{LP}(@)$-algebra $\mathfrak{A}$. Then*

(i) $\Diamond a \leq \Diamond b$ *or* $\Diamond b \leq \Diamond a$,

(ii) $\Diamond(a \wedge \neg\Diamond b) = \Diamond(\Diamond a \wedge \neg\Diamond b)$, *and*

(iii) $\Diamond b < \Diamond a$ *implies* $\Diamond(a \wedge \neg\Diamond b) = \Diamond a$.

Proof. The proof of this is similar to that of Lemma 5.1.11. □

Now, for the main result of this section. We prove the lemmas needed after the proof of this theorem.

Theorem 5.1.26. *The named fragment of any hybrid logic* $\mathbf{LP}(@)\Sigma$ *has the strong finite hybrid algebra property.*

Proof. Suppose $@_{\mathbf{i}}\varphi \notin Name(\mathbf{LP}(@)\Sigma)$. By Theorem, 5.1.19, there is a well-connected $\mathbf{LP}(@)$-algebra $\mathfrak{A} = (\mathbf{A}, X_A)$ and an assignment ι such that $\mathfrak{A} \models \Sigma^{\approx}$ but $\mathfrak{A}, \iota \not\models @_{\mathbf{i}}\varphi \approx \top$. We now let S_0 be the set of elements of $\mathfrak{A}$ used in the evaluation of $@_{\mathbf{i}}\varphi$ and $\top$ under ι. Then let $S_1 = S_0 \cup \{x_0\}$ for some arbitrary $x_0 \in X_A$. Furthermore, let $X_S = X_A \cap S_1$, and finally, let $S = S_1 \cup \{\Diamond x \mid x \in X_S\}$. Define $\mathbf{B}_S$ as the Boolean subalgebra of $\mathfrak{A}$ generated by S. Since S is a finite subset of A, $\mathbf{B}_S$ is finite. Also, $\mathbf{B}_S$ clearly preserves all Boolean operations. Further, define $\Diamond^S$ as in (5.1) in Section 5.1.1, and, for $x \in X_S$, let

$$@^S_x b = \begin{cases} \top & x \leq b \\ \bot & \text{otherwise.} \end{cases}$$

Finally, we let $\mathfrak{B}_S = (\mathbf{B}_S, \Diamond^S, @^S, X_S)$. Since $x_0 \in X_S$, we know that $X_S \neq \varnothing$. To show that $\mathfrak{B}_S \not\models @_{\mathbf{i}}\varphi \approx \top$, we have to show that $\Diamond^S$ extends $\Diamond$, and that $@^S$ extends $@$. Well we know that R satisfies (R), so $\Diamond^S$ extends $\Diamond$. Furthermore, by Lemma 5.1.27, for any $x \in X_S$ and $b \in B_S$, $@^S_x b = @_x b$. Finally, by Lemma 5.1.28, $\mathfrak{B}_S \models \mathbf{LP}(@)\Sigma^{\approx}$.

To conclude this proof, we calculate an upper bound for the number of elements in the algebra $\mathfrak{B}_S$. First, let us calculate an upper bound for the number of elements in S. Let $l(\varphi)$ be the length of the formula φ. Then the length of $@_{\mathbf{i}}\varphi$ is at most $l(\varphi)+1$. So S_0 contains at most $l(\varphi)+2$ elements, which means that S_1 has at most $l(\varphi)+2+1 = l(\varphi)+3$ elements. Now, we also know that $X_A \cap S_1$ contains at most $l(\varphi)+1$ atoms, so $\{\Diamond x \mid x \in X_S\}$ contains at most $l(\varphi)+1$ elements. Hence,

$$|S| \leq l(\varphi) + 3 + l(\varphi) + 1 = 2l(\varphi) + 4.$$

Therefore, $\mathfrak{B}_S$ contains at most $2^{2l(\varphi)+4}$ atoms, and hence, at most $2^{2^{2l(\varphi)+4}}$ elements. □

For the lemma below, let $\mathfrak{A}$ and $\mathfrak{B}_S$ be the algebras in the proof of Theorem 5.1.26.

Lemma 5.1.27. *If x is an element of X_S and b an element of B_S, then we have $@^S_x b = @_x b$, and therefore, $@^S$ is a normal modal operator.*

Proof. Let $x \in X_S$ and $b \in B_S$, and assume $x \leq b$. Then $x \in X_A$ and $b \in A$, and so $@_x b = \top$ by Proposition 2.2.5. But we also know that $@^S_x b = \top$. Similarly, if $x \not\leq b$, $@_x b = \bot = @^S_x b$. To show that $@^S$ is a normal modal operator, first not that it is clear that $@^S$ is normal. For the additivity, let $x \in X_S$ and $a, b \in B_S$. We know that B_S is closed under the Boolean operators, so $a \vee b \in B_S$, so $@^S_x(a \vee b) = @_x(a \vee b)$. Now, since $@$ is a modal operator, $@_x(a \vee b) = @_x a \vee @_x b$. But $@_x a \vee @_x b = @^S_x a \vee @^S_x b$, so $@^S_x(a \vee b) = @^S_x a \vee @^S_x b$. □

Lemma 5.1.28. *$\mathfrak{B}_S$ can be embedded into $\mathfrak{A}$.*

Proof. Let $b_1, \ldots, b_n$ be the atoms of $\mathfrak{B}_S$. Now, appealing to Lemma 5.1.25, order the atoms of $\mathfrak{B}_S$ in the same way as in Lemma 5.1.13. Note that in this case we have $a_1 = a_2 = \top$. Let $\theta\colon B_S \to A$ be defined in the same way as in Lemma 5.1.13. From here the proof is similar to that of Lemma 5.1.13, all we have to check is that θ respects @. For simplicity, we will drop all the i's denoting the 'piece' we are working in. Now, let $x \in X_S$ and $b \in B_S$, and assume $b = \bot$. Then

$$\theta(@^S_x \bot) = \theta(\bot) = \bot$$

and

$$@_{\theta(x)}\theta(\bot) = @_x\bot = \bot.$$

Next, assume b is an atom of $\mathfrak{B}$. Then we have the following cases:

Case 1: $b = b_{k(i)+j}$, $1 \leq i \leq m$ and $k(i) \leq k(i) + j \leq k(i+1) - 1$ such that $b_{k(i)+j} \in X_S$ and $x \leq b_{k(i)+j}$. Then

$$\theta(@^S_x b_{k(i)+j}) = \theta(\top) = \top$$

and

$$@_{\theta(x)}\theta(b_{k(i)+j}) \quad = \quad @_x b_{k(i)+j} = \top.$$

Case 2: $b = b_{k(i)+j}$, $1 \leq i \leq m$ and $k(i) \leq k(i) + j \leq k(i+1) - 1$ such that $b_{k(i)+j} \notin X_S$ or $b_{k(i)+j} \in X_S$, and $x \nleq b_{k(i)+j}$. Since $x \nleq b_{k(i)+j}$, $x \wedge b_{k(i)+j} = \bot$. But θ respects the Boolean operators, so

$$\theta(x) \wedge \theta(b_{k(i)+j}) = \theta(x \wedge b_{k(i)+j}) = \theta(\bot) = \bot.$$

This means that

$$@_{\theta(x)}\theta(x) \wedge @_{\theta(x)}\theta(b_{k(i)+j}) = @_{\theta(x)}(\theta(x) \wedge \theta(b_{k(i)+j})) = @_{\theta(x)}\bot = \bot.$$

Hence, since $@_{\theta(x)}\theta(x) = @_x x = \top$, $@_{\theta(x)}\theta(b_{k(i)+j}) = \bot$. Conversely,

$$\theta(@^S_x b_{k(i)+j}) = \theta(\bot) = \bot.$$

Note that the case $b_{k(i)+j} \notin X_S$ and $x \leq b_{k(i)+j}$, $1 \leq i \leq m$ and $k(i) \leq k(i)+j \leq k(i+1)-1$, does not occur since $x \leq b_{k(i)+j}$ implies that x and $b_{k(i)+j}$ are the same atom but $x \in X_S$.

Now, finally, for any $x \in X_S$ and $b \in B_S$,

$$\begin{aligned}
@_{\theta(x)}\theta(b) &= @_{\theta(x)} \bigvee_{b_j \leq b} \theta(b_j) && \text{(by the definition of } \theta\text{)} \\
&= \bigvee_{b_j \leq b} @_{\theta(x)}\theta(b_j) && \text{(since @ is a normal operator)} \\
&= \bigvee_{b_j \leq b} \theta(@^S_x b_j) && \text{(by Case 1 and Case 2 above)} \\
&= \theta\left(\bigvee_{b_j \leq b} @^S_x b_j\right) && \text{(as in Claim 9 in the proof of Lemma 5.1.13)} \\
&= \theta\left(@^S_x\left(\bigvee_{b_j \leq b} b_j\right)\right) && \text{(by Lemma 5.1.27)} \\
&= \theta(@^S_x b) && \text{(since } \mathfrak{B}_S \text{ is atomic)} \quad \square
\end{aligned}$$

Corollary 5.1.29. *If Σ is finite, then the named fragment of any hybrid logic* $\mathbf{LP}(@)\Sigma$ *is decidable.*

Earlier we made the claim that the natural way of constructing a finite hybrid @-algebra from a permeated hybrid @-algebra that is also permeated fails. Let us now explain why this is the case. Let $\mathfrak{A} = (\mathbf{A}, X_A)$ be a permeated hybrid algebra, and let S be a finite subset of elements of A. Consider the Boolean subalgebra $\mathbf{A}_S$ generated by S. Furthermore, let $X_S = \{x_0\}$, where $x_0 \in S \cap X_A$, and then define $\mathfrak{A}_S = (\mathbf{A}_S, \diamond^S, X_S)$. We know that if $S = \{a_1, \dots, a_n\}$ and a_i^0 and a_i^1 denotes $\neg a_i$ and a_i, respectively, then

$$At\mathbf{A}_S = \{a_1^{f(1)} \wedge \cdots a_n^{f(n)} \mid f : \{1, \dots, n\} \to \{0, 1\}\} - \{\bot\}.$$

But

$$\neg x_0 \wedge \bigwedge\left(\{a_i^{f(i)} \mid f : \{1, \dots, n\} \to \{0, 1\}\} - \{x_0, \neg x_0\}\right) \neq \bot,$$

so for $\mathfrak{A}_S$ to be permeated, we have to find a designated atom $y \in X_S$ such that

$$y \leq \neg x_0 \wedge \bigwedge\left(\{a_i^{f(i)} \mid f : \{1, \dots, n\} \to \{0, 1\}\} - \{x_0, \neg x_0\}\right).$$

However, $x_0 \not\leq \neg x_0 \wedge \bigwedge\left(\{a_i^{f(i)} \mid f : \{1, \dots, n\} \to \{0, 1\}\} - \{x_0, \neg x_0\}\right)$. But we know that $\mathfrak{A}$ is permeated, so there is some $x_1 \in X$ such that

$$x_1 \leq \neg x_0 \wedge \bigwedge\left(\{a_i^{f(i)} \mid f : \{1, \dots, n\} \to \{0, 1\}\} - \{x_0, \neg x_0\}\right).$$

So add x_1 to S, and denote this set by S'. Further, add x_1 to X_S, and denote this set of designated atoms by $X_{S'}$. Now, consider the subalgebra $\mathbf{A}'_S$ generated by S'. Then we have

$$\neg x_0 \wedge \neg x_1 \wedge \bigwedge\left(\{a_i^{f(i)} \mid f : \{1, \dots, n\} \to \{0, 1\}\} - \{x_0, \neg x_0, x_1, \neg x_1\}\right) \neq \bot,$$

But both

$$x_0 \not\leq \neg x_0 \wedge \neg x_1 \wedge \bigwedge \left(\{a_i^{f(i)} \mid f : \{1, \ldots, n\} \to \{0,1\}\} - \{x_0, \neg x_0, x_1, \neg x_1\} \right)$$

and

$$x_1 \not\leq \neg x_0 \wedge \neg x_1 \wedge \bigwedge \left(\{a_i^{f(i)} \mid f : \{1, \ldots, n\} \to \{0,1\}\} - \{x_0, \neg x_0, x_1, \neg x_1\} \right),$$

so to fix this, we add another designated atom from X_A to S' and $X_{S'}$ and repeat the whole process again. We then see that this results in a snow ball effect, and the only way to make $\mathfrak{A}_S$ permeated is to add all the designated atoms from X_A to S. However, then we need not get a finite algebra, as required.

5.1.3 An analogue of Bull's Theorem for $\mathcal{H}(\mathsf{E})$

In this subsection, we want to know to what extent Bull's Theorem holds for extensions of the logic obtained by adding the axioms (T), (4) and $(.3)$ to $\mathbf{H}(\mathsf{E})$ (denoted by **HS4.3**). Once again this generalization is not straighforward. As for $\mathcal{H}(@)$, the truth of $\mathcal{H}(\mathsf{E})$-formulas is not transferred from the supermodel to the submodel when taking point-generated submodels. The good news here is that we can enforce well-connectedness axiomatically. But first, we claim that the frame property that every two states have a common predecessor implies well-connectedness. To see this, let $\mathfrak{g} = (W, R, A, B)$ be a two sorted general frame such that every two states in W have a common predecessor. We will show that $\mathfrak{g}^* = (A, \cap, \cup, -, \varnothing, W, \langle R \rangle)$ is well-connected. So let $a, b \in A$ such that $a \neq \varnothing$ and $b \neq \varnothing$. Then there are v and w in W such that $v \in a$ and $w \in b$. But since every two states in W have a common predecessor, there is a state u such that uRv and uRw. Hence, since $v \in a$ and $w \in b$, $u \in \langle R \rangle a$ and $u \in \langle R \rangle b$, which means that $u \in \langle R \rangle a \cap \langle R \rangle b$. Thus, $\langle R \rangle a \cap \langle R \rangle b \neq \varnothing$.

Now, it turns out that the formula $\mathsf{E}p \wedge \mathsf{E}q \to \mathsf{E}(\Diamond p \wedge \Diamond q)$ defines the class of frames with this frame property. To prove this, we have to show that $\mathfrak{F}$ is a frame in which any two states have a common predecessor iff $\mathfrak{F} \Vdash \mathsf{E}p \wedge \mathsf{E}q \to \mathsf{E}(\Diamond p \wedge \Diamond q)$. For the left-to-right direction, let $\mathfrak{F} = (W, R)$ be a frame in which any two states have a common predecessor, and assume that $\mathfrak{F} \nVdash \mathsf{E}p \wedge \mathsf{E}q \to \mathsf{E}(\Diamond p \wedge \Diamond q)$. Then there is some valuation V and some state $s \in W$ such that $(\mathfrak{F}, V), s \Vdash \mathsf{E}p \wedge \mathsf{E}q$ but $(\mathfrak{F}, V), s \nVdash \mathsf{E}(\Diamond p \wedge \Diamond q)$. From $(\mathfrak{F}, V), s \Vdash \mathsf{E}p \wedge \mathsf{E}q$ we know that there are states t and u in W such that $t \Vdash p$ and $u \Vdash q$. But we know that t and u have a common predecessor v, i.e., vRt and vRu. This means that $(\mathfrak{F}, V), u \Vdash \Diamond p \wedge \Diamond q$, and so $(\mathfrak{F}, V), s \Vdash \mathsf{E}(\Diamond p \wedge \Diamond q)$, a contradiction. For the converse, let $\mathfrak{F}$ be a frame such that the states s and t have no common predecessor. Now, let $V(p) = \{s\}$ and $V(q) = \{t\}$. Then we have $(\mathfrak{F}, V), s \Vdash \mathsf{E}p \wedge \mathsf{E}q$, however, since s and t have no common predecessor, $(\mathfrak{F}, V), s \nVdash \mathsf{E}(\Diamond p \wedge \Diamond q)$.

So instead of working with extensions of $\mathbf{H}(\mathsf{E})\mathbf{S4.3}$, we will work with extensions of the logic $\mathbf{H}(\mathsf{E})\mathbf{S4.3}$ together with the axiom $\mathsf{E}p \wedge \mathsf{E}q \to \mathsf{E}(\Diamond p \wedge \Diamond q)$. We will denote this axiom by (cpa), which stands for common predecessor axiom. We can show that the common predecessor axiom together with (T), (4) and $(.3)$ defines the class of transitive and connected frames, so we can also view the frames of this logic as chains of clusters. For this reason we will denote this logic by $\mathbf{LP}(\mathsf{E})$, where the LP is short for linear order of pre-orders.

Definition 5.1.30. The logic $\mathbf{LP}(\mathsf{E})$ is the smallest set of formulas containing all propositional tautologies, the axioms in Table 5.3 and which is closed under the inference rules in Table 5.3.

Axioms:	
$(Taut)$	$\vdash \varphi$ for all classical propositional tautologies.
(K)	$\vdash \Box(p \to q) \to (\Box p \to \Box q)$
$(Dual)$	$\vdash \Diamond p \leftrightarrow \neg\Box\neg p$
(K_{A})	$\vdash \mathsf{A}(p \to q) \to (\mathsf{A}p \to \mathsf{A}q)$
$(Dual_{\mathsf{A}})$	$\vdash \mathsf{E}p \leftrightarrow \neg\mathsf{A}\neg p$
$(Incl_{\mathbf{j}})$	$\vdash \mathsf{E}\mathbf{j}$
(Nom_{E})	$\vdash \mathsf{E}(\mathbf{i} \wedge p) \to \mathsf{A}(\mathbf{i} \to p)$
$(T\mathsf{E})$	$\vdash p \to \mathsf{E}p$
$(4\mathsf{E})$	$\vdash \mathsf{E}\mathsf{E}p \to \mathsf{E}p$
$(B\mathsf{E})$	$\vdash p \to \mathsf{A}\mathsf{E}p$
$(Incl_{\Diamond})$	$\vdash \Diamond p \to \mathsf{E}p$
(T)	$\vdash p \to \Diamond p$
(4)	$\vdash \Diamond\Diamond p \to \Diamond p$
$(.3)$	$\vdash \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(q \wedge \Diamond p)$
(cpa)	$\vdash \mathsf{E}p \wedge \mathsf{E}q \to \mathsf{E}(\Diamond p \wedge \Diamond q)$
Rules of inference:	
$(Modus\ ponens)$	If $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$.
$(Sorted\ substitution)$	$\vdash \varphi'$ whenever $\vdash \varphi$, where φ' is obtained from φ by sorted substitution.
(Nec)	If $\vdash \varphi$, then $\vdash \Box\varphi$.
(Nec_{A})	If $\vdash \varphi$, then $\vdash \mathsf{A}\varphi$.

Table 5.3: Axioms and inference rules of $\mathbf{LP}(\mathsf{E})$

Let us now give the algebras we will be working with in this section.

Definition 5.1.31. A *hybrid closure* E-*algebra* is a hybrid E-algebra satisfying the following conditions:

(*refl*) $a \leq \Diamond a$, and

(*trans*) $\Diamond\Diamond a \leq \Diamond a$.

Definition 5.1.32. A $\mathbf{H}(\mathsf{E})\mathbf{S4.3}$-*algebra* is a hybrid closure E-algebra satisfying in addition

(.3) $\Diamond a \wedge \Diamond b \leq \Diamond(a \wedge \Diamond b) \vee \Diamond(b \wedge \Diamond b) \vee \Diamond(a \wedge b)$.

The logic $\mathbf{LP}(\mathsf{E}) \oplus \Sigma$ is characterized by the class of well-connected $\mathbf{H}(\mathsf{E})\mathbf{S4.3}$-algebras. But before we prove this, we first prove the following lemma:

Lemma 5.1.33. *Let $\mathfrak{A}$ be a hybrid E-algebra. Then $\mathfrak{A}$ is well-connected iff $\mathfrak{A}$ validates*

$$\mathsf{E}a \wedge \mathsf{E}b \leq \mathsf{E}(\Diamond a \wedge \Diamond b).$$

Proof. For the left-to-right direction, assume $\mathfrak{A}$ is well-connected. Consider the following cases:

Case 1: $a = \bot$ and $b = \bot$. Then $\mathsf{E}a = \bot$ and $\mathsf{E}b = \bot$, which means that $\mathsf{E}a \wedge \mathsf{E}b \leq \mathsf{E}(\Diamond a \wedge \Diamond b)$.

Case 2: $a = \bot$ and $b > \bot$, or vice versa. We then clearly have $\mathsf{E}a \wedge \mathsf{E}b = \bot$, and so $\mathsf{E}a \wedge \mathsf{E}b \leq \mathsf{E}(\Diamond a \wedge \Diamond b)$.

Case 3: $a > \bot$ and $b > \bot$. But $\mathfrak{A}$ is well-connected, so $\Diamond a \wedge \Diamond b > \bot$. This means that $\mathsf{E}(\Diamond a \wedge \Diamond b) = \top$, and hence, $\mathsf{E}a \wedge \mathsf{E}b \leq \mathsf{E}(\Diamond a \wedge \Diamond b)$.

For the other direction, suppose $\mathfrak{A}$ validates $\mathsf{E}a \wedge \mathsf{E}b \leq \mathsf{E}(\Diamond a \wedge \Diamond b)$. Now, let $a, b \in A$, and assume $a \neq \bot$ and $b \neq \bot$. Then $\mathsf{E}a = \top$ and $\mathsf{E}b = \top$, so $\top = \mathsf{E}a \wedge \mathsf{E}b \leq \mathsf{E}(\Diamond a \wedge \Diamond b)$. Hence, $\mathsf{E}(\Diamond a \wedge \Diamond b) = \top$, which gives $\Diamond a \wedge \Diamond b \neq \bot$ by definition. □

Theorem 5.1.34. *Every normal hybrid logic $\mathbf{LP}(\mathsf{E}) \oplus \Sigma$ is sound and complete with respect to the class of all well-connected $\mathbf{H}(\mathsf{E})\mathbf{S4.3}$-algebras validating Σ.*

Proof. This result follows immediately from Theorem 3.3.1 and Lemma 5.1.33. □

As in the previous two sections, we have the following lemma for $\mathbf{H}(\mathsf{E})\mathbf{S4.3}$-algebras:

Lemma 5.1.35. *Let a and b be elements of a well-connected $\mathbf{H}(\mathsf{E})\mathbf{S4.3}$-algebra $\mathfrak{A}$. Then*

(i) $\Diamond a \leq \Diamond b$ *or* $\Diamond b \leq \Diamond a$,

(ii) $\Diamond(a \wedge \neg\Diamond b) = \Diamond(\Diamond a \wedge \neg\Diamond b)$, *and*

(iii) $\Diamond b < \Diamond a$ *implies* $\Diamond(a \wedge \neg\Diamond b) = \Diamond a$.

Proof. Similar to that of Lemma 5.1.11. □

Now for the main result. The lemmas needed to prove this theorem follow afterwards.

Theorem 5.1.36. *Every normal hybrid logic $\mathbf{LP}(\mathsf{E}) \oplus \Sigma$ has the strong finite hybrid algebra property.*

Proof. Suppose $\varphi \notin \mathbf{LP}(\mathsf{E}) \oplus \Sigma$. By Theorem, 5.1.34, there is a well-connected $\mathbf{H}(\mathsf{E})\mathbf{S4.3}$-algebra $\mathfrak{A} = (\mathbf{A}, X_A)$ and an assignment ι such that $\mathfrak{A} \models \Sigma^{\approx}$ but $\mathfrak{A}, \iota \not\models \varphi \approx \top$. We now let S_0 be the set of elements of $\mathfrak{A}$ used in the evaluation of φ and $\top$ under ι. Then let $S_1 = S_0 \cup \{x_0\}$ for some arbitrary $x_0 \in X_A$. Furthermore, let $X_S = X_A \cap S_1$, and finally, let $S = S_1 \cup \{\Diamond x \mid x \in X_S\}$. Define $\mathbf{B}_S$ as the Boolean subalgebra of $\mathfrak{A}$ generated by S. Since S is a finite subset of A, $\mathbf{B}_S$ is finite. Also, $\mathbf{B}_S$ preserves all Boolean operations. Further, define $\Diamond^S b$ as in (5.1) in Section 5.1.1, and for $b \in B$, let

$$\mathsf{E}^S b = \begin{cases} \top & b > \bot \\ \bot & \text{otherwise.} \end{cases}$$

Now, we let $\mathfrak{B}_S = (\mathbf{B}_S, \Diamond^S, \mathsf{E}^S, X_S)$. We have to make sure that $\mathfrak{B}_S$ is the right algebra. First, $X_S \neq \varnothing$, since $x_0 \in X_S$. Next, we know that $\Diamond^S$ extends $\Diamond$, and furthermore, by Lemma 5.1.37, E^S extends E, so $\mathfrak{B}_S \not\models \varphi \approx \top$. Finally, by Lemma 5.1.38, $\mathfrak{B}_S \models \mathbf{LP}(\mathsf{E})\Sigma^{\approx}$. We therefore also have that $\mathfrak{B}_S$ is well-connected by Lemma 5.1.33.

Let us now calculate an upper bound for the number of elements in the algebra $\mathfrak{B}_S$. First, we calculate an upper bound for the number of elements in S. Let $l(\varphi)$ be the length of the formula φ. Then S_0 contains at most $l(\varphi) + 1$ elements. This means that S_1 has at most $l(\varphi) + 1 + 1 = l(\varphi) + 2$ elements. Now, we also know that $X_A \cap S_1$ contains at most $l(\varphi) + 1$ atoms, so $\{\Diamond x \mid x \in X_S\}$ contains at most $l(\varphi) + 1$ elements. Hence,

$$|S| \leq l(\varphi) + 2 + l(\varphi) + 1 = 2l(\varphi) + 3.$$

We can thus conclude that $\mathfrak{B}_S$ contains at most $2^{2l(\varphi)+3}$ atoms, and hence, at most $2^{2^{2l(\varphi)+3}}$ elements. □

For the remainder of this section, let $\mathfrak{A}$ and $\mathfrak{B}_S$ be the hybrid algebras in the proof of Theorem 5.1.36. We now show that E^S is a normal modal operator extending E.

Lemma 5.1.37. *If $b \in B_S$, then $\mathsf{E}^S b = \mathsf{E}b$, and therefore, E^S is a normal modal operator.*

Proof. Let $b \in B_S$, and assume $b > \bot$. Then $b \in A$, and so $\mathsf{E}b = \top$ by the definition of a hybrid E-algebra. But we also know that $\mathsf{E}^S b = \top$. Similarly, if $b = \bot$, $\mathsf{E}b = \bot = \mathsf{E}^S b$. Clearly, E^S is normal. To show that E^S is additive, let $a, b \in B_S$. But we know that B_S is closed under the Boolean operators, so $a \vee b \in B_S$. This means that $\mathsf{E}^S(a \vee b) = \mathsf{E}(a \vee b)$. Now, since E is a modal operator, $\mathsf{E}(a \vee b) = \mathsf{E}a \vee \mathsf{E}b$. Hence, since $\mathsf{E}a \vee \mathsf{E}b = \mathsf{E}^S a \vee \mathsf{E}^S b$, $\mathsf{E}^S(a \vee b) = \mathsf{E}^S a \vee \mathsf{E}^S b$. □

Lemma 5.1.38. *$\mathfrak{B}_S$ can be embedded into $\mathfrak{A}$.*

Proof. Let $b_1, \ldots, b_n$ be the atoms of $\mathfrak{B}_S$. Now, appealing to Lemma 5.1.35, order the atoms of $\mathfrak{B}_S$ in the same way as in Lemma 5.1.13. In this case, we have $a_1 = a_2 = \top$. Let $\theta\colon B_S \to A$ be defined in the same way as in Lemma 5.1.13. From here the proof is the same as the proof of Lemma 5.1.13, so all we have to check is that θ respects E. If $b = \bot$, then $\theta(\mathsf{E}^S b) = \theta(\bot) = \bot$ and $\mathsf{E}\theta(b) = \mathsf{E}\bot = \bot$. Next, assume b is an atom of $\mathfrak{B}_S$. We then consider the following cases:

Case 1: $b \in X_S$ and $b = b_{k(j)}$ for some $1 \leq j \leq m$ and $k(j) \leq k(j) + l \leq k(j+1) - 1$. Since $b_{k(j)} > \bot$, $\theta(\mathsf{E}^S b_{k(j)}) = \theta(\top) = \top$ and $\mathsf{E}\theta(b_{k(j)}) = \mathsf{E}b_{k(j)} = \top$.

Case 2: $b \notin X_S$ and $b = b_{k(j)}$ for some $1 \leq j \leq m$. As in the previous case, since $b_{k(j)} > \bot$, $\theta(\mathsf{E}^S b) = \theta(\top) = \top$. On the other hand, since $b_{k(j)}$ is not designated,

$$\begin{aligned}\mathsf{E}\theta(b_{k(j)}) &= \mathsf{E}\left(\Diamond b_{k(j)} \wedge \neg\left(b_{k(j)+1} \vee \cdots b_{k(j+1)-1}\right) \wedge \neg\Diamond b_{k(j-1)}\right)\\ &\geq \mathsf{E}\left(b_{k(j)} \wedge \neg\left(b_{k(j)+1} \vee \cdots b_{k(j+1)-1}\right) \wedge \neg\Diamond b_{k(j-1)}\right)\\ &= \mathsf{E}(b_{k(j)} \wedge \neg\Diamond b_{k(j-1)}).\end{aligned}$$

We now claim that $b_{k(j)} \wedge \neg\Diamond b_{k(j-1)} > \bot$. To see this, suppose otherwise, then $b_{k(j)} \leq \Diamond b_{k(j-1)}$, which means that $\Diamond b_{k(j)} \leq \Diamond\Diamond b_{k(j-1)} = \Diamond b_{k(j-1)}$, contradicting our ordering. Hence, $\mathsf{E}(b_{k(j)} \wedge \neg\Diamond b_{k(j-1)}) = \top$, which means that $\mathsf{E}\theta(b_{k(j)}) \geq \top$. We thus have $\mathsf{E}\theta(b_{k(j)}) = \top$, as required.

Case 3: $b \notin X_S$ and $b = b_{k(j)+l}$ for some $1 \leq j \leq m$ and $k(j) + 1 \leq k(j) + l \leq k(j+1) - 1$. As in Case 1, $\theta(\mathsf{E}^S b_{k(j)+l}) = \theta(\top) = \top$. Now, we know that $b_{k(j)+j} \wedge \Diamond b_{k(j-1)} = \bot$, for if not, $b_{k(j)+l} \leq \Diamond b_{k(j-1)}$, which means $\Diamond b_{k(j)} = \Diamond b_{k(j)+l} \leq \Diamond\Diamond b_{k(j-1)} = \Diamond b_{k(j-1)}$, again contradicting our ordering. Hence,

$$\mathsf{E}\theta(b_{k(j)+l}) = \mathsf{E}(b_{k(j)+1} \wedge \neg\Diamond b_{k(j-1)}) = \top.$$

Finally, for any $b > \bot$,

$$\begin{aligned}\mathsf{E}\theta(b) &= \mathsf{E}\bigvee_{b_j \leq b} \theta(b_j) && \text{(by the definition of } \theta)\\ &= \bigvee_{b_j \leq b} \mathsf{E}\theta(b_j) && \text{(since } \mathsf{E} \text{ is a normal modal operator)}\\ &= \bigvee_{b_j \leq b} \theta(\mathsf{E}^S b_j) && \text{(by Cases 1, 2 and 3 above)}\\ &= \theta(\bigvee_{b_j \leq b} \mathsf{E}^S b_j) && \text{(as in Claim 9 in the proof of Lemma 5.1.13)}\\ &= \theta(\mathsf{E}^S \bigvee_{b_j \leq b} b_j) && \text{(by Lemma 5.1.37)}\\ &= \theta(\mathsf{E}^S b) && \text{(since } \mathfrak{B}_S \text{ is atomic)}\end{aligned}$$

□

Corollary 5.1.39. *If Σ is finite, then* $\mathbf{LP}(\mathsf{E}) \oplus \Sigma$ *is decidable.*

5.2 Hybrid extensions of S4 with the finite model property

In [18], Bull characterized a class of axiomatic extensions of the normal modal logic **S4** with the finite model property. This result takes the form of a syntactic characterization of a class of formulas that may be added as axioms to **S4**, somewhat in the spirit of Sahlqvist's theorem in modal correspondence theory. As for his result in [20], he was able to restrict his attention to well-connected closure algebras. He then constructed a finite Boolean subalgebra from such a well-connected closure algebra in the usual way, and defined a modal operator that preserves all existing operators. The syntactically defined class of formulas is so defined that the finite algebra obtained by the construction validates all the formulas that are validated by the original algebra.

In this section, we extend this result to our hybrid languages $\mathcal{H}, \mathcal{H}(@)$ and $\mathcal{H}(\mathsf{E})$.

5.2.1 Hybrid extensions of S4 in the language $\mathcal{H}$ with the finite model property

Here we show how to extend Bull's result in [18] to the language $\mathcal{H}$. First, we define the formulas we will work with, which will be called Γ-*formulas*.

Γ-formulas are recursively defined by the following inductive rules:

$$\begin{array}{rcl} \alpha & ::= & \bot \mid \top \mid \Box^m p \mid \Box^n \mathbf{i} \mid \alpha_1 \wedge \alpha_2 \mid \alpha_1 \vee \alpha_2 \ (m, n > 0) \\ \beta & ::= & \alpha \mid \beta_1 \wedge \beta_2 \mid \beta_1 \vee \beta_2 \mid \Box(\alpha \to \beta) \mid \Box\neg\alpha \\ \gamma & ::= & \Box\neg\mathbf{i} \mid \alpha \mid \Box(\beta \to \alpha) \mid \Box\neg\beta \mid \gamma_1 \wedge \gamma_2 \mid \gamma_1 \vee \gamma_2 \mid \Box\gamma \end{array}$$

Note that this definition extends Bull's definition in [18] with the clauses $\Box^n\mathbf{i}$ and $\Box\neg\mathbf{i}$ for nominals.

Let us give some examples of Γ-formulas that are important in the study of modal and hybrid logic.

Example 5.2.1. The following formulas are Γ-formulas:

(i) the formula $\Diamond\Box p \to \Box p \equiv \Box\neg\Box p \vee \Box p$ defining the class of *euclidean frames* (i.e., the class of frames in which the accessibility relation satisfies $\forall x \forall y \forall z(xRy \wedge xRz \to yRz)$),

(ii) the formula $\Diamond\Diamond\mathbf{i} \to \neg\Diamond\mathbf{i} \equiv \Box\Box\neg\mathbf{i} \vee \Box\neg\mathbf{i}$ defining intransitivity, and

(iii) the formula $\Diamond\mathbf{i} \to \Box\mathbf{i} \equiv \Box\neg\mathbf{i} \vee \Box\mathbf{i}$ defining *determinism* (i.e., the class of frames in which the accessibility relation satisfies $\forall x \forall y \forall z(xRy \wedge xRz \to y = z)$).

We now give the main result of this section. The lemmas needed to prove this result follow afterwards.

Theorem 5.2.2. *Let* Σ *be a set of* Γ*-formulas. Then every normal hybrid logic* $\mathbf{HS4} \oplus \Sigma$ *has the strong finite hybrid algebra property.*

Proof. Suppose $\varphi \notin \mathbf{S4H} \oplus \Sigma$. By Theorem 5.1.5, there is a piecewise well-connected hybrid closure algebra $\mathfrak{A} = (\mathbf{A}, X_A)$ and an assignment ν such that $\mathfrak{A} \models \Sigma^{\approx}$ but $\mathfrak{A}, \nu \not\models \varphi \approx \top$. We also know that there are D_1 and D_2 in A that satisfy the conditions of Definition 5.1.4 (possibly $D_1 = D_2 = \top$). Now, let S_0 be the set of elements of $\mathfrak{A}$ used in the evaluation of φ and $\top$ under ν. Then let $S_1 = S_0 \cup \{x_0\} \cup \{D_1, D_2\}$, for some arbitrary atom $x_0 \in X_A$.

Finally, let $S = S_1 \cup \{\Box x \mid x \in X_A \cap S_1\}$. Define $\mathbf{B}_S$ as the Boolean subalgebra of $\mathfrak{A}$ generated by S. Since S is a finite subset of A, $\mathbf{B}_S$ is finite. Also, $\mathbf{B}_S$ clearly preserves all Boolean operations. Further, for $b \in B_S$, let $(b\uparrow)_C = \{a \in B_S \mid a = \Diamond a \text{ and } b \leq a\}$ and define

$$\Diamond^S b := \bigwedge (b\uparrow)_C.$$

Finally, let $\mathfrak{B}_S = (\mathbf{B}_S, \Diamond^S, X_S)$, where $X_S = X_A \cap S_1$. Now, by Lemma 5.2.4, $\Diamond^S$ extends $\Diamond$, so $\mathfrak{B}_S \not\models \varphi \approx \top$. Furthermore, by Proposition 5.2.5, $\mathfrak{B}_S$ is a hybrid closure algebra, and finally, by Lemma 5.2.8, $\mathfrak{B}_S \models \Sigma^{\approx}$.

To calculate an upper bound for $\mathfrak{B}_S$, first note that $|S| \leq 2l(\varphi) + 5$, so $\mathfrak{B}_S$ contains at most $2^{l(\varphi)+5}$ atoms, and hence, at most $2^{2^{l(\varphi)+5}}$ elements. $\Box$

Unless stated otherwise, in what follows, $\mathfrak{A}$ and $\mathfrak{B}_S$ will be the algebras in the proof of Theorem 5.2.2. We need the following lemma to show that $\Diamond^S$ extends $\Diamond$ and that $\mathfrak{B}_S$ is a hybrid closure algebra.

Lemma 5.2.3. *$\Diamond b \leq \Diamond^S b = \Diamond\Diamond^S b$ for all $b \in B_S$, and therefore, $\Box\Box^S b = \Box^S b \leq \Box b$ for all $b \in B_S$.*

Proof. First, we prove that $\Diamond b \leq \Diamond^S b$. By the definition of $\Diamond^S$, $b \leq a$ and $a = \Diamond a$ for all $a \in (b\uparrow)_C$. Hence, $\Diamond b \leq \Diamond a$, which means $\Diamond b \leq a$ for all $a \in (b\uparrow)_C$. This gives

$$\Diamond b \leq \bigwedge (b\uparrow)_C,$$

and so $\Diamond b \leq \Diamond^S b$. For the equality $\Diamond^S b = \Diamond\Diamond^S b$, note that the left-to-right inequality follows from (*refl*). Conversely, $(b\uparrow)_C$ is finite, so

$$\begin{aligned}
\Diamond\Diamond^S b &= \Diamond \bigwedge (b\uparrow)_C \\
&\leq \bigwedge \{\Diamond a \mid b \leq a \in B_S \text{ and } a = \Diamond a\} \\
&= \bigwedge (b\uparrow)_C \\
&= \Diamond^S b,
\end{aligned}$$

where the third step follows from the fact that $\Diamond a = a$. Thus, since $b \in B_S$ implies $\neg b \in B_S$, we have $\Diamond\neg b \leq \Diamond^S \neg b = \Diamond\Diamond^S \neg b$. Hence, $\neg\Diamond\Diamond^S \neg b = \neg\Diamond^S \neg b \leq \neg\Diamond\neg b$, and so $\Box\Box^S b = \Box^S b \leq \Box b$. $\Box$

We now show that $\Diamond^S$ extends $\Diamond$.

Lemma 5.2.4. *Let b be an element in B_S. If $\Diamond b \in B_S$, then $\Diamond^S b = \Diamond b$, and therefore, if $\Box b \in B_S$, then $\Box^S b = \Box b$.*

Proof. First, $\Diamond b \leq \Diamond^S b$ by Lemma 5.2.3. For the converse inequality, note that $\Diamond b \in B_S$, $b \leq \Diamond b$ and $\Diamond\Diamond b = \Diamond b$, so $\Diamond b \in (b\uparrow)_C$. But then

$$\bigwedge (b\uparrow)_C \leq \Diamond b,$$

which means that $\Diamond^S b \leq \Diamond b$. Hence, $\Diamond^S b = \Diamond b$. For the second part, assume $\Box b \in B_S$. Then $\neg\Diamond\neg b \in B_S$, and so, since B_S is closed under the Boolean operators, $\Diamond\neg b \in B_S$. Now, by the first part, $\Diamond^S \neg b = \Diamond\neg b$. Hence, $\neg\Diamond^S\neg b = \neg\Diamond\neg b$, and so $\Box^S b = \Box b$. $\Box$

The second part of Lemma 5.2.4 tells us that $\Box^S x = \Box x$ for $x \in X_S$. To see this, note that if $x \in X_S$, $x \in X_A \cap S_1$, and so, by the definition of S, $\Box x \in S$. But this means that $\Box x \in B_S$, and the result follows.

The next proposition shows that $\mathfrak{B}_S$ is a hybrid closure algebra.

Proposition 5.2.5. *$\mathfrak{B}_S$ is a finite hybrid closure algebra.*

Proof. First, $X_S \neq \varnothing$ since $x_0 \in X_S$. For (*refl*), note that $\Diamond b \leq \Diamond^S b$ by Lemma 5.2.3. But we know that $b \leq \Diamond b$, so $b \leq \Diamond^S b$.

To prove (*trans*), first note that $\Diamond\Diamond^S b = \Diamond^S b$ by Lemma 5.2.3, so $\Diamond\Diamond^S b \in B_S$. Hence, by Lemma 5.2.4, $\Diamond\Diamond^S b = \Diamond^S\Diamond^S b$, which means that $\Diamond^S\Diamond^S b = \Diamond^S b$.

Next, we show that $\Diamond^S$ is a normal modal operator. We know that $\bot \in B_S$, $\bot \leq \bot$ and $\Diamond\bot = \bot$, so $\bot \in (\bot\uparrow)_C$. But then $\Diamond^S\bot \leq \bot$, which means that $\Diamond^S\bot = \bot$.

For the additivity, first let $a \in (b \vee b'\uparrow)_C$. Then $a = \Diamond a$ and $b \vee b' \leq a$. But $b \vee b' \leq a$ implies $b \leq a$ and $b' \leq a$. Hence, since $\Diamond a = a$, we have $a \in (b\uparrow)_C$ and $a \in (b'\uparrow)_C$, and therefore, $(b \vee b'\uparrow)_C \subseteq (b\uparrow)_C$ and $(b \vee b'\uparrow)_C \subseteq (b'\uparrow)_C$. This means that $\Diamond^S b \leq \Diamond^S(b \vee b')$ and $\Diamond^S b' \leq \Diamond^S(b \vee b')$, so $\Diamond^S b \vee \Diamond^S b' \leq \Diamond^S(b \vee b')$. For the converse inequality, note that

$$\Diamond^S b \vee \Diamond^S b' = \bigwedge(b\uparrow)_C \vee \bigwedge(b'\uparrow)_C = \bigwedge\{a \vee a' \mid a \in (b\uparrow)_C \text{ and } a' \in (b'\uparrow)_C\}.$$

Now, let $c \in \{a \vee a' \mid a \in (b\uparrow)_C \text{ and } a' \in (b'\uparrow)_C\}$. Then $c = a_0 \vee a_0'$ for some $a_0 \in (b\uparrow)_C$ and $a_0' \in (b'\uparrow)_C$. We thus have that $\Diamond a_0 = a_0$ and $b \leq a_0$, as well as $\Diamond a_0' = a_0'$ and $b' \leq a_0'$. Hence, $b \vee b' \leq a_0 \vee a_0' = c$ and $\Diamond c = \Diamond(a_0 \vee a_0') = \Diamond a_0 \vee \Diamond a_0' = a_0 \vee a_0' = c$, which means that $c \in (b \vee b'\uparrow)_C$. Therefore, $\{a \vee a' \mid a \in (b\uparrow)_C \text{ and } a' \in (b'\uparrow)_C\} \subseteq (b \vee b\uparrow)_C$, so

$$\Diamond^S(b \vee b') \leq \bigwedge\{a \vee a' \mid a \in (b\uparrow)_C \text{ and } a' \in (b'\uparrow)_C\} = \Diamond^S b \vee \Diamond^S b'. \qquad \Box$$

Before we prove the final lemma, which forms the crux of this subsection, we need two lemmas. But first we introduce the following terminology: an element a of a BAO is called *closed*, if $\Diamond a = a$; similarly a is called *open*, if $\Box a = a$. The following lemma tells us that for any assignment ν on a hybrid closure algebra and any Γ-formula γ, $\nu(\gamma)$ is an open element of the hybrid algebra.

Lemma 5.2.6. *Let $\mathfrak{A}$ be a hybrid closure algebra, let ν be an assignment on $\mathfrak{A}$, and let γ be a Γ-formula. Then $\Box\widetilde{\nu}(\gamma) = \widetilde{\nu}(\gamma)$.*

Proof. The proof is by structural induction on the α, β and γ-formulas. We first consider the α-formulas. The base steps are (i) – (iv) and the inductive steps (v) – (vi).

(i) Assume α is $\bot$. By (*refl*), we have that $\Box\bot = \bot$, so

$$\Box\widetilde{\nu}(\bot) = \Box\bot = \bot = \widetilde{\nu}(\bot).$$

(ii) Assume α is $\top$. Then we have

$$\Box\widetilde{\nu}(\top) = \Box\top = \top = \widetilde{\nu}(\top).$$

(iii) Assume α has the form $\Box^m p$ $(m > 0)$. Then

$$\begin{aligned}
\Box\widetilde{\nu}(\Box^m p) &= \Box\Box^m\widetilde{\nu}(p) && \text{(by Definition 2.1.8)}\\
&= \Box^m\widetilde{\nu}(p) && \text{(from the fact that } m > 0 \text{ and by } (\mathit{refl}) \text{ and } (\mathit{trans}))\\
&= \widetilde{\nu}(\Box^m p) && \text{(by Definition 2.1.8)}
\end{aligned}$$

(iv) Assume α has the form $\Box^n\mathbf{i}$ $(n > 0)$. Then

$$\begin{aligned}
\Box\widetilde{\nu}(\Box^n\mathbf{i}) &= \Box\Box^n\widetilde{\nu}(\mathbf{i}) && \text{(by Definition 2.1.8)}\\
&= \Box^n\widetilde{\nu}(\mathbf{i}) && \text{(from the fact that } n > 0 \text{ and by } (\mathit{refl}) \text{ and } (\mathit{trans}))\\
&= \widetilde{\nu}(\Box^n\mathbf{i}) && \text{(by Definition 2.1.8)}
\end{aligned}$$

(v) Assume α has the form $\alpha_1 \wedge \alpha_2$. Then we have

$$\begin{aligned}
\Box\widetilde{\nu}(\alpha_1 \wedge \alpha_2) &= \Box(\widetilde{\nu}(\alpha_1) \wedge \widetilde{\nu}(\alpha_2))\\
&= \Box\widetilde{\nu}(\alpha_1) \wedge \Box\widetilde{\nu}(\alpha_2) && \text{(since } \Box \text{ distributes over finite meets)}\\
&= \widetilde{\nu}(\alpha_1) \wedge \widetilde{\nu}(\alpha_2) && \text{(by the inductive hypothesis)}\\
&= \widetilde{\nu}(\alpha_1 \wedge \alpha_2)
\end{aligned}$$

(vi) Assume α has the form $\alpha_1 \vee \alpha_2$. First, by (*refl*), $\Box\widetilde{\nu}(\alpha_1 \vee \alpha_2) \leq \widetilde{\nu}(\alpha_1 \vee \alpha_2)$. Conversely,

$$\begin{aligned}
\Box\widetilde{\nu}(\alpha_1 \vee \alpha_2) &= \Box(\widetilde{\nu}(\alpha_1) \vee \widetilde{\nu}(\alpha_2))\\
&\geq \Box\widetilde{\nu}(\alpha_1) \vee \Box\widetilde{\nu}(\alpha_2) && \text{(by the monotonicity of } \Box)\\
&= \widetilde{\nu}(\alpha_1) \vee \widetilde{\nu}(\alpha_2) && \text{(by the inductive hypothesis)}\\
&= \widetilde{\nu}(\alpha_1 \vee \alpha_2)
\end{aligned}$$

For the β-formulas, the base steps are (i) – (ii) and the inductive steps (iii) – (iv).

(i) Assume β is α. Then $\Box\widetilde{\nu}(\alpha) = \widetilde{\nu}(\alpha)$ by the first part for the α-formulas.

(ii) If β has the form $\Box\neg\alpha$, then

$$\begin{aligned}
\Box\widetilde{\nu}(\Box(\neg\alpha)) &= \Box\Box\widetilde{\nu}(\neg\alpha) && \text{(by Definition 2.1.8)}\\
&= \Box\widetilde{\nu}(\neg\alpha) && \text{(by } (\mathit{refl}) \text{ and } (\mathit{trans}))\\
&= \widetilde{\nu}(\Box\neg\alpha) && \text{(by Definition 2.1.8)}
\end{aligned}$$

(iii) The Boolean cases are the same as for the α-formulas.

(iv) Assume β has the form $\Box(\alpha \to \beta)$. Then

$$\begin{aligned}
\Box\widetilde{\nu}(\Box(\alpha \to \beta)) &= \Box\Box\widetilde{\nu}(\alpha \to \beta) && \text{(by Definition 2.1.8)}\\
&= \Box\widetilde{\nu}(\alpha \to \beta) && \text{(by } (\textit{refl}) \text{ and } (\textit{trans})\text{)}\\
&= \widetilde{\nu}(\Box(\alpha \to \beta)) && \text{(by Definition 2.1.8)}
\end{aligned}$$

Finally, we consider the γ-formulas. The base steps are (i) – (iv) and the inductive steps (v) – (vi).

(i) If α has the form $\Box\neg\mathbf{i}$, then

$$\begin{aligned}
\Box\widetilde{\nu}(\Box\neg\mathbf{i}) &= \Box\Box\widetilde{\nu}(\neg\mathbf{i}) && \text{(by Definition 2.1.8)}\\
&= \Box\widetilde{\nu}(\neg\mathbf{i}) && \text{(by } (\textit{refl}) \text{ and } (\textit{trans})\text{)}\\
&= \widetilde{\nu}(\Box\neg\mathbf{i}) && \text{(by Definition 2.1.8)}
\end{aligned}$$

(ii) Assume γ is α. Then $\Box\widetilde{\nu}(\alpha) = \widetilde{\nu}(\alpha)$ by the first part for the α-formulas.

(iii) Assume γ has the form $\Box(\beta \to \alpha)$. Then

$$\begin{aligned}
\Box\widetilde{\nu}(\Box(\beta \to \alpha)) &= \Box\Box\widetilde{\nu}(\beta \to \alpha) && \text{(by Definition 2.1.8)}\\
&= \Box\widetilde{\nu}(\beta \to \alpha) && \text{(by } (\textit{refl}) \text{ and } (\textit{trans})\text{)}\\
&= \widetilde{\nu}(\Box(\beta \to \alpha)) && \text{(by Definition 2.1.8)}
\end{aligned}$$

(iv) If γ has the form $\Box\neg\beta$, then we have

$$\begin{aligned}
\Box\widetilde{\nu}(\Box(\neg\beta)) &= \Box\Box\widetilde{\nu}(\neg\beta) && \text{(by Definition 2.1.8)}\\
&= \Box\widetilde{\nu}(\neg\beta) && \text{(by } (\textit{refl}) \text{ and } (\textit{trans})\text{)}\\
&= \widetilde{\nu}(\Box\neg\beta) && \text{(by Definition 2.1.8)}
\end{aligned}$$

(v) The Boolean cases are the same as for the α-formulas.

(vi) If γ has the form $\Box\gamma$, then

$$\begin{aligned}
\Box\widetilde{\nu}(\Box(\gamma)) &= \Box\Box\widetilde{\nu}(\gamma) && \text{(by Definition 2.1.8)}\\
&= \Box\widetilde{\nu}(\gamma) && \text{(by } (\textit{refl}) \text{ and } (\textit{trans})\text{)}\\
&= \widetilde{\nu}(\Box\gamma) && \text{(by Definition 2.1.8)}
\end{aligned}$$

□

We can also show that the D_i's in $\mathfrak{A}$ are open:

Lemma 5.2.7. $\Box D_i = D_i$ *for each* $i = 1, 2$.

Proof. First, recall from the definition of a piecewise well-connected hybrid algebra that for each $j = 1, 2$, $\Diamond D_j = D_j$. Hence, $\Box\neg D_j = \neg D_j$. But we know that $D_1 \wedge D_2 = \bot$ and $D_1 \vee D_2 = \top$, so $D_i \leq \neg D_j$ and $\neg D_j \leq D_i$, $i, j = 1, 2$ and $i \neq j$. This means that $\neg D_j = D_i$, $i, j = 1, 2$ and $i \neq j$, which gives $\Box D_i = D_i$. $\square$

Lemma 5.2.8. *Let γ be a Γ-formula. If $\mathfrak{A} \models \gamma \approx \top$, then $\mathfrak{B}_S \models \gamma \approx \top$.*

Proof. We have two cases, namely (i) D_1 and D_2 are disjoint 'pieces', or (ii) $D_1 = D_2 = \top$. We will only prove the lemma for the first case as the proof of the second case is similar. Let θ be an assignment on $\mathfrak{B}_S$, and define an assignment ρ: $\mathsf{PROP} \cup \mathsf{NOM} \to A$ by $\rho(p) = \Box^S\theta(p)$ for all $p \in \mathsf{PROP}$ and $\rho(\mathbf{i}) = \theta(\mathbf{i})$ for all $\mathbf{i} \in \mathsf{NOM}$. Now, for each $i = 1, 2$, define $\theta_i(p) = \theta(p) \wedge D_i$ and $\theta_i(\mathbf{i}) = \theta(\mathbf{i}) \wedge D_i$. Similarly, for each $i = 1, 2$, define $\rho_i(p) = \rho(p) \wedge D_i$ and $\rho_i(\mathbf{i}) = \rho(\mathbf{i}) \wedge D_i$. We now prove the following claims:

Claim 1. For each $i = 1, 2$ and any α-formula α, $\widetilde{\rho_i}(\alpha) = \widetilde{\theta_i}(\alpha)$.

Proof of claim. The proof of this claim is by structural induction on α. The base steps are (i) – (iv) and the inductive steps (v) – (vi).

(i) Assume α is $\bot$. Then $\widetilde{\rho_i}(\bot) = \bot = \widetilde{\theta_i}(\bot)$.

(ii) If α has the form $\top$, then $\widetilde{\rho_i}(\top) = \top = \widetilde{\theta_i}(\top)$.

(iii) Assume α has the form $\Box^m p$ $(m > 0)$. Then we have

$$\begin{aligned}
\widetilde{\rho_i}(\Box^m p) &= \Box^m \rho_i(p) && \text{(by Definition 2.1.8)}\\
&= \Box\rho_i(p) && \text{(by (\textit{refl}) and (\textit{trans}))}\\
&= \Box(\rho(p) \wedge D_i) && \text{(by the definition of } \rho_i)\\
&= \Box\rho(p) \wedge \Box D_i && \text{(since } \Box \text{ distributes over finite meets)}\\
&= \Box\Box^S\theta(p) \wedge \Box D_i && \text{(by the definition of } \rho)\\
&= \Box^S\theta(p) \wedge \Box^S D_i && \text{(by Lemmas 5.2.3 and 5.2.4)}\\
&= \Box^S(\theta(p) \wedge D_i) && \text{(since } \Box^S \text{ distributes over finite meets)}\\
&= \Box^S\theta_i(p) && \text{(by the definition of } \theta_1)\\
&= (\Box^S)^m\theta_i(p) && \text{(by the fact that } \mathfrak{B}_S \text{ validates (\textit{refl}) and (\textit{trans}))}\\
&= \widetilde{\theta_i}(\Box^n p) && \text{(by Definition 2.1.8)}
\end{aligned}$$

(iv) If α has the form $\Box^n \mathbf{i}$ $(n > 0)$, then

$$\begin{aligned}
\widetilde{\rho_i}(\Box^n \mathbf{i}) &= \Box^n \rho_i(\mathbf{i}) \\
&= \Box \rho_i(\mathbf{i}) && \text{(by } (\mathit{refl}) \text{ and } (\mathit{trans})\text{)} \\
&= \Box(\rho(\mathbf{i}) \wedge D_i) && \text{(by the definition of } \rho_i\text{)} \\
&= \Box\rho(\mathbf{i}) \wedge \Box D_i && \text{(since } \Box \text{ distributes over finite meets)} \\
&= \Box\theta(\mathbf{i}) \wedge \Box D_i && \text{(by the definition of } \rho\text{)} \\
&= \Box^S\theta(\mathbf{i}) \wedge \Box^S D_i && (\theta(\mathbf{i}) \in X_S \text{ and Lemma 5.2.4}) \\
&= \Box^S(\theta(\mathbf{i}) \wedge D_i) && \text{(since } \Box^S \text{ distributes over finite meets)} \\
&= \Box^S\theta_i(\mathbf{i}) && \text{(by the definition of } \theta_i\text{)} \\
&= (\Box^S)^n\theta_i(\mathbf{i}) && \text{(by } (\mathit{refl}) \text{ and } (\mathit{trans})\text{)} \\
&= \widetilde{\theta_i}(\Box^n \mathbf{i})
\end{aligned}$$

(v) Assume α has the form $\alpha_1 \wedge \alpha_2$. Then

$$\begin{aligned}
\widetilde{\rho_i}(\alpha_1 \wedge \alpha_2) &= \widetilde{\rho_i}(\alpha_1) \wedge \widetilde{\rho_i}(\alpha_2) && \text{(by Definition 2.1.8)} \\
&= \widetilde{\theta_i}(\alpha_1) \wedge \widetilde{\theta_i}(\alpha_2) && \text{(by the inductive hypothesis)} \\
&= \widetilde{\theta_i}(\alpha_1 \wedge \alpha_2) && \text{(by Definition 2.1.8)}
\end{aligned}$$

(vi) If α is of the form $\alpha_1 \vee \alpha_2$, then

$$\begin{aligned}
\widetilde{\rho_i}(\alpha_1 \vee \alpha_2) &= \widetilde{\rho_i}(\alpha_1) \vee \widetilde{\rho_i}(\alpha_2) && \text{(by Definition 2.1.8)} \\
&= \widetilde{\theta_i}(\alpha_1) \vee \widetilde{\theta_i}(\alpha_2) && \text{(by the inductive hypothesis)} \\
&= \widetilde{\theta_i}(\alpha_1 \vee \alpha_2) && \text{(by Definition 2.1.8)}
\end{aligned}$$

Claim 2. $\widetilde{\rho}(\alpha) = \widetilde{\theta}(\alpha)$ for any α-formula α.

Proof of claim.

$$\begin{aligned}
\widetilde{\rho}(\alpha) &= \widetilde{\rho}(\alpha) \wedge \top \\
&= \widetilde{\rho}(\alpha) \wedge (D_1 \vee D_2) && \text{(by definition of a piecewise well-connected algebra)} \\
&= (\widetilde{\rho}(\alpha) \wedge D_1) \vee (\widetilde{\rho}(\alpha) \wedge D_2) && \text{(by distributivity)} \\
&= \widetilde{\rho_1}(\alpha) \vee \widetilde{\rho_2}(\alpha) && \text{(by the definitions of } \rho_1 \text{ and } \rho_2\text{)} \\
&= \widetilde{\theta_1}(\alpha) \vee \widetilde{\theta_2}(\alpha) && \text{(by Claim 1)} \\
&= (\widetilde{\theta}(\alpha) \wedge D_1) \vee (\widetilde{\theta}(\alpha) \wedge D_2) && \text{(by the definitions of } \theta_1 \text{ and } \theta_2\text{)} \\
&= \widetilde{\theta}(\alpha) \wedge (D_1 \vee D_2) && \text{(by distributivity)} \\
&= \widetilde{\theta}(\alpha) \wedge \top && \text{(by definition of a piecewise well-connected algebra)} \\
&= \widetilde{\theta}(\alpha)
\end{aligned}$$

Claim 3. For each $i = 1, 2$ and any β-formula β, $\widetilde{\theta}_i(\beta) \leq \widetilde{\rho}_i(\beta)$ for any β-formula β.

Proof of claim. The proof of this claim is also by structural induction. The base step are (i) – (ii) and the inductive steps (iii) – (iv).

(i) The case where β is α follows from Claim 1.

(ii) If β is of the form $\Box\neg\alpha$, then

$$\begin{aligned}
\widetilde{\rho}_i(\Box\neg\alpha) &= \Box\neg\widetilde{\rho}_i(\alpha) \\
&= \Box\neg\widetilde{\theta}_i(\alpha) && \text{(by Claim 1)} \\
&\geq \Box^S\neg\widetilde{\theta}_i(\alpha) && \text{(by Lemma 5.2.3)} \\
&= \widetilde{\theta}_i(\Box\neg\alpha)
\end{aligned}$$

(iii) The Boolean case are similar to that of the α-formulas.

(iv) Assume β is of the form $\Box(\alpha \to \beta)$. Here

$$\begin{aligned}
\widetilde{\rho}_i(\Box(\alpha \to \beta)) &= \Box\widetilde{\rho}_i(\alpha \to \beta) \\
&= \Box\widetilde{\rho}_i(\neg\alpha \vee \beta) \\
&= \Box(\neg\widetilde{\rho}_i(\alpha) \vee \widetilde{\rho}_i(\beta)) \\
&\geq \Box(\neg\widetilde{\theta}_i(\alpha) \vee \widetilde{\theta}_i(\beta)) && \text{(by Claim 1 and the inductive hypothesis)} \\
&= \Box(\widetilde{\theta}_i(\neg\alpha \vee \beta)) \\
&= \Box\widetilde{\theta}_i(\alpha \to \beta) \\
&\geq \Box^S\widetilde{\theta}_i(\alpha \to \beta) && \text{(by Lemma 5.2.3)} \\
&= \widetilde{\theta}_i(\Box(\alpha \to \beta))
\end{aligned}$$

Claim 4. $\widetilde{\theta}(\beta) \leq \widetilde{\rho}(\beta)$ for any β-formula β.

Proof of claim.

$$\begin{aligned}
\widetilde{\theta}(\beta) &= \widetilde{\theta}(\beta) \wedge \top \\
&= \widetilde{\theta}(\beta) \wedge (D_1 \vee D_2) && \text{(by definition of a piecewise well-connected algebra)} \\
&= (\widetilde{\theta}(\beta) \wedge D_1) \vee (\widetilde{\theta}(\beta) \wedge D_2) && \text{(by distributivity)} \\
&= \widetilde{\theta}_1(\beta) \vee \widetilde{\theta}_2(\beta) && \text{(by the definitions of } \theta_1 \text{ and } \theta_2\text{)} \\
&\leq \widetilde{\rho}_1(\beta) \vee \widetilde{\rho}_2(\beta) && \text{(by Claim 3)} \\
&= (\widetilde{\rho}(\beta) \wedge D_1) \vee (\widetilde{\rho}(\beta) \wedge D_2) && \text{(by the definitions of } \rho_1 \text{ and } \rho_2\text{)} \\
&= \widetilde{\rho}(\beta) \wedge (D_1 \vee D_2) && \text{(by distributivity)} \\
&= \widetilde{\rho}(\beta) \wedge \top && \text{(by definition of a piecewise well-connected algebra)} \\
&= \widetilde{\rho}(\beta)
\end{aligned}$$

Claim 5. For each $i = 1, 2$ and any Γ-formula γ, $\widetilde{\rho}_i(\gamma) = D_i$ implies $\widetilde{\theta}_i(\gamma) = D_i$.

Proof of claim. Note that we only need to show that $\widetilde{\rho}_i(\gamma) \geq D_i$ implies $\widetilde{\theta}_i(\gamma) \geq D_i$ since both $\widetilde{\rho}_i(\gamma) \leq D_i$ and $\widetilde{\theta}_i(\gamma) \leq D_i$. We use structural induction on γ. The base steps are (i) – (iv) and the inductive steps (v) – (vii).

(i) If γ has the form $\Box\neg\mathbf{i}$, then

$$
\begin{aligned}
&\widetilde{\rho}_i(\Box\neg\mathbf{i}) \geq D_i \\
&\Longrightarrow \Box\widetilde{\rho}_i(\neg\mathbf{i}) \geq D_i \\
&\Longrightarrow \widetilde{\rho}_i(\neg\mathbf{i}) \geq D_i && \text{(by } (\mathit{refl})\text{)} \\
&\Longrightarrow \neg\widetilde{\rho}_i(\mathbf{i}) \geq D_i \\
&\Longrightarrow \neg(\widetilde{\rho}(\mathbf{i}) \wedge D_i) \geq D_i && \text{(by the definition of } \rho_i\text{)} \\
&\Longrightarrow \neg(\widetilde{\theta}(\mathbf{i}) \wedge D_i) \geq D_i && \text{(by the definition of } \rho\text{)} \\
&\Longrightarrow \neg\widetilde{\theta}_i(\mathbf{i}) \geq D_i && \text{(by the definition of } \theta_i\text{)} \\
&\Longrightarrow \widetilde{\theta}_i(\neg\mathbf{i}) \geq D_i \\
&\Longrightarrow \Box^S\widetilde{\theta}_i(\neg\mathbf{i}) \geq D_i && \text{(by monotonicity of } \Box^S \text{ and Lemmas 5.2.7 and 5.2.4)} \\
&\Longrightarrow \widetilde{\theta}_i(\Box\neg\mathbf{i}) \geq D_i
\end{aligned}
$$

(ii) If γ is α, the claim immediately follows from Claim 1.

(iii) Assume that γ has the form $\Box(\beta \to \alpha)$. Then

$$
\begin{aligned}
&\widetilde{\rho}_i(\Box(\beta \to \alpha)) \geq D_i \\
&\Longrightarrow \Box\widetilde{\rho}_i(\beta \to \alpha) \geq D_i \\
&\Longrightarrow \widetilde{\rho}_i(\beta \to \alpha) \geq D_i && \text{(by } (\mathit{refl})\text{)} \\
&\Longrightarrow \widetilde{\rho}_i(\neg\beta \vee \alpha) \geq D_i \\
&\Longrightarrow \neg\widetilde{\rho}_i(\beta) \vee \widetilde{\rho}(\alpha) \geq D_i \\
&\Longrightarrow \neg\widetilde{\theta}_i(\beta) \vee \widetilde{\theta}_i(\alpha) \geq D_i && \text{(by Claims 1 and 3)} \\
&\Longrightarrow \widetilde{\theta}_i(\neg\beta \vee \alpha) \geq D_i \\
&\Longrightarrow \widetilde{\theta}_i(\beta \to \alpha) \geq D_i \\
&\Longrightarrow \Box^S\widetilde{\theta}_i(\beta \to \alpha) \geq D_i && \text{(by monotonicity of } \Box^S \text{ and Lemmas 5.2.7 and 5.2.4)} \\
&\Longrightarrow \widetilde{\theta}_i(\Box(\beta \to \alpha) \geq D_i
\end{aligned}
$$

(iv) If γ has the form $\Box\neg\beta$, then

$$\begin{aligned}
&\widetilde{\rho}_i(\Box\neg\beta) \geq D_i \\
&\Longrightarrow \Box\widetilde{\rho}_i(\neg\beta) \geq D_i \\
&\Longrightarrow \widetilde{\rho}_i(\neg\beta) \geq D_i && \text{(by } (\textit{refl})\text{)} \\
&\Longrightarrow \neg\widetilde{\rho}_i(\beta) \geq D_i && \text{(by Definition 2.1.8)} \\
&\Longrightarrow \neg\widetilde{\theta}_i(\beta) \geq D_i && \text{(by Claim 3)} \\
&\Longrightarrow \widetilde{\theta}_i(\neg\beta) \geq D_i && \text{(by Definition 2.1.8)} \\
&\Longrightarrow \Box^S\widetilde{\theta}_i(\neg\beta) \geq D_i && \text{(by monotonicity of } \Box^S \text{ and Lemmas 5.2.7 and 5.2.4)} \\
&\Longrightarrow \widetilde{\theta}_i(\Box\neg\beta) \geq D_i
\end{aligned}$$

(v) Assume γ has the form $\gamma_1 \wedge \gamma_2$. Then we have

$$\begin{aligned}
&\widetilde{\rho}_i(\gamma_1 \wedge \gamma_2) \geq D_i \\
&\Longrightarrow \widetilde{\rho}_i(\gamma_1) \wedge \widetilde{\rho}_i(\gamma_2) \geq D_i \\
&\Longrightarrow \widetilde{\rho}_i(\gamma_1) \geq D_i \text{ and } \widetilde{\rho}_i(\gamma_2) \geq D_i \\
&\Longrightarrow \widetilde{\theta}_i(\gamma_1) \geq D_i \text{ and } \widetilde{\theta}_i(\gamma_2) \geq D_i && \text{(by the inductive hypothesis)} \\
&\Longrightarrow \widetilde{\theta}_i(\gamma_1) \wedge \theta(\gamma_2) \geq D_i \\
&\Longrightarrow \widetilde{\theta}_i(\gamma_1 \wedge \gamma_2) \geq D_i
\end{aligned}$$

(vi) Assume γ has the form $\gamma_1 \vee \gamma_2$, and suppose $\widetilde{\rho}_i(\gamma_1 \vee \gamma_2) = D_i$. Then $\widetilde{\rho}_i(\gamma_1) \vee \widetilde{\rho}_i(\gamma_2) = D_i$. But we know that $\Box\widetilde{\rho}(\gamma_1) = \widetilde{\rho}(\gamma_1)$ by Lemma 5.2.6. We also know from Lemma 5.2.7 that for each $i = 1, 2$, $\Box D_i = D_i$. We therefore have

$$\Box\widetilde{\rho}_i(\gamma_1) = \Box(\widetilde{\rho}(\gamma_1) \wedge D_i) = \Box\widetilde{\rho}(\gamma_1) \wedge \Box D_i = \widetilde{\rho}(\gamma_1) \wedge D_i = \widetilde{\rho}_i(\gamma_1)$$

Similarly, $\Box\widetilde{\rho}_i(\gamma_2) = \widetilde{\rho}_i(\gamma_2)$. Hence, $\Box\widetilde{\rho}_i(\gamma_1) \vee \Box\widetilde{\rho}_i(\gamma_2) = D_i$, and so, for $j \neq i$,

$$\Diamond\neg\widetilde{\rho}_i(\gamma_1) \wedge \Diamond\neg\widetilde{\rho}_i(\gamma_2) = D_j.$$

But then $\Diamond\neg\widetilde{\rho}_i(\gamma_1) \wedge \Diamond\neg\widetilde{\rho}_i(\gamma_2) \wedge D_i = \bot$, which means that

$$\Diamond\neg\widetilde{\rho}_i(\gamma_1) \wedge \Diamond D_i \wedge \Diamond\neg\widetilde{\rho}_i(\gamma_2) \wedge \Diamond D_i = \bot.$$

Hence, by the monotonicity of $\Diamond$, $\Diamond(\neg\widetilde{\rho}_i(\gamma_1) \wedge D_i) \wedge \Diamond(\neg\widetilde{\rho}_i(\gamma_2) \wedge D_i) = \bot$, and so, since $\neg\widetilde{\rho}_i(\gamma_1) \wedge D_i \leq D_i$ and $\neg\widetilde{\rho}_i(\gamma_2) \wedge D_i \leq D_i$, by the definition of a piecewise well-connected hybrid algebra, $\neg\widetilde{\rho}_i(\gamma_1) \wedge D_i = \bot$ or $\neg\widetilde{\rho}_i(\gamma_2) \wedge D_i = \bot$. We then get $D_i \leq \widetilde{\rho}_i(\gamma_1)$ or $D_i \leq \widetilde{\rho}_i(\gamma_2)$. But $\widetilde{\rho}_i(\gamma_1) \leq D_i$ and $\widetilde{\rho}_i(\gamma_2) \leq D_i$, so $\widetilde{\rho}_i(\gamma_1) = D_i$ or $\widetilde{\rho}_i(\gamma_2) = D_i$. Now, by the inductive hypothesis, $\widetilde{\theta}_i(\gamma_1) = D_i$ or $\widetilde{\theta}_i(\gamma_2) = D_i$, which gives $\widetilde{\theta}_i(\gamma_1) \vee \widetilde{\theta}_i(\gamma_2) = D_i$. Therefore, $\widetilde{\theta}_i(\gamma_1 \vee \gamma_2) = D_i$.

(vii) If γ has the form $\Box\gamma_1$, then

$$\begin{aligned}
&\widetilde{\rho}_i(\Box\gamma_1) \geq D_i \\
&\implies \Box\widetilde{\rho}_i(\gamma_1) \geq D_i \\
&\implies \widetilde{\rho}_i(\gamma_1) \geq D_i && \text{(by } (\mathit{refl})) \\
&\implies \widetilde{\theta}_i(\gamma_1) \geq D_i && \text{(by the inductive hypothesis)} \\
&\implies \Box^S\widetilde{\theta}_i(\gamma_1) \geq D_i && \text{(by the monotonicity of } \Box^S \text{ and Lemmas 5.2.7 and 5.2.4)} \\
&\implies \widetilde{\theta}_i(\Box\gamma_1) \geq D_i
\end{aligned}$$

Claim 6. For any Γ-formula γ, $\widetilde{\rho}(\gamma) = \top$ implies $\widetilde{\theta}(\gamma) = \top$.

Proof of claim.

$$\begin{aligned}
&\widetilde{\rho}(\gamma) = \top \\
&\implies \widetilde{\rho}(\gamma) \wedge D_1 = D_1 \text{ and } \widetilde{\rho}(\gamma) \wedge D_2 = D_2 \\
&\implies \widetilde{\rho_1}(\gamma) = D_1 \text{ and } \widetilde{\rho_2}(\gamma) = D_2 && \text{(by the definitions of } \rho_1 \text{ and } \rho_2) \\
&\implies \widetilde{\theta_1}(\gamma) = D_1 \text{ and } \widetilde{\theta_2}(\gamma) = D_2 && \text{(by Claim 5)} \\
&\implies \widetilde{\theta}(\gamma) \wedge D_1 = D_1 \text{ and } \widetilde{\theta}(\gamma) \wedge D_2 = D_2 && \text{(by the definitions of } \theta_1 \text{ and } \theta_2) \\
&\implies (\widetilde{\theta}(\gamma) \wedge D_1) \vee (\widetilde{\theta}(\gamma) \wedge D_2) = D_1 \vee D_2 = \top \\
&\implies \widetilde{\theta}(\gamma) \wedge (D_1 \vee D_2) = \top \\
&\implies \widetilde{\theta}(\gamma) = \top
\end{aligned}$$

Now, since $\mathfrak{A} \models \gamma \approx \top$, we have $\mathfrak{B}_S \models \gamma \approx \top$ by Claim 6. □

Corollary 5.2.9. *Let Σ be a finite set of Γ-formulas. Then* $\mathbf{HS4} \oplus \Sigma$ *is decidable.*

5.2.2 Hybrid extensions of S4 in the language $\mathcal{H}(@)$ with the finite model property

In this section, we prove a result analogous to Bull's result in [18] for the logic obtained by adding the axioms (4) and (T) to the logic $\mathbf{H}(@)$ (denoted $\mathbf{H}(@)\mathbf{S4}$). But first we define the inductive class of formulas we will work in this subsection.

We define $\Gamma(@)$-*formulas* by the following inductive rules:

$$\begin{aligned}
\alpha &::= \bot \mid \top \mid \mathbf{i} \mid \Box^m p \mid \Box^n \mathbf{i} \mid \neg\alpha \mid \alpha_1 \wedge \alpha_2 \mid \alpha_1 \vee \alpha_2 \mid @_{\mathbf{i}}\alpha \ (m, n > 0) \\
\beta &::= \alpha \mid \beta_1 \wedge \beta_2 \mid \beta_1 \vee \beta_2 \mid \alpha \to \beta \mid \Box\beta \mid @_{\mathbf{i}}\beta \\
\gamma &::= \alpha \mid \Diamond\alpha \mid \neg\beta \mid \Diamond\neg\beta \mid \beta \to \alpha \mid \gamma_1 \wedge \gamma_2 \mid \Box\gamma
\end{aligned}$$

In comparing this to the definition of α, β and γ-formulas in Subsection 5.2.1, notice the absence of the clause $\gamma_1 \vee \gamma_2$ in the above definition. Recall that in Subsection 5.2.1, we used well-connectedness for the inductive step in Lemma 5.2.8 where γ is of the form $\gamma_1 \vee \gamma_2$. So if we add this clause to the above definition, we need well-connectedness. Like in Subsection 5.1.2, we would have to add the axiom $@_{\mathbf{i}}\Diamond\mathbf{j} \vee @_{\mathbf{j}}\Diamond\mathbf{i}$ to the logic $\mathbf{H}(@)\mathbf{S4}$ to get well-connectedness. However, as in Subsection 5.1.2, we would still only be able to prove a finite hybrid algebra property for the named fragment of the logic $\mathbf{H}(@)\mathbf{S4}$ extended with $@_{\mathbf{i}}\Diamond\mathbf{j} \vee @_{\mathbf{j}}\Diamond\mathbf{i}$ and $\Gamma(@)$-formulas. So instead we drop the clause $\gamma_1 \vee \gamma_2$, which in turn allows us to add all nominals and all formulas of the form $\neg\alpha$ to the set of α-formulas, as well as all formulas of the form $\Diamond\alpha$ and $\Diamond\neg\beta$ to the set of γ-formulas. Also notice the absence of boxes in some of the clauses. The reason we can do this is that we dropped the clause $\gamma_1 \vee \gamma_2$, and therefore don't need a lemma similar to Lemma 5.2.6 here.

Before we give the main results of this section, we give some examples of $\Gamma(@)$-formulas important in the study of hybrid logic.

Example 5.2.10. The following are $\Gamma(@)$-formulas:

(i) the formula $\Box p \to \Box\Box p$ defining transitivity,

(ii) the formula $\Box\Box p \to \Box p$ defining density,

(iii) the formula $\mathbf{i} \to \Diamond\mathbf{i} \equiv \Box\neg\mathbf{i} \to \neg\mathbf{i}$ defining reflexivity,

(iv) the formula $\Diamond\mathbf{i}$ defining universality, and

(v) the formula $@_{\mathbf{i}}(\neg\mathbf{j} \wedge \neg\mathbf{k}) \to @_{\mathbf{j}}\mathbf{k}$ defining the class of frames with at most two states.

Recall that Bull used the fact that every normal extension of $\mathbf{S4}$ is sound and complete with respect to the corresponding class of closure algebras validating its axioms to obtain his result. We have a similar result for the normal hybrid logic $\mathbf{H}(@)\mathbf{S4}$:

Theorem 5.2.11. *The logic $\mathbf{H}(@)\mathbf{S4} \oplus \Sigma$ is sound and complete with respect to the class of all hybrid closure @-algebras validating Σ.*

Proof. This result follows immediately from Theorem 3.2.1. $\square$

We now give the main result, and subsequently prove the results needed in the proof.

Theorem 5.2.12. *Let Σ be a set of $\Gamma(@)$-formulas. Then the logic $\mathbf{H}(@)\mathbf{S4} \oplus \Sigma$ has the strong finite hybrid algebra property.*

Proof. Suppose $\varphi \notin \mathbf{H}(@)\mathbf{S4} \oplus \Sigma$. By Theorem 5.2.11, there is a hybrid closure @-algebra $\mathfrak{A} = (\mathbf{A}, X_A)$ and an assignment ν such that $\mathfrak{A} \models \Sigma^{\approx}$ but $\mathfrak{A}, \nu \not\models \varphi \approx \top$. Now, let S_0 be the set of elements of $\mathfrak{A}$ used in the evaluation of φ and $\top$ under ν. Then let $S_1 = S_0 \cup \{x_0\}$, where x_0 is an arbitrary atom in X_A, and let $X_S = X_A \cap S_1$. We then let $S = S_1 \cup \{\Box x \mid x \in X_S\}$. Define $\mathbf{B}_S$ as the Boolean subalgebra of $\mathfrak{A}$ generated by S. Next, for $b \in B_S$, let $(b\uparrow)_C = \{a \in B_S \mid a = \Diamond a \text{ and } b \leq a\}$ and define

$$\Diamond^S b := \bigwedge (b\uparrow)_C.$$

Furthermore, for any $x \in X_S$ and $b \in B_S$, let

$$@^S_x b = \begin{cases} \top & x \leq b \\ \bot & \text{otherwise.} \end{cases}$$

Finally, let $\mathfrak{B}_S = (\mathbf{B}_S, \Diamond^S, @^S, X_S)$. We have to show that $\mathfrak{B}_S$ is the right kind of algebra. First, by Proposition 5.2.16, $\mathfrak{B}_S$ is a hybrid closure @-algebra. By Lemma 5.2.4, $\Diamond^S$ extends $\Diamond$. Furthermore, by Lemma 5.2.15, $@^S_x b = @_x b$ for all $x \in X_S$ and $b \in B_S$. Hence, $\mathfrak{B}_S \not\models \varphi \approx \top$. Finally, for all $\Gamma(@)$-formulas γ in Σ, $\mathfrak{B}_S \models \gamma \approx \top$ by Lemma 5.2.17.

To conclude this proof, note that $|S| \leq 2l(\varphi) + 3$, so $\mathfrak{B}_S$ contains at most $2^{l(\varphi)+3}$ atoms, and hence, at most $2^{2^{l(\varphi)+3}}$ elements. $\square$

For the remainder of this section, $\mathfrak{A}$ and $\mathfrak{B}_S$ will be the algebras in the proof of Theorem 5.2.12.

Lemma 5.2.13. *$\Diamond b \leq \Diamond^S b = \Diamond\Diamond^S b$ for all $b \in B_S$, and therefore, $\Box\Box^S b = \Box^S b \leq \Box b$ for all $b \in B_S$.*

Proof. The proof is the same as that of Lemma 5.2.3. $\square$

Lemma 5.2.14. *Let b be an element in B_S. If $\Diamond b \in B_S$, then $\Diamond^S b = \Diamond b$, and therefore, if $\Box b \in B_S$, then $\Box^S b = \Box b$.*

Proof. The proof is the same as that of Lemma 5.2.4 $\square$

Our next lemma shows that $@^S$ is a normal modal operator extending @.

Lemma 5.2.15. *If x is an element of X_S and b an element of B_S, then we have $@^S_x b = @_x b$.*

Proof. The proof is similar to that of Lemma 5.1.27. $\square$

Proposition 5.2.16. *$\mathfrak{B}_S$ is a hybrid closure @-algebra.*

Proof. First, $x_0 \in X_S$, so $X_S \neq \varnothing$. Proving that $\Diamond^S$ is a normal modal operator satisfying (*refl*) and (*trans*) is done in the same way as in Proposition 5.2.5. Finally, the fact that $@^S$ satisfies ($K@$), (*self-dual*), (*agree*), (*ref*), (*introduction*), and (*back*) follows from Proposition 2.2.5. $\square$

Lemma 5.2.17. *Let γ be a $\Gamma(@)$-formula. If $\mathfrak{A} \models \gamma \approx \top$, then $\mathfrak{B}_S \models \gamma \approx \top$.*

Proof. Let θ be an assignment on $\mathfrak{B}_S$, and define an assignment ρ: $\mathsf{PROP} \cup \mathsf{NOM} \to A$ by $\rho(p) = \Box^S\theta(p)$ for all $p \in \mathsf{PROP}$ and $\rho(\mathbf{i}) = \theta(\mathbf{i})$ for all $\mathbf{i} \in \mathsf{NOM}$. We now prove the following claims:

Claim 1. $\widetilde{\theta}(\alpha) = \widetilde{\rho}(\alpha)$ for any α-formula α.

Proof of claim. The proof of this claim is by structural induction on α. The base steps are (i) – (v) and the inductive steps (vi) – (ix).

(i) Assume α is $\bot$. Then $\widetilde{\rho}(\bot) = \bot = \widetilde{\theta}(\bot)$.

(ii) If α has the form $\top$, then $\widetilde{\rho}(\top) = \top = \widetilde{\theta}(\top)$.

(iii) Assume α has the form $\mathbf{i}$. Then we have

$$\begin{aligned}
\widetilde{\rho}(\mathbf{i}) &= \rho(\mathbf{i}) && \text{(by Definition 2.2.10)}\\
&= \theta(\mathbf{i}) && \text{(by the definition of } \rho)\\
&= \widetilde{\theta}(\mathbf{i}) && \text{(by Definition 2.2.10)}
\end{aligned}$$

(iv) Assume α has the form $\Box^m p$ $(m > 0)$. Then

$$\begin{aligned}
\widetilde{\rho}(\Box^m p) &= \Box^m \rho(p) \\
&= \Box^m \Box^S \theta(p) && \text{(by the definition of } \rho)\\
&= \Box\Box^S \theta(p) && \text{(from the fact that } m > 0 \text{ and by } (\textit{refl}) \text{ and } (\textit{trans}))\\
&= \Box^S \theta(p) && \text{(by Lemma 5.2.13)}\\
&= (\Box^S)^m \theta(p) && \text{(from the fact that } m > 0 \text{ and by } (\textit{refl}) \text{ and } (\textit{trans}))\\
&= \widetilde{\theta}(\Box^m p)
\end{aligned}$$

(v) If α has the form $\Box^n \mathbf{i}$, then we have

$$\begin{aligned}
\widetilde{\rho}(\Box^n \mathbf{i}) &= \Box^n \rho(\mathbf{i}) \\
&= \Box^n \theta(\mathbf{i}) && \text{(by the definition of } \rho)\\
&= \Box \theta(\mathbf{i}) && \text{(from the fact that } n > 0 \text{ and by } (\textit{refl}) \text{ and } (\textit{trans}))\\
&= \Box^S \theta(\mathbf{i}) && (\theta(\mathbf{i}) \in X_S \text{ and Lemma 5.2.14})\\
&= (\Box^S)^n \theta(\mathbf{i}) && \text{(from the fact that } n > 0 \text{ and by } (\textit{refl}) \text{ and } (\textit{trans}))\\
&= \widetilde{\theta}(\Box^n \mathbf{i})
\end{aligned}$$

(vi) Assume α has the form $\neg\alpha_1$. Then

$$\begin{aligned}
\widetilde{\rho}(\neg\alpha_1) &= \neg\widetilde{\rho}(\alpha_1) && \text{(by Definition 2.2.10)}\\
&= \neg\widetilde{\theta}(\alpha_1) && \text{(by the inductive hypothesis)}\\
&= \widetilde{\theta}(\neg\alpha_1) && \text{(by Definition 2.2.10)}
\end{aligned}$$

(vii) Assume α has the form $\alpha_1 \wedge \alpha_2$. Then

$$\begin{aligned}
\widetilde{\rho}(\alpha_1 \wedge \alpha_2) &= \widetilde{\rho}(\alpha_1) \wedge \widetilde{\rho}(\alpha_2) && \text{(by Definition 2.2.10)}\\
&= \widetilde{\theta}(\alpha_1) \wedge \widetilde{\theta}(\alpha_2) && \text{(by the inductive hypothesis)}\\
&= \widetilde{\theta}(\alpha_1 \wedge \alpha_2) && \text{(by Definition 2.2.10)}
\end{aligned}$$

(viii) If α is of the form $\alpha_1 \vee \alpha_2$, then

$$\begin{aligned}
\widetilde{\rho}(\alpha_1 \vee \alpha_2) &= \widetilde{\rho}(\alpha_1) \vee \widetilde{\rho}(\alpha_2) && \text{(by Definition 2.2.10)}\\
&= \widetilde{\theta}(\alpha_1) \vee \widetilde{\theta}(\alpha_2) && \text{(by the inductive hypothesis)}\\
&= \widetilde{\theta}(\alpha_1 \vee \alpha_2) && \text{(by Definition 2.2.10)}
\end{aligned}$$

(ix) Assume α has the form $@_{\mathbf{i}}\alpha$. Then we have

$$\begin{aligned}
\widetilde{\rho}(@_{\mathbf{i}}\alpha) &= @_{\rho(\mathbf{i})}\widetilde{\rho}(\alpha) \\
&= @_{\theta(\mathbf{i})}\widetilde{\theta}(\alpha) && \text{(by the inductive hypothesis and the definition of } \rho) \\
&= @^{S}_{\theta(\mathbf{i})}\widetilde{\theta}(\alpha) && \text{(by Lemma 5.2.15)} \\
&= \widetilde{\theta}(@_{\mathbf{i}}\alpha)
\end{aligned}$$

Claim 2. $\widetilde{\theta}(\beta) \leq \widetilde{\rho}(\beta)$ for any β-formula β.

Proof of claim. The proof of this claim is also by structural induction. The base step are (i) – (ii) and the inductive steps (iii) – (v).

(i) The case where β is α follows from Claim 1.

(ii) The Boolean case are similar to that of the α-formulas.

(iii) Assume β is of the form $\alpha \to \beta$. Here

$$\begin{aligned}
\widetilde{\rho}(\alpha \to \beta) &= \widetilde{\rho}(\neg\alpha \vee \beta) \\
&= \neg\widetilde{\rho}(\alpha) \vee \widetilde{\rho}(\beta) && \text{(by Definition 2.2.10)} \\
&\geq \neg\widetilde{\theta}(\alpha) \vee \widetilde{\theta}(\beta) && \text{(by Claim 1 and the inductive hypothesis)} \\
&= \widetilde{\theta}(\neg\alpha \vee \beta) && \text{(by Definition 2.2.10)} \\
&= \widetilde{\theta}(\alpha \to \beta)
\end{aligned}$$

(iv) Assume β has the form $\Box\beta$. Then

$$\begin{aligned}
\widetilde{\rho}(\Box\beta) &= \Box\widetilde{\rho}(\beta) \\
&\geq \Box\widetilde{\theta}(\beta) && \text{(by the inductive hypothesis and the monotonicity of } \Box) \\
&\geq \Box^{S}\widetilde{\theta}(\beta) && \text{(by Lemma 5.2.13)} \\
&= \widetilde{\theta}(\Box\beta)
\end{aligned}$$

(v) If β is of the form $@_{\mathbf{i}}\beta$, then

$$\begin{aligned}
\widetilde{\rho}(@_{\mathbf{i}}\beta) &= @_{\rho(\mathbf{i})}\widetilde{\rho}(\beta) \\
&\geq @_{\theta(\mathbf{i})}\widetilde{\theta}(\beta) && \text{(definition of } \rho\text{, monotonicity of @, and inductive hypothesis)} \\
&= @^{S}_{\theta(\mathbf{i})}\widetilde{\theta}(\beta) && \text{(by Lemma 5.2.15)} \\
&= \widetilde{\theta}(@_{\mathbf{i}}\beta)
\end{aligned}$$

Claim 3. For any Γ-formula γ, $\widetilde{\rho}(\gamma) = \top$ implies $\widetilde{\theta}(\gamma) = \top$.

Proof of claim. We use structural induction on γ. The base steps are (i) – (v) and the inductive steps (vi) – (vii).

(i) If γ is α, the claim immediately follows from Claim 1.

(ii) Assume γ has the form $\Diamond\alpha$. Then

$$\begin{aligned}
\widetilde{\rho}(\Diamond\alpha) = \top &\Longrightarrow \Diamond\widetilde{\rho}(\alpha) = \top \\
&\Longrightarrow \Diamond\widetilde{\theta}(\alpha) = \top && \text{(by Claim 2)} \\
&\Longrightarrow \Diamond^{S}\widetilde{\theta}(\alpha) = \top && \text{(by Lemma 5.2.13)} \\
&\Longrightarrow \widetilde{\theta}(\Diamond\alpha) = \top
\end{aligned}$$

(iii) If γ has the form $\neg\beta$, then

$$\begin{aligned}
\widetilde{\rho}(\neg\beta) = \top &\Longrightarrow \neg\widetilde{\rho}(\beta) = \top \\
&\Longrightarrow \neg\widetilde{\theta}(\beta) = \top && \text{(by Claim 2)} \\
&\Longrightarrow \widetilde{\theta}(\neg\beta) = \top
\end{aligned}$$

(iv) Assume γ has the form $\Diamond\neg\beta$. We then have

$$\begin{aligned}
\widetilde{\rho}(\Diamond\neg\beta) = \top &\Longrightarrow \Diamond\neg\widetilde{\rho}(\beta) = \top \\
&\Longrightarrow \Diamond\neg\widetilde{\theta}(\beta) = \top && \text{(by Claim 2 and the monotonicity of } \Diamond) \\
&\Longrightarrow \Diamond\widetilde{\theta}(\neg\beta) = \top \\
&\Longrightarrow \Diamond^{S}\widetilde{\theta}(\neg\beta) = \top && \text{(by Lemma 5.2.13)} \\
&\Longrightarrow \widetilde{\theta}(\Diamond\neg\beta) = \top
\end{aligned}$$

(v) If γ has the form $\beta \to \alpha$, then

$$\begin{aligned}
\widetilde{\rho}(\beta \to \alpha) = \top &\Longrightarrow \widetilde{\rho}(\neg\beta \vee \alpha) = \top \\
&\Longrightarrow \neg\widetilde{\rho}(\beta) \vee \widetilde{\rho}(\alpha) = \top \\
&\Longrightarrow \neg\widetilde{\theta}(\beta) \vee \widetilde{\theta}(\alpha) = \top && \text{(by Claims 1 and 2)} \\
&\Longrightarrow \widetilde{\theta}(\neg\beta \vee \alpha) = \top \\
&\Longrightarrow \widetilde{\theta}(\beta \to \alpha) = \top
\end{aligned}$$

(vi) Assume γ has the form $\gamma_1 \wedge \gamma_2$. Then we have

$$\begin{aligned}
\widetilde{\rho}(\gamma_1 \wedge \gamma_2) = \top &\Longrightarrow \widetilde{\rho}(\gamma_1) \wedge \widetilde{\rho}(\gamma_2) = \top \\
&\Longrightarrow \widetilde{\rho}(\gamma_1) = \top \text{ and } \widetilde{\rho}(\gamma_2) = \top \\
&\Longrightarrow \widetilde{\theta}(\gamma_1) = \top \text{ and } \widetilde{\theta}(\gamma_2) = \top && \text{(by the inductive hypothesis)} \\
&\Longrightarrow \widetilde{\theta}(\gamma_1) \wedge \theta(\gamma_2) = \top \\
&\Longrightarrow \widetilde{\theta}(\gamma_1 \wedge \gamma_2) = \top
\end{aligned}$$

(vii) If γ has the form $\Box\gamma_1$, then

$$\begin{aligned}
\widetilde{\rho}(\Box\gamma_1) = \top &\Longrightarrow \Box\widetilde{\rho}(\gamma_1) = \top \\
&\Longrightarrow \widetilde{\rho}(\gamma_1) = \top && \text{(by } (\mathit{refl})\text{)} \\
&\Longrightarrow \widetilde{\theta}(\gamma_1) = \top && \text{(by the inductive hypothesis)} \\
&\Longrightarrow \Box^S\widetilde{\theta}(\gamma_1) = \Box^S\top \\
&\Longrightarrow \Box^S\widetilde{\theta}(\gamma_1) = \top && \text{(Lemma 5.2.14)} \\
&\Longrightarrow \widetilde{\theta}(\Box\gamma_1) = \top
\end{aligned}$$

Now, since $\mathfrak{A} \models \gamma \approx \top$, we have $\mathfrak{B}_S \models \gamma \approx \top$ by Claim 3. $\Box$

Corollary 5.2.18. *Let Σ be a finite set of $\Gamma(@)$-formulas. Then $\mathbf{H}(@)\mathbf{S4} \oplus \Sigma$ is decidable.*

5.2.3 Hybrid extensions of S4 in the language $\mathcal{H}(\mathsf{E})$ with the finite model property

Here we show how to extend Bull's result in [18] to the logic obtained by adding the axioms (4) and (T) to the hybrid logic $\mathbf{H}(\mathsf{E})$ denoted $\mathbf{H}(\mathsf{E})\mathbf{S4}$. As in Section 5.1.3, we have to enforce well-connectedness axiomatically. However, not having well-connectedness is not the end of the world. As we stated in Section 5.2.2, if we remove the clause $\gamma_1 \vee \gamma_2$, we do not need well-connectedness. But more on this at the end of the section. For now we will work with the collection of $\Gamma(\mathsf{E})$-*formulas* defined below.

$\Gamma(\mathsf{E})$-formulas are recursively defined by the following inductive rules:

$$\begin{aligned}
\alpha &::= \bot \mid \top \mid \Box^m p \mid \Box^n \mathbf{i} \mid \mathsf{A}p \mid \mathsf{A}\mathbf{i} \mid \alpha_1 \wedge \alpha_2 \mid \alpha_1 \vee \alpha_2 \mid \mathsf{A}\alpha \ (m, n > 0) \\
\beta &::= \alpha \mid \Box\neg\alpha \mid \beta_1 \wedge \beta_2 \mid \beta_1 \vee \beta_2 \mid \Box(\alpha \to \beta) \mid \mathsf{A}\beta \\
\gamma &::= \Box\neg\mathbf{i} \mid \mathsf{A}\neg\mathbf{i} \mid \alpha \mid \Box(\beta \to \alpha) \mid \Box\neg\beta \mid \mathsf{A}\neg\beta \mid \mathsf{A}(\beta \to \alpha) \mid \gamma_1 \wedge \gamma_2 \mid \gamma_1 \vee \gamma_2 \mid \Box\gamma \mid \mathsf{A}\gamma
\end{aligned}$$

The above definition clearly extends the definition of the Γ-formulas, and therefore also extends Bull's definition in [18].

Example 5.2.19. The following formulas are $\Gamma(\mathsf{E})$-formulas:

(i) the formula $\Diamond\Box p \to \Box p \equiv \Box\neg\Box p \vee \Box p$ defining the class of euclidean frames,

(ii) the formula $\Diamond\Diamond\mathbf{i} \to \neg\Diamond\mathbf{i} \equiv \Box\Box\neg\mathbf{i} \vee \Box\neg\mathbf{i}$ defining intransitivity,

(iii) the formula $\Diamond\mathbf{i} \to \Box\mathbf{i} \equiv \Box\neg\mathbf{i} \vee \Box\mathbf{i}$ defining determinism,

(iv) the formula $\mathsf{A}\Diamond\top \equiv \mathsf{A}\neg\Box\neg\top$ defining the class of frames in which every state has a successor,

(v) the formula $\neg\mathsf{E}\Diamond\top \equiv \mathsf{A}\Box\bot$ defining the class of frames in which no state has a successor,

(vi) the formula $\neg\mathsf{E}\Diamond\mathbf{i} \equiv \mathsf{A}\Box\neg\mathbf{i}$ defining the class of frames in which no state has a named successor, and

(vii) the formula $\mathsf{A}(\Diamond\mathbf{i} \to \Box\mathbf{i}) \equiv \mathsf{A}(\Box\neg\mathbf{i} \vee \Box\mathbf{i})$ defining the class of frames in which every state has at most one successor.

Before we give the main result, recall that the logic **S4** is characterized by well-connected closure algebras. We have a similar result for the logic $\mathbf{H}(\mathsf{E})\mathbf{S4} \oplus \Sigma$.

Theorem 5.2.20. *Let Σ be a set of $\mathcal{H}(\mathsf{E})$-formulas containing the common predecessor axiom. Then the logic $\mathbf{H}(\mathsf{E})\mathbf{S4}\oplus\Sigma$ is sound and complete with respect to the class of all well-connected hybrid closure E-algebras validating Σ.*

Proof. This result follows from Theorem 3.3.1 and Lemma 5.1.33. $\square$

For the remainder of this section Σ will be a set of $\Gamma(\mathsf{E})$-formulas together with the common predecessor axiom $\mathsf{E}p \wedge \mathsf{E}q \to \mathsf{E}(\Diamond p \wedge \Diamond q)$.

We now give the main result. Once again we will prove the lemmas needed for the proof of this theorem afterwards.

Theorem 5.2.21. *The logic $\mathbf{H}(\mathsf{E})\mathbf{S4} \oplus \Sigma$ has the strong finite hybrid algebra property.*

Proof. Suppose $\varphi \notin \mathbf{H}(\mathsf{E})\mathbf{S4}\oplus\Sigma$. By Theorem 5.2.20, there is a well-connected hybrid closure E-algebra $\mathfrak{A} = (\mathbf{A}, X_A)$ and an assignment ν such that $\mathfrak{A}, \nu \not\models \varphi \approx \top$. Now, let S_0 be the set of elements of $\mathfrak{A}$ used in the evaluation of φ and $\top$ under ν. Then let $S_1 = S_0 \cup \{x_0\}$, where x_0 is an arbitrary atom in X_A, and let $S = S_1 \cup \{\Box x \mid x \in X_A \cap S_1\}$. Define $\mathbf{B}_S$ as the Boolean subalgebra of $\mathfrak{A}$ generated by S. Since S is a finite subset of A, $\mathbf{B}_S$ is finite. Furthermore, $\mathbf{B}_S$ clearly preserves all Boolean operations. Next, for $b \in B_S$, let $(b\uparrow)_C = \{a \in B_S \mid a = \Diamond a \text{ and } b \leq a\}$ and define

$$\Diamond^S b := \bigwedge (b\uparrow)_C.$$

Also, for any $b \in B_S$, let

$$\mathsf{E}^S b = \begin{cases} \top & b > \bot \\ \bot & \text{otherwise.} \end{cases}$$

Finally, let $\mathfrak{B}_S = (\mathbf{B}_S, \Diamond^S, \mathsf{E}^S, X_S)$, where $X_S = X_A \cap S_1$. We have to show that $\mathfrak{B}_S$ is the right kind of hybrid algebra. First, $X_S \neq \varnothing$ since $x_0 \in X_S$. By Lemma 5.2.23, $\Diamond^S$ extends $\Diamond$. Furthermore, by Lemma 5.2.24, $\mathsf{E}^S b = \mathsf{E}b$ for all $b \in B_S$. Hence, $\mathfrak{B}_S \not\models \varphi \approx \top$. Proving that $\Diamond^S$ is a normal modal operator satisfying (*refl*) and (*trans*) is done in the same way as in Proposition 5.2.5. From this and the definition of E^S, we can conclude that $\mathfrak{B}_S$ is a hybrid closure E-algebra. To show that $\mathsf{E}^S a \wedge \mathsf{E}^S b \leq \mathsf{E}^S(\Diamond^S a \wedge \Diamond^S b)$, let $a, b \in B_S$. Then $\mathsf{E}^S a \wedge \mathsf{E}^S b = \mathsf{E}a \wedge \mathsf{E}b$ by Lemma 5.2.24. But since $\mathfrak{A}$ validates the common predecessor axiom, $\mathsf{E}^S a \wedge \mathsf{E}^S b = \mathsf{E}a \wedge \mathsf{E}b \leq \mathsf{E}(\Diamond a \wedge \Diamond b)$. Now, by Lemma 5.2.22, $\mathsf{E}^S a \wedge \mathsf{E}^S b \leq \mathsf{E}(\Diamond^S a \wedge \Diamond^S b)$, and so, by Lemma 5.2.24, $\mathsf{E}^S a \wedge \mathsf{E}^S b \leq \mathsf{E}^S(\Diamond^S a \wedge \Diamond^S b)$. We therefore also know from Lemma 5.1.33 that $\mathfrak{B}_S$ is well-connected. Finally, for all Γ-formulas γ in Σ, $\mathfrak{B}_S \models \gamma \approx \top$ by Lemma 5.2.26.

In conclusion, $|S| \leq 2l(\varphi) + 3$, which means that $\mathfrak{B}_S$ contains at most $2^{l(\varphi)+3}$ atoms, and hence, at most $2^{2^{l(\varphi)+3}}$ elements. $\square$

Unless stated otherwise, for the remainder of this section, $\mathfrak{A}$ and $\mathfrak{B}_S$ will be the algebras in the proof of Theorem 5.2.21.

Lemma 5.2.22. *$\Diamond b \leq \Diamond^S b = \Diamond\Diamond^S b$ for all $b \in B_S$, and therefore, $\Box\Box^S b = \Box^S b \leq \Box b$ for all $b \in B_S$.*

Proof. The proof is the same as that of Lemma 5.2.3. $\Box$

Lemma 5.2.23. *Let b be an element of B_S. If $\Diamond b \in B_S$, then $\Diamond^S b = \Diamond b$, and therefore, if $\Box b \in B_S$, then $\Box^S b = \Box b$.*

Proof. The same as that of Lemma 5.2.4 $\Box$

Lemma 5.2.24. *If $b \in B_S$, then $\mathsf{E}^S b = \mathsf{E} b$, and therefore, if $b \in B_S$, $\mathsf{A}^S b = \mathsf{A} b$.*

Proof. The first part of this proof is the same as that of Lemma 5.1.37. For the second part, let $b \in B_S$. But we know that $\mathfrak{B}_S$ is closed under the Boolean operators, so $\neg b \in B_S$. Hence, by the first part of this lemma, $\mathsf{E}^S \neg b = \mathsf{E}\neg b$, and so $\neg\mathsf{E}^S\neg b = \neg\mathsf{E}\neg b$. We therefore have $\mathsf{A}^S b = \mathsf{A} b$. $\Box$

We now show that for any assignment ν on a hybrid closure E-algebra and any Γ-formula γ, $\nu(\gamma)$ is an open element of the hybrid E-algebra.

Lemma 5.2.25. *Let $\mathfrak{A}$ be a hybrid closure E-algebra, let ν be an assignment on $\mathfrak{A}$, and let γ be a $\Gamma(\mathsf{E})$-formula. Then $\Box\widetilde{\nu}(\gamma) = \widetilde{\nu}(\gamma)$.*

Proof. As for Lemma 5.2.6, the proof is by structural induction on the α, β and γ-formulas. We only consider the case where α has the form $\mathsf{A}\mathbf{i}$ as the other cases not involving A can be proved in the same way as in the proof of Lemma 5.2.6, and the cases involving A are similar to this case. So assume α has the form $\mathsf{A}\mathbf{i}$. First, $\Box\widetilde{\nu}(\mathsf{A}\mathbf{i}) \leq \widetilde{\nu}(\mathsf{A}\mathbf{i})$ by (*refl*). Conversely,

$$\begin{aligned}
\Box\widetilde{\nu}(\mathsf{A}\mathbf{i}) &= \Box\mathsf{A}\widetilde{\nu}(\mathbf{i}) && \text{(by Definition 2.3.7)}\\
&\geq \mathsf{A}\mathsf{A}\widetilde{\nu}(\mathbf{i}) && \text{(by } (incl_\Diamond) \text{ of Proposition 2.3.4)}\\
&= \mathsf{A}\widetilde{\nu}(\mathbf{i}) && \text{(by } (refl_\mathsf{E}) \text{ and } (trans_\mathsf{E}) \text{ of Proposition 2.3.4)}\\
&= \widetilde{\nu}(\mathsf{A}\mathbf{i}) && \text{(by Definition 2.3.7)}
\end{aligned}$$

$\Box$

Lemma 5.2.26. *Let γ be a $\Gamma(\mathsf{E})$-formula. If $\mathfrak{A} \models \gamma \approx \top$, then $\mathfrak{B}_S \models \gamma \approx \top$.*

Proof. Let θ be an assignment on $\mathfrak{B}_S$, and define an assignment $\rho\colon \mathsf{PROP} \cup \mathsf{NOM} \to A$ by $\rho(p) = \Box^S\theta(p)$ for all $p \in \mathsf{PROP}$ and $\rho(\mathbf{i}) = \theta(\mathbf{i})$ for all $\mathbf{i} \in \mathsf{NOM}$. We now prove the following claims:

Claim 1. $\widetilde{\theta}(\alpha) = \widetilde{\rho}(\alpha)$ for any α-formula α.

Proof of claim. The proof of this claim is by structural induction on α. The base steps are (i) – (v) and the inductive steps (vi) – (viii).

(i) Assume α is $\bot$. Then $\widetilde{\rho}(\bot) = \bot = \widetilde{\theta}(\bot)$.

(ii) If α has the form $\top$, then $\widetilde{\rho}(\top) = \top = \widetilde{\theta}(\top)$.

(iii) Assume α has the form $\Box^m p$ $(m > 0)$. Then

$$\begin{aligned}
\widetilde{\rho}(\Box^m p) &= \Box^m \rho(p) \\
&= \Box^m \Box^S \theta(p) && \text{(by the definition of } \rho) \\
&= \Box \Box^S \theta(p) && \text{(by (\textit{refl}) and (\textit{trans}))} \\
&= \Box^S \theta(p) && \text{(by Lemma 5.2.22)} \\
&= (\Box^S)^m \theta(p) && \text{(by (\textit{refl}) and (\textit{trans}))} \\
&= \widetilde{\theta}(\Box^m p)
\end{aligned}$$

(iv) If α has the form $\Box^n \mathbf{i}$ $(n > 0)$, then we have

$$\begin{aligned}
\widetilde{\rho}(\Box^n \mathbf{i}) &= \Box^n \rho(\mathbf{i}) \\
&= \Box^n \theta(\mathbf{i}) && \text{(by the definition of } \rho) \\
&= \Box \theta(\mathbf{i}) && \text{(by (\textit{refl}) and (\textit{trans}))} \\
&= \Box^S \theta(\mathbf{i}) && (\theta(\mathbf{i}) \in X_S \text{ and Lemma 5.2.23}) \\
&= (\Box^S)^n \theta(\mathbf{i}) && \text{(by (\textit{refl}) and (\textit{trans}))} \\
&= \widetilde{\theta}(\Box^n \mathbf{i})
\end{aligned}$$

(v) Assume α has the form $\mathsf{A}\mathbf{i}$. Then we have

$$\begin{aligned}
\widetilde{\rho}(\mathsf{A}\mathbf{i}) &= \mathsf{A}\widetilde{\rho}(\mathbf{i}) \\
&= \mathsf{A}\theta(\mathbf{i}) && \text{(by the definition of } \rho) \\
&= \mathsf{A}^S \theta(\mathbf{i}) && \text{((by Lemma 5.2.24)} \\
&= \widetilde{\theta}(\mathsf{A}\mathbf{i})
\end{aligned}$$

(vi) Assume α has the form $\alpha_1 \wedge \alpha_2$. Then

$$\begin{aligned}
\widetilde{\rho}(\alpha_1 \wedge \alpha_2) &= \widetilde{\rho}(\alpha_1) \wedge \widetilde{\rho}(\alpha_2) \\
&= \widetilde{\theta}(\alpha_1 \wedge \alpha_2)
\end{aligned}$$

(vii) If α is of the form $\alpha_1 \vee \alpha_2$, then

$$\begin{aligned}
\widetilde{\rho}(\alpha_1 \vee \alpha_2) &= \widetilde{\rho}(\alpha_1) \vee \widetilde{\rho}(\alpha_2) \\
&= \widetilde{\theta}(\alpha_1) \vee \widetilde{\theta}(\alpha_2) && \text{(by the inductive hypothesis)} \\
&= \widetilde{\theta}(\alpha_1 \vee \alpha_2)
\end{aligned}$$

(viii) Assume α has the form $\mathsf{A}\alpha$. Then we have

$$\begin{aligned}\widetilde{\rho}(\mathsf{A}\alpha) &= \mathsf{A}\widetilde{\rho}(\alpha) \\ &= \mathsf{A}\theta(\alpha) && \text{(by the inductive hypothesis)} \\ &= \mathsf{A}^S\theta(\alpha) && \text{(by Lemma 5.2.24)} \\ &= \widetilde{\theta}(\mathsf{A}\alpha)\end{aligned}$$

Claim 2. $\widetilde{\theta}(\beta) \leq \widetilde{\rho}(\beta)$ for any β-formula β.

Proof of claim. The proof of this claim is also by structural induction. The base step are (i) – (ii) and the inductive steps (iii) – (v).

(i) The case where β is α follows from Claim 1.

(ii) If β is of the form $\Box\neg\alpha$, then

$$\begin{aligned}\widetilde{\rho}(\Box\neg\alpha) &= \Box\neg\widetilde{\rho}(\alpha) \\ &= \Box\neg\widetilde{\theta}(\alpha) && \text{(by Claim 1)} \\ &\geq \Box^S\neg\widetilde{\theta}(\alpha) && \text{(by Lemma 5.2.22)} \\ &= \widetilde{\theta}(\Box\neg\alpha)\end{aligned}$$

(iii) The Boolean cases are similar to that of the α-words.

(iv) Assume β is of the form $\Box(\alpha \to \beta)$. Here

$$\begin{aligned}\widetilde{\rho}(\Box(\alpha \to \beta)) &= \Box\widetilde{\rho}(\alpha \to \beta) \\ &= \Box\widetilde{\rho}(\neg\alpha \vee \beta) \\ &= \Box(\neg\widetilde{\rho}(\alpha) \vee \widetilde{\rho}(\beta)) \\ &\geq \Box(\neg\widetilde{\theta}(\alpha) \vee \widetilde{\theta}(\beta)) && \text{(by Claim 1 and the inductive hypothesis)} \\ &= \Box\widetilde{\theta}(\neg\alpha \vee \beta) \\ &= \Box\widetilde{\theta}(\alpha \to \beta) \\ &\geq \Box^S\widetilde{\theta}(\alpha \to \beta) && \text{(by Lemma 5.2.22)} \\ &= \widetilde{\theta}(\Box(\alpha \to \beta))\end{aligned}$$

(v) If β is of the form $\mathsf{A}\beta$, then

$$\begin{aligned}\widetilde{\rho}(\mathsf{A}\beta) &= \mathsf{A}\widetilde{\rho}(\beta) \\ &\geq \mathsf{A}\widetilde{\theta}(\beta) && \text{(by the inductive hypothesis and monotonicity of A)} \\ &= \mathsf{A}^S\widetilde{\theta}(\beta) && \text{(by Lemma 5.2.24)} \\ &= \widetilde{\theta}(\mathsf{A}\beta)\end{aligned}$$

Claim 3. For any Γ-formula γ, $\widetilde{\rho}(\gamma) = \top$ implies $\widetilde{\theta}(\gamma) = \top$.

Proof of claim. We use structural induction on γ. The base steps are (i) – (vi) and the inductive steps (vii) – (ix).

(i) If γ has the form $\Box\neg\mathbf{i}$, then

$$\begin{aligned}
\widetilde{\rho}(\Box\neg\mathbf{i}) = \top &\Longrightarrow \Box\widetilde{\rho}(\neg\mathbf{i}) = \top && \\
&\Longrightarrow \widetilde{\rho}(\neg\mathbf{i}) = \top && \text{(by } (\mathit{refl})\text{)} \\
&\Longrightarrow \neg\widetilde{\rho}(\mathbf{i}) = \top && \\
&\Longrightarrow \neg\widetilde{\theta}(\mathbf{i}) = \top && \text{(by the definition of } \rho\text{)} \\
&\Longrightarrow \widetilde{\theta}(\neg\mathbf{i}) = \top && \\
&\Longrightarrow \Box^S\widetilde{\theta}(\neg\mathbf{i}) = \Box^S\top && \\
&\Longrightarrow \Box^S\widetilde{\theta}(\neg\mathbf{i}) = \top && \text{(by Lemma 5.2.23)} \\
&\Longrightarrow \widetilde{\theta}(\Box\neg\mathbf{i}) = \top &&
\end{aligned}$$

(ii) Assume γ has the form $\mathsf{A}\neg\mathbf{i}$. Then we have

$$\begin{aligned}
\widetilde{\rho}(\mathsf{A}\neg\mathbf{i}) = \top &\Longrightarrow \mathsf{A}\widetilde{\rho}(\neg\mathbf{i}) = \top && \text{(by Definition 2.3.7)} \\
&\Longrightarrow \widetilde{\rho}(\neg\mathbf{i}) = \top && \text{(by } (\mathit{refl}_{\mathsf{E}}) \text{ of Proposition 2.3.4)} \\
&\Longrightarrow \neg\widetilde{\rho}(\mathbf{i}) = \top && \\
&\Longrightarrow \neg\widetilde{\theta}(\mathbf{i}) = \top && \text{(by the definition of } \rho\text{)} \\
&\Longrightarrow \widetilde{\theta}(\neg\mathbf{i}) = \top && \\
&\Longrightarrow \mathsf{A}^S\widetilde{\theta}(\neg\mathbf{i}) = \mathsf{A}^S\top && \\
&\Longrightarrow \mathsf{A}^S\widetilde{\theta}(\neg\mathbf{i}) = \top && \text{(by Proposition 2.3.3)} \\
&\Longrightarrow \widetilde{\theta}(\mathsf{A}\neg\mathbf{i}) = \top &&
\end{aligned}$$

(iii) If γ is α, the claim immediately follows from Claim 1.

(iv) Assume that γ has the form $\Box(\beta \to \alpha)$. Then

$$\begin{aligned}
\widetilde{\rho}(\Box(\beta \to \alpha)) = \top &\Longrightarrow \Box\widetilde{\rho}(\beta \to \alpha) = \top \\
&\Longrightarrow \widetilde{\rho}(\beta \to \alpha) = \top && \text{(by } (\mathit{refl})\text{)} \\
&\Longrightarrow \widetilde{\rho}(\neg\beta \vee \alpha) = \top \\
&\Longrightarrow \neg\widetilde{\rho}(\beta) \vee \widetilde{\rho}(\alpha) = \top \\
&\Longrightarrow \neg\widetilde{\theta}(\beta) \vee \widetilde{\theta}(\alpha) = \top && \text{(by Claims 1 and 2)} \\
&\Longrightarrow \widetilde{\theta}(\neg\beta \vee \alpha) = \top \\
&\Longrightarrow \widetilde{\theta}(\beta \to \alpha) = \top \\
&\Longrightarrow \Box^S\widetilde{\theta}(\beta \to \alpha) = \Box^S\top \\
&\Longrightarrow \Box^S\widetilde{\theta}(\beta \to \alpha) = \top && \text{(by Lemma 5.2.23)} \\
&\Longrightarrow \widetilde{\theta}(\Box(\beta \to \alpha)) = \top
\end{aligned}$$

(v) If γ has the form $\Box\neg\beta$, then

$$\begin{aligned}
\widetilde{\rho}(\Box\neg\beta) = \top &\Longrightarrow \Box\widetilde{\rho}(\neg\beta) = \top \\
&\Longrightarrow \widetilde{\rho}(\neg\beta) = \top && \text{(by } (\mathit{refl})\text{)} \\
&\Longrightarrow \neg\widetilde{\rho}(\beta) = \top \\
&\Longrightarrow \neg\widetilde{\theta}(\beta) = \top && \text{(by Claim 2)} \\
&\Longrightarrow \widetilde{\theta}(\neg\beta) = \top \\
&\Longrightarrow \Box^S\widetilde{\theta}(\neg\beta) = \Box^S\top \\
&\Longrightarrow \Box^S\widetilde{\theta}(\neg\beta) = \top && \text{(by Lemma 5.2.23)} \\
&\Longrightarrow \widetilde{\theta}(\Box\neg\beta) = \top
\end{aligned}$$

(vi) Assume γ has the form $\mathsf{A}\neg\beta$. We then have

$$\begin{aligned}
\widetilde{\rho}(\mathsf{A}\neg\beta) = \top &\Longrightarrow \mathsf{A}\widetilde{\rho}(\neg\beta) = \top && \text{(by Definition 2.3.7)} \\
&\Longrightarrow \widetilde{\rho}(\neg\beta) = \top && \text{(by } (\mathit{refl}) \text{ of Proposition 2.3.4)} \\
&\Longrightarrow \neg\widetilde{\rho}(\beta) = \top \\
&\Longrightarrow \neg\widetilde{\theta}(\beta) = \top && \text{(by Claim 2)} \\
&\Longrightarrow \widetilde{\theta}(\neg\beta) = \top \\
&\Longrightarrow \mathsf{A}^S\widetilde{\theta}(\neg\beta) = \mathsf{A}^S\top \\
&\Longrightarrow \mathsf{A}^S\widetilde{\theta}(\neg\beta) = \top && \text{(by Proposition 2.3.3)} \\
&\Longrightarrow \widetilde{\theta}(\mathsf{A}\neg\beta) = \top
\end{aligned}$$

(vii) Assume γ has the form $\gamma_1 \wedge \gamma_2$. Then we have

$$\begin{aligned}
\widetilde{\rho}(\gamma_1 \wedge \gamma_2) = \top &\Longrightarrow \widetilde{\rho}(\gamma_1) \wedge \widetilde{\rho}(\gamma_2) = \top \\
&\Longrightarrow \widetilde{\rho}(\gamma_1) = \top \text{ and } \widetilde{\rho}(\gamma_2) = \top \\
&\Longrightarrow \widetilde{\theta}(\gamma_1) = \top \text{ and } \widetilde{\theta}(\gamma_2) = \top \qquad \text{(by the inductive hypothesis)} \\
&\Longrightarrow \widetilde{\theta}(\gamma_1) \wedge \theta(\gamma_2) = \top \\
&\Longrightarrow \widetilde{\theta}(\gamma_1 \wedge \gamma_2) = \top
\end{aligned}$$

(viii) Assume γ has the form $\gamma_1 \vee \gamma_2$, and suppose $\widetilde{\rho}(\gamma_1 \vee \gamma_2) = \top$. Then $\widetilde{\rho}(\gamma_1) \vee \widetilde{\rho}(\gamma_2) = \top$. But we know that $\Box\widetilde{\rho}(\gamma_1) = \widetilde{\rho}(\gamma_1)$ and $\Box\widetilde{\rho}(\gamma_2) = \widetilde{\rho}(\gamma_2)$ by Lemma 5.2.25, so

$$\Box\widetilde{\rho}(\gamma_1) \vee \Box\widetilde{\rho}(\gamma_2) = \top.$$

Hence, $\Diamond\neg\widetilde{\rho}(\gamma_1) \wedge \Diamond\neg\widetilde{\rho}(\gamma_2) = \bot$, and so, by the well-connectedness of $\mathfrak{A}$, $\neg\widetilde{\rho}(\gamma_1) = \bot$ or $\neg\widetilde{\rho}(\gamma_2) = \bot$. This gives $\widetilde{\rho}(\gamma_1) = \top$ or $\widetilde{\rho}(\gamma_2) = \top$. Therefore, by the inductive hypothesis, $\widetilde{\theta}(\gamma_1) = \top$ or $\widetilde{\theta}(\gamma_2) = \top$, which means that $\widetilde{\theta}(\gamma_1) \vee \widetilde{\theta}(\gamma_2) = \top$. We thus have $\widetilde{\theta}(\gamma_1 \vee \gamma_2) = \top$.

(ix) If γ has the form $\Box\gamma_1$, then

$$\begin{aligned}
\widetilde{\rho}(\Box\gamma_1) = \top &\Longrightarrow \Box\widetilde{\rho}(\gamma_1) = \top \\
&\Longrightarrow \widetilde{\rho}(\gamma_1) = \top \qquad \text{(by } (\mathit{refl})) \\
&\Longrightarrow \widetilde{\theta}(\gamma_1) = \top \qquad \text{(by the inductive hypothesis)} \\
&\Longrightarrow \Box^S\widetilde{\theta}(\gamma_1) = \Box^S\top \\
&\Longrightarrow \Box^S\widetilde{\theta}(\gamma_1) = \top \qquad \text{(Lemma 5.2.23)} \\
&\Longrightarrow \widetilde{\theta}(\Box\gamma_1) = \top
\end{aligned}$$

Now, since $\mathfrak{A} \models \gamma \approx \top$, we have $\mathfrak{B}_S \models \gamma \approx \top$ by Claim 3. □

Corollary 5.2.27. *If Σ is finite, then* $\mathbf{H}(\mathsf{E})\mathbf{S4} \oplus \Sigma$ *is decidable.*

To conclude, as we stated at the beginning of the section, not having well-connectedness is not the end of the world. Even if we do not have well-connectedness, we can still obtain a positive result. However, we would then have to make our definition of Γ-formulas smaller. To see this, recall that in induction step (vi) of Claim 3 in the proof of Theorem 5.2.8, we use the fact that the algebra we started with is well-connected. But the well-connectedness follows from the common predecessor axiom, so if we remove this axiom, we have to remove all Γ-formulas of the form $\gamma_1 \vee \gamma_2$ from our definition of Γ-formulas. In this case, we have the following: if Σ is a set of Γ-formulas, the logic $\mathbf{H}(\mathsf{E})\mathbf{S4} \oplus \Sigma$ is sound and complete with respect to the class of all finite hybrid closure E-algebras validating Σ. Note, however, that this would exclude formulas like $\Diamond\Box p \to \Box p$, $\Diamond\Diamond\mathbf{i} \to \neg\Diamond\mathbf{i}$ and $\Diamond\mathbf{i} \to \Box\mathbf{i}$ in Example 5.2.19.

Conclusion

In this thesis, we approach hybrid logic with *algebraic methods* — methods using tools and techniques from universal algebra. The idea is to associate, with any logic, a class of algebras such that logical properties of the logic correspond to algebraic properties of the class of algebras. In the case of hybrid logics, we found that there are two possible algebraic semantics for hybrid logics. The first, called *orthodox interpretations*, is the class of Boolean algebras with operators in which the nominals are interpreted as constants. The second, called *hybrid algebras*, consists of a Boolean algebra with operators together with a non-empty subset of the atoms of the BAO over which the nominals range. However, we prefer to work with hybrid algebras for the following reasons:

(i) The orthodox interpretations are not dual to the intended relational semantics for hybrid languages in the appropriate way.

(ii) The rule (*Sorted substitution*) is *not* sound in orthodox interpretations.

(iii) The @ operator does not completely behave like it should in orthodox interpretations of $\mathcal{H}(@)$.

Hybrid algebras do not suffer from the same shortcomings.

As for the modal counterpart of hybrid logic, adding these new algebraic tools to the hybrid logic toolbox proved to be very useful in solving open problems in the field of hybrid logic, as well as confirming existing results. In particular, we made the following contributions to the field of hybrid logic:

(i) We obtained general completeness results with respect to hybrid algebras for the better known hybrid logics. These results also coincide with the informal definition of algebraizability in the literature.

(ii) We developed a *hybrid* Sahlqvist theory. More specifically, we extended the definition of inductive formulas in the literature to the hybrid language with satisfaction operators and showed that every formula in this class has a first-order frame correspondent. We also showed that certain subclasses of the hybrid inductive formulas are respectively preserved under canonical extensions and Dedekind MacNeille completions of certain hybrid algebras, which ensures that these formulas axiomatize relationally complete hybrid logics.

(iii) We proved analogues of Bull's Theorem for some of the well-known hybrid logics. In particular, the finite algebra property is transferred from the modal logic **S4.3** to its hybrid companion obtained by adding nominals to **S4.3** and to all its axiomatic extensions. It is not clear at this stage if this is also true when we add satisfaction operators or the global modality in addition to nominals. However, the finite algebra property is transferred from **S4.3** and all its axiomatic extensions to the hybrid logic obtained by adding nominals and the global modality as well as an axiom enforcing well-connectedness.

(iv) We provide sufficient conditions for a class of axiomatic extensions of the modal logic **S4** to have the finite model property. This result takes the form of a syntactic characterization of a class of formulas that may be added as axioms to **S4**.

Although hybrid logics have been researched since the nineties, many questions about them still remain unanswered. The research done in this thesis is expected to have an impact in the field of hybrid logic by providing new insights for research communities focusing on hybrid logics. It is possible that this research might have further impact through the applications of hybrid logics to areas like knowledge representation, artificial intelligence, as well as formal specification and verification of hardware and software.

We conclude with a list of open questions as well as some directions on what the way forward could hold:

(i) What does Blok and Pigozzi's definition of algebraizability in [16] look like for hybrid logics, and in particular, where exactly do the hybrid logics investigated in this thesis lie in the Leibniz hierarchy?

(ii) It is a well-known fact that the global consequence relation of the basic normal modal logic is algebraizable. Is this the case for the global consequence relation of the logics investigated in this thesis?

(iii) Unlike for model logics, the finite hybrid algebra property investigated in Chapter 5 is not enough to give us a finite relational model, so the question naturally arises: can we get finite relational models for the hybrid logics investigated in this thesis?

(iv) Is the finite model property transferred from **S4.3** and all its axiomatic extensions to its hybrid companion obtained by adding nominals and satisfaction operators to it?

(v) The intuitionistic versions of the hybrid logics investigated in this thesis. In particular:

  (a) What do the axiomatizations of these intuitionistic hybrid logics look like?
  (b) What does the relational semantics look like?
  (c) Can the results obtained in this thesis be generalized to these intuitionistic hybrid logics?

Bibliography

[1] C. Areces, P. Blackburn, and M. Marx. A road-map on complexity of hybrid logics. *Proc. of the 13th CSL, 1999*, LNCS **1683**, 307–321, 1999.

[2] C. Areces, P. Blackburn, and M. Marx. Hybrid Logics: Characterization, Interpolation and Complexity. *Journal of Symbolic Logic*, 66: 977–1010, 2001.

[3] C. Areces, P. Blackburn, and M. Marx. The computational complexity of hybrid temporal logics. *Logic Journal of the IGPL*, 8: 653–679, 2000.

[4] C. Areces, H. de Nivelle, and M. de Rijke. Prefixed resolution: a resolution method for modal and description logics. In *Automated Deduction—CADE-16 (Trento, 1999)*, pages 187–201. Springer, Berlin, 1999.

[5] C. Areces, H. de Nivelle, and M. de Rijke. Resolution in modal, description and hybrid logic. *Journal of Logic and Computation*, 11(5): 717–736, 2001.

[6] C. Areces and D. Gorìn. Ordered resolution with selection for h(@). In F. Baader and A. Voronkov, editors, *Proceedings of LPAR 2004*, volume 3452 of *LNCS*, pages 125–141, Montevideo, Uruguay, 2005. Springer.

[7] N. Bezhanishvili and B. ten Cate. Transfer results for hybrid logic. Part 1: the case without satisfaction operators. *Journal of Logic and Computation*, 16(2): 177–197, 2006.

[8] G. Birkhoff. Sub-direct unions in universal algebras. *Bulletin of the American Mathematical Society*, 50: 764–768, 1944.

[9] P. Blackburn. Representation, Reasoning, and Relational Structures: a Hybrid Logic Manifesto. *Logic Journal of the IGPL*, 8: 339–365, 2000.

[10] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, Cambridge, UK, 2001.

[11] P. Blackburn and M. Marx. Tableaux for quantified hybrid logic. In U. Egly and C. Fermüller, editors, *Automated reasoning with analytic tableaux and related methods, TABLEAUX 2002*, volume 2381 of *LNAI*, pages 38–52. Springer Verlag, 2002.

[12] P. Blackburn and J. Seligman. Hybrid languages. *Journal of Logic, Language and Information*, 4(3): 251–272, 1995. Special issue on decompositions of first-order logic.

[13] P. Blackburn and B. ten Cate. Pure extensions, proof rules, and hybrid axiomatics. *Studia Logica*, 84: 277–322, 2006.

[14] P. Blackburn and M. Tzakova. Hybrid languages and temporal logic. *Logic Journal of the IGPL*, 7(1): 27–54, 1999.

[15] P. Blackburn, J. van Benthem and F. Wolter. Handbook of Modal Logic, Volume 3 (Studies in Logic and Practical Reasoning). Elsevier Science Inc. New York, NY, USA, 2006

[16] W. J. Blok and D. Pigozzi. Algebraizable logics. *Memoirs of the American Mathematical Society*, 77, 396, 1989.

[17] T. Braüner. Natural deduction for hybrid logic. *Journal of Logic and Computation*, 14(3): 329–3353, 2004.

[18] R. A. Bull. A class of extensions of the modal system S4 with the finite model property. *Zeitschrift für mathematische logik und grundlagen der mathematik*, 11: 127–132, 1965.

[19] R. A. Bull. An approach to tense logic. *Theoria*, 36: 282–300, 1970.

[20] R. A. Bull. That all normal extensions of S4.3 have the finite model property. *Zeitschr. f. math. Logik und Grundlagen d. Math.* 12: 341–344, 1966.

[21] S. Burris and H.P. Sankappanavar. *A Course in Universal Algebra*. Graduate Texts in Mathematics. Springer, 1981.

[22] A. Chagrov and L. A. Chagrova. The truth about algorithmic problems in correspondence theory. In G. Governatori, I. Hodkinson, and Y. Venema, editors, *Advances in Modal Logic*, volume 6, pages 121-138. College Publications, 2006.

[23] A. Chagrov and M. Zakharyaschev. *Modal Logic*, volume 35 of *Oxford Logic Guides*. Oxford University Press, 1997.

[24] L. A. Chagrova. An undecidable problem in correspondence theory. *Journal of Symbolic Logic*, 56:1261-1272, 1991.

[25] C.C. Chang and H. J. Keisler. *Model Theory*. North-Holland, 1990.

[26] C. Chen and I. Lin. The complexity of propositional modal theories and the complexity of consistency of propositional modal theories. In A. Nerode and Y. Matiyasevich, editors, *Proceedings of the Third International Symposium on Logical Foundations of Computer Science*, volume 813 of *Lecture Notes in Computer Science*, pages 69–80. Springer, 1994.

[27] W. E. Conradie. Algorithmic Correspondence and Completeness in Modal Logic. PhD thesis, University of the Witwatersrand, 2006.

[28] W. E. Conradie. Completeness and correspondence in hybrid logic via an extension of SQEMA. *Electronic Notes in Theoretical Computer Science*, 2007.

[29] W. E. Conradie. Definability and changing perspectives the Beth property for three extensions of modal logic. Masters thesis, University of Amsterdam, 2002.

[30] W. E. Conradie and A. Craig. Canonicity results for mu-calculi: an algorithmic approach. To appear.

[31] W. Conradie, Y. Fomatatiand, A. Palmigiano and S. Sourabh. Algorithmic correspondence for intuitionistic modal mu-caculus. *Theoretical Computer Science*, (2014). To appear.

[32] W. Conradie, S. Ghilardi and A. Palmigiano. Unified Correspondence. In Johan van Benthem on Logic and Information Dynamics, A. Baltag and S. Smets, eds., vol. 5 of Outstanding Contributions to Logic, Springer International Publishing, 933–975, 2014.

[33] W. Conradie, V. Goranko and D. Vakarelov. Algorithmic correspondence and completeness in modal logic I: The core algorithm SQEMA. *Logical Methods in Computer Science*, 2(1:5), 2006.

[34] W. Conradie, W. Morton and C. van Alten. An Algebraic Look at Filtrations in Modal Logic. To appear in *Logic Journal of the IGPL.*

[35] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for distributive modal logic. *Annals of Pure and Applied Logic*, 163: 338–376, 2012.

[36] B. Davey and H. Priestley. Introduction to Lattices and Order. *Cambridge University Press*, 2002.

[37] L. Esakia. Topological Kripke models. *Soviet Math. Dokl.*, 15:147–151, 1974.

[38] K. Fine. Normal forms in modal logic. *Notre Dame Journal of Formal Logic*, 16(2): 229–237, 1975.

[39] K. Fine. The logics containing S4.3. *Zeitschrift. f. math. Logik und Grundlagen d. Math.* 17: 371–376, 1971.

[40] G. Gargov and V. Goranko. Modal Logic with Names. *Journal of Philosophical Logic*, 22: 607–636, 1993.

[41] G. Gargov, S. Passy and T. Tinchev. Modal environment for Boolean speculations. In D. Skordev, editor, Mathematical Logic and its Applications. *Proceedings of the Summer School and Conference dedicated to the 80th Anniversary of Kurt Gödel (Druzhba, 1986)*, pages 253–263, 1987.

[42] M. Gehrke, Y. Nagahashi and H. Venema. A Sahlqvist theorem for distributive modal logic. *Annals of Pure and Applied Logic*, 131: 65–102, 2005.

[43] M. Gehrke and J. Vosmaer. A View of Canonical Extension. In Nick Bezhanishvili, Sebastian Löbner, Kerstin Schwabe, and Luca Spada, editors, *TbiLLC*, volume 6618 of *Lecture Notes in Computer Science*, pages 77–100. Springer, 2009.

[44] S. Ghilardi and M. Zawadowski. Sheaves, games and model completions. Trends in logic. Kluwer Academic, 2002.

[45] S. Givant and Y. Venema. The preservation of Sahlqvist equations in completions of Boolean algebras with operators. *Alg. Univ.*, 41(1): 47–84, 1999.

[46] R. I. Goldblatt. Mathematics of Modality. *CSLI Lecture Notes No. 43*, 1993.

[47] R. I. Goldblatt. Metamathematics of modal logic I. *Reports on Mathematical Logic*, 6:41–78, 1976.

[48] R. Goldblatt. The McKinsey axiom is not canonical. *Journal of Symbolic Logic*, 56(2): 554–562, 1991.

[49] R.Goldblatt. Varieties of complex algebras. *Annals of Pure and Applied Logic*, 44: 173–242, 1989.

[50] R.I. Goldblatt and S.K. Thomason. Axiomatic classes in propositional modal logic. In J. Crossley, editor, *Algebra and Logic*, pages 163173. Springer-Verlag, 1974.

[51] V. Goranko. Temporal logic with reference pointers. In *Temporal Logic*, volume 827 of *Lecture Notes in Artificial Intelligence*, pages 133–148. Springer-Verlag, 1994.

[52] V. Goranko and S. Passy. Using the universal modality: gains and questions. *Journal of Logic and Computation*, 2(1): 5–30, 1990.

[53] V. Goranko and D. Vakarelov. Sahlqvist formulae in hybrid polyadic modal languages. *Journal of Logic and Computation*, 11(5): 737–254, 2001.

[54] V. Goranko and D. Vakarelov. Sahlqvist formulas unleashed in polyadic modal languages. In F. Wolter, H. Wansing, M. de Rijke, and M. Zakharyaschev, editors, *Advances in Modal Logic*, 3: 221–240, Singapore, 2002. World Scientific.

[55] V. Goranko and D. Vakarelov. Elementary canonical formulae: Extending Sahlqvist theorem. *Annals of Pure and Applied Logic*, 141(1-2):180–217, 2006.

[56] W. Hodges. *Model Theory.* Cambridge University Press, 1993.

[57] I. Hodkinson. Hybrid Formulas and Elementarily Generated Modal Logics. *Notre Dame Journal of Formal Logic*, 47(4): 443–478, 2006.

[58] M. Kracht. Tools and Techniques in Modal Logic. Elsevier, 1999.

[59] R. E. Ladner. The computational complexity of provability in systems of modal propositional logic. SIAM J. Comput. 6: 467–480, 1977.

[60] E. J. Lemmon. Algebraic Semantics for Modal Logics I. *Journal of Symbolic Logic*, 31: 46–65, 1966.

[61] T. Litak. Algebraization of hybrid logic with binders. In *Proceedings of RelMiCS/AKA 9*, pages 281–295, 2006.

[62] J. C. C. McKinsey and A. Tarski. Some theorems about the sentential calculi of Lewis and Heyting. *The Journal of Symbolic Logic*, 13(1): 1–15, 1948.

[63] A. Prior. Past, Present and Future. Oxford University Press, 1967.

[64] H. Sahlqvist. Correspondence and completeness in the first and second-order semantics for modal logic. In S. Kanger, editor, *Proceedings of the 3rd Scandinavian Logic Symposium, Uppsala* 1973, pages 110-143, Amsterdam, 1975. Springer-Verlag.

[65] A. Schaerf. Reasoning with individuals in concept languages. *Data and Knowledge Engineering*, 13(2): 141–176, 1994.

[66] J. Seligman. Internalization: The case of hybrid logics. *Journal of Logic and Computation*, 11(5): 671–689, 2001. Special Issue on Hybrid Logics. Areces, C. and Blackburn, P. (eds.).

[67] J. Seligman. The logic of correct description. In M. de Rijke, editor, *Advances in Intensional Logic*, pages 107–135. Kluwer, 1997.

[68] R. Sikorski. Boolean algebras, second edition. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 25: x + 237, Springer Verlag, Berlin, 1964.

[69] E. Spaan. Complexity of modal logics. Ph.D. thesis, ILLC, University of Amsterdam, 1993.

[70] A. Tarski. Grundzüge des Systemenkalk üls. *Ester Teil.* Fund. Math., 25: 503–525, 1935.

[71] A. Tarski. Logic, Semantics, and Metamathematics. Papers from 1923 to 1938. Trans. J. H. Woodger, 2nd Ed. J. Corcoran, Hackett Pub. Co., Indianapolis, Indiana, 1983.

[72] B. ten Cate. *Model theory for extended modal languages.* PhD thesis, University of Amsterdam, 2005. ILLC Dissertation Series DS–2005–01.

[73] B. ten Cate, M. Marx and P. Viana. Hybrid logics with Sahlqvist axioms. *Logic Journal of the IGPL*, 13(3): 293–300, 2005.

[74] M. Tzakova. Tableaux calculi for hybrid logics. In N. Murray, editor, *Proceedings of the Conference on Tableaux Calculi and Related Methods (TABLEAUX), Saratoga Springs, USA*, volume 1617 of *LNAI*, pages 278–292. Springer Verlag, 1999.

[75] J. van Benthem. Modal frame correspondences and fixed-points. *Studia Logica*, 83: 133–155, 2006.

[76] J. van Benthem. Modal formulas are either elementary or not $\Sigma\Delta$-elementary. *Journal of Symbolic Logic*, 41:436-438, 1976.

[77] J. van Benthem. Modal Logic and Classical Logic. Bibliopolis, 1983.

[78] Y. Venema. Algebras and Coalgebras.

[79] A. Visser. Uniform interpolation and layered bisimulation. In Godel 1996, volume 6 of Lecture Notes Logic, pages 139–164, 1996.

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