METHOD OF LINES BASED ON FINITE ELEMENT TECHNIQUE TO ANALYSE ELECTROMAGNETIC PROBLEMS

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Abstract

A hybrid scheme called finite element method of lines is described and proposed for modelling and analysis of generalized computational electromagnetic problems with emphasis on a number of irregular waveguide. This finite element based method of lines is developed by combining finite element method and the method of lines, so that it not only has high flexibility to treat geometrically and compositionally complex problems but also maintains high accuracy of semi-analytical technique. Analytical and numerical algorithmic building blocks of this new scheme are discussed such as geometry discretization, element mapping, element trial functions, reformulation and computational issues of non-linear ordinary differential equations. The results therefore show that this new technique is able to efficiently solve complex problems as compared with the conventional method of lines. MATLAB was used to compute the solutions of various problems.

Keywords
Method of lines; computational electromagnetics; guided-wave problems; differential equation methods; variational analysis

I: Introduction

The dynamic technological progress that are occurring in telecommunication and information systems in the recent years, including high frequency telecommunications and navigation, radar systems and computer networks created a very big demand for better and accurate modelling of electromagnetic systems. We cannot run away from electromagnetic processes since they are everywhere in our daily life thus in generators, transformers and motors, converting mechanical to electric energy and vice versa. Electromagnetic waves in free space also enable wireless communication and also a vastly growing application in electromagnetics is in optics.

According to Chen et al, 1998, microwave technology is a novel finite element method of lines to analyse electromagnetic problems. Optical fibers allow the transport of light pulses over much longer distances than achieved by electric signals through cables. Short light pulses are generated by laser resonators. Optical multiplexers realized by photonic crystals have obtained much attraction over recent years. All these applications are rather complex, hence for further technical developments and optimization a deeper insight into electromagnetic processes is necessary, Microwave and Optical Technology, 1998. Recent research in computational electromagnetic has been widely focused on the development of general-purpose solution methods for electromagnetic problems such as scattering, dielectric cavity resonators, dielectric waveguides, integrated optical waveguides, EMI and EMC studies, VLSI chips and packages, and computer-aided design. Similar to many other physical and technical effects such as solid and fluid mechanics, heat transfer, quantum mechanics, geosciences, astrophysics, etc). Electromagnetic phenomena are modelled by partial differential equations (PDEs). This is the basis for the mathematical analysis and numerical treatment.

The main idea of the MOL is to replace the spatial (boundary-value) derivatives in the PDE with algebraic approximations. Then the spatial derivatives are no longer stated explicitly in terms of the spatial independent variables meaning that only the initial-value variable, typically time in a physical problem, remains. In other words, with only one remaining independent variable, we have a system of ODEs that approximate the original PDE. The challenge, then, is to formulate the approximating system of ODEs. Once this is done, we can apply any integration algorithm for initial-value ODEs to compute an approximate numerical solution to the PDE. Thus, one of the salient features of the MOL is the use of existing, and generally well-established, numerical methods for ODEs.

A: Aim

- to demonstrate the use of method of lines based on finite element technique to analyse computational electromagnetic problems

B: Objectives

- To solve irregular electromagnetic problems
- To generate a reasonably accurate solution on the basis of a semi-discrete scheme
- To analyse and compare selected numerical properties of different operator projection
- To perform numerical validation of the proposed methods applied to solving selected electromagnetic problems.

II: OVERVIEW

Classical electromagnetics treats electric and magnetic macroscopic phenomena including their interaction. Electric fields which vary in time cause magnetic fields and vice versa. James Clark Maxwell described these phenomena in his "Treatise on Electricity and Magnetism" in 1862. The classic theory mainly involves the following four time- and space-dependent vector fields:

- the electric field intensity denoted by E [V /m],
- the magnetic field intensity H [A/m],
- the electric displacement field (electric flux) D [As/m^2],
- The magnetic induction field (magnetic flux) B [V s/m^2].

The sources of electromagnetic fields are electric charges and currents described by:

- the charge density ρ [As/m^3]) and
- the current density function j [A/m^2],

Where the SI units denotes meter (m), seconds (s), Ampere (A), Volt (V).

A: The finite element method (FEM)

It is a popular numerical technique for obtaining approximate solutions to boundary value problems of mathematical physics. However, FEM fully discretizes a problem into a system of algebraic equations with discrete nodal/ edge values as the basic unknowns and thus discretization error is introduced. It is, of course, desirable to solve boundary-value problems analytically whenever possible

B: The method of lines (MOL)

It is a technique for solving **partial differential equations** (PDEs) in which all but one dimension is discretized. The method of lines (MOL) is semi-discrete / semi-analytical and has found great application in multi-layers planar circuits.

But the above method of lines uses finite difference (FD) technique to discretize the problem into a system of **ordinary differential equations** (ODEs) with nodal line functions as the basic unknowns.

MOL allows standard, general-purpose methods and software, developed for the numerical integration of ODEs and DAEs, to be used. A large number of integration routines have been developed over the years in many different programming languages, and some have been published as open source resources.

The method of lines most often refers to the construction or analysis of numerical methods for partial differential equations that proceeds by first discretizing the spatial derivatives only and leaving the time variable continuous. This leads to a system of ordinary differential equations to which a numerical method for initial value ordinary equations can be applied. The method of lines in this context dates back to at least the early 1960s. Many papers discussing the accuracy and stability of the method of lines for various types of partial differential equations have appeared since. E. Schiesser of Lehigh University is one of the major proponents of the method of lines, having published widely in this field.

C: Finite element method of lines (FEMOL)

Different from the conventional MOL, the finite element based method of lines (FEMOL) uses **finite element** (FE) technique in semi-discretization. As a semi-analytical or semi-discrete method, FEMOL make itself distinguished from standard FEM in several aspects. This finite element based method of lines is developed by combining finite element method and the method of lines, so that it not only has high flexibility to treat geometrically and compositionally complex problems but also maintains high accuracy of semi-analytical technique.

Partial differential equations (PDEs) defined on some particular domains even though they may be arbitrary are semi-discretized into a system of ODEs defined on discrete mesh lines (straight or curved) via variational principles, and then the resulting ODE system can directly be solved using a standard and robust ODE solver. Due to the efficient adaptivity and the super-convergence capability built into today's ODE solvers, highly reliable and accurate solutions of the ODE system can be obtained numerically, and hence the semi-analytical characteristics featured in this method are well preserved. As a semi-analytical or semi-discrete method, the FEMOL makes itself distinguished from the standard FEM procedures in view of several aspects. Intuitively, this method should be powerful and efficient, in particular, for those field problems for which solutions, for example in a two-dimensional (2D) case, may exhibit quite 'wild' behaviour in one direction and rather 'mild' in the other. For the standard line meshes, both FEMOL and FEM share the same convergence order in the formulated energy norms. But the errors in a FEMOL solution are independent of the true solution variation in the mesh line direction. In other words, no matter how naughty the true solution behaves along the mesh line direction, we will be able to gain an equally or consistently accurate FEMOL solution as long as the behaviours of the true solutions are similar to that of the FEM along the discrete direction. This characteristic is very attractive when there is a singularity in the computational domain. On the other hand, the use of robust ODE solvers makes the solution of the resulting ODEs (generally with variable coefficients) unified, efficient, accurate and effortless as compared with the conventional MOL in which a linear transformation is required to de-couple the coupled ODEs

RELATED WORK

III: Efficient hybrid scheme of finite element method of lines for modelling computational electromagnetic problems

The research was done by Chen et al, 2004 in which a hybrid scheme called FEMOL was introduced for efficiently solving irregular electromagnetic problems. Basic algorithmic concepts and theoretical frameworks of this new approach were described in detail through modelling and analysis of elliptical problems governed by Helmholtz equation. Results of three selected examples with arbitrary cross-sections were presented to show the efficiency and accuracy of this new method. According to their paper, there seemed to be nothing special rather than the combination of two trivial techniques. The detailed description showed a number of original and interesting concepts proposed in this new scheme leading to a powerful and efficient semi-analytical numerical algorithm. In particular, the application of semi-discrete FE rather than finite difference (FD) usually implemented in the conventional MOL allowed a parametric element mapping even with arbitrarily curved nodal lines and end-sides. Therefore, they concluded that the FEMOL is an easy and convenient approach in dealing with a problem defined on arbitrary domain. As compared with other familiar approaches such as FEM, MOL and FD techniques, the proposed FEMOL shows a resemblance in algorithmic behaviour to the FEM and MOL but exhibits more features over its counterparts. This method could easily be extended and applied to model 3D electromagnetic field boundary value problems with complex and irregular geometry.

IV: High Order Finite Element Methods for Electromagnetic Field Computation

Linz and Juli (2006) dealt with the higher-order Finite Element Method (FEM) for computational electromagnetics. The hpversion of FEM combined local mesh refinement (h) and local increase of the polynomial order of the approximation space (p). A key tool in the design and the analysis of numerical methods for electromagnetic problems used was the de Rham Complex relating the function spaces H 1 (Ω), H (curl, Ω), H (div, Ω), and L2 (Ω) and their natural differential operators. A short outline of the construction is as follows. The gradient fields of higher-order H 1-conforming shape functions were H (curl)-conforming and were chosen explicitly as shape functions for H (curl). In the next step the gradient functions were extended to a hierarchical and conforming basis of the desired polynomial space. An analogous principle was used for the construction of H (div)-conforming basis functions. By separate treatment of edge-based, face-based, and cell-based functions, and by including the corresponding gradient function, the local exact sequence property was established: the subspaces corresponding to a single edge, a single face or a single cell formed an exact sequence. A main advantage is that an arbitrary polynomial order on each edge could be chosen, face, and cell without destroying the global exact sequence. The main difficulty in the construction of efficient and parameter-robust preconditioners for electromagnetic problems is indicated by the different scaling of solenoid and irrotational fields in the curl-curl problem. Robust Schwarz-type methods for Maxwell's equations rely on a FE-space splitting, which also has to provide a correct splitting of the kernel of the curl operator. Due to the local exact sequence property this is already satisfied for simple splitting strategies. Numerical examples illustrate the robustness and performance of the method.

By this transformation, the time-harmonic Maxwell equations can be stated as

$$\mathrm{curl}\, E(x) + i\omega\mu H(x) = 0$$

$$\mathrm{div}\, \mu H(x) = 0$$

$$\mathrm{curl}\, H(x) - (i\omega\varrho + \sigma)E(x) = j\, i(x)$$

,

$$\mathrm{Div}\, \varrho E(x) = \rho(x).$$

Methodology

Analytical and numerical algorithmic building blocks of this new scheme will be done following the steps below:

1. We start with a general PDE system in three dimensions (3D) that, with some simplifying assumptions, is reduced to a 1D linear PDE. Here is some terminology:
    a) The starting point of this application is a classic PDE system, Maxwell's equations of electromagnetic (EM) field theory.
    b) After reduction of these equations to 1D, followed by some additional simplifications, we arrive at the *damped wave equation* (DWE).
2. The uses of coordinate-free PDEs that can then be specialized to a particular coordinate system; for the following analysis, this is Cartesian coordinates.
3. Spatial convergence of the DWE numerical solution by h- and p-refinement.
4. The effect of the method of lines (MOL) finite-difference (FD) approximations on the bandwidth and sparsity of the ordinary differential equation (ODE) Jacobian matrix.
5. A general method for the construction of PDE test problems is illustrated by a specific example for the DWE.
6. The physical significance of the DWE (which we can say without exaggeration is profound).

We start the analysis with the *differential form of Maxwell's equations for EM field*:

$$\nabla \times H = J + J_d = J + \frac{\partial D}{\partial t}$$
(1.1)

$$\nabla \times E = -\frac{\partial B}{\partial t}$$
(1.2)

$$\nabla \bullet D = \rho$$
(1.3)

$$\nabla \bullet B = 0$$
(1.4)

$$\frac{\partial \rho}{\partial t} = \nabla \bullet J = 0$$
(1.5)

Where

H Magnetic field intensity (amps/m)
E Electric field intensity (volts/m)
D Electric flux density (coulombs/m)
B Magnetic flux density (Weber/m)
J Electric current density (amps/m)
J_d Displacement current density (amps/m)
ρ Charge density (coulombs/m)
t Time (s)
× Curl vector operator
• Vector dot product
∇ Del vector operator (1/m)

Equation system above clearly has more dependent variables than equations. We therefore use some *constitutive equations* to provide the required additional relationships between the dependent variables:

$$D = \epsilon E$$
(1.6)

$$B = \mu H$$
(1.7)

$$J = \sigma E$$ (1.8)

where

ϵ Capacitivity or permittivity (farads/m)
μ Inductivity or permeability (henrys/m)
σ Conductivity (mohs/m)

Equations (1.1) to (1.8) are a complete set of PDEs (number of dependent variables =number of equations). We now proceed to combine these equations and finally obtain a single equation in **E**

If Eqs. (1.6), (1.7), and (1.8) are substituted into Eq. (1.1),

$$\left(\frac{1}{\mu}\right) \nabla \times B = \sigma E + \epsilon \frac{\partial E}{\partial t}$$
(2.0)

Differentiation of Eq. (2.0) with respect to *t* (assuming a linear, homogeneous, isotropic medium) gives

$$\left(\frac{1}{\mu}\right) \nabla \times \frac{\partial B}{\partial t} = \sigma \frac{\partial E}{\partial t} + \epsilon \frac{\partial^2 E}{\partial t^2}$$
(3.0)

Where the order of the LHS differentiation with respect to *t* and ∇ (space) has been interchanged. (3.0)

Substitution of Eq. (1.2) into Eq. (3.0) gives

$$-\nabla \times \nabla \times E = \mu\sigma \frac{\partial E}{\partial t} + \mu\epsilon \frac{\partial^2 E}{\partial t^2}$$
(4.0)

The identity

$$\nabla \times (\nabla \times J) = \nabla(\nabla \bullet J) - \nabla^2 J$$

Substitution into Eq. (4.0) finally gives a single equation for **E**

$$\mu\epsilon \frac{\partial^2 E}{\partial t^2} + \mu\sigma \frac{\partial E}{\partial t} = \nabla^2 E$$
(5.0)

Equation (5.0) is the time-dependent Maxwell equation for the electric field, **E**. It also applies to **H** and **J** in place of **E**. Equation (1.8) represents Ohm's law. For the case of a non-conductor where σ= 0, Eq. (5.0) reduces to the wave equation. Equation (5.0) is both hyperbolic (from $\frac{\partial^2 E}{\partial t^2}$ and$\nabla^2 E$) and parabolic from ∂**E**/∂*t* and ∇**E**). As expressed in terms of ∇, it is coordinate independent. If it is reduced to 1D in Cartesian coordinates, we have

$$\mu\epsilon \frac{\partial^2 E}{\partial t^2} + \mu\sigma \frac{\partial E}{\partial t} = \frac{\partial^2 E}{\partial x^2}$$
(6.0)

Equation (6.0), a linear, constant-coefficient *PDE*, is the starting point for the MOL analysis. It is second order in t and x. We take as the two required initial conditions (ICs)

$$E(x, t = 0) \cos(\pi x)$$
(7.1)

$$\frac{\partial E(x=0,t)}{\partial t} = 0$$
(7.2)

The two required boundary conditions (BCs) we take to be **homogeneous Neumann** are

$$\frac{\partial E(x=0,t)}{\partial x} = 0$$
(8.1)

$$\frac{\partial E(x=1,t)}{\partial x} = 0$$
(8.2)

Equations (6.0) to (8.2) are the complete specification of the PDE problem for the following MOL analysis.

The main program therefore had the following:

1. A *global* area should be specified to share parameters with other routines. A *for* loop then steps through three cases for FDs of second, fourth, and sixth order. The parameters of Eq. (6.0) are set numerically and passed to the ODE routine as global parameters c1 and c2. We will later return to the significance of these parameters.
2. In accordance with the usual naming convention, the dependent variable is *u* rather than the *E* of Eqs. (6.0) to (8.2) in the subsequent programming. The boundaries and number of grid points in *x* are set. Then the grid in *x* is used to defined the ICs (Eqs. 7.0). Note that $u(x, t = 0)$ is stored in array u0 (i) for $1 = i = n$ and the derivative $\partial u(x, t = 0)/\partial t$ is also stored in u0 (i) for $n + 1 = i = 2n$, so that the 1D array (vector) of dependent variables is initialized as required by ode15s.
3. The *t* interval is $0 = t = 1$ with an output interval of 0.2 so that six outputs are displayed (including the initial point $t = 0$). For the three cases ncase=1, 2, 3 (from the for loop at the beginning), ndss is passed as a global parameter to select one of the differentiation routines dss042, dss044, dss046, respectively (called by the ODE routine pde 1 discussed next). The sparse matrix version of ode15s integrates the n= (2) (101) =202 ODEs using the ODE routine pde1.
4. The array *u* returned by ode15s with the ODE solutions is transformed into two 1D arrays (with $u(x, t)$ and $\partial u(x, t)/\partial t$) to facilitate the handling of the output.
5. Selected tabular output and plots of the solution are then produced.

The overall code is shown in appendix 1

V: Examples and discussions

A: Example 1

The example demonstrates the *skin effect* when AC current is carried by a wire with circular cross section. The conductivity of copper is $57 \cdot 10^6$, and the permeability is 1, i.e., $\mu = 4\pi 10^{-7}$. At the line frequency (50 Hz) the $\omega^2\varepsilon$-term is negligible. Due to the induction, the current density in the interior of the conductor is smaller than at the outer surface where it is set to $J_S = 1$, a Dirichlet condition for the electric field, $E_c = 1/\sigma$. For this case an analytical solution is available,

$$J = J_s \frac{J_0(kr)}{J_0(kR)}$$

Where

$$k = \sqrt{J\omega\mu\sigma}$$

R is the radius of the wire, *r* is the distance from the centre line, and $J_0(x)$ is the first Bessel function of zeroth order.

Solution

After using pde tool box a solution was reached. The solution of the AC power electromagnetics equation is very complex. The plots show the real part of the solution, but the solution vector, is the full complex solution. On that note the solution vector can be exported to the main workspace and you can plot various properties of the complex solution by using the user entry. The skin effect is an AC phenomenon whereby decreasing the frequency of the alternating current results in a decrease of the skin effect. Approaching DC conditions, the current density is close to uniform (experiment using different angular frequencies).

The solution is simulated at appendix two.

B: Example 2

The same criteria can also be used to simulate a guided wave with the square cross section area. Instead of using the pdetool, i used the command window instead.

Given that t is the vector of time and x is the spatial discretization, we solve this problem using a mesh of 20 nodes and request the solution at five values of t. Finally, we extract and plot the first component of the solution. PDEPE function which solves partial differential equations in one space variable and time was used. The code for the guided wave is as follows:

```
x = linspace(0,1,20);
t = [0 0.5 1 1.5 2];
sol =
pdepe(0,@pdex1pde,@pdex1ic,@pdex1bc,x
,t);
u1 = sol(:,:,1);
Surf(x, t, u1);
xlabel('x');ylabel('t');zlabel('u');
hold on
u1 = sol(:,:,1);
surf(x,t,u1);
xlabel('x'); ylabel('t'); zlabel('u');
end
```

solution

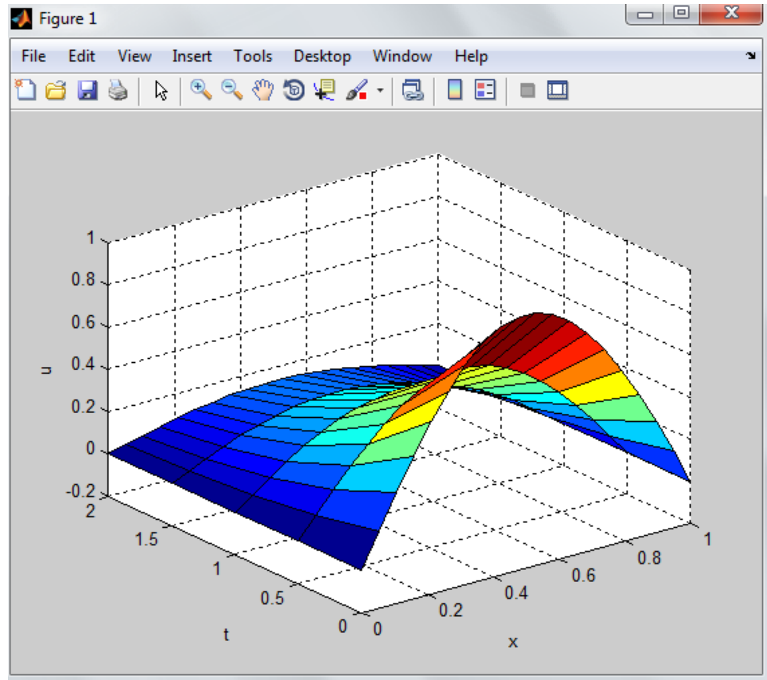

Fig 1.0: One space variable and time for the computation of guided wave guides mainly use in wireless power transmission.

C: Example 3
General solution to BVP

Solving boundary value problems using the command window is much easier than other means since MATLAB can understand some other programming codes e.g. C++. BVP4C function solves boundary value problems for ordinary differential equations for electromagnetic problems. The example function TWOODE has a differential equation written as a system of two first order ODEs. For a TWOODE situation, a TWOBC function has to be called to evaluate the residual in the boundary conditions for TWOBVP. Prior to using BVP4C, we have to provide a guess for the solution we want represented at a mesh. The solver then adapts the mesh as it refines the solution.

BVPINIT assembles the initial guess in the form that the function BVP4C will need. For an initial mesh of [0 1 2 3 4] and a constant guess of y(x) = 1, y'(x) = 0 we need BVPINIT

The code solution are shown in the appendix 3

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Biography

Tawanda Mushiri is a PhD student at the University of Johannesburg in the field of fuzzy logic systems and maintenance, is a Lecturer at the University of Zimbabwe teaching Machine Dynamics and Machine Design. He is a holder of BSc Honors in Mechanical Engineering and MSc in Manufacturing Systems and Operations Management (MSOM).

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