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A Note on the Multi-stage Spectral Relaxation Method for Chaos Control and Synchronization

Abstract: In this study, we present and apply a new, accurate and easy to implement numerical method to realize and verify the synchronization between two identical chaotic Lorenz, Genesio-Tesi, Rössler, Chen and Rikitake systems. The proposed method is called the multi-stage spectral relaxation method (MSRM). We utilize the active control technique for the synchronization of these systems. To illustrate the effectiveness of the method, simulation results are presented and compared with results obtained using the Runge-Kutta (4, 5) based MATLAB solver, ode45.

Keywords: multistage spectral relaxation method, chaos control and synchronization

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1 Introduction

Since the introduction of the Lorenz system [17], the concept of chaos has been extensively studied by various researchers. Chaos has numerous applications in a variety of systems such as electrical circuits, lasers, fluid dynamics, mechanical devices, population growth and many other areas of scientific applications. In recent years, scientists have focused on the control and synchronization of chaotic systems. Synchronization of chaos is a process in which two or more chaotic systems adjust a given property of their motion to a common behavior, due to coupling or forcing [4]. Chaos synchronization is used in applications such as secure communication, chaotic broad band radio, encryption, etc. The idea of chaos control and synchronization was first introduced by Pecora and Carroll [24] by synchronizing two identical chaotic systems with different initial conditions. Since then chaos control and synchronization has received increasing attention. After Pecora and Carroll's method a wide variety of control approaches have been proposed for the synchronization of chaotic systems such as the active control method [1–3, 36], adaptive control method [7, 10, 15], backstepping method [37], sliding mode control method [14], linear and nonlinear feedback control method [13, 28].

In this work, we utilise the active control method for the chaos control and synchronization of two identical chaotic systems. Bai [3] used the active control method for the synchronization of two identical Lorenz systems. The synchronization of the Rössler and Chen systems by the active control method was considered by Agiza and Yassen [2]. Vincent [32] and Umut [31] applied the active control method to synchronize the Rikitake and the Genesio-Tesi system respectively. Chaos synchronization between two different chaotic systems by the active control method has also been considered (see [1, 36]). The active control systems consists of two coupled chaotic systems, one called the *drive/master* system and the other called the *response/slave* system. The output of the drive system is used to control the response system so that the output of the response system tracks the output of the drive system asymptotically. Then the drive and response systems are said to be synchronized. Consequently computing the solutions of the chaos control systems involves more tedious computations since two coupled chaotic systems are involved.

Wang et al. [34] used the multi-stage homotopy pertubation method to find analytical solutions of active control systems to realize and verify the synchronization between the Lorenz and Chen systems.

The main objective of this work is to apply a new multi-stage iterative scheme called the multi-stage spectral relaxation method (MSRM) to solve active control systems and verify the synchronization of two identical chaotic systems. The MSRM have been successfully applied in solving chaotic and hyperchaotic systems by Motsa et al. [22, 23]. The aim of this work is to determine

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if the MSRM is applicable for larger chaotic systems like the control systems. We consider control systems for the chaotic Lorenz, Genesio-Tesi, Rössler, Chen and Rikitake systems. This paper is organized as follows, Section 2 gives a brief description of the proposed MSRM algorithm. A brief description of how the active control method is used for the synchronization of chaotic systems is given in Section 3. In Section 4 we present the numerical implementation of the MSRM on several examples of chaos control systems. In Section 5 we present and discuss numerical results to demonstrate the effectiveness of the proposed MSRM. Finally, Section 6 contains a summarized conclusion of the results.

2 Multi-stage spectral relaxation method

In this section, we give a brief description of how the multi-stage spectral relaxation method (MSRM) algorithm is developed for the solution of common chaotic systems governed by nonlinear systems of first order IVPs. A chaotic system can be expressed as a system of n nonlinear first order differential equations of the form

$$\dot{x}_r(t) = \sum_{k=1}^{n} \alpha_{r,k} x_k(t) + f_r[x_1(t), x_2(t), \ldots, x_n(t)] + g_r, \quad r = 1, 2, \ldots, n \tag{1}$$

subject to the initial conditions

$$x_r(0) = x_r^*, \tag{2}$$

where x_r are the unknown variables and x_r^* are the corresponding initial conditions. $\alpha_{r,k}$ and g_r are known constant input parameters and f_r is the nonlinear component of the rth equation and the dot denotes differentiation with respect to time t.

The scheme computes the solution of Eq. (1) in a sequence of equal sub-intervals that makes the entire interval. So, first the interval $\Omega = [0, T]$ is divided into a sequence of non-overlapping sub-intervals $\Omega_i = [t_{i-1}, t_i]$ $(i = 1, 2, 3, \ldots, f)$ where $t_0 = 0$ and $t_f = T$. We denote the solution of (1) in the first sub-interval $[t_0, t_1]$ as $x_r^1(t)$ and the solutions in the subsequent sub-intervals $[t_{i-1}, t_i]$ $(i = 2, 3, \ldots, f)$ as $x_r^i(t)$. For obtaining the solution in the first interval $[t_0, t_1]$, Eq. (2) is used as the initial condition. By using the continuity condition between neighbouring sub-intervals the obtained solution in the interval $[t_0, t_1]$ is used to obtain the initial condition for the next sub-interval $[t_1, t_2]$. This is applied over the f successive sub-intervals, that is, the obtained solution for each sub-interval $[t_{i-1}, t_i]$ is used to obtain the initial condition for the next sub-interval $[t_i, t_{i+1}]$ $(i = 1, 2, \ldots, f-1)$. Thus, in each interval $[t_{i-1}, t_i]$ we must solve

$$\dot{x}_r^i = \alpha_{r,r} x_r^i + (1 - \delta_{rs}) \sum_{k=1}^{n} \alpha_{r,k} x_k^i + f_r[x_1(t), x_2(t), \ldots, x_n(t)] + g_r, \tag{3}$$

subject to

$$x_r^i(t_{i-1}) = x_r^{i-1}(t_{i-1}) \tag{4}$$

where δ_{rs} is the Kronecker delta. The main idea behind the MSRM scheme is decoupling the system of nonlinear IVPs using the Gauss-Siedel idea of decoupling systems of algebraic equations.

For an n equation IVP system, the proposed MSRM iteration scheme for the solution in the interval $\Omega_i = [t_{i-1}, t_i]$ is given as

$$\begin{aligned} &\dot{x}_{1,s+1}^i - \alpha_{1,1} x_{1,s+1}^i \\ &\quad = \alpha_{1,2} x_{2,s}^i + \alpha_{1,3} x_{3,s}^i + \cdots + \alpha_{1,n} x_{n,s}^i \\ &\qquad + f_1[x_{1,s}^i, x_{2,s}^i, \ldots, x_{n,s}^i] + g_1, \end{aligned} \tag{5}$$

$$\begin{aligned} &\dot{x}_{2,s+1}^i - \alpha_{2,2} x_{2,s+1}^i \\ &\quad = \alpha_{2,1} x_{1,s+1}^i + \alpha_{2,3} x_{3,s}^i + \alpha_{2,4} x_{4,s}^i \\ &\qquad + \cdots + \alpha_{2,n} x_{n,s}^i \\ &\qquad + f_2[x_{1,s+1}^i, x_{2,s}^i, x_{3,s}^i, \ldots, x_{n,s}^i] + g_2, \end{aligned} \tag{6}$$

$$\vdots$$

$$\begin{aligned} &\dot{x}_{n,s+1}^i - \alpha_{n,n} x_{n,s+1}^i \\ &\quad = \alpha_{n,1} x_{1,s+1}^i + \alpha_{n,2} x_{2,s+1}^i \\ &\qquad + \cdots + \alpha_{n,n-1} x_{n-1,s+1}^i \\ &\qquad + f_n[x_{1,s+1}^i, x_{2,s+1}^i, \ldots, x_{n-1,s+1}^i, x_{n,s}^i] + g_n \end{aligned} \tag{7}$$

subject to the initial conditions

$$x_{r,s+1}^i(t_{i-1}) = x_r^{i-1}(t_{i-1}), \quad r = 1, 2, \ldots, n, \tag{8}$$

where $x_{r,s}$ is the estimate of the solution after s iterations. A suitable initial guess to start the iteration scheme (5–7) is one that satisfies the initial condition (8). A convenient choice of initial guess that was found to work in the numerical experiments considered in this work is

$$x_{r,0}^i(t) = \begin{cases} x_r^* & \text{if } i = 1 \\ x_r^{i-1}(t_{i-1}) & \text{if } 2 \le i \le f \end{cases} \tag{9}$$

The Chebyshev spectral method is used to solve Eqs. (5)–(7) on each interval $[t_{i-1}, t_i]$. First, the region $[t_{i-1}, t_i]$ is transformed to the interval $[-1, 1]$ on which the spectral method is defined by using the linear transformation

$$t = \frac{(t_i - t_{i-1})\tau}{2} + \frac{(t_i + t_{i-1})}{2} \tag{10}$$

in each interval $[t_{i-1}, t_i]$ for $i = 1, \ldots, f$. We then discretize the interval $[t_{i-1}, t_i]$ using the Chebyshev-Gauss-Lobatto collocation points [6, 30]

$$\tau_j^i = \cos\left(\frac{\pi j}{N}\right), \quad j = 1, 2, \ldots, N \tag{11}$$

which are the extrema of the Nth order Chebyshev polynomial

$$T_N(\tau) = \cos(N \cos^{-1} \tau). \tag{12}$$

The Chebyshev spectral collocation method is based on the idea of introducing a differentiation matrix D which is used to approximate the derivatives of the unknown variables $x_{r,s+1}^i(t)$ at the collocation points as the matrix vector product

$$\frac{dx_{r,s+1}^i}{dt} = \sum_{k=0}^{N} \mathbf{D}_{jk} x_{r,s+1}^i = \mathbf{D}\mathbf{X}_{r,s+1}^i, \quad j = 1, 2, \ldots, N \tag{13}$$

where $\mathbf{D} = 2D/(t_i - t_{i-1})$ and $\mathbf{X}_{r,s+1}^i = [x_{r,s+1}^i(\tau_0^i), x_{r,s+1}^i(\tau_1^i), \ldots, x_{r,s+1}^i(\tau_N^i)]$ are the vector functions at the collocation points τ_j^i.

Applying the Chebyshev spectral collocation method in Eqs. (5)–(7) gives

$$\mathbf{A}_r \mathbf{X}_{r,s+1}^i = \mathbf{B}_r^i, \quad \mathbf{X}_{r,s+1}^i(\tau_N^{i-1}) = \mathbf{X}_r^{i-1}(\tau_N^{i-1}), \quad r = 1, 2, \ldots, n. \tag{14}$$

with

$$\mathbf{A}_r = \mathbf{D} - \alpha_{r,r}\mathbf{I}, \tag{15}$$

and

$$\mathbf{B}_1^i = \alpha_{1,2}\mathbf{X}_{2,s}^i + \alpha_{1,3}\mathbf{X}_{3,s}^i + \cdots + \alpha_{1,n}\mathbf{X}_{n,s}^i + f_1[(\mathbf{X}_{1,s}^i, \mathbf{X}_{2,s}^i, \ldots, \mathbf{X}_{n,s}^i] + g_1, \tag{16}$$

$$\mathbf{B}_2^i = \alpha_{2,1}\mathbf{X}_{1,s+1}^i + \alpha_{2,3}\mathbf{X}_{3,s}^i + \alpha_{2,4}\mathbf{X}_{4,s}^i + \cdots + \alpha_{2,n}\mathbf{X}_{n,s}^i + f_2[\mathbf{X}_{1,s+1}^i, \mathbf{X}_{2,s}^i, \mathbf{X}_{3,s}^i, \ldots, \mathbf{X}_{n,s}^i] + g_2, \tag{17}$$

$$\vdots$$

$$\mathbf{B}_n^i = \alpha_{n,1}\mathbf{X}_{1,s+1}^i + \alpha_{n,2}\mathbf{X}_{2,s+1}^i + \cdots + \alpha_{n,n-1}\mathbf{X}_{n-1,s+1}^i + f_n[\mathbf{X}_{1,s+1}^i, \mathbf{X}_{2,s+1}^i, \ldots, \mathbf{X}_{n-1,s+1}^i, \mathbf{X}_{n,s}^i] + g_n, \tag{18}$$

where $\mathbf{I}$ is an identity matrix of order $N+1$. Thus, starting from the initial approximation (9), the recurrence formula

$$\mathbf{X}_{r,s+1}^i = \mathbf{A}_r^{-1}\mathbf{B}_r^i, \quad r = 1, 2, \ldots, n. \tag{19}$$

can be used to obtain the solution $x_r^i(t)$ in the interval $[t_{i-1}, t_i]$. The solution approximating $x_r(t)$ in the entire interval $[t_0, t_F]$ is given by

$$x_r(t) = \begin{cases} x_r^1(t), & t \in [t_0, t_1] \\ x_r^2(t), & t \in [t_1, t_2] \\ \vdots & \\ x_r^F(t), & t \in [t_{f-1}, t_f] \end{cases} \tag{20}$$

3 Active control method

In this section we give a basic principle behind chaos synchronization of chaotic systems using the active control method. The active control systems consists of two coupled chaotic systems, one called the drive system and the other called the response system. The output of the drive system is used to control the response system so that the output of the response system tracks the output of the master system asymptotically. Then the drive and response systems are said to be synchronized [1, 29]. As already mentioned above the drive system can be expressed as a system of n equations of the form

$$\dot{x}_r(t) = \sum_{k=1}^{n} \alpha_{r,k} x_k(t) + f_r[x_1(t), x_2(t), \ldots, x_n(t)] + g_r, \quad r = 1, 2, \ldots, n \tag{21}$$

and the response system is given by

$$\dot{x}_{n+r}(t) = \sum_{k=1}^{n} \alpha_{r,k} x_{n+k}(t) + f_r[x_{n+1}(t), x_{n+2}(t), \ldots, x_{2n}(t)] + u_r(t) + g_r, \quad r = 1, 2, \ldots, n \tag{22}$$

where x_r and x_{n+r} are the unknown variables. $\alpha_{r,k}$ and g_r are known constant input parameters and f_r is the nonlinear component of the rth equation. $u_r(t)$ are the active control functions. The drive and response systems achieve synchronization if appropriate active control functions are chosen. The drive and response systems are said to have been synchronized if

$$\lim_{t\to\infty} \|e_r\| = \lim_{t\to\infty} \|x_{n+r} - x_r\| = 0 \tag{23}$$

where $e_r = x_{n+r} - x_r$ are the error states. Therefore, the error dynamics are given by

$$\begin{aligned}\dot{e}_r = \dot{x}_{n+r} - \dot{x}_r = \sum_{k=1}^{n} \alpha_{r,k} e_k \\ + F_r[x_1(t), x_2(t), \ldots, x_{2n}(t)] + u_r(t)\end{aligned} \tag{24}$$

where $F_r[x_1, x_2, \ldots, x_{2n}]$ are the nonlinear and non-common parts in the drive and response systems.

Since the error vectors should converge to zero, appropriate controllers should eliminate nonlinear terms, i.e.

$$u_r(t) = V_r(t) - F_r \tag{25}$$

where $V_r(t) = \sum_{k=1}^{n} h_{r,k} e_k$ are the linear controllers and $h_{r,k}$ are constant parameters.

Substituting Eq. (25) in Eq. (24) we obtain

$$\dot{e}_r = \left(\sum_{k=1}^{n} \alpha_{r,k} e_k + \sum_{k=1}^{n} h_{r,k} e_k \right) \tag{26}$$

which can be written in compact form as

$$\dot{\mathbf{e}} = (\mathbf{A} + \mathbf{H})\mathbf{e} \tag{27}$$

where $\mathbf{e} = [e_1, e_2, \ldots, e_n]$. $\mathbf{A}$ and $\mathbf{H}$ are $n \times n$ constant matrices whose entries are $\alpha_{r,k}$ and $h_{r,k}$ respectively. If the eigenvalues of the matrix $(\mathbf{A} + \mathbf{H})$ are negative, then the error state vectors asymptotically converge to zero. That is, the drive and response systems asymptotically synchronize.

There are many possible choices of $V_r(t)$ satisfying

$$\begin{pmatrix} V_1(t) \\ V_2(t) \\ \vdots \\ V_n(t) \end{pmatrix} = \mathbf{H} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} \tag{28}$$

The elements $h_{r,k}$ of the matrix $\mathbf{H}$ should be chosen such that the eigenvalues of the matrix $(\mathbf{A} + \mathbf{H})$ are negative.

4 Numerical examples

In this section we demonstrate the applicability of the MSRM in solving chaotic control systems. First we utilise the active control method to control and synchronize systems of IVPs exhibiting chaotic behavior. To verify the chaos control and synchronization of these chaotic systems, we then use the MSRM algorithm to solve the resulting control systems. We consider the Lorenz, Genesio-Tesi, Rossler, Chen, and the Rikitake systems.

4.1 Lorenz system

The Lorenz system is a dynamical system commonly used to explore chaos. It is given by the following set of autonomous differential equations

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = -x_1 x_3 + b x_1 - x_2, \\ \dot{x}_3 = x_1 x_2 - c x_3. \end{cases} \tag{29}$$

The Lorenz systems models a three mode approximation to the motion of a layer of fluid heated from below. Bai [3] considered the synchronization of the Lorenz system using the active control method. System (29) is taken as the drive system and the response is defined by

$$\begin{cases} \dot{x}_4 = a(x_5 - x_4) + u_1(t), \\ \dot{x}_5 = -x_4 x_6 + b x_4 - x_5 + u_2(t), \\ \dot{x}_6 = x_4 x_5 - c x_6 + u_3(t). \end{cases} \tag{30}$$

Applying the active control method, the control functions are defined as

$$u_1(t) = V_1(t), \tag{31}$$

$$u_2(t) = x_4 x_6 - x_1 x_3 + V_2(t), \tag{32}$$

$$u_3(t) = -x_4 x_5 + x_1 x_2 + V_3(t). \tag{33}$$

The matrix $\mathbf{H}$ can be chosen as

$$\mathbf{H} = \begin{pmatrix} a-1 & -a & 0 \\ -b & 0 & 0 \\ 0 & 0 & c-1 \end{pmatrix} \tag{34}$$

and hence

$$\begin{cases} \dot{x}_4 = (1-a)x_1 + a x_2 - x_4, \\ \dot{x}_5 = b x_1 - x_5 - x_1 x_3, \\ \dot{x}_6 = (1-c)x_3 - x_6 + x_1 x_2. \end{cases} \tag{35}$$

The systems (29) and (35) were solved for the parameters $a = 10$, $b = 28$, $c = 8/3$ and the initial conditions $x_1(0) = -10$, $x_2(0) = -5$, $x_3(0) = 35$ and $x_4(0) = 0$, $x_5(0) = 0$, $x_6(0) = 0$ for the drive and response systems respectively. In this example, the parameters used in the MSRM iteration

$$\begin{aligned}&\alpha_{11} = -a, \quad \alpha_{12} = a, \quad \alpha_{21} = b, \quad \alpha_{22} = -1, \quad \alpha_{33} = -c,\\ &\alpha_{41} = 1-a, \quad \alpha_{42} = a, \quad \alpha_{44} = -1, \quad \alpha_{51} = b, \quad \alpha_{55} = -1,\\ &\alpha_{63} = 1-c, \quad \alpha_{66} = -1, \quad f_2 = -x_1x_3, \quad f_3 = x_1x_2,\\ &f_5 = -x_1x_3, \quad f_6 = x_1x_2 \end{aligned} \tag{36}$$

4.2 Genesio-Tesi system

The Genesio-Tesi system, proposed by Genesio and Tesi [12] is described by the following simple three-dimensional autonomous system with only one quadratic nonlinear term:

$$\begin{cases} \dot{x}_1 = x_2,\\ \dot{x}_2 = x_3,\\ \dot{x}_3 = -ax_1 - bx_2 - cx_3 + x_1^2. \end{cases} \tag{37}$$

where $a, b, c < 0$ are parameters. The Genesio-Tesi exhibits chaotic behavior when $a = -6$, $b = -2.92$, $c = -1.2$. System (37) is taken as the drive system and the response is defined by

$$\begin{cases} \dot{x}_4 = x_5 + u_1(t),\\ \dot{x}_5 = x_6 + u_2(t),\\ \dot{x}_6 = -ax_4 - bx_5 - cx_6 + x_4^2 + u_3(t). \end{cases} \tag{38}$$

Umut [31] applied the active control method for the synchronization of the Genesio-Tesi system. The active control functions are defined as

$$u_1(t) = V_1(t), \tag{39}$$

$$u_2(t) = V_2(t), \tag{40}$$

$$u_3(t) = x_1^2 - x_4^2 + V_3(t). \tag{41}$$

The matrix **H** can be chosen as

$$\mathbf{H} = \begin{pmatrix} -1 & -1 & 0\\ 0 & -1 & -1\\ a & b & c-1 \end{pmatrix} \tag{42}$$

and hence

$$\begin{cases} \dot{x}_4 = x_1 + x_2 - x_4,\\ \dot{x}_5 = x_2 + x_3 - x_5,\\ \dot{x}_6 = -ax_1 - bx_2 - (c-1)x_3 - x_6 + x_1^2. \end{cases} \tag{43}$$

The systems (37) and (43) were solved for the parameters $a = 6$, $b = 2.92$, $c = 1.2$ and the initial conditions $x_1(0) = 0.2$, $x_2(0) = -0.3$, $x_3(0) = 0.1$ and $x_4(0) = 0$, $x_5(0) = 0$, $x_6(0) = 0$ for the drive and response systems respectively. In this example, the parameters used in the MSRM iteration

$$\begin{aligned}&\alpha_{12} = 1, \quad \alpha_{23} = 1, \quad \alpha_{31} = -a, \quad \alpha_{32} = -b, \quad \alpha_{33} = -c,\\ &\alpha_{41} = 1, \quad \alpha_{42} = 1, \quad \alpha_{44} = -1, \quad \alpha_{52} = 1, \quad \alpha_{53} = 1,\\ &\alpha_{55} = -1, \quad \alpha_{61} = -a, \quad \alpha_{62} = -b, \quad \alpha_{63} = 1-c,\\ &\alpha_{66} = -1, \quad f_3 = x_1^2, \quad f_6 = x_1^2. \end{aligned} \tag{44}$$

4.3 Rössler system

The Rössler system was originally studied by Rössler [26] in 1976. The drive Rössler system is defined by

$$\begin{cases} \dot{x}_1 = -(x_2 + x_3),\\ \dot{x}_2 = x_1 + ax_2,\\ \dot{x}_3 = b + x_1x_3 - cx_3, \end{cases} \tag{45}$$

and the response is defined by

$$\begin{cases} \dot{x}_1 = -(x_5 + x_6) + u_1(t),\\ \dot{x}_2 = x_4 + ax_5 + u_2(t),\\ \dot{x}_3 = b + x_4x_6 - cx_6 + u_3(t). \end{cases} \tag{46}$$

Agiza and Yassen [2] applied the active control technique for the synchronization of the Rössler system. The control functions are defined as

$$u_1(t) = V_1(t), \tag{47}$$

$$u_2(t) = V_2(t), \tag{48}$$

$$u_3(t) = x_1x_3 - x_4x_5 + V_3(t). \tag{49}$$

The matrix **H** can be chosen as

$$\mathbf{H} = \begin{pmatrix} -1 & 1 & 1\\ -1 & -(1+a) & 0\\ 0 & 0 & c-1 \end{pmatrix} \tag{50}$$

and hence

$$\begin{cases} \dot{x}_4 = x_1 - x_2 - x_3 - x_4,\\ \dot{x}_5 = x_1 + (1+a)x_2 - x_5,\\ \dot{x}_6 = b - (c-1)x_3 - x_6 + x_1x_3. \end{cases} \tag{51}$$

The Rössler system exhibits chaotic behavior if the parameters are chosen as $a = 0.2$, $b = 0.2$, $c = 5.7$. The system was solved for the initial conditions $x_1(0) = 0.5$, $x_2(0) = 1$, $x_3(0) = 1.5$ and $x_4(0) = 6$, $x_5(0) = 3.5$, $x_6(0) = 5$ for the drive and response systems respectively. In this example, the parameters used in the MSRM iteration

$$\begin{aligned}&\alpha_{12} = -1, \quad \alpha_{13} = -1, \quad \alpha_{21} = 1, \quad \alpha_{22} = a, \quad \alpha_{33} = -c,\\ &\alpha_{41} = 1, \quad \alpha_{42} = -1, \quad \alpha_{43} = -1, \quad \alpha_{44} = -1, \quad \alpha_{51} = 1,\\ &\alpha_{52} = 1 + a, \quad \alpha_{55} = -1, \quad \alpha_{61} = -a, \quad \alpha_{63} = 1 - c,\\ &\alpha_{66} = -1, \quad f_3 = x_1x_3, \quad f_6 = x_1x_3, \quad g_3 = b, \quad g_6 = b.\end{aligned} \tag{52}$$

4.4 Chen system

In 1999 Chen [8] found a similar but topologically non-equivalent chaotic attractor to the Lorenz attractor called the Chen attractor. Lü et al. [18] state that it was proven that the Chen system is dual to the Lorenz system. The drive and response systems for the Chen system are defined as follows:

$$\begin{cases}\dot{x}_1 = a(x_2 - x_1),\\ \dot{x}_2 = (c - a)x_1 - x_1x_3 + cx_2,\\ \dot{x}_3 = x_1x_2 - bx_3.\end{cases} \tag{53}$$

and

$$\begin{cases}\dot{x}_1 = a(x_2 - x_1) + u_1(t),\\ \dot{x}_2 = (c - a)x_1 - x_1x_3 + cx_2 + u_2(t),\\ \dot{x}_3 = x_1x_2 - bx_3 + u_3(t).\end{cases} \tag{54}$$

The Chen system has a chaotic attractor when $a = 35$, $b = 3$, $c = 28$. The synchronization of the Chen system using the control method was considered by Agiza and Yassen [2]. Applying the active control method, the control functions are defined as

$$u_1(t) = V_1(t), \tag{55}$$

$$u_2(t) = x_4x_6 - x_1x_3 + V_2(t), \tag{56}$$

$$u_3(t) = x_1x_2 - x_4x_5 + V_3(t). \tag{57}$$

The matrix $\mathbf{H}$ can be chosen as

$$\mathbf{H} = \begin{pmatrix} a-1 & -a & 0\\ a-c & -(1+c) & 0\\ 0 & 0 & b-1\end{pmatrix} \tag{58}$$

and hence

$$\begin{cases}\dot{x}_4 = -(a-1)x_1 + ax_2 - x_4,\\ \dot{x}_5 = -(a-c)x_1 + (1+c)x_2 - x_5 - x_1x_3,\\ \dot{x}_6 = -(b-1)x_3 - x_6 + x_1x_2.\end{cases} \tag{59}$$

The systems (53) and (59) were solved for the parameters $a = 35$, $b = 3$, $c = 28$ and the initial conditions $x_1(0) = 0.5$, $x_2(0) = 1$, $x_3(0) = 1$ and $x_4(0) = 10.5$, $x_5(0) = 20$, $x_6(0) = 38$ for the drive and response systems respectively. In this example, the parameters used in the MSRM iteration

$$\begin{aligned}&\alpha_{11} = -a, \quad \alpha_{12} = a, \quad \alpha_{21} = c - a, \quad \alpha_{22} = c, \quad \alpha_{33} = -b,\\ &\alpha_{41} = 1 - a, \quad \alpha_{42} = a, \quad \alpha_{44} = -1, \quad \alpha_{51} = c - a,\\ &\alpha_{52} = (1 + c), \quad \alpha_{55} = -1, \quad \alpha_{63} = 1 - b, \quad \alpha_{66} = -1,\\ &f_2 = -x_1x_3, \quad f_3 = x_1x_2, \quad f_5 = -x_1x_3, \quad f_6 = x_1x_2.\end{aligned} \tag{60}$$

4.5 Rikitake system

The Rikitake system is a two-disc dynamo system which is a simple mechanical model used to study the reversals of the earth's magnetic field [16]. The system was idealised by Rikitake [25]. It is governed by the following three dimensional system of nonlinear differential equations

$$\begin{cases}\dot{x}_1 = -bx_1 + x_2x_3,\\ \dot{x}_2 = -bx_2 + x_1(x_3 - a),\\ \dot{x}_3 = 1 - x_1x_2,\end{cases} \tag{61}$$

where $a, b > 0$. The Rikitake system exhibits chaotic behavior for $a = 5$ and $b = 2$. System (61) is taken as the drive system and the response system is given by

$$\begin{cases}\dot{x}_4 = -bx_4 + x_5x_6 + u_1(t),\\ \dot{x}_5 = -bx_5 + x_4(x_6 - a) + u_2(t),\\ \dot{x}_6 = 1 - x_4x_5 + u_3(t),\end{cases} \tag{62}$$

The snchronization of the Rikitake system using active control was considered by Vincent [32]. Applying the active control method, the control functions are defined as

$$u_1(t) = -x_5x_6 + x_2x_3 + V_1(t), \tag{63}$$

$$u_2(t) = -x_4x_6 + x_1x_3 + V_2(t), \tag{64}$$

$$u_3(t) = x_4x_5 - x_1x_2 + V_3(t). \tag{65}$$

The matrix $\mathbf{H}$ can be chosen as

$$\mathbf{H} = \begin{pmatrix} -(1-b) & 0 & 0\\ a & -(1-b) & 0\\ 0 & 0 & -1\end{pmatrix} \tag{66}$$

and hence

$$\begin{cases} \dot{x}_4 = (1-b)x_1 - x_4 + x_2x_3, \\ \dot{x}_5 = -ax_1 + (1-b)x_2 - x_5 + x_1x_3, \\ \dot{x}_6 = 1 + x_3 - x_6 - x_1x_2. \end{cases} \tag{67}$$

The systems (61) and (67) were solved for the parameters $a = 5$, $b = 2$ and the initial conditions $x_1(0) = -4$, $x_2(0) = 2.5$, $x_3(0) = 1$ and $x_4(0) = -2$, $x_5(0) = 0$, $x_6(0) = 5$ for the drive and response systems respectively. In this example, the parameters used in the MSRM iteration

$$\begin{aligned} &\alpha_{11} = -b, \quad \alpha_{21} = -a, \quad \alpha_{22} = -b, \quad \alpha_{41} = (1-b), \\ &\alpha_{44} = -1, \quad \alpha_{51} = -a, \quad \alpha_{52} = (1-b), \quad \alpha_{55} = -1, \\ &\alpha_{63} = 1, \quad \alpha_{66} = -1, \quad f_1 = x_2x_3, \quad f_2 = x_1x_3, \\ &f_3 = -x_1x_2, \quad f_4 = x_2x_3, \quad f_5 = x_1x_3, \quad f_6 = -x_1x_2, \\ &g_3 = 1, \quad g_6 = 1. \end{aligned} \tag{68}$$

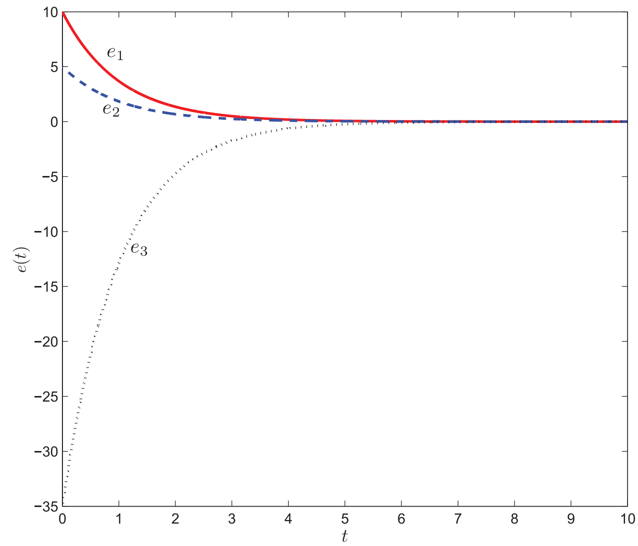

Fig. 2: Errors for the Lorenz system using the MSRM

5 Results and discussion

In this section we present the numerical results of the implementation of the multi-stage spectral relaxation method (MSRM) to the examples mentioned above. The results obtained were compared to those from the MATLAB in-built solver, ode45. The ode45 solver integrates a system of ordinary differential equations using explicit 4th and 5th Runge-Kutta formula.

Figure 1 illustrates a comparison of the results obtained using MSRM and ode45. A good agreement of the results is observed. From Figure 1 it can be seen that the active controllers have synchronized the Lorenz system since as the time progresses the response system quickly follows the drive system. Figure 2 shows the time responses of the error vectors. It is clear that after control functions are activated, the error vectors converge to zero rapidly. Results of the Genesio-Tesi control system obtained by the MSRM and ode45 are shown in Figure 3. Again the two sets of results are in good agreement. As time increases, the response system quickly follows the drive system which shows that the Genesio-Tesi system

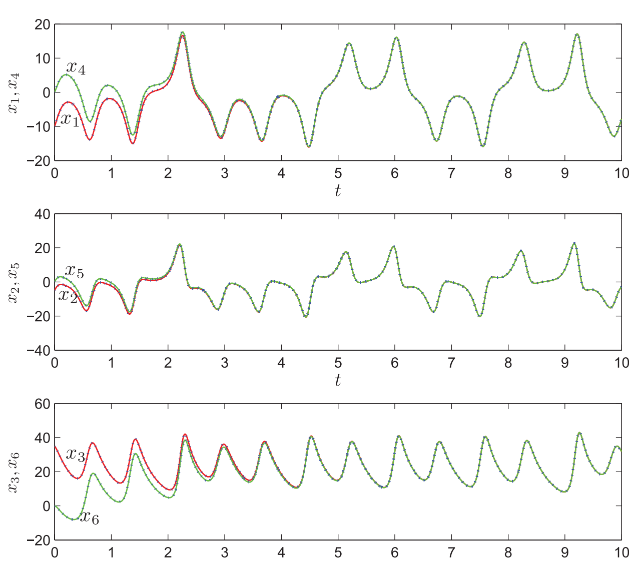

Fig. 1: Comparison between the MSRM (solid line) and `ode45` (dots) results for the Lorenz system

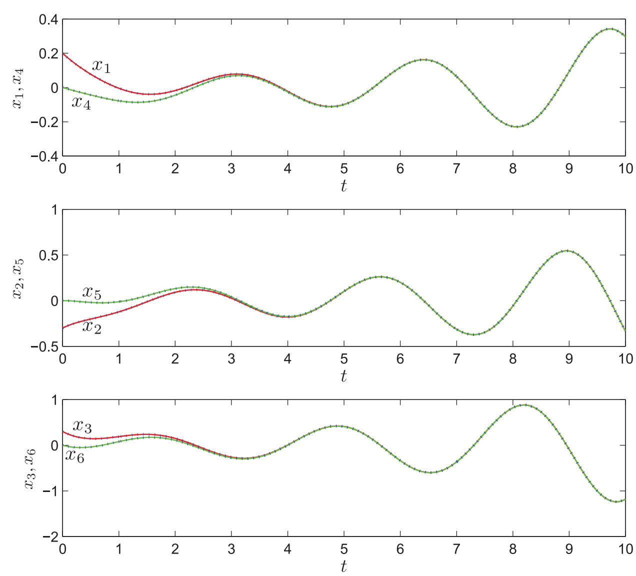

Fig. 3: Comparison between the MSRM (solid line) and `ode45` (dots) results for the Genesio-Tesi system

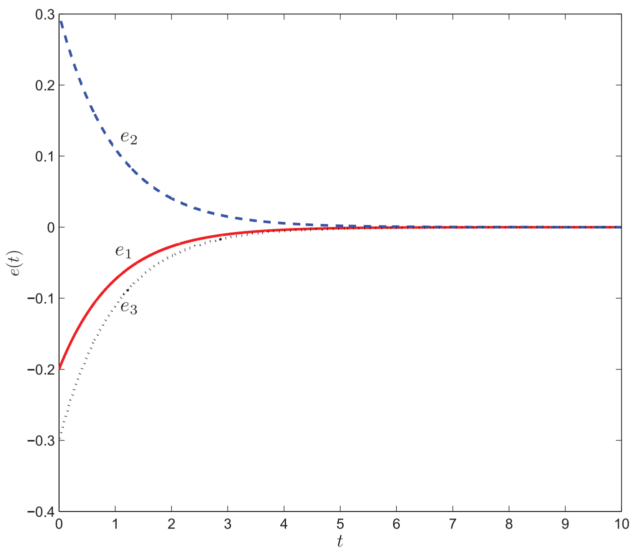

Fig. 4: Errors for the Genesio-Tesi system using the MSRM

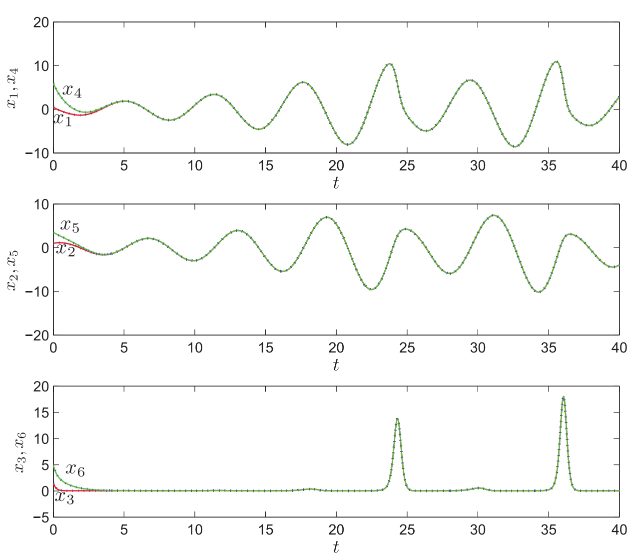

Fig. 5: Comparison between the MSRM (solid line) and `ode45` (dots) results for the Rössler system

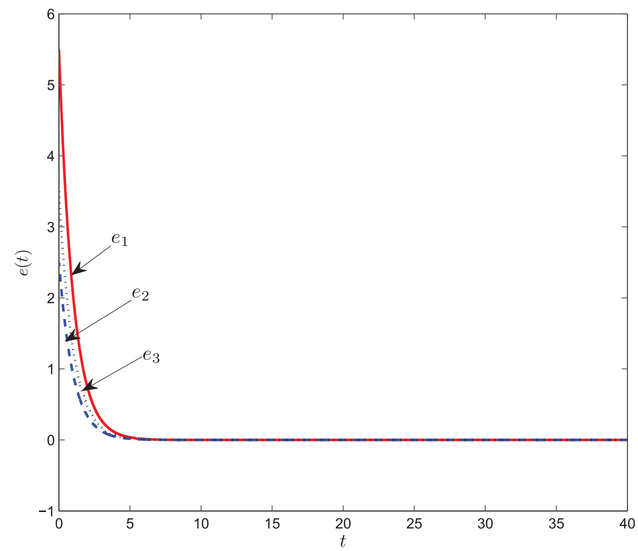

Fig. 6: Errors for the Rössler system using the MSRM

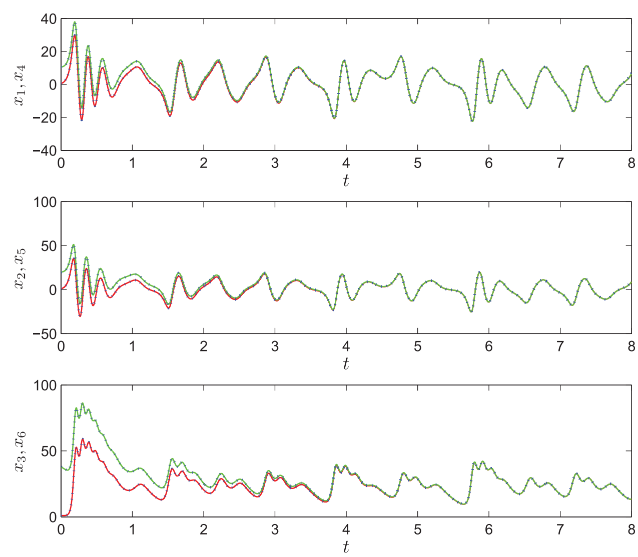

Fig. 7: Comparison between the MSRM (solid line) and `ode45` (dots) results for the Chen system

have been synchronized. Figure 4 shows the time responses of the error vectors. It is clear that after control functions are activated, the error vectors converge to zero rapidly. We also observe good agreement of results for the solution of the Rössler system obtained using the MSRM and ode45 as shown in Figure 5. It can also be depicted from Figure 5 that the active controllers have synchronized the Rössler system since the response system pursue the drive system as the time increases. This can also be gathered from Figure 6 which shows the state errors approaches zero quickly after the control is effected. Figure 7 illustrates a comparison of the results obtained using MSRM and ode45. The results obtained are the same. From Figure 7 it can be seen that the active controllers have synchronized the Chen system since as the time progresses the response system quickly follows the drive system. Figure 8 shows the time responses of the error vectors. It is clear that after control functions are activated, the error vectors converge to zero quickly which confirms that the Chen system has been synchronized. Again the MSRM results agree with the ode45 results for the Rikitake system. The comparison of the results is shown in Figure 9. As the time progresses the response system quickly follows the drive system which

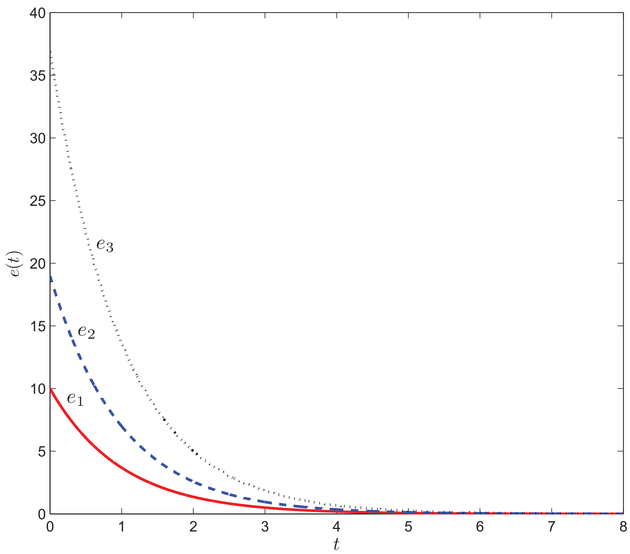

Fig. 8: Errors for the Chen system using the MSRM

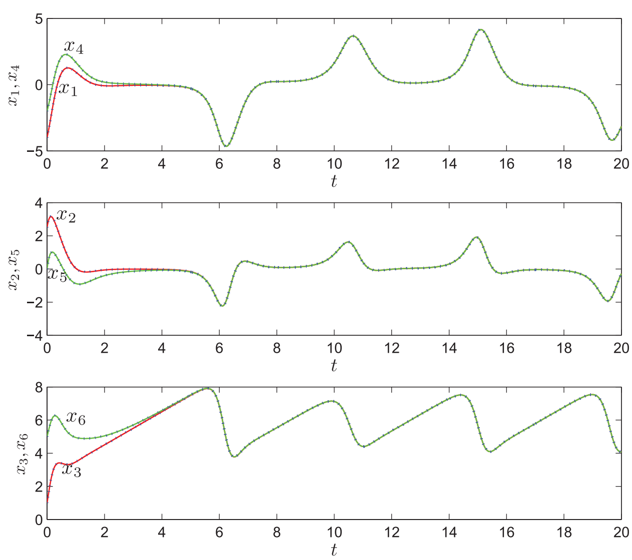

Fig. 9: Comparison between the MSRM (solid line) and `ode45` (dots) results for the Rikitake system

shows that the Rikitake system has been synchronized. Figure 10 shows the time responses of the error vectors. It is clear that after control functions are activated, the error vectors converge to zero quickly.

6 Conclusion

In this work we have applied a new method called the multi-stage spectral relaxation method (MSRM) for the solutions of control systems of the chaotic Lorenz,

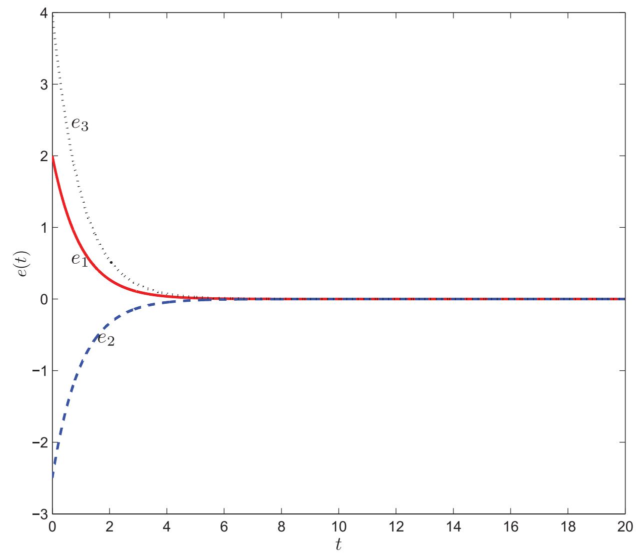

Fig. 10: Errors for the Rikitake system using the MSRM

Genesio-Tesi, Rössler, Chen and Rikitake systems. For the chaos control and synchronization of these systems we utilized the active control method. The obtained results of the MSRM are comparable to results obtained by the Runge-Kutta (4, 5) based MATLAB built-in solver, ode45. The advantage of the MSRM over other multi-stage methods is that it decouples the systems based on the Gauss-Siedel approach and, therefore is easy to implement. In addition the MSRM does not require any derivatives or linearization. It is simply based on a systematic rearrangement of the governing equations and the subsequent solution in a sequential manner. Thus we can safely say we have presented an accurate, easy to implement, and reliable method for solving large systems which exhibit chaotic behavior.

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