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The effects of fractional order on a 3-D quadratic autonomous system with four-wing attractor

Zenghui Wang · Yanxia Sun · Guoyuan Qi · Barend Jacobus van Wyk

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Abstract In this paper, a fractional 3-dimensional (3-D) 4-wing quadratic autonomous system (Qi system) is analyzed. Time domain approximation method (Grunwald–Letnikov method) and frequency domain approximation method are used together to analyze the behavior of this fractional order chaotic system. It is found that the decreasing of the system order has great effect on the dynamics of this nonlinear system. The fractional Qi system can exhibit chaos when the total order less than 3, although the regular one always shows periodic orbits in the same range of parameters. In some fractional order, the 4 wings are decayed to a scroll using the frequency domain approximation method which is different from the result using time domain approximation method. A surprising finding is that the phase diagrams display a character of local self-similar in the 4-wing attractors of this fractional Qi system using the frequency approximation method even though the number and the characteristics of equilibria are not changed. The frequency spectrums show that there is some shrinking tendency of the bandwidth with the falling of the system states order. However, the change of fractional order has little effect on the bandwidth of frequency spectrum using the time domain approximation method. According to the bifurcation analysis, the fractional order Qi system attractors start from sink, then period bifurcation to some simple periodic orbits, and chaotic attractors, finally escape from chaotic attractor to periodic orbits with the increasing of fractional order α in the interval $[0.8, 1]$. The simulation results revealed that the time domain approximation method is more accurate and reliable than the frequency domain approximation method.

Keywords Chaos · Fractional order · Four-wing attractor · Bifurcation · Lyapunov exponent

Z. Wang (✉)
Department of Electrical and Mining Engineering,
University of South Africa, Florida 0719, South Africa
e-mail: wangzengh@gmail.com

Z. Wang · Y. Sun · G. Qi · B.J. van Wyk
French South African Technical Institute in Electronics
(F'SATIE), Department of Electrical Engineering,
Tshwane University of Technology, Pretoria 0001,
South Africa

Y. Sun
e-mail: sunyanxia@gmail.com

G. Qi
e-mail: guoyuanqi@gmail.com

B.J. van Wyk
e-mail: vanwykb@gmail.com

Y. Sun
Mathématiques et Sciences et Technologies
de l 'Information et de la Communication,
Universite Paris-Est, Paris 94010, France
e-mail: sunyanxia@gmail.com

1 Introduction

The derivative theory of fractional order goes back to a question raised in the year 1695 by L' Hopital to G.W. Leibniz, in which the meaning of derivative of order of $1/2$ was discussed. The subject of fractional calculus has gained considerable popularity and attention during the past three decades or so, mainly due to its widespread applications in the fields of science and engineering [1, 2]. Many systems are known to display fractional-order dynamics, such as viscoelastic systems [3–6], dielectric polarization [7], quantitative finance [8–11], and quantum evolution of a complex system [12]. The dynamics of the fractional nonlinear system have also been studied extensively during recent years. According to the Poincaré–Bendixson theorem (for a review, see [13]), chaos cannot occur in continuous-time autonomous systems of order less than three, which is based on the usual integer order concepts. However, some fractional-order nonlinear systems display chaos when the total order less than three [14–21]. Most of the researches on the fractional chaos or hyperchaos systems are finding the lowest total order of some well-known nonlinear systems, which are chaos or hyperchaos when the system orders are integer, and have not paid much attention to the rich dynamics caused by the fractional order.

Recently, Qi et al. [24] has proposed a new 3-D autonomous system with five equilibria, in which each equation contains a single quadratic term. The system can generate two coexisting single-wing chaotic attractors and a pair of diagonal double-wing chaotic attractors. It is amazing to find that a real four-wing chaotic attractor can be generated when the two diagonal double-wing chaotic attractors merge together in some way which is different from Lorenz systems.

In this paper, we investigate the dynamics of the fractional four-wing 3-D quadratic autonomous system (in this paper, the system is called the Qi system), and find that the fractional Qi system exhibits chaos although the normal Qi system is not chaotic using the same parameters. Time domain approximation method (Grunwald–Letnikov method) and frequency domain approximation method are used together to analyze the behavior of the fractional order chaotic system. The phase diagrams and frequency analysis show that the dynamics of fractional Qi system is different from that of the integer order Qi system.

2 Introduction to fractional calculus

Fractional calculus is a generalization of integration and differentiation to noninteger order fundamental operator ${}_aD_t^\alpha$, where a and t are the limits of the operation. The continuous integro-differential operator is defined as [17]:

$$ {}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & \alpha > 0, \\ 1, & \alpha = 0, \\ \int_a^t (d\tau)^{-\alpha}, & \alpha < 0. \end{cases} $$

There are several definitions for fractional derivatives. The most commonly used definitions are the Grunwald–Letnikov, Riemann–Liouville and Caputo definitions [29]. The Grunwald–Letnikov is given as

$$ {}_aD_t^\alpha f(t) = \lim_{x\to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} f(t-jh), \quad (1) $$

where $[\cdot]$ means the integer part and h is the time step. The Riemann–Liouville definition is

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (2) $$

where $n-1 < \alpha < n$ and $\Gamma(\cdot)$ is the *Gamma* function.

Another definition is the Caputo definition whose properties are similar to the Riemann–Liouville [1]. Caputo's derivative of order α and with the lower limit 0 can be viewed as regularization of the Riemann–Liouville derivative and is defined as

$$ {}_0D_t^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (3) $$

where $\Gamma(\cdot)$ is the *Gamma* function and $n-1 \le \alpha < n$. The main advantage of the Caputo fractional derivative (3) is a formal generalization of the integer derivative under Laplace transformation [1]. Considering all the initial conditions to be zero, the Laplace transformation of (3) becomes the more expected and conforming form,

$$ L\left(\frac{d^\alpha f(x)}{dt^\alpha}\right) = s^\alpha L\big(f(t)\big). \quad (4) $$

Thus, the fractional integral operator of order α can be represented by the transfer function $F(s) = 1/s^\alpha$

in the frequency domain. The standard definition of the fractional differ-integral does not allow direct implementation of the fractional operators in the time-domain. An efficient method to solve this problem is to approximate fractional operators by using standard integer order operators [22]. The approximation of $1/s^{0.95}$ with error about an 1 dB is given in [21] by

$$\frac{1}{s^{0.95}} \approx \frac{1.2831s^2 + 18.6004s + 2.0833}{s^3 + 18.4738s^2 + 2.6754s + 0.003}, \tag{5}$$

and the approximations of $1/s^{0.8}$ and $1/s^{0.9}$ are given in [14], as shown in the following:

$$\frac{1}{s^{0.9}} \approx \frac{1.766s^2 + 38.27s + 4.914}{s^3 + 36.15s^2 + 7.789s + 0.01}, \tag{6}$$

$$\frac{1}{s^{0.8}} \approx \frac{5.235s^3 + 1453s^2 + 5306s + 254.9}{s^4 + 658.1s^3 + 5700s^2 + 658.2s + 1}. \tag{7}$$

In the following simulations, we will use these approximations as a frequency domain approximation method.

However, the frequency domain approximation methods are not always reliable, especially in detecting chaotic behavior in nonlinear systems [30–32]. Although the time domain methods are complicated and require long simulation time, the time domain methods are more accurate and more reliable than the frequency based approximation [30]. Therefore, it is necessary to use the time domain method to get the solution of fractional differential equation. There are several time domain methods such as the predictor–corrector based methods [33, 34], FIR form method [17], and so on. In this paper, the Grunwald–Letnikov method [17] is used. The formula is derived from (1). The details for Grunwald–Letnikov method will be given in next section.

To analyze the character of the equilibria of fractional order system, the preliminary knowledge is about the stability of the linear time invariant fractional-order systems. A fractional-order linear time invariant system can be represented in the following state-space form:

$$\begin{cases} D^\alpha x = Ax + Bu, \\ y = Cx, \end{cases} \tag{8}$$

where $x \in R^n$, $u \in R^r$, $y \in R^p$ are states and input and output vectors of the system and $A \in R^{n\times n}$, $B \in R^{n\times r}$, $C \in R^{p\times n}$, and α is the fractional commensurate order. It has been shown that the autonomous system $D^\alpha x = Ax$, $x(0) = x_0$ is asymptotically stable if the following conditions is satisfied [36]:

$$\left|\arg\left(\mathrm{eig}(A)\right)\right| > \alpha\pi/2, \tag{9}$$

where $0 < \alpha < 1$ and $\mathrm{eig}(A)$ represents the eigenvalues of matrix A.

We consider the following commensurate fractional order system:

$$D^\alpha x = f(x), \tag{10}$$

where $0 < \alpha < 1$ and $x \in R^n$. The equilibrium points of system (10) are calculated by the following equation:

$$f(x) = 0. \tag{11}$$

These equilibria are locally asymptotically stable if all the eigenvalues of the Jacobian matrix $J = \partial f/\partial x$ evaluated at these equilibria satisfy [36]:

$$\left|\arg\left(\mathrm{eig}(J)\right)\right| > \alpha\pi/2. \tag{12}$$

3 A fractional Qi system

Qi et al. [24] has proposed a 3-D quadratic autonomous system, which can generate a four-wing chaotic attractor with very complicated topological structures over a large range of parameters. This nonlinear system is described by a system of first-order ordinary differential equations:

$$\begin{aligned} \dot{x} &= a(y - x) + eyz, \\ \dot{y} &= cx + dy - xy, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{13}$$

where $a, b, c, e \in R^+$ and $c \in R$ are constant parameters of the system. To study the effect of fractional derivatives on the dynamics of the Qi system, the integer derivative is replaced by a fractional derivative, as follows:

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= a(y - x) + eyz, \\ \frac{d^\alpha y}{dt^\alpha} &= cx + dy - xz, \\ \frac{d^\alpha z}{dt^\alpha} &= -bz + xy, \end{aligned} \tag{14}$$

where α is the fractional order. When $\alpha = 1$, (14) is equivalent to the classical integer-order Qi equation (13).

To compute the states of the fractional system using frequency domain approximation method, the fractional derivatives are transformed into frequency domain, then the fractional operators are approximated by using standard integer order operators [22]; finally, the standard integer order operators are transformed back to time domain. For example, when $\alpha = 0.95$, the system (14) becomes

$$\begin{aligned}
\frac{d^{0.95}x}{dt^{0.95}} &= a(y-x) + eyz, \\
\frac{d^{0.95}y}{dt^{0.95}} &= cx + dy - xz, \\
\frac{d^{0.95}z}{dt^{0.95}} &= -bz + xy.
\end{aligned} \tag{15}$$

The approximation of $1/s^{0.95}$ is (5). Finally, we can get the approximation system in the form of first-order ordinary differential equations:

$$\begin{aligned}
\dot{x} &= x_2, \\
\dot{x_2} &= x_3, \\
\dot{x_3} &= -18.4738x_3 - 2.6574x_2 - 0.003x \\
&\quad + 1.2831\big[a(y_3 - x_3) \\
&\quad + e(y_3 z + 2y_2 z_2 + y z_3)\big] + 18.6004\big[a(y_2 - x_2) \\
&\quad + e(y_2 z + y z_2)\big] + 2.0833\big[a(y-x) + eyz\big], \\
\dot{y} &= y_2, \\
\dot{y_2} &= y_3, \\
\dot{y_3} &= -18.4738y_3 - 2.6574y_2 - 0.003y \\
&\quad + 1.2831[cx_3 + dy_3 - x_3 z - 2x_2 z_2 - x z_3] \\
&\quad + 18.6004(cx_2 + dy_2 - x_2 z - x z_2) \\
&\quad + 2.0833(cx + dy - xz), \\
\dot{z} &= z_2, \\
\dot{z_2} &= z_3, \\
\dot{z_3} &= -18.4738z_3 - 2.6574z_2 - 0.003z \\
&\quad + 1.2831(-bz_3 + x_3 y + 2x_2 y_2 + x y_3) \\
&\quad + 18.6004(-bz_2 + x_2 y + x y_2) \\
&\quad + 2.0833(-bz + xy).
\end{aligned} \tag{16}$$

For the time domain approximation method, we use the Grunwald–Letnikov method [17] to get the numerical solution of the fractional order Qi system (14). The iterative formula is

$$\begin{aligned}
x(k) &= \big(a\big(y(k-1) - x(k-1)\big) \\
&\qquad + ey(k-1)z(k-1)\big)h^\alpha - \sum_{j=1}^{k} c_j^{(\alpha)} x(k-j), \\
y(k) &= \big(cx(k) + dy(k-1) - x(k)z(k-1)\big)h^\alpha \\
&\qquad - \sum_{j=1}^{k} c_j^{(\alpha)} y(k-j), \\
z(k) &= \big(-bz(k-1) + x(k)y(k)\big)h^\alpha \\
&\qquad - \sum_{j=1}^{k} c_j^{(\alpha)} z(k-j),
\end{aligned} \tag{17}$$

where T_{sim} is the simulation time, h is the time step, $N = [T_{\text{sim}}/h]$, $k = 1, 2, 3, \ldots, N$, and $(x(0), y(0), z(0))$ is the initial conditions. The binomial coefficients $c_j^{(\alpha)}$ are calculated according to

$$c_0^{(\alpha)} = 1, \qquad c_j^{(\alpha)} = \left(1 - \frac{1+\alpha}{j}\right) c_{j-1}^{(\alpha)}. \tag{18}$$

3.1 Fractional order $\alpha = 0.95$

For chaotic systems, it is proved that wings (or scrolls) are generated only around the saddle points of index 2. Moreover, saddle points of index 1 are responsible only for connecting wings [26–28]. A necessary stability condition for the fractional order system (14) to remain chaotic is keeping at least one eigenvalue λ of the Jacobian matrix of (14), which is evaluated at the equilibrium, in the unstable region [17, 32]. This means

$$\alpha > \frac{2}{\pi} \arctan\left(\frac{|\text{Im}(\lambda)|}{\text{Re}(\lambda)}\right), \tag{19}$$

which can be derived from (12). The number of saddle points and eigenvalues for one-scroll, double-scroll, and multiscroll attractors was described in [30].

When the fractional order $\alpha = 0.95$ and the system parameters $a = 14$, $b = 43$, $c = -4$, $d = 16$, $e = 4$, the equilibria of this fractional order Qi system (14) are

$$\begin{aligned} &S_0 = (0, 0, 0),\\ &S_1 = (-32.6840, 14.7844, -11.2375),\\ &S_2 = (32.6840, -14.7844, -11.2375),\\ &S_3 = (18.2299, 8.8159, 3.7375),\\ &S_4 = (-18.2299, -8.8159, 3.7375), \end{aligned} \tag{20}$$

which are same with the equilibria of the original Qi system. The Jacobian matrix of system (14), evaluated at (x^*, y^*, z^*), is

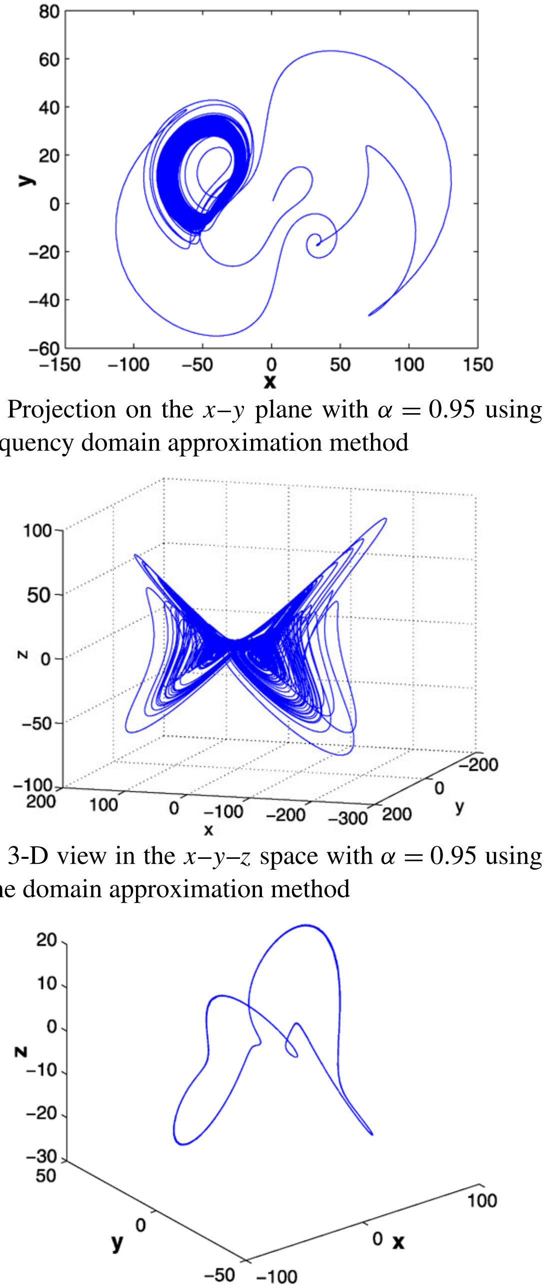
$$J = \begin{bmatrix} -14 & 14 + 4z^* & 4y^* \\ -4 - z^* & 16 & -x^* \\ y^* & x^* & x^* \end{bmatrix}. \tag{21}$$

For the given parameters, the eigenvalues corresponding to different equilibria are

$$\begin{aligned} S_0:&\quad \lambda_1 = -12, \lambda_2 = 14, \lambda_3 = -43,\\ S_{1,2}:&\quad \lambda_1 = -56.9522, \lambda_{2,3} = 7.9761 \pm j30.8679,\\ S_{3,4}:&\quad \lambda_1 = -49.9942, \lambda_{2,3} = 4.497 \pm j19.1020. \end{aligned}$$

According to (19), $\alpha > 0.8528$ is the necessary condition for the fractional order Qi system (14) when $a = 14$, $b = 43$, $c = -4$, $d = 16$ and $e = 4$. Therefore, it is possible that the fractional order Qi system (14) shows chaotic dynamics when $\alpha = 0.95$, $a = 14$, $b = 43$, $c = -4$, $d = 16$ and $e = 4$.

From the system equation (14), it is seen that the total order is 2.85 when $\alpha = 0.95$. If the frequency domain method is used, which means that (16) is used, the phase portrait is shown in Fig. 1(a) with parameters $a = 14$, $b = 43$, $c = -4$, $d = 16$, and $e = 4$. The frequency domain approximated system displays chaos with the largest Lyapunov exponent 5.6. In this paper, we used the Wolf algorithm to calculate the Lyapunov exponents [23]. For the time domain approximation method, the time step $h = 0.0005s$ and run time $T_{\text{sim}} = 50s$ in this paper. If time domain approximation method is used, the 3-D phase diagram is shown in Fig. 1(b) which reveals 4-wing chaotic dynamics. However, the regular Qi system shows periodic orbits as can be seen from Fig. 1(c). Comparing the subdiagrams in Fig. 1, it is easy to find that the decreasing of the system order has great effect on the dynamics of the nonlinear systems and the dynamical behaviors using the time-domain approximation method are different from the result using frequency-domain approximation method. The fractional Qi system can exhibit chaos when the total order less than 3, although the regular one always shows periodic orbits in the same range of parameters. It means that

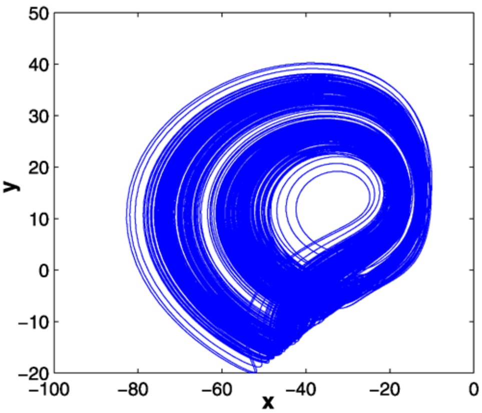


(a) Projection on the x–y plane with $\alpha = 0.95$ using frequency domain approximation method

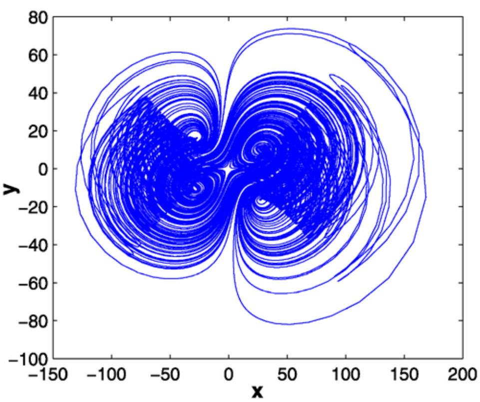
(b) 3-D view in the x–y–z space with $\alpha = 0.95$ using time domain approximation method

(c) 3-D view in the x–y–z space with $\alpha = 1$

Fig. 1 The phase diagram of the Qi system, with $a = 14$, $b = 43$, $c = -4$, $d = 16$, and $e = 4$



(a) Projection on the x–y plane with $\alpha = 0.95$



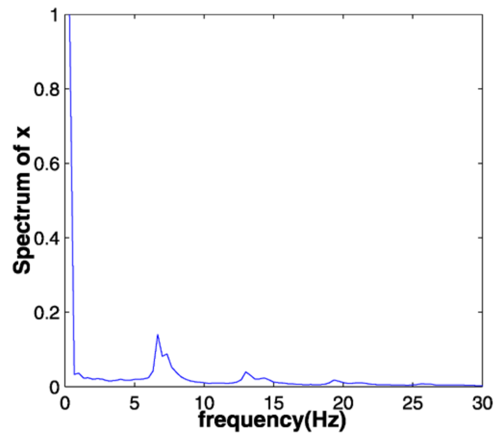
(b) Projection on the x–y plane with $\alpha = 1$

Fig. 2 The phase diagram of the Qi system, with $a = 14$, $b = 43$, $c = 0$, $d = 16$, and $e = 4$

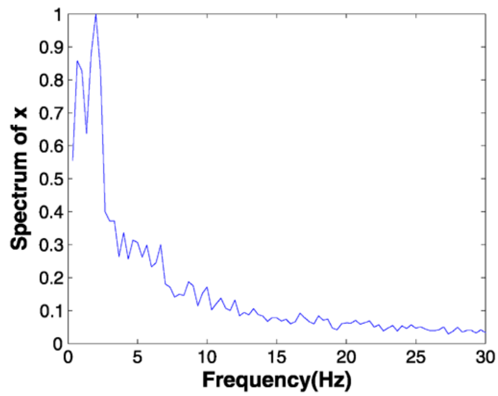
although the system total order decreases, the system dynamics maybe become more sensitive which is contrary to the result in [20].

Qi system generates a four-wing chaotic attractor with parameters $a = 14$, $b = 43$, $c = 0$, $d = 16$, and $e = 4$, whose phase portrait is shown in Fig. 2(b). Four wings attractor of the Qi system are decayed to one scroll using the same parameters when the system order α decrease to 0.95 using time domain frequency approximation method, which is shown in Fig. 2(a). However, the fractional order Qi system, using the time domain approximation method, shows four wings chaotic attractors similar to Fig. 2(b) which means the dynamical behaviors using the time-domain approximation method is different from the result using frequency-domain approximation method.

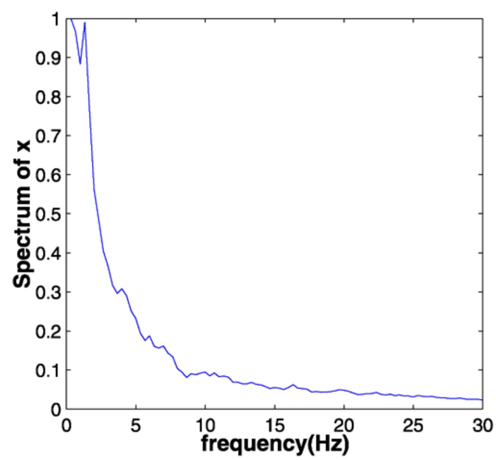
Now we analyze the effects on the bandwidth of the frequency spectrum. Figures 3(a), 3(b), and 3(c) are the frequency spectrums of the fractional Qi sys-



(a) The frequency spectrum when $\alpha = 0.95$ using frequency domain approximation method



(b) The frequency spectrum when $\alpha = 0.95$ using time domain approximation method



(c) The frequency spectrum when $\alpha = 1$

Fig. 3 The frequency spectrum of the Qi system, with $a = 14$, $b = 43$, $c = -1$, $d = 16$, and $e = 4$

tem using frequency domain approximation method, fractional Qi system using time domain approximation method, and the Qi system, respectively. The spectral averages are 3, and all the spectra are normalized.

As can be seen from Figs. 3(a) and 3(c), the bandwidth of frequency spectrums shrinks with the reduction of the system order α from 1 to 0.95 using the frequency domain approximation method. However, the bandwidth of frequency spectrums when fractional order $\alpha = 0.95$, using the time domain approximation method, are similar to the original Qi system.

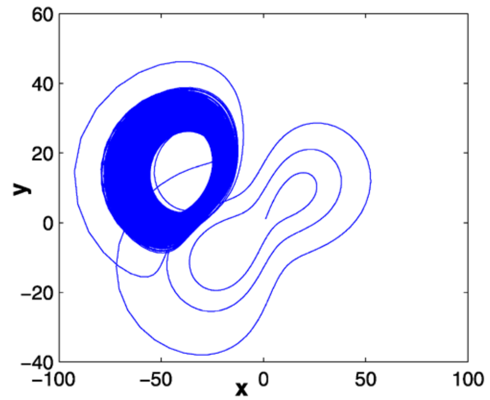
3.2 Fractional order $\alpha = 0.9$

The system total order is 2.7 according to (14) when $\alpha = 0.9$. When the "fractional order Qi system" exhibits chaos with $\alpha = 0.9$ using frequency domain approximation method, the phase diagrams are one scroll, which is similar to that of fractional order $\alpha = 0.95$. Figures 4(a) and 4(b) are phase diagrams of the "fractional order Qi system" with initial conditions $(1, 1, 1)$ and $(0.1, 1.1, 1)$ using frequency domain approximation method, respectively.

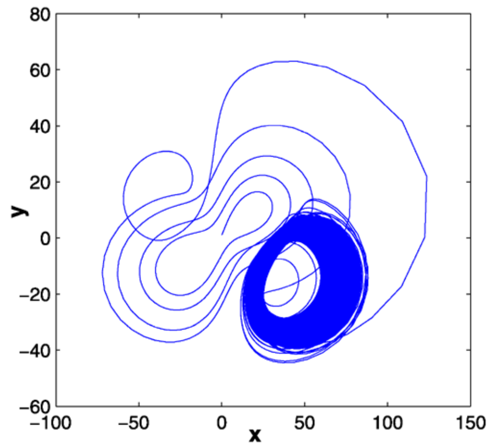
Figure 5(a) and 5(b) are the frequency spectrums of the fractional Qi system variable x and y using the frequency domain approximation method, respectively. As can be seen from Fig. 3(a), the frequency spectrums of "fractional order Qi system" with $\alpha = 0.9$ are similar to that with $\alpha = 0.95$. Figure 6 is the frequency spectrums of fractional order Qi system variable x using time domain approximation method. As can be seen from Figs. 6, 3(b), and 3(c), the change of fractional order has little effect on the bandwidth of frequency spectrum using the time domain approximation method.

3.3 Fractional order $\alpha = 0.8$

The total order of the Qi system is 2.4 according to (14) when $\alpha = 0.8$. With parameters $a = 14$, $b = 43$, $c = -4$, $d = 16$, and $e = 4$, and the frequency domain approximation method is used, Figs. 7(a)–7(c) show the observed projections on different phase planes. Figure 7(d) shows the system states in 3-D space. As can be seen from Fig. 7, the system displays 4-wing chaotic attractor when we look at the diagrams as a whole. However, there are several subattractors in each of the four wings when the attractors are magnified. If the top-left subattractor is magnified with x and y being limited to $[-35, -31]$ and $[13, 16.5]$, respectively,



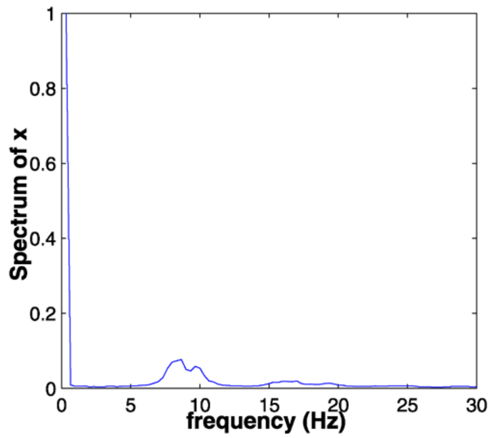
(a) Projection on the x–y plane, $c = -4$



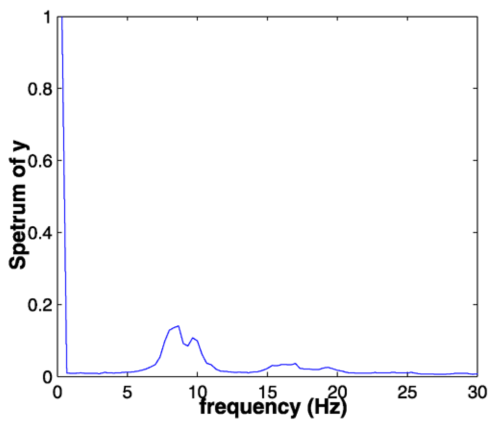
(b) Projection on the x–y plane, $c = -3$

Fig. 4 The phase diagrams of the "fractional order Qi system" using the frequency domain approximation method, with $a = 14$, $b = 43$, $d = 16$, and $e = 4$

there are 3 lower level subattractors as shown in Fig. 8. If the bottom-right attractor in Fig. 8 is magnified farther with x and y being limited to $[-33.3, -32.2]$ and $[14.2, 14.9]$, respectively, there are 2 lower level subattractors as shown in Fig. 9. As can be seen from Figs. 8 and 9, there are 4 subattractors in the top-left wing when the running time is 1100 s. If other wings are also magnified, it can be found that there are about 13 subattractors altogether in the fractional Qi system with $\alpha = 0.8$, $a = 14$, $b = 43$, $c = -4$, $d = 16$, and $e = 4$ when the running time is 1100 s. Seen from these phase diagrams, the "fractional order Qi system" exhibits a character of self-similar in the local, which differs from the integer Qi systems. This kind of diagram is similar to some fractal images (for example, Julia sets) [25]. However, most of the fractal images



(a) The frequency spectrum of x



(b) The frequency spectrum of y

Fig. 5 The frequency spectrum of the Qi system, with $\alpha = 0.9$, $a = 14$, $b = 43$, $c = -1$, $d = 16$, and $e = 4$

are created by two of the most important well-known methods which are iterated function systems and Julia sets, and both of them are discrete methods. The Qi system is a determinate continuous nonlinear system with fractional order, which is different from the fractal images that we have ever seen.

There is a logical question that whether the character of local self-similar is same with the global 4 scrolls. For chaotic systems, it is proved that wings (or scrolls) are generated only around the saddle points of index 2. Moreover, saddle points of index 1 are responsible only for connecting wings [26–28]. It is necessary to analyze the equilibria of the original Qi system and the "fractional order Qi system" using the frequency approximation method. For the original Qi system, there are five equilibria when $a = 14$, $b = 43$, $c = -4$, $d = 16$, and $e = 4$. They are (20). For the

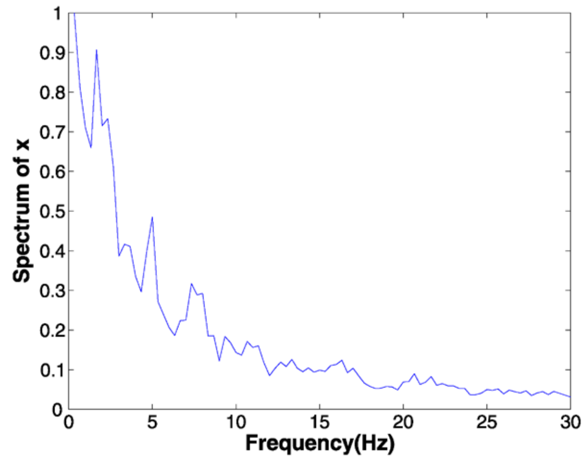


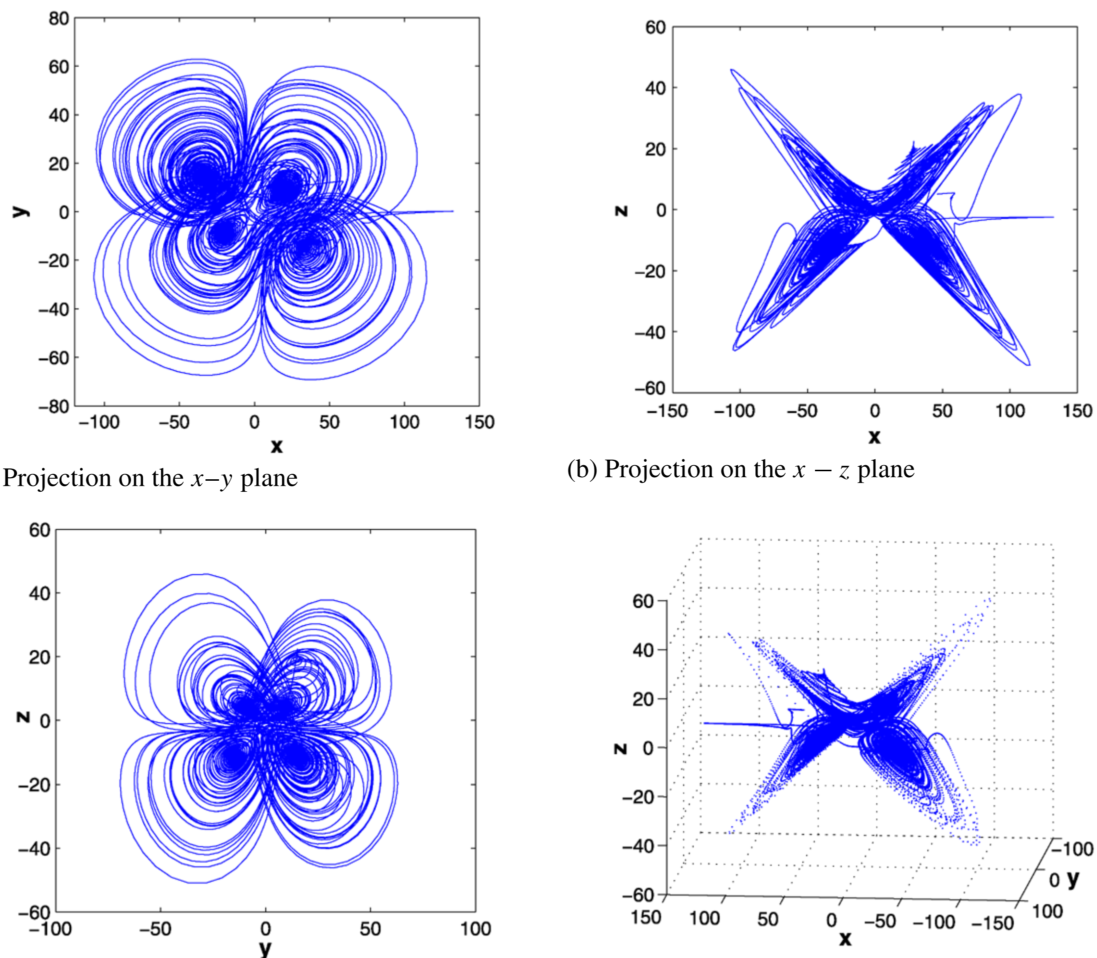
Fig. 6 The frequency spectrum of the Qi system using the time domain approximation method, with $\alpha = 0.9$, $a = 14$, $b = 43$, $c = -1$, $d = 16$, and $e = 4$

"fractional order Qi system" using frequency approximation method, the equilibria are

$$\begin{aligned}
\bar{S}_0 &= (0, 0, 0),\\
\bar{S}_1 &= (32.6814, -14.7871, -11.2376),\\
\bar{S}_2 &= (-32.6814, 14.7871, -11.2376),\\
\bar{S}_3 &= (-18.2287, -8.8176, 3.7376),\\
\bar{S}_4 &= (18.2287, 8.8176, 3.7376).
\end{aligned} \tag{22}$$

It is easy to find that S_0 is a saddle point of index 1; and S_1, S_2, S_3, and S_4 are the saddle points of index 2 using the Jacobian matrix of the original Qi system at its equilibria. We can also get that $\bar{S}_0$ is a saddle point of index 1; and $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$, and $\bar{S}_4$ are the saddle points of index 2 using the Jacobian matrix of the "fractional order Qi system," which uses frequency approximated method, at its equilibria. According to the analysis, the number of these equilibria is the same and the locations of these equilibria of the "fractional order Qi system" are near the equilibria of the original integral chaotic system. For the original integral chaotic system, each of four saddle points is connected with one wing which is similar to the global four wings of the "fractional order Qi system." Therefore, the local self-similar is not same as the global 4-scroll, but they should be caused by the collective dynamics of these four equilibria "fractional order Qi system."

Figures 10(a) and 10(b) are the frequency spectrums of the fractional Qi system states x and y using



(a) Projection on the x–y plane

(b) Projection on the $x - z$ plane

(c) Projection on the y–z plane

(d) 3-D view by '.' in the x–y–z space

Fig. 7 Fractional Qi system chaotic attractor, with $\alpha = 0.8$, $a = 14$, $b = 43$, $c = -4$, $d = 16$, and $e = 4$

the frequency domain approximation method, respectively. The spectral averages are 3, and all the spectra are normalized. As can be seen from Figs. 2, 4, and 7, the decreasing of the system order has a great effect on the orbits of the Qi system using frequency domain approximation method. It can make the system exhibit chaos although the regular Qi system is periodic and becomes one scroll from 4-wing attractors. The "fractional order Qi system", using the frequency domain approximation method, displays self- similar character which is different from the integer Qi system. According to Figs. 3, 5, and 10, the frequency spectrums show that there are shrinking tendencies of the bandwidth with the falling of the system states order.

However, the chaotic dynamics is not the real dynamics of the fractional order Qi system since $\alpha > 0.8528$ is the necessary condition for the fractional order Qi system (14) when $a = 14$, $b = 43$, $c = -4$, $d = 16$, and $e = 4$. Using time domain approximation method, the orbits of the fractional order Qi system is convergent as shown in the phase diagram Fig. 11. This means the time domain approximation method is more accurate and reliable than the frequency domain approximation method.

3.4 Bifurcation with respect to the fractional order α

It is easily to find that the decreasing of the system order has great effect on the dynamics of the nonlinear systems. Using the bifurcation diagram with respect to the fractional order, it would be easy to find the fractional order effects on the dynamical behaviors of

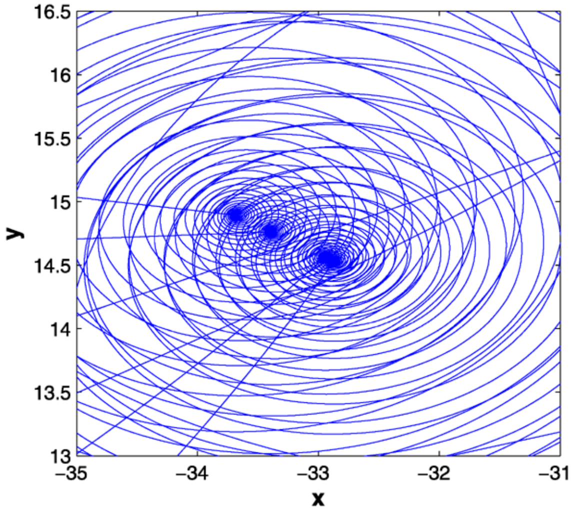


Fig. 8 The magnified phase diagram project on x–y plane with $x \in [-35, -31]$ and $y \in [13, 16.5]$

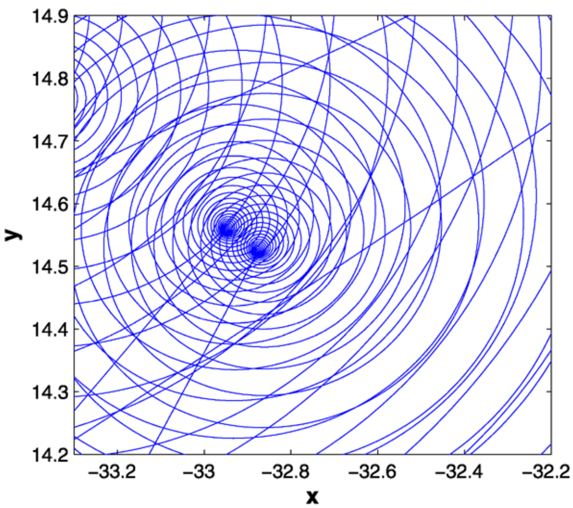
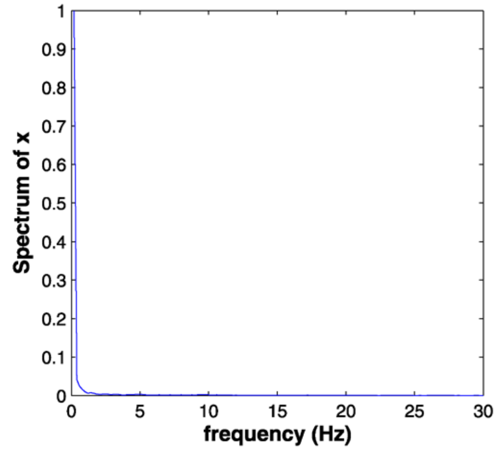
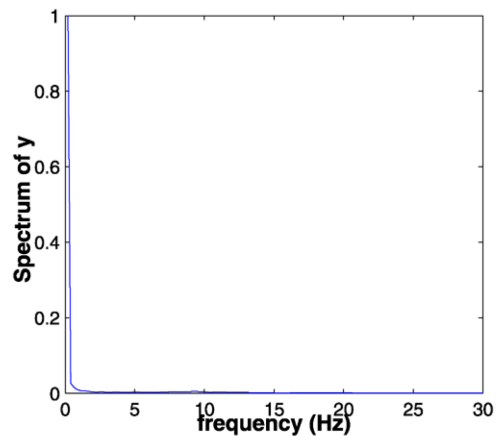


Fig. 9 The magnified phase diagram project on x–y plane with $x \in [-33.3, -32.2]$ and $y \in [14.2, 14.9]$

fractional order systems. As the time domain approximation method is more accurate and reliable, only the time domain approximation method is used to obtain the bifurcation diagram. When the system parameters $a = 14$, $b = 43$, $c = -4$, $d = 16$, and $e = 4$, Fig. 12 is the bifurcation diagram of the fractional order Qi system state variable x with respect to fractional order α. As can be seen from Fig. 12, the fractional order Qi system attractors start from sink, then some simple periodic orbits and chaotic attractors finally escape from



(a) The frequency spectrum of x



(b) The frequency spectrum of y

Fig. 10 The frequency spectrum of the fractional Qi system, with $\alpha = 0.8$, $a = 14$, $b = 43$, $c = -4$, $d = 16$, and $e = 4$

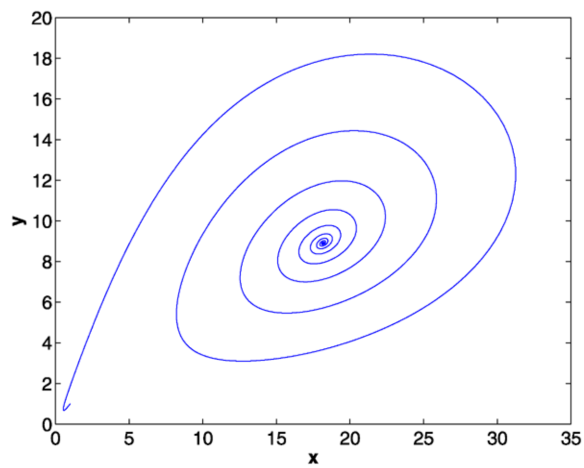


Fig. 11 Phase diagram of the fractional order Qi system using time domain approximation method, and with $\alpha = 0.8$, $a = 14$, $b = 43$, $c = -4$, $d = 16$, and $e = 4$

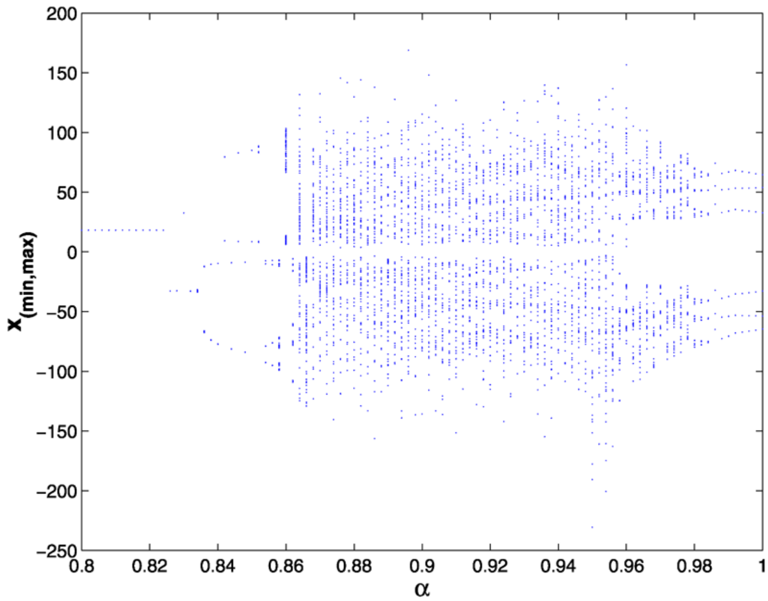


Fig. 12 The bifurcation diagram of the fractional order Qi system with respect to fractional order α, and with $a = 14$, $b = 43$, $c = -4$, $d = 16$, and $e = 4$

chaotic attractor to periodic orbits with the increasing of α in the interval [0.8, 1].

4 Conclusion

In this work, we analyzed the fractional Qi system. It is found that the fractional Qi system could exhibit chaos when the total order less than 3, although the regular one always shows periodic orbits in the same range of parameters. This result is different from that the fractional order differential equations are at least as stable as their integer order counterparts [35]. An interesting finding is that the phase diagrams show local self- similar in the fractional 4-wing system using frequency domain approximation method even though the number and the characteristics of equilibria are not changed. The fractional Qi system exhibits rich and interesting dynamics which is very different from the result of the standard integer-order dynamics. According to analysis of the bifurcation, the fractional order Qi system attractors start from sink, then some simple periodic orbits and chaotic attractors finally escape from chaotic attractor to periodic orbits with the increasing of α in the interval [0.8, 1]. The simulation results revealed that the time domain approximation method is more accurate and reliable than the frequency domain approximation method. What is the fundamental reason of the different dynamics? It may come from the fractional derivative nonlocal character which is different from the integer dynamics. It is important to systematically develop some methods for the analysis of the fractional system. The rich dynamics caused by fractional order are also important topics for future studies.

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